STUDIES ON TAILORING OF THERMOMECHANICAL PROPERTIES OF COMPOSITES

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Layered composite materials consisting of thin orthotropic layers offer for a designer many possibilities to tailor the structure: the behaviour and properties of the structure can be influenced not only by varying the geometry and thicknesses of the structure but also by varying the lay-up of the laminate. As new orthotropic materials having high specific strength and stiffness are used in structures, the tailoring is essential to utilize all the benefits of these materials. In this thesis tailoring and optimization of thermomechanical properties of layered composite structures are considered.

The tailoring problem is formulated and solved as a constrained nonlinear optimization problem. Different types of global thermomechanical properties, such as stiffnesses, coefficients of thermal expansion and natural frequencies and buckling loads of composite plates, as well as layer-wise properties, such as stresses and strains in a certain lamina, are considered. Also, coupled thermal-structural problems are studied.

When lay-up parameters, i.e. number of layers, and their orientations and thicknesses, are employed as design variables, global as well as layer-wise properties of the laminate can be considered. As relations between thermomechanical properties and lay-up parameters are highly nonlinear, optimization may suffer from various local optima. However, in tailoring the global minima or maxima are not the points of interest but rather the points of design space, where appropriate values for considered properties are achieved.

In the thesis optimization of global thermomechanical properties is presented also by applying so-called lamination parameters as design variables. The lamination parameters are defined as integrals of the functions, which consist of sines and cosines of the lay-up angles of different layers multiplied by the powers of the thickness co-ordinate $z$, through the thickness of the laminate. Thus, information of the lay-up of the laminate can be compressed into these parameters and only twelve lamination parameters are needed to describe the behaviour of a common laminate. The use of these parameters as design variables is advantageous, because the number of parameters needed is small and often formulating a convex optimization problem is possible. After finding optimal lamination parameters, a procedure is needed to generate a lay-up corresponding to these parameters. Explicit equations are derived for generating lay-ups having optimal bending lamination parameters. For creating a laminate having both optimal in-plane and bending lamination parameters, a new optimization problem searching laminates having lamination parameters as close as possible to the optimal ones is formulated. In that problem, also layer-wise properties and restrictions of manufacturing are taken into account. A genetic algorithm search is employed for solving that later problem as the value of the objective function can be computed efficiently. Also, often the thicknesses and orientations of different layers can have only discrete values, which can be handled easily in the GA search, where all design variables are discrete in character.

Keywords: optimization, layered laminates, orthotropic materials, lamination parameters
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Oulu, 8 November 1999

Maija Autio
List of original papers

This dissertation comprises an introductory report and the five following papers:


IV Autio M (1999) Optimization of coupled thermal-structural problems of laminated plates with lamination parameters. (Accepted for publication in Structural Optimization)

V Autio M (1999) Determining the real lay-up of a laminate corresponding to optimal lamination parameters by genetic search. (Accepted for publication in Structural Optimization)

Paper I has two co-authors. The formulation, programming and testing of general tailoring problems were carried by the author, while the implementation of the stability analysis of laminated plates was made by Tech.Lic. Laitinen. The manuscript of paper I was prepared jointly by Prof. Pramila, Tech.Lic. Laitinen and the author.
**Original features**

The following features of this thesis are believed to be original:

1. Consideration of different types of thermomechanical properties of laminates and laminated structures (stiffnesses and CTEs of laminates, properties of laminated structures, strength of laminates) in same tailoring problem

2. Tailoring and optimization of coupled thermal-structural problems of laminated structures.

3. Formulation of coupled thermal-structural problem as a function of lamination parameters.

4. Explicit equations for determining the lay-up corresponding to bending lamination parameters.

5. Formulation and solution of the optimization problem of laminated structures in two phases: first global properties are considered in lamination parameter space and then a lay-up corresponding to optimal lamination parameters is sought also taking layer-wise properties into account.
Symbols and abbreviations used in this thesis and enclosed papers are given in the following. Italic letters indicate scalars. Matrices are denoted by bold upper case letters and vectors by lower case bold letters. The numbers of the sections or the appendices given in parentheses refer to the first usage of symbols.

## Latin letters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>in-plane stiffness matrix of a laminate (App. I)</td>
</tr>
<tr>
<td>$A_1, A_2$</td>
<td>material invariants (App. IV)</td>
</tr>
<tr>
<td>$a, b$</td>
<td>side lengths of a plate (App. I)</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>coupling stiffness matrix of a laminate (App. I)</td>
</tr>
<tr>
<td>$B$</td>
<td>derivatives of element shape functions (App. III)</td>
</tr>
<tr>
<td>$B$</td>
<td>approximation of the Hessian matrix of the Lagrangian function (App. I)</td>
</tr>
<tr>
<td>$b$</td>
<td>boundary point of feasible domain (App. IV)</td>
</tr>
<tr>
<td>$C_{mn}, W_{mn}$</td>
<td>amplitudes in Ritz method (App. I, App. II)</td>
</tr>
<tr>
<td>$c$</td>
<td>specific heat (Sec. 4.2)</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>bending stiffness matrix of a laminate (App. I)</td>
</tr>
<tr>
<td>$D_{th}$</td>
<td>conductivity matrix (App. III)</td>
</tr>
<tr>
<td>$d$</td>
<td>search direction vector in SQP (App. I)</td>
</tr>
<tr>
<td>$F$</td>
<td>general thermomechanical property (Chapter 2)</td>
</tr>
<tr>
<td>$F$</td>
<td>target value of thermomechanical property (Chapter 2)</td>
</tr>
<tr>
<td>$F$</td>
<td>lower bound of thermomechanical property (Chapter 2)</td>
</tr>
<tr>
<td>$F$</td>
<td>upper bound of thermomechanical property (Chapter 2)</td>
</tr>
<tr>
<td>$F_{th}$</td>
<td>thermal loading vector (App. III)</td>
</tr>
<tr>
<td>$f_{	ext{lay}_{ij}}$</td>
<td>general layer property (Sec. 3.3)</td>
</tr>
<tr>
<td>$f_{	ext{lay}_{ij}}$</td>
<td>lower bound of a layer property (Sec. 3.3)</td>
</tr>
</tbody>
</table>
\( f_{lay,j} \) upper bound of a layer property (Sec. 3.3)

\( f(x) \) objective function (App. I), fitness function in GA (App. V)

\( f_1 \) lowest natural frequency (App. IV)

\( G \) conductivity matrix (App. III)

\( G \) matrix of first derivatives of element shape functions (App. IV)

\( G^B \) coupling conductivity matrix (App. III)

\( g(x) \) constraint function (App. I)

\( h \) total thickness of the laminate (Sec. 4.2)

\( \bar{h} \) total relative thickness of the laminate (App. V)

\( h_i \) thickness of a lamina layer (Chapter 1)

\( h_{ud} \) thickness of sequential layers having same orientation (App. V)

\( h_{lim} \) limit thickness of sequential layers having same orientation (App. V)

\( K \) stiffness matrix (App. III)

\( K^B \) bending stiffness matrix (App. IV)

\( K^G \) geometric stiffness matrix (App. IV)

\( K^S \) shear stiffness matrix (App. IV)

\( K_1, K_2 \) material invariants (Sec. 4.2)

\( K_{ij} \) shear correction factors (App. IV)

\( k \) coefficient of thermal conduction (Sec. 4.2)

\( M \) mass matrix (App. IV)

\( M \) resultant moments (App. I)

\( M^T, M^{th} \) thermal resultant moments (App. I, App. III)

\( m, n \) number of terms in displacement approximation function (Chapter 4)

