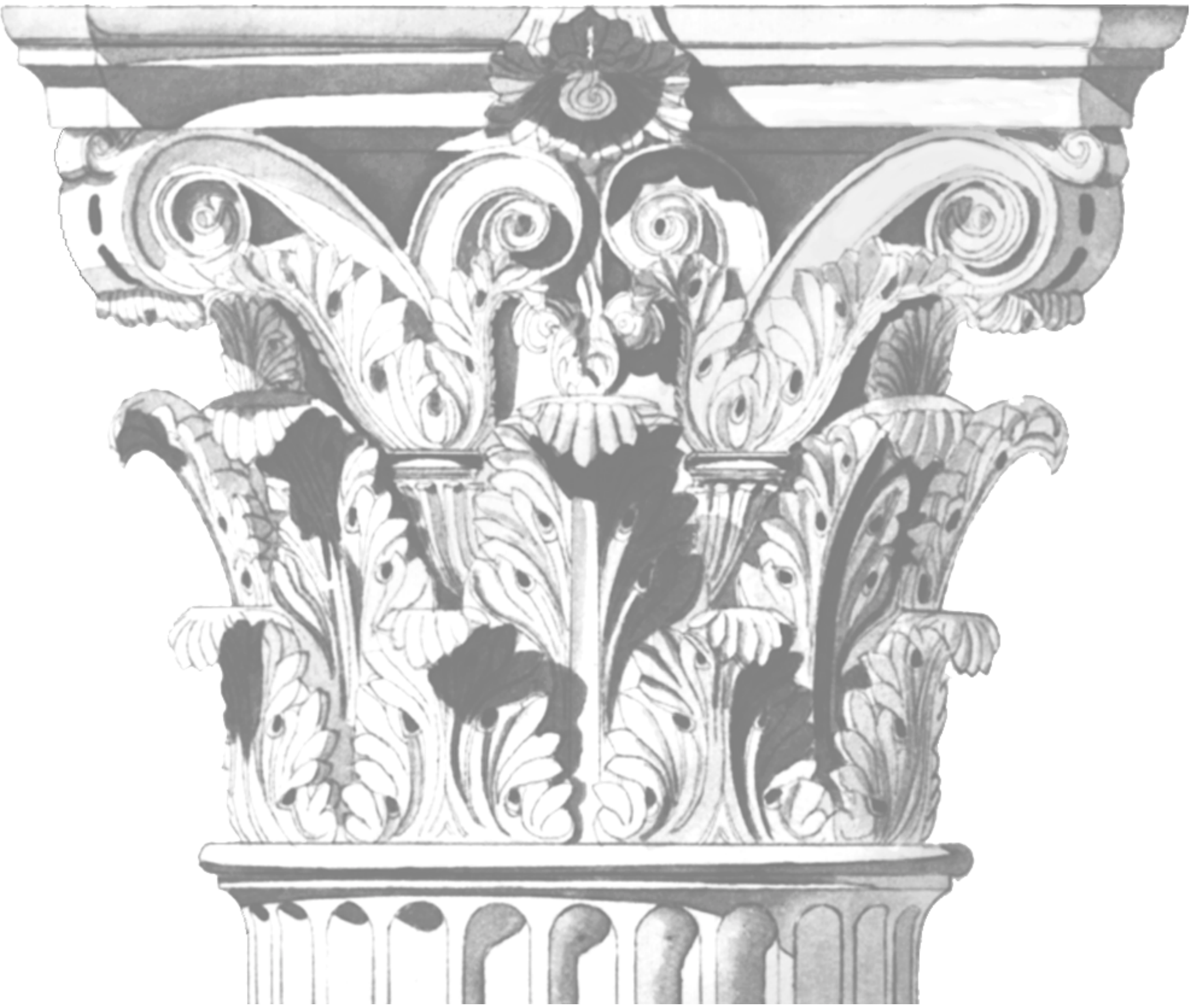
SOME BOUNDARY ELEMENT METHODS FOR HEAT CONDUCTION PROBLEMS

MARTTI HAMINA

Mathematics Division

OULU 2000



MARTTI HAMINA

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Abstract

This thesis summarizes certain boundary element methods applied to some initial and boundary value problems. Our model problem is the two-dimensional homogeneous heat conduction problem with vanishing initial data. We use the heat potential representation of the solution. The given boundary conditions, as well as the choice of the representation formula, yield various boundary integral equations. For the sake of simplicity, we use the direct boundary integral approach, where the unknown boundary density appearing in the boundary integral equation is a quantity of physical meaning.