\( N \) resultant forces (App. I)

\( N^T, N_{th} \) thermal resultant forces (App. I, App. III)

\( N_x, N_y, N_{xy} \) plane stresses (App. IV)

\( n \) number of layers (Chapter 1)

\( p_i \) weighting coefficient (Chapter 2)

\( p_{thick} \) weighting coefficient of total relative thickness (App. V)

\( p_{thick} \) weighting coefficient of total thickness (App. V)

\( p_{e1} \) penalty factor in GA (App. V)

\( Q \) matrix of material coefficients in \( xyz \)-system (App. I)

\( Q_s \) matrix of shear material coefficients in \( xyz \)-system (App. IV)

\( Q \) material coefficients in material coordinate system (App. I)

\( r \) penalty parameter vector in SQP (App. I)

\( T \) temperature distribution (Sec. 4.2)

\( T \) kinetic energy (App. I)

\( T^B \) prescribed boundary temperatures (App. III)

\( T_g \) glass transition temperature (Chapter 6)

\( T(x) \) positive functional of constraints in flexible tolerance method (App. I)
\[ TWC \] value of Tsai-Wu failure criterion function (App. V)
\[ TWC_{lim} \] limit value of Tsai-Wu failure criterion function (App. V)
\[ TWC_{max} \] maximum value of Tsai-Wu failure criterion function (App. V)
\[ U \] strain energy (App. III)
\[ U_1 \ldots U_7 \] material invariants (Sec. 3.2)
\[ u \] displacement vector (App. III)
\[ u \] Lagrangian multiplier vector (App. I)
\[ u_n \] nodal displacement (App. III)
\[ V \] potential energy of in-plane loads (App. I)
\[ w \] transverse displacement (Chapter 4)
\[ W_{1,2} \] normalized thicknesses of in-plane case (App. IV)
\[ \mathbf{x} \] design variable vector (Chapter 2)
\[ x, y, z \] Cartesian coordinates
\[ z^* \] normalized through the thickness coordinate (Sec. 3.2)
\[ z_1 \] normalized thickness coordinate of two layered laminate (App. IV)

**Greek letters**

\[ \alpha \] coefficient of thermal expansion vector in material direction (App. I)
\[ \hat{\alpha} \] coefficient of thermal expansion vector in \( xyz \)-coordinates (App. III)
\[ \alpha \] coefficient in linear combination (App. IV)
\[ \beta \] step length in SQP (App. I)
\[ \Delta T \] temperature difference (App. I)
\[ \delta \] angle between two consecutive layers (App. IV)
\[ \varepsilon \] strain vector (App. I)
\[ \varepsilon^0 \] mid-plane strain vector (App. I)
\[ \varepsilon^{th} \] thermal strain vector (App. III)
\[ \Theta \] orientation angle of entire laminate (App. IV)
\[ \theta \] orientation of a lamina layer (Chapter 1)
\[ \kappa \] curvature vector (App. I)
\[ \lambda \] vector of approximations of Lagrangian multipliers in SQP (App. I)
\[ \lambda \] buckling factor, eigenvalue (App. IV)
\[ \xi_1, \xi_2, \xi_3, \xi_4 \] lamination parameters (Sec. 3.2)
\[ \xi_{m1 \ldots m7} \] parameters containing material invariants (Sec. 3.2)
\[ \xi_1, \hat{\xi}_2, \hat{\xi}_3 \] rotated lamination parameters (App. IV)
\[ \Pi \] potential energy (App. I)
\[ \rho \] mass per unit area (App. I), density (Sec. 4.2)
\(\rho r\) rate of internal energy generation (Sec. 4.2)
\(\sigma\) stress vector (App. I)
\(\phi\) buckling mode, eigenmode (App. IV)
\(\Phi\) value of flexible tolerance criterion in FT (App. I)
\(\phi\) orientation of first layer (App. IV)
\(\phi(x)\) merit function of SQP (App. I)
\(\Psi(x,A)\) augmented Lagrangian function in SQP (App. I)
\(\Omega\) area of an element (App. III)

### Superscripts

- \(A, B, D\) in-plane, coupling, bending of a laminate
- \(T, th\) thermal
- \(*\) optimal value, normalized coordinates
- \(-\) in \(xyz\)-coordinate system

### Subscripts

- \(i\) \(i\)th layer
- \(e\) element level
- \(k\) \(k\)th iteration
- \(x, y, z\) in \(xyz\)-coordinate system
- \(1, 2\) in material coordinate system of a lamina

### Abbreviations

- CLT classical lamination theory
- CPT classical plate theory
- CTC coefficient of thermal conduction
- CTE coefficient of thermal expansion
- FEM finite element method
- FSDPT first-order shear-deformation plate theory
- FT flexible tolerance method
- GA genetic algorithm
- SQP successive quadratic programming
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1. Introduction

Composites are materials where two or more materials are joined together at a macroscopic level. For example, plywood is a composite consisting of thin birch veneers glued together and concrete is a composite mixture of cement, sand, stone, and water. In this thesis polymer composites consisting of a polymer matrix and fibrous reinforcements are considered. In a sense, composite materials can be considered as structures consisting of fibres and matrix materials.

Polymer composites are usually light materials having the high stiffness and strength to weight ratio. Thus, they are applied widely in aircraft, space and vehicle industry. They also have good chemical resistance, and are used in chemical and process engineering in pipes, tanks etc. As composites have a high capability of energy absorption, they are also applied in different kinds of safety equipment such as helmets and waistcoats. On the other hand, as composites are heterogeneous and often also orthotropic, the analysis and design of composite structures is more complex than structures made of traditional isotropic materials, like steel. Moreover, the manufacturing of composite structures is more complicated, and thus, in general, the use of composite materials is more expensive than that of traditional materials.

The layered composites considered in this thesis are made of thin orthotropic plies, which consist of stiff fibres embedded in softer plastic matrix material. Thus, the properties of a lamina are highly directional and anisotropic, and the properties of structures made of these laminae can be modified to fulfill specific needs by varying the orientations and thicknesses of the plies, for example. As the number of possible variables affecting the behaviour of the structure is large and as relations between design variables and thermomechanical properties of the structure are complicated, the traditional trial-and-error method does not give reasonable results, and a systematic procedure is needed for creating lay-ups providing prescribed or optimal thermomechanical properties of the structure.

Designers do not always take advantage of all the possibilities of tailoring the lay-up, and structures are often oversized. In this case, the saving in weight which would be possible by using materials with high specific stiffness and strength may be reduced. Also, the anisotropic and heterogeneous nature of composite materials places extra demands on design. For instance, composite parts warp easily after production and their dimensional stability may be low, which causes difficulties in the assembly phase, see Gutowski (1997). By tailoring, the dimensional stability in the critical direction may be increased,
and manufacturing costs thus reduced.

By tailoring layered composites, structures having peculiar behaviour, which cannot be achieved with isotropic materials, can be generated. For example, Pramila (1990) has derived analytical expressions for designing symmetric and balanced $[+\theta/-\theta]_s$ laminates having a zero coefficient of thermal expansion (CTE) in one direction. He also derived angles giving extrema for CTEs and showed that those angles are independent of CTEs of the laminae used. Ishikawa et al. (1989) applied so-called lamination parameters to create laminates with almost null CTEs. As they dealt with symmetric and balanced laminates, only two lamination parameters are involved in the tailoring problem, and the problem can be solved graphically. By tailoring a topology of a microstructure of a material, an even wider variation of thermomechanical properties can be achieved. For example, Sigmund (1994) applied the homogenization method and inverse solution technique for creating arbitrary materials, including materials with Poisson’s ratio -1.0.

Various researchers have studied the optimization of stiffness (or stiffness related) and strength properties of laminated structures. One of the earliest papers published in this area is the work of Schmit & Farshi (1973). They investigated the minimization of the weight of laminates with respect to thicknesses of layers having prescribed fibre orientations and subject to in-plane stiffness and strength constraints. Later (Schmit & Farshi 1977) buckling constraints of a simply supported rectangular plate were also included in the problem. Mesquita & Kamat (1987a,b) considered the optimization of stiffened laminated plates subject to frequency constraints and with respect to two sets of design variables: number of plies in a predefined orientation, i.e. these variables obtain discrete values, and stiffener areas, i.e. these are continuous design variables. Thus their problem belongs to the class of nonlinear mixed integer programming, and Dakin’s algorithm was applied in the solution. Miki (1982, 1985) introduced lamination parameters as design variables in the optimization problems dealing with in-plane and bending stiffnesses of laminated plates. He studied symmetric and balanced laminates, and thus the number of design variables was two and a graphical solution of the problems was possible. Lamination parameters were also employed by Fukunaga & Vanderplaats (1991), Grenestedt (1990) and Fukunaga et al. (1994), for example, in various problems considering the optimization of global properties of laminated structures. Zhang & Evans (1992) developed a design program for the tailoring of stiffness properties of laminates, where they employed the orientations and thicknesses of the layers as design variables. Thanedar & Chamis (1995) introduced a procedure for the tailoring of layered composites subject to probabilistic constraints. In their work the probabilistic nature of the strength values of unidirectional layers was taken into account at a micromechanical level before the tailoring, and thus evaluating the probability of failure in the iteration process was not necessary.

As the number of types of possible design variables is large, new heuristic optimization algorithms are widely used in designing of composite structures. Fang & Springer (1993) applied the Monte Carlo procedure for optimizing the weight and strength of composite laminates with respect to a laminate configuration, the design variables being the number of ply groups, the ply orientation and the material in each ply group, and the thickness and material of core layers. Sadagopan & Pitchumani (1998) employed simulated annealing for the tailoring of composite structures. The decision variables considered in their work are the reinforcement morphology, the material of the matrix and reinforcements, the ar-
architecture of a reinforcement arrangement and the volume fraction of reinforcements, i.e. most variables have a discrete character. Le Riche & Gaudin (1998) applied evolutionary optimization for designing dimensionally stable composites under thermal, hygroscopic and mechanical loads. They showed that \[ ([\pm \theta_1]_{n_1/2} [\pm \theta_2]_{n_2/2}] \] stacking sequences are sufficient for in-plane design of symmetric and balanced laminates, and divided the problem into three dependent subproblems: the minimization of the number of layers in the laminate, the determination of fibre orientations, and the stacking arrangement of the plies, which were solved by employing enumerative and evolutionary algorithms.

This thesis treats the tailoring and optimization of thermomechanical properties of layered composites reinforced by unidirectional continuous fibres. The structure of a laminate and notation used are shown in Fig. 1. Typical reinforcement fibres considered are continuous glass or carbon fibres and matrix materials used are plastics, such as polyester, vinylester or epoxy. The properties of an orthotropic lamina layer are assumed to be known a priori. Various thermomechanical properties are tailored with respect to lay-up parameters or lamination parameters. In this thesis the term lay-up parameters means the orientations and thicknesses of layers, and the number of layers in a laminate. These can be applied directly as design variables, but then the optimization often suffers from various local minima. More attention has been paid to the use of lamination parameters as design variables, as with these certain advantages can be obtained. However, using lamination parameters is not as straightforward as using lay-up parameters directly.

The outline of this thesis is the following: in Chapter 2 the general formulation of the tailoring problem is considered and in Chapter 3 different design variables are discussed. In Chapter 4 different thermomechanical properties considered in this work are presented with an emphasis on the thermal and coupled thermal-structural problems. The algorithms applied in the solution of optimization problems are briefly reviewed in Chapter 5, and finally Chapters 6 and 7 contain discussion and conclusions.

Fig. 1. Structure of a layered laminate.
2. Formulation of the tailoring problem

In the tailoring of a composite structure, values of the selected design variables are varied to find out a structure that deforms in a specific way for given loads or, more generally, so that the structure has prescribed thermoelastic, strength, stability and dynamic properties. That kind of problems can be called *inverse problems* to distinguish them from the problems where the minimum or maximum of certain property is sought. Although an inverse problem is formulated and solved as an optimization problem, the global minima or maxima are not the points of interest, but the points of design space, where reasonable values for considered properties are achieved, are sufficient.

The definition and solution of an inverse problem can be divided into three steps, see Tarantola (1987). Firstly, the physical system in question has to be parametrized, i.e. those parameters, whose values completely characterize the system from the considered point of view, are defined. The model parameters are the parameters for which the values are to be determined, and observed parameters are the parameters which can be measured or which are known a priori. Secondly, in forward modelling the physical relations between the model and observed parameters are established and finally, in inverse modelling the values of observed parameters are applied to infer the actual values of model parameters.

A solution of an inverse problem can seldom be obtained with an analytical expression, or seldom the number of design variables and feasible areas of desired values are that small that model space can be explored systematically. Usually the problem is solved by using iteration algorithms. Mathematically an inverse problem can be formulated as an optimization problem

$$
\min_\mathbf{x} \sum_{i=1}^{m} p_i \left( F_i(\mathbf{x}) - \bar{F}_i \right)^2
$$

subjected to

$$
\bar{F}_i \leq F_i(\mathbf{x}) \leq \bar{F}_i \quad i = m + 1, M
$$

(1)

where $F_i$ are the considered properties, i.e. observed parameters, for which the exact prescribed values $ar{F}_i$ are sought by varying the design variables $\mathbf{x}$, the number of considered properties being $M$ and the number of properties dealt in the objective function being $m$. If some properties are more significant compared with others, they can be emphasized with the weighting coefficients $p_i$. For some properties only lower or upper limits, $\bar{F}_i$.
and \( F_i \), may be given, and such properties can be handled as constraint functions. As the relations between the design variables \( x \) and the thermomechanical properties \( F_i \) are often nonlinear, solution algorithms suitable for nonlinear constrained problems should be employed, see Chapter 5.

As different properties may have different magnitudes, like stiffnesses and coefficients of thermal expansion, the properties considered in Eq. (1) should be scaled to have the same magnitude. Another way would be to employ the relative differences between values of the properties in a current design point and the target values. In examples shown in Appendix I the values of all considered properties are scaled to have the magnitude about one.
3. Design variables

3.1. Designing with respect to lay-up parameters

As the composite material itself can be considered as a structure, the number of possible design variables in designing of structures made of composites is larger than in designing of structures made of isotropic materials. Lay-up parameters, i.e. the number of layers and orientations and thicknesses of the layers, for example, can be applied directly as design variables. Consequently, all kinds of thermomechanical properties can be considered in the same problem: layer properties, such as values of failure criterion function, as well as global properties, such as stiffnesses of the laminate. The limitations of manufacturing can be taken into account in tailoring, and after the design the manufacture of the structure is more or less straightforward.