We consider two different sets of boundary conditions, the Dirichlet problem, where the boundary temperature is given and the Neumann problem, where the heat flux across the boundary is given. Even a nonlinear Neumann condition satisfying certain monotonicity and growth conditions is possible. The approach yields a nonlinear boundary integral equation of the second kind.

In the stationary case, the model problem reduces to a potential problem with a nonlinear Neumann condition. We use the spaces of smoothest splines as trial functions. The nonlinearity is approximated by using the L^2-orthogonal projection. The resulting collocation scheme retains the optimal L^2-convergence. Numerical experiments are in agreement with this result. This approach generalizes to the time dependent case. The trial functions are tensor products of piecewise linear and piecewise constant splines. The proposed projection method uses interpolation with respect to the space variable and the orthogonal projection with respect to the time variable. Compared to the Galerkin method, this approach simplifies the realization of the discrete matrix equations. In addition, the rate of the convergence is of optimal order.

On the other hand, the Dirichlet problem, where the boundary temperature is given, leads to a single layer heat operator equation of the first kind. In the first approach, we use tensor products of piecewise linear splines as trial functions with collocation at the nodal points. Stability and suboptimal L^2-convergence of the method were proved in the case of a circular domain. Numerical experiments indicate the expected quadratic L^2-convergence.

Later, a Petrov-Galerkin approach was proposed, where the trial functions were tensor products of piecewise linear and piecewise constant splines. The resulting approximative scheme is stable and convergent. The analysis has been carried out in the cases of the single layer heat operator and the hypersingular heat operator. The rate of the convergence with respect to the L^2-norm is also here of suboptimal order.

Keywords: collocation, boundary integrals, heat conduction

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Finally, I wish to express my warmest thanks to my parents for their constant support.

Oulu, March 2000 Martti Hamina

List of the original articles

This thesis is a summary of the work published in the following five articles:

I Hamina M & Saranen J (1994) On the spline collocation method for the single-layer heat operator equation. Math Comp 62: 41-64.

II Hamina M (1997) A collocation type projection method for the single layer heat operator equation. Preprint, University of Oulu, December 1997.

III Hamina M (2000) An approximation method for the hypersingular heat operator equation. J Comput Appl Math 115: 229-243.

IV Hamina M, Ruotsalainen K & Saranen J (1992) The numerical approximation of the solution of a nonlinear boundary integral equation with the collocation method. J Integral Equations Appl 4: 95-115.

V Hamina M (1997) On the numerical solution of a non-linear heat conduction problem. Integral Methods in Science and Engineering, Vol 2, (Constanda C, Saranen J, Seikkala S eds), Addison Wesley Longman Inc: 93-98.

These articles are referred to in the text by their Roman numerals.

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1 Introduction

Initial and boundary value problems are seldom analytically solvable. Therefore numerical methods are of great importance. The Finite Element Method has been among the most popular computational techniques. However, in some instances the Boundary Element Method has evolved as a powerful alternative for finite elements. Compared to the finite element method, the most important feature of the boundary element method is that it only requires discretization of the boundary rather than that of the whole volume. Thus, the boundary element method is advantageous when the domain extends to infinity or the shape of the boundary is complex. The most severe restriction of the boundary element approach is that it is applicable only to such problems where the fundamental solution is available.

The boundary and initial value problem of the heat equation is the most usual model problem of the so-called parabolic boundary and initial value problems. This work is devoted to the boundary element solution of the homogeneous heat equation. An exellent classical treatment of partial differential equations of the parabolic type is given by Friedman [22]. Pogorzelski's book [45] serves as a classical introduction to integral equation methods. Some textbooks concerning the modern treatment of integral equation methods and boundary element methods are Atkinson [8], and Chen & Zhou [13].