Most thermomechanical properties depend on lay-up parameters in a nonlinear way, and optimization with respect to lay-up parameters often suffers on various local optima. For example in Fig. 2 the coefficient of thermal expansion in $x$-direction and the first natural frequency of a rectangular carbon/epoxy plate are shown as a function of layer orientations $\theta_1$ and $\theta_2$, the laminate of the plate being symmetric and balanced $[\pm \theta_1/ \pm \theta_2]_s$. It can be seen that even with a simple laminate and only two design variables, the response surface of a property can be complicated. However, in tailoring, where some reasonable level of properties or some target values for properties are sought, a local minimum is sufficient, although it cannot be known whether the optimum found is either a local or global one.
Some design variables, such as the number of layers and layer thicknesses, have a discrete character, whereas some, such as the layer orientation, may have a continuous character. This makes optimization difficult and only few algorithms can handle both continuous and integer design variables at the same time. Often all design variables are handled as continuous ones and the optimal values are rounded off to the nearest acceptable discrete values. However, in problems with a large number of design variables, it might be difficult to find a discrete design set that does not violate the constraints.

3.2. Lamination parameters as design variables

Tsai & Pagano (1968) derived lamination parameters with which the terms of sines and cosines appearing in rotation formulae of an orthotropic material can be replaced. The parameters are defined as the weighted integrals over the thickness of the laminate

\[
\xi_{\{2,3,4\}}^A = \frac{1}{2} \int_{-1}^{1} [\cos(2\theta_i) \cos(4\theta_i) \sin(2\theta_i) \sin(4\theta_i)] \, dz^*
\]

\[
\xi_{\{2,3,4\}}^B = \frac{1}{2} \int_{-1}^{1} [\cos(2\theta_i) \cos(4\theta_i) \sin(2\theta_i) \sin(4\theta_i)] \, z^* \, dz^*
\] (2)

\[
\xi_{\{2,3,4\}}^D = \frac{3}{2} \int_{-1}^{1} [\cos(2\theta_i) \cos(4\theta_i) \sin(2\theta_i) \sin(4\theta_i)] \, z^{*2} \, dz^*
\]
where $\theta_i$ is the orientation of $i$th layer, $z^*$ is the dimensionless thickness co-ordinate scaled over the range -1...1. Thus all information from the lay-up of the laminate can be compressed into these lamination parameters. Twelve lamination parameters are sufficient to describe the behaviour of a general laminate and many thermomechanical properties, such as in-plane, coupling and bending stiffnesses of a laminate, can be expressed as linear functions of lamination parameters.

The lamination parameters are clearly dependent on each other. The trigonometric relations among the four in-plane, coupling or bending parameters define a feasible domain in the lamination parameter space $\xi_i \in R^4$

$$
2(\xi_1^j)^2(1 - \xi_2^j) + 2(\xi_3^j)^2(1 + \xi_2^j) + (\xi_4^j)^2 + (\xi_4^j)^2 - 4\xi_1^j\xi_3^j\xi_4^j \leq 1
$$

$$
(\xi_1^j)^2 + (\xi_2^j)^2 \leq 1
$$

$$
-1 \leq \xi_2^j \leq 1
$$

$$
\xi_4^j = A; B; D
$$

These relations are given by Avellaneda & Milton (1989) and Fukunaga & Sekine (1992), for example. From Fig. 3, where the feasible domain is plotted in $R^3$ by choosing $\xi_4 = 0$, i.e. rotating the entire laminate so that $\xi_4 = 0$, it can be seen that the domain defined by Eq. (3) is also closed and convex in $R^4$. A feasible domain among all twelve lamination parameters is not known thus far. However, in many cases the laminates considered are symmetrical, and thus in lamination parameter space the in-plane and bending behaviour can be handled in different problems.

As the number of lamination parameters is sufficiently small and as many thermomechanical properties are linear in the lamination parameters, they are useful as design
variables in optimizing global properties of laminates. Then the number of design variables is independent of the number of layers and all variables have continuous character. The feasible region of the lamination parameters in Eq. (3) can be handled with constraint functions. The most attractive benefit of using lamination parameters as design variables is that optimization problems often become convex, and thus the convergence to the local optima can be avoided. Grenestedt & Gudmundson (1993) have presented the convexity proof of various potential functions by utilizing the convexity of feasible domain and the linear dependency of stiffness matrices on lamination parameters. Lipton (1994a) has examined the optimal compliance design of plates and has given the proof of convexity and the conditions for existence of a saddle point in compliance optimization. Hammer et al. (1997) applied these results in the optimization of layered laminate plates and showed the efficiency of applying the saddle point theorem.

Drawback in using lamination parameters as design variables is that only the global properties of the laminate, such as the natural frequencies, can be presented as functions of lamination parameters. If properties of layers, such as values of failure criterion functions, are considered, the problem must still be expressed with traditional lay-up parameters. One way to solve optimization problems concerning both global and layer properties is to divide the optimization into two problems: first, the lamination parameters giving optimum values for global properties are solved, and then another optimization problem, in which the aim is to find a lay-up giving optimal lamination parameters and meeting the demands of layer properties, is formulated and solved.

Only lay-ups consisting of layers, in which all the material properties are equal or multiples of each other, can be presented by means of twelve lamination parameters. In hybrid composites material invariants can be included into the design variables, see Kalamkarov & Kolpakov (1997). For example the in-plane stiffnesses of a hybrid laminate

\[
\begin{align*}
A_{11} &= \sum_{i=1}^{n} U_{1i} h_i + \sum_{i=1}^{n} U_{2} \cos 2\theta_i h_i + \sum_{i=1}^{n} U_{3} \cos 4\theta_i h_i \\
A_{12} &= \sum_{i=1}^{n} U_{1i} h_i - \sum_{i=1}^{n} U_{3} \cos 4\theta_i h_i \\
A_{22} &= \sum_{i=1}^{n} U_{1i} h_i - \sum_{i=1}^{n} U_{2} \cos 2\theta_i h_i + \sum_{i=1}^{n} U_{3} \cos 4\theta_i h_i \\
A_{66} &= \sum_{i=1}^{n} U_{2} \cos 4\theta_i h_i \\
A_{16} &= \frac{1}{2} \sum_{i=1}^{n} U_{2} \sin 2\theta_i h_i + \sum_{i=1}^{n} U_{3} \sin 4\theta_i h_i \\
A_{26} &= \frac{1}{2} \sum_{i=1}^{n} U_{2} \sin 2\theta_i h_i - \sum_{i=1}^{n} U_{3} \sin 4\theta_i h_i
\end{align*}
\]
can be described with seven parameters:

\[
\begin{align*}
\xi_{m5} &= \sum_{i=1}^{n} U_i^1 h_i \\
\xi_{m6} &= \sum_{i=1}^{n} U_i^2 h_i \\
\xi_{m7} &= \sum_{i=1}^{n} U_i^4 h_i \\
\xi_{m1} &= \sum_{i=1}^{n} U_i^5 \cos 2\theta_i h_i \\
\xi_{m2} &= \sum_{i=1}^{n} U_j^3 \cos 4\theta_i h_i \\
\xi_{m3} &= \sum_{i=1}^{n} U_j^2 \sin 2\theta_i h_i \\
\xi_{m4} &= \sum_{i=1}^{n} U_j^3 \sin 4\theta_i h_i
\end{align*}
\]  

(5)

where \( U_1, \ldots, U_5 \) are the material invariants, see Jones (1975). However, in optimization a feasible domain among seven variables should be known, and values of material invariants should be taken into account in the definition of the feasible domain.

During the optimization with lamination parameters the iteration search may go to a point where the stiffness matrix of the laminate is not positive-definite and the value of the objective function (or constraint) cannot be defined. This may happen especially when the optimum point lies on the boundary of the feasible domain. The latter can be avoided for example by applying the interior penalty function algorithm, where the minimum point is approached from the interior of feasible area. In gradient-based methods, where the search direction and step length are solved separately, the step length can be reduced until the iteration point is in the feasible area.