1.1 Recent history

The mathematical theory of the boundary element method has developed vastly during the last twenty years. The most familiar discretization methods are the spline Galerkin and spline collocation methods. Other well known approaches are the Nyström method, trigonometric collocation and quadrature methods. During 1976-1983 a rather complete theory of the Galerkin boundary element method on smooth boundaries was developed. The theory was based on the concept of strongly elliptic pseudodifferential operators. Thus, the analysis covered very large classes of boundary element methods. The practical implementation, as well as the effect of the numerical integration, has been discussed in references [26], [27] and [35], for example. In the presence of singularities (domains with corners, mixed boundary conditions) theoretical progress has been based on the works of Costabel, Stephan, Chandler and several other authors.

In the Galerkin approach, the boundary integral equation is satisfied in the average sense. From the practical point of view, this means double integration over the boundary of the domain. Another approach, easier to implement, is the collocation method where the boundary integral equation is fulfilled on certain, so-called, collocation points. However, the theoretical analysis of the collocation method is much more difficult. There are two main lines of research. One is based on the equivalence between the collocation method and a certain Galerkin method. The other relies on direct Fourier analysis.

The equivalent Galerkin method is available only when the trial functions are splines of an odd degree. The analysis allows quasi-uniform meshes and yields asymptotic error estimates with respect to some nonstandard Sobolev norms, which depend on the order of the operator and on the degree of the trial functions. The method was developed by Arnold & Wendland [5] for the one-dimensional case and extended by Arnold & Saranen [4] for biperiodic problems. These ideas generalize also to the case of certain parabolic boundary integral equations, as was shown by Hamina & Saranen in [24], [25] and paper I.

The direct Fourier analysis is applicable if the meshes are uniform. This approach has been applied to the one-dimensional spline collocation in papers [6], [56], [54]. We emphasize that the convergence of the spline collocation in higher dimensions is not generally known for quasi-uniform meshes, even in the case of elliptic problems. There are results in some special cases where the equation is given on a torus [4], [15], [47], square or cube [17], [46]. Recently Costabel & Saranen [18] have obtained optimal order error estimates for convolutional parabolic boundary integral equations by using Fourier techniques.

For time dependent problems, the papers by Arnold & Noon [3] and Noon [37] were of crucial importance. They discovered that the single layer heat operator is coercive in certain anisotropic Sobolev spaces. Their analysis applied to three dimensional heat conduction problems. Corresponding results in two space dimensions were given by Hsiao & Saranen [28] and Costabel [14]. The coercivity property was at once a rigorous basis for the analysis of Galerkin boundary element methods. The fundamental mapping properties of the associated boundary integral operators as well as the solvability theory was developed by Costabel [14] and Hsiao & Saranen [29].

The first theoretical result concerning the spline collocation approximation of the single layer heat operator equation was in a conference paper by Hamina & Saranen, [24]. The given results cover the case where the spatial domain is a disk. Later, Hamina & Saranen in paper I analysed a method where the trial functions were tensor products of piecewise linear splines defined on a quasi-uniform space-time mesh. Stability and suboptimal convergence of the method as well as numerical results were reported. The extension of the approach to more general boundary curves remained to be done. In their conference article, Hämäläinen & Saranen [32] reported a compact perturbation technique which generalizes the analysis to more general boundary curves, provided that the parametrization is with respect to the arc length parameter. A complete discussion of this topic is presented by Hämäläinen [31].

In paper II Hamina suggested a Petrov-Galerkin approach, where interpolation was used with respect to the space variable, and orthogonal projection was used with respect to the time variable. He applied the method to the single layer heat operator equation as well as to the hypersingular heat operator equation. In both cases, quasi-uniform meshes are allowed. For the single layer heat operator equation (paper II), the trial functions were tensor products of piecewise linear (space) and piecewise constant splines. The hypersingular heat operator decreases smoothness. Therefore, numerical approximation of the solution of the hypersingular heat operator equation (paper III) requires a smoother boundary density. The lowest order trial functions are tensor products of piecewise cubic (space) and piecewise linear splines. In articles II and III, stability and suboptimal convergence of the resulting Petrov-Galerkin method was proved. In the case of the hypersingular heat operator equation, a stability condition between the mesh parameters appears in contrast to the case of the single layer heat operator equation which is unconditionally stable.