3.3. From lamination parameters to real lay-up

After finding the optimal lamination parameters, the question how to define an ordinary lay-up corresponding to the optimal lamination parameters arises. Lipton (1994b) has shown that all sets of four in-plane lamination parameters can be obtained by using three layers at most. The entire laminate described by four lamination parameters can be rotated into the position where one lamination parameter is equal to zero. As the feasible region of the rest of parameters is a convex hull, any point inside the feasible domain can be expressed as a linear combination of one corner and one boundary point. A boundary point of the feasible domain can be proved to correspond to a laminate having layers in two different orientations, and a corner point of the feasible domain corresponds to a one-layered laminate. Thus each interior point of that feasible domain corresponds to a three-layered laminate and all combinations of lamination parameters can be expressed with layers in three different orientations at most. Lipton (1994b) has derived explicit formulae for generating that three-layered laminate in terms of in-plane lamination parameters and the corresponding formulas are derived for bending lamination parameters in Appendix IV.

In practical applications, manufacturing, available materials etc. may place limitations on gathering the suitable lay-up, which cannot be taken into account by explicit equations. Also, both the values of in-plane and bending lamination parameters of a laminate may be desired to be as close as possible to the optimal ones. Then a new optimization problem searching lay-ups having lamination parameters close to the optimal ones and, also fulfilling the demands of layer properties and the restrictions of manufacturing, may be formulated. In this problem, the lay-up parameters are applied as design variables, and the difference between the lamination parameters of the lay-up in the current design point
and the target lamination parameters is minimized:

$$\min_{n, \theta_i, h_i} \sum_{k=1}^{4} \left[ (\xi_k^A - \xi_k^A)^2 + (\xi_k^B - \xi_k^B)^2 + (\xi_k^D - \xi_k^D)^2 \right]$$

subject to \( \hat{f}_{lay_{i,j}} \leq f_{lay_{i,j}} \leq \bar{f}_{lay_{i,j}} \) (6)

As the detailed lay-up of the laminate is now known, also layer properties \( f_{lay_{i,j}} \), such as stresses and strains in plies, can be considered, \( \hat{f}_{lay_{i,j}} \) and \( \bar{f}_{lay_{i,j}} \) being the lower and upper limits of the \( j \)th considered property of the \( i \)th layer.

In Appendix V genetic algorithm (GA) search is applied in the solution of the optimization problem shown in Eq. (6). Genetic algorithm is well suited for the solution of this kind of problem, because the value of the objective function can be computed efficiently without using time-consuming FEM solutions. Moreover, in practice the thicknesses and orientations of different layers can often only have discrete values which can be easily handled in GA search, as in GA all design variables have discrete character. In Appendix V the thickness of the unidirectional layers in the laminate and the value of the Tsai-Wu failure criterion function in strain space are considered during the iteration in order to avoid convergence into lay-ups which are susceptible to failure. During the GA iteration, strains in the midplane of the laminate are not solved by the FEM but instead the strains computed with optimal lamination parameters are used. Thus the value of the failure criterion can be estimated only with lay-ups which have lamination parameters rather close to optimal ones. In some examples presented in Appendix V the generation of laminates having both optimal in-plane and bending lamination parameters was rather easy, but in some cases, as when dealing with buckling factors for temperature loading, finding a lay-up with optimal lamination parameters was impossible. Then the lay-ups providing the highest possible buckling factor were obtained by using weighting factors.
4. Thermomechanical properties considered

Most properties of a laminate or a laminated structure depend in one way or other on the lay-up of the laminate. In Appendices I and II the tailoring of thermomechanical properties of laminates and laminated plates and cylinders are considered. Coupled thermal-structural problems of laminated plates are presented and tailored in Appendices III and IV.

As during optimization values of considered properties are computed frequently, computing the properties must be efficient. If the properties of a laminate are points of interest, the values of the properties can be efficiently found by simple point analysis, but if the properties of laminated structures are considered, approximative methods are usually employed in the solution, which often requires working with large structural stiffness matrices. The Ritz method can be useful in the analysis of simple plate and cylinder geometries, whereas by using finite element method (FEM) complex geometries can be considered. Because one shape function per displacement distribution over the whole structure is applied in the Ritz discretization, the number of degrees of freedom, i.e. the size of the stiffness matrix of the structure, is rather small compared to cases where the structure is discretized with the FEM, see Laitinen (1994). Leissa & Narita (1989) have studied the vibration of rectangular laminated plates with the Ritz method and shown that good convergence of the results can be obtained using 144 terms \((m, n = 12)\) in the shape function of the displacement \(w\). As the number of terms in the shape function can be easily increased during the optimization, in the beginning, where the iteration point is usually far from optimum, the number of terms can be rather small \((m, n = 8)\), and as the iteration progresses, the number of terms in the shape functions can be increased. Also in FEM the element mesh can be refined as the iteration gets nearer to the optimum.

In order to exploit all possibilities of tailoring over the whole structure, the lay-up of the laminate can be allowed to vary from point to point throughout the structure, i.e. the lay-up of the laminate is in-plane non-homogeneous. Of course the manufacturing of that kind of laminate is complex, but it can be put into practice by using numerically controlled filament winding or automated fibre placement systems, for example. In tailoring the variation of lay-up can be implemented easily by FEM, for example by using lay-up parameters of each element as design variables, see e.g. Appendix III. In the Ritz discretization the variation of lay-up parameters should be taken into account during the integration of energy terms, see Laitinen (1994).
4.1. Mechanical properties

Mechanical properties considered in the computer code (LAMINV) presented in Appendices I and II are listed in Table 1. The properties of a laminate are calculated by employing point analysis and classical lamination theory (CLT), see Jones (1975), for example. In computing natural frequencies, buckling loads and deflections of laminated plates, the Ritz method and classical plate theory (CPT) or first-order shear-deformation plate theory (FSDPT) are applied. The properties of cylindrical shells are determined with the shear deformable Sander’s shell theory and the Ritz method, see Laitinen (1994).

In Appendix IV the lowest natural frequency of a laminated plate is maximized with respect to four bending and two in-plane lamination parameters. The finite element method with a first-order shear deformation element is applied in the discretization of the plate.

Table 1. Considered properties in LAMINV-code.

<table>
<thead>
<tr>
<th>Property</th>
<th>Loads, geometry, BCs</th>
<th>Calculation method</th>
</tr>
</thead>
<tbody>
<tr>
<td>total thickness or square mass of the laminate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>in-plane and bending stiffnesses of a laminate</td>
<td></td>
<td>CLT, point analysis</td>
</tr>
<tr>
<td>in-plane and bending coefficients of thermal expansion</td>
<td></td>
<td>CLT, point analysis</td>
</tr>
<tr>
<td>value of Tsai-Hill criterion (value of certain layer or max. among all layers)</td>
<td>in-plane loads and moments, $\Delta T$</td>
<td>CLT, point analysis</td>
</tr>
<tr>
<td>strains of a lamina</td>
<td></td>
<td></td>
</tr>
<tr>
<td>curvatures of a laminate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>max. deflection of a rectangular plate</td>
<td>uniform pressure, in-plane loads and moments, $\Delta T$</td>
<td>CPT or FSDPT, Ritz method</td>
</tr>
<tr>
<td>max. value of Tsai-Hill criterion</td>
<td></td>
<td></td>
</tr>
<tr>
<td>max. strains in a lamina</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(continued)
Table 1. Considered properties in LAMINV-code (continued).

<table>
<thead>
<tr>
<th>Property</th>
<th>Loads, geometry, BCs</th>
<th>Calculation method</th>
</tr>
</thead>
<tbody>
<tr>
<td>lowest natural frequencies of a rectangular plate</td>
<td><img src="image" alt="Rectangular Plate Diagram" /> BC: SS/SS SS/CS CS/CS</td>
<td>CPT or FSDPT, Ritz method</td>
</tr>
<tr>
<td>buckling loads of a rectangular plate</td>
<td><img src="image" alt="Rectangular Plate Diagram" /> BC: SS/SS SS/CS CS/CS uniaxial/multiaxial compression and shear loads</td>
<td>CPT or FSDPT, Ritz method</td>
</tr>
<tr>
<td>max. deflection of a cylindrical shell</td>
<td><img src="image" alt="Cylindrical Shell Diagram" /> BC: SS/SS SS/CS CS/CS uniform pressure, in-plane loads and moments, $\Delta T$</td>
<td>Sander's shell theory, Ritz method</td>
</tr>
<tr>
<td>max. value of Tsai-Hill criterion</td>
<td><img src="image" alt="Cylindrical Shell Diagram" /> BC: SS/SS SS/CS CS/CS max. value of Tsai-Hill criterion</td>
<td>Sander's shell theory, Ritz method</td>
</tr>
<tr>
<td>max. strains in a lamina</td>
<td><img src="image" alt="Cylindrical Shell Diagram" /> BC: SS/SS SS/CS CS/CS max. strains in a lamina</td>
<td>Sander's shell theory, Ritz method</td>
</tr>
<tr>
<td>lowest natural frequencies of a cylindrical shell</td>
<td><img src="image" alt="Cylindrical Shell Diagram" /> BC: (open): SS/SS SS/CS CS/CS BC: (closed): SS CS</td>
<td>Sander's shell theory, Ritz method</td>
</tr>
<tr>
<td>buckling loads of a cylindrical shell</td>
<td><img src="image" alt="Cylindrical Shell Diagram" /> BC: (open): SS/SS SS/CS CS/CS BC: (closed): SS CS uniaxial/multiaxial compression and shear loads (open) axial force, pressure and torque (closed)</td>
<td>Sander's shell theory, Ritz method</td>
</tr>
</tbody>
</table>

Boundary conditions (BCs) of plates: all sides simply supported (SS/SS), two opposite sides simply supported and two other clamped (SS/CS), all sides clamped (CS/CS).

Boundary conditions (BCs) of closed cylindrical shells: ends at $x = 0$ and $x = x_l$ are simply supported (SS) or clamped (CS).
4.2. Thermal problems

As the coefficients of thermal conduction (CTCs) depend on the lay-up of the laminate, the tailoring procedure can also be applied to thermal problems. Laminate plates having prescribed or almost prescribed temperature distribution are generated by Autio & Pramila (1995).

The equation of heat conduction for a heterogeneous anisotropic laminate in in-plane is

$$\frac{\partial}{\partial x} \left( k_{xx} \frac{\partial T(x, y)}{\partial x} + k_{xy} \frac{\partial T(x, y)}{\partial y} \right) + \frac{\partial}{\partial y} \left( k_{xy} \frac{\partial T(x, y)}{\partial x} + k_{yy} \frac{\partial T(x, y)}{\partial y} \right) + \rho r = \rho c \frac{\partial T(x, y)}{\partial t}$$  \hspace{1cm} (7)

where $k_{xx}$, $k_{yy}$ and $k_{xy}$ are the CTCs of the laminate in its global $xy$-coordinate system, $\rho$ is the density, $c$ the specific heat and $\rho r$ the rate of internal energy generation. The CTCs of the laminate can be expressed as the function of lay-up parameters or as the function of lamination parameters

$$k_{xx} = \frac{1}{h} \sum_{i=1}^{n} (k_{i}^1 \cos^2 \theta_i + k_{i}^2 \sin^2 \theta_i) h_i$$  \hspace{1cm} (8)

$$k_{yy} = \frac{1}{h} \sum_{i=1}^{n} (k_{i}^1 \sin^2 \theta_i + k_{i}^2 \cos^2 \theta_i) h_i$$  \hspace{1cm} (8)

$$k_{xy} = \frac{1}{h} \sum_{i=1}^{n} (k_{i}^1 - k_{i}^2) \sin \theta_i \cos \theta_i h_i$$  \hspace{1cm} (8)

where $k_{i}^1$ and $k_{i}^2$ are the CTCs of the $i$th orthotropic ply in principal directions, $h_i$ is the thickness of the $i$th ply and $h$ the total thickness of the laminate. In lamination parameter formulation all layers of the laminate are assumed to be made of the same material.

If the problem is supposed to be time-independent, the right-hand part of Eq. (7) is zero. Also, $\rho r$ can be neglected if no heat generation occurs. Then, for a one layered laminate, deriving an analytical equation between the orientation of the layer and the temperature field $T(x, y)$ is possible:

$$\frac{k_1}{k_2} = -\frac{\frac{\partial^2 T(x, y)}{\partial x^2} \sin^2 \theta - 2 \frac{\partial^2 T(x, y)}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 T(x, y)}{\partial y^2} \cos^2 \theta}{\frac{\partial^2 T(x, y)}{\partial x^2} \cos^2 \theta + 2 \frac{\partial^2 T(x, y)}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 T(x, y)}{\partial y^2} \sin^2 \theta}$$  \hspace{1cm} (9)

With multi-layered laminates the situation is more complicated and the analytical relation between the lay-up and the temperature field cannot be generated. The field Eq. (7) over the structure can be solved with the FEM or finite difference method. After obtaining the discrete nodal solutions, a corresponding continuous temperature field can be computed by using a linear regression model. Then the problem of finding a lay-up giving the prescribed temperature distribution over the structure can be formulated as a nonlinear optimization problem, where the difference between these regression coefficients and their target values are minimized with respect to the lay-up of a laminate.
In the lamination parameter formulation the relation between parameters $\xi_1^A$ and $\xi_3^A$ as function of temperature field $T(x, y)$ can be expressed as

$$\frac{K_1}{K_2} = -\frac{\left(\frac{\partial^2 T(x, y)}{\partial x^2} - \frac{\partial^2 T(x, y)}{\partial y^2}\right) \xi_1^A + 2 \frac{\partial^2 T(x, y)}{\partial x \partial y} \xi_3^A}{\left(\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2}\right)}$$

if $\frac{\partial^2 T(x, y)}{\partial x^2} \neq \frac{\partial^2 T(x, y)}{\partial y^2}$

$$\xi_3^A = -\frac{K_1}{K_2} \left(\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2}\right)$$

else

where $K_1 = 0.5(k_1 + k_2)$ and $K_2 = 0.5(k_1 - k_2)$.

**EXAMPLE:** Let the target temperature field be $T(x, y) = 10x^2 + 10y^2 - 50xy + 20$ and the CTCs of the layer $k_1 = 18.0$ and $k_2 = 0.5$, i.e. the ratio of CTCs in the fibre direction and in the direction perpendicular to the fibres is 36. The target distribution can be obtained with a one layered laminate, whose orientation is

$$\theta = \frac{1}{2} \arcsin \left( \frac{-k_1/k_2 + 1}{k_1/k_2 - 1} \frac{\partial^2 T(x, y)}{\partial x^2} \right) = 12.5^\circ$$

The corresponding temperature distribution over the $1 \times 1$ plate is shown in Fig. 4(b), whereas in Fig. 4(a) the temperature distribution over the same plate is shown with a value $\theta = -45^\circ$.

Next the same plate is supposed to consist of layers in two different orientations. The solution of a nonlinear minimization problem

$$\min_{\theta_1, \theta_2} \left[ (\beta_0 - 20)^2 + (\beta_1 - 10)^2 + (\beta_2 - 10)^2 + (\beta_3 + 50)^2 \right]$$

$$0^\circ \leq \theta_1, \theta_2 \leq 180^\circ$$

where $\beta_{0,3}$ are the regression coefficients, gives the orientations $\theta_1 = 90^\circ$ and $\theta_2 = 28.3^\circ$, among others. The corresponding temperature distribution over the plate is given in Fig. 4(c).

From the lamination parameter formulation it can be noticed that the temperature distribution is a function of the CTCs and $\xi_3^A$ only, since $\partial^2 T(x, y)/\partial x^2 = \partial^2 T(x, y)/\partial y^2$. The target temperature distribution can be generated with all lay-ups having $\xi_3^A = 0.423$, i.e. various laminates meet the target.
4.3. Coupled thermal-structural problems

In most cases the displacements, stresses etc. due to temperatures in a structure are important from the design point of view. By coupling thermal and structural problems, the displacements, stresses, strains etc. due to given boundary temperatures can be considered in tailoring problems. Then the analysis of one design point consists of two phases: first the temperature distribution over the structure is solved and then the displacements and strains etc. due to the temperature distribution are computed. Detailed equations for coupled thermal-structural problems in terms of lay-up parameters and lamination parameters are given in Appendices III and IV.

In Appendix III the minimization or maximization of strain energy and the minimization of selected displacements of a thin laminated plate due to given boundary temperatures in the thermal problem and given displacement boundary conditions in the structural problem are considered. The design variables employed in tailoring are the orientations of the layers. The structure is discretized by using the FEM, and the SQP-algorithm is applied in the solution of the optimization problem. As the analysis of one design point consists of two FEM-solutions, the iteration time is reduced by applying the analytical sensitivity analysis.

The lamination parameter formulation is applied in the solution of the above-mentioned problem in Appendix IV, where the buckling of a laminated plate due to given boundary temperatures is also considered. In in-plane cases, i.e. in cases considering strain energy and displacements, considered properties can be shown to be concave functions in the lamination parameters. As in addition to this the feasible domain of in-plane parameters is convex, a convex optimization problem can be formulated and a global optimum found. The buckling factors of a laminated plate loaded by given boundary temperatures are the functions of four in-plane lamination parameters and four bending lamination parameters. In lamination parameter space it can be shown that the function of the buckling factors...
is concave with respect to bending lamination parameters, but convex with respect to in-plane lamination parameters. As the feasible domain among four in-plane as well as four bending parameters is convex, see Eq. (3), the optimal bending lamination parameters lie inside the feasible domain of $\xi^{D}_{1,4}$ and the optimal in-plane parameters can be found at a boundary point of the feasible domain of $\xi^{A}_{1,4}$. These points are also the result of the maximization of the lowest buckling factor, because the feasible domain among all eight parameters is unknown and omitted in the optimization. A boundary point of a feasible domain of lamination parameters can be shown to correspond to a laminate having layers with two different orientations, whereas an inside point corresponds to a laminate consisting of layers in three different orientations at the most. Thus a laminate having both optimal in-plane and bending lamination parameters cannot be generated and lay-ups having lamination parameters as close as possible to the optimal ones are sought by using GA, see Appendix V.
5. Solving the tailoring problem

Usually the tailoring problem is formulated and solved like an optimization problem. If the parameters of lay-up are applied directly as design variables, the objective function and constraints are usually nonlinear, or if the lamination parameters are employed as design variables, the constraints given by Eq. (3) are nonlinear. Thus, solution algorithms suitable for a constrained nonlinear programming should be employed. In Appendix I the flexible tolerance method (FT) and successive quadratic programming (SQP) algorithms are applied in tailoring the properties of laminated structures, and in Appendices III and IV the SQP is utilized in the solution of coupled thermal-structural optimization problems.

The FT improves the value of the objective function by using the information provided by feasible points, as well as certain non feasible ones called near-feasible points, and no gradients of the objective function or constraints are needed, see Himmelblau (1972). However, the FT is not useful in problems with many design variables, because the method is rather slow. The SQP is a gradient-based algorithm, where the search direction and the step length of an iteration are solved separately, see Schittkowski (1986). The SQP has been robust in various applications and it has been successful in tests comparing different algorithms suitable for nonlinear problems (Arora 1990, Ragsdell 1984).

The gradients needed during the SQP-iteration can be computed by using the difference method or analytically. Computing the gradients with the difference method is straightforward and does not require any extra programming work. However, when the number of design variables increases, the iteration may become computationally expensive, because in the central difference method, for example, in addition to computing the value of the considered property in a current design point, the value has to be solved twice for each design variable. Furthermore, gradients computed with the difference method are only approximations and not exact derivatives. The sensitivities of properties can be computed efficiently by using analytical equations, see Haftka et al. (1990). Derivatives of already solved quantities with respect to design variables can be determined in a simple manner by applying the chain rule of differentiation and analytical derivatives, but this involves more programming work than the application of the finite difference method. Mateus et al. (1991) have presented the sensitivities of in-plane, bending and coupling stiffness matrices of a laminate with respect to the orientation and thickness design variables. Equations for computing sensitivities with respect to design variables in coupled thermal-structural problems are given in Appendix III.
6. Discussion and limitations

The limitations of tailoring presented in this thesis, as well as subjects open to further study, are discussed in the following.

In this thesis only the thermomechanical properties of layered structures are considered. However, the moisture effect should also be taken into account in design, if the structure is exposed to humidity. As the phenomenon of moisture diffusion is similar to that of heat conduction, and as the moisture expansion of a lamina can be analysed with analogy to the thermal expansion, models for analysing thermal and coupled thermal-structural behaviour can be employed directly in moisture and coupled moisture-structural problems. The increase in temperature is known to increase the moisture diffusion. This coupling could be included into equations of heat conduction and moisture diffusion by using cross-coupling coefficients. However, this would lead to nonlinear problems, which would increase computing time.

The material properties of unidirectional layers are supposed to be temperature independent, but in fact, properties are often temperature dependent. However, with most plastic composites, the change in properties as a function of temperature is negligible, if the temperature range considered is well below the glass transition temperature \( T_g \) of the matrix material. With temperature ranges close to \( T_g \), the temperature dependency of material parameters and the nonlinearity in the heat conduction equation arising from this reason should be taken into account. The material properties are also time dependent, as the polymers are viscoelastic in nature. The amount of time dependent deformations increases as the temperature increases towards \( T_g \). Thus, the effect of viscoelasticity can be omitted in temperatures well below \( T_g \).

In the tailoring of thermal properties, the temperature distribution through the thickness of the laminate is supposed to be constant and only the in-plane model is considered. This is reasonable for thin laminates, and in cases where the changes in temperature are so slow that the temperature through the thickness can be considered a steady-state. If the stresses etc. due to the thermal loads in thick laminates are considered, the three-dimensional model should be applied.

Applied design variables are lay-up or lamination parameters, and the thermomechanical properties of a unidirectional ply are assumed to be known a priori. These design variables are sufficient in the tailoring of stiffnesses and stiffness related properties of layered structures, as the stiffness properties of a lamina can be determined reliably, and
available theories describing the relation between design variables and stiffness properties give reliable results compared with the measurements. On the other hand, tailoring the strength properties and failure behaviour with respect to lay-up parameters may not give reasonable results, because the models predicting the strength or the failure of a laminate give unreliable and dissimilar results, see for example Soden et al. (1998). Moreover, the strength properties of a unidirectional ply cannot be determined as reliably as the stiffness properties.

In optimization with respect to lamination parameters, only the feasible domain among four in-plane, coupling or bending parameters is known. Thus only those problems, where in-plane and bending behaviour can be separated, can be handled in lamination parameter optimization.

In Appendices IV and V the optimization of thermomechanical properties is broken down into two phases: First the optimization is carried out in the lamination parameter space with explicit optimization method and then in the lay-up parameter space with a probabilistic optimization algorithm. This makes programming work more tedious than in cases where only one type of design variables are employed. However, the solution of the problem in the two phases is efficient, as the problems in the lamination parameter space are usually convex. The efficiency of GA-search could be increased by tuning the values of GA-parameters.
7. Conclusions

This thesis treats the tailoring of thermomechanical properties of layered composites. As the material properties of unidirectional plies are assumed to be known a priori, lay-up parameters, i.e. number of layers and layer orientations and thicknesses, or lamination parameters are used as design variables. Various thermomechanical properties and responses of layered laminates due to external forces and temperature loads are considered.

By varying the lay-up of the laminate the thermomechanical properties of composite structures can be tailored within certain limits. As the thermal properties, such as coefficients of thermal conduction and coefficients of thermal expansion, are functions of the lay-up of the laminate, the temperature distributions, displacements etc. due to given boundary temperatures must also be considered in the tailoring problems. Then the analysis of one design point includes the solution of a coupled thermal-structural problem: first the temperature distribution over the structure due to boundary temperatures is computed and then displacements, stresses etc. due to the temperature difference and displacement boundary conditions are solved.

If the lay-up parameters of the laminate are applied directly as design variables, both layer properties and global properties of the laminate can be considered. On the other hand, as the relation between thermomechanical properties and lay-up parameters is non-linear, various local optima exist in design space, and it cannot be known, whether the optimum found is the global or local one. However, this may be acceptable in tailoring problems where prescribed values of the properties are sought or certain limits for the properties are given.

By applying the lamination parameters in tailoring, the number of design variables can be reduced and formulating a convex optimization problem is often possible. Thus, in most cases a global minimum can be found. The drawbacks of a lamination parameter formulation are that only the global properties of the laminate can be considered, and that another procedure is needed to generate a real lay-up corresponding to a set of lamination parameters. If only in-plane or bending behaviour of the laminate is considered, a lay-up having prescribed lamination parameters can be found by explicit equations. In practical applications, a second optimization problem may be formulated for gathering a lay-up having values of lamination parameters close to optimal ones and also fulfilling the demands of layer properties and the limitations of manufacturing. A genetic algorithm search seems to be suitable for the solution of this later optimization task.
8. Summaries of the appended papers

**Paper I (App. I)** presents a general formulation of a tailoring or inverse problem. Common thermomechanical properties of a laminate structure, such as in-plane and bending stiffnesses, coefficients of thermal expansion, strength, strains and curvatures due to in-plane forces and moments, deflections, buckling loads and natural frequencies of a laminated plate, are tailored with respect to ply orientations and/or ply thicknesses. The relations between design variables and considered thermomechanical properties are formulated by using classical lamination theory and Ritz procedure. As these relations are nonlinear, minimization algorithms suitable for nonlinear constrained problems are employed in solution. In paper I flexible tolerance method (FT) and successive quadratic programming algorithm (SQP) are applied, and a short overview of these algorithms is presented. Examples for finding laminates having prescribed thermomechanical properties, or laminates having minimum or maximum values of certain thermomechanical properties with constraints for other ones, are given. As the problems are nonlinear, different starting points of iteration give different results.

**In Paper II (App. II)** the optimization and tailoring of laminated plates are studied. Classical plate theory (CPT) or first-order shear-deformation plate theory (FSDPT) with the Ritz method is employed in the solution of deflections, natural frequencies and buckling factors of rectangular plates. With plates having two opposite sides simply supported, the interpolation function of deflection $w$ consists of double sine series, whereas with plates having two opposite sides clamped, interpolation functions consisting of cosine terms or hyperbolic terms are employed. With clamped plates the effect of the applied plate theory and different interpolation functions are tested by comparing the results computed by the Ritz method with the results computed by FEM. With thin plates ($a/h \approx 100$) the classical plate theory with the hyperbolic interpolation function gave best results, whereas FSDPT should naturally be employed with thick plates. The differences between results computed with interpolation functions consisting of cosine terms or hyperbolic terms were small. Two examples for tailoring the properties of laminated plates are given.

**Paper III (App. III)** considers a coupled thermal-structural problem of laminated plates. The behaviour of a laminated plate with given boundary temperatures and displacement constraints is tailored with respect to orientations of different layers. Finite element discretization is applied to the analysis of coupled thermal-structural problems, which now consists of two phases: first the solution of temperature distribution over the
structure and then the computation of displacements, stresses and strains. Detailed equations for solving the coupled problems with the FEM discretization are given. In tailoring the SQP algorithm is employed. As the analysis of one point in design space consists of two time-consuming FEM solutions, an analytical sensitivity analysis is applied in the iteration. Equations employed in the sensitivity analysis are derived and presented. In examples the strain energy of the structure is optimized or the sum of squares of certain nodal displacements is minimized, the laminate of the structure being one-layered or multi-layered. In the last example a one-layered laminate with discretized fibre orientations over the element mesh is considered.

In Paper IV (App. IV) the coupled thermal-structural problem of a laminated plate is expressed in terms of lamination parameters. The strain energy, or certain displacements of the laminated plate due to given boundary temperatures and displacement constraints is optimized with respect to in-plane lamination parameters. Also buckling of the plate due to thermal loads is considered. The buckling factors are expressed as a function of four in-plane and four bending lamination parameters, and the smallest factor is maximized with respect to these parameters. Besides these thermal problems, the natural frequencies of the laminated plate, which are functions of two in-plane and four bending lamination parameters, are studied. Detailed equations for solving the coupled problems with FEM discretization as function of lamination parameters are given. When the lamination parameters are employed as design variables, the optimization problem is convex in most cases, and thus the global optimum can be found. After finding optimal lamination parameters, a procedure to generate a real lay-up corresponding to these parameters is needed. In Paper IV explicit equations for finding a lay-up corresponding to optimal in-plane parameters derived by Lipton (1994b) are used, and new equations are derived for bending parameters.

Paper V (App. V) presents how a real lay-up of a laminate corresponding to optimal lamination parameters can be determined by genetic search. Optimal lamination parameters computed in paper IV are used as target lamination parameters and the difference between target lamination parameters and lamination parameters of current design is minimized by varying the lay-up of the laminate. As the time-consuming FEM solutions are not needed during the iteration and, as often in reality, orientations and thicknesses can have only discrete values, the genetic algorithm search is employed in minimization. Two ways of coding the design variables are employed in the GA search: either orientations and relative thicknesses are coded separately or available uni/multiaxial plies are coded. With GA search laminates having both in-plane, as well as bending lamination parameters as close as possible to optimal ones, can be created, which is not possible (at least this far) with explicit equations. Also layer-wise properties, such as values of failure criterion functions of a lamina, can be taken into account in GA search. In paper V a limit for the value of the Tsai-Wu failure criterion function in strain space is included as a penalty in the fitness function of GA search in order to avoid the convergence to lay-ups having a high tendency of failure. Also, to decrease the probability of the matrix cracking the total thickness of contiguous plies in the same direction is limited.
List of references