It is remarkable here that, in the one-dimensional case, optimal order error estimates with respect to the L^2-norm are available, in contrast to the multidimensional and time-dependent cases where the proof of optimal order error estimates is not known.

Another line of research started when Ruotsalainen & Wendland [53] converted a nonlinear Neumann boundary value problem for the Laplacian to a nonlinear boundary integral equation. They proved that the resulting nonlinear integral equation is uniquely solvable, and the spline Galerkin boundary element method is stable and convergent. Their approach was based on the theory of monotone operators. Later, the idea was extended to the collocation method by Ruotsalainen & Saranen [52], and to L^p-theory by Eggermont & Saranen [20]. In their conference paper Hamina, Ruotsalainen & Saranen [23] proposed an approximate collocation scheme such that the nonlinearity is replaced by its L^2-orthogonal projection. An asymptotic error estimate of order $\mathcal{O}(h^{\frac{3}{2}})$ with respect to the $H^{\frac{1}{2}}$-norm was presented for the solution of the resulting approximate collocation equation. A numerical example was also given. The proofs, as well as several extensions, - optimal order L^2-convergence, the effect of the numerical integration - were published in paper IV. The proof of the L^2-convergence is based on the results by Saranen [55].

Ruotsalainen [48] considered a nonlinear mixed boundary value problem for the Laplacian in the plane. He converted the problem into a system of boundary integral equations and proved quasi-optimality estimates for the Galerkin boundary element solutions for this nonlinear operator equation. In article [49] Ruotsalainen adapted the analysis to cases where the nonlinearity satisfies a polynomial growth condition. He considered the indirect formulation which transforms the boundary value problem for the Laplacian into a nonlinear boundary integral equation. The results on stability and optimal order error estimates for the Galerkin method in L^p spaces are given. The corresponding convergence results for the collocation method are presented in Ruotsalainen [50].

Analogously to the elliptic case, the Neumann problem for the homogeneous heat equation with vanishing initial data can be reduced to a nonlinear boundary integral equation. Therefore, it was natural to extend the linear theory to certain nonlinear problems by applying the theory of monotone operators to boundary integral operators acting on anisotropic Sobolev spaces. Hsiao & Saranen [30] obtained stability and optimal order convergence for the Galerkin approximation. Ruotsalainen & Saranen [52] presented convergence results for the Galerkin approximation when the nonlinearity had a polynomial growth condition. In paper V Hamina proposed a method where interpolation is used with respect to the space variable and orthogonal projection is used with respect to the time variable. The trial functions are tensor products of piecewise linear (space) and piecewise constant splines. Hamina obtained stability and optimal order convergence for the resulting Petrov-Galerkin approximation. In contrast to the above mentioned Petrov-Galerkin method, stability and convergence of the collocation method is still an open problem.

1.2 Boundary integral formulations

Considering the heat conduction problem, we first introduce a boundary integral approach for solution of the homogeneous heat equation with the given nonlinear Neumann type boundary condition and vanishing initial data. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the smooth boundary $\Gamma = \partial\Omega$. With $0 < T < \infty$, we have the heat conduction problem, (see [30], [52] and paper V)

$$\begin{cases} \partial_t \Phi - \Delta\Phi = 0, & \text{in } Q_T = \Omega \times (0,T), \\ \partial_{\mathbf{n}}\Phi(\mathbf{x},t) = -F(\mathbf{x},t,\Phi(\mathbf{x},t)) + g_{N\Gamma}(\mathbf{x},t), & \text{on } \Sigma_T = \Gamma \times (0,T), \\ \Phi(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega. \end{cases} \tag{1.1}$$

In the stationary case, the model problem reduces to the following potential equation with a given nonlinear Neumann type boundary condition (see [53], [51] and paper IV)

$$\begin{cases} \Delta\Phi = 0, & \text{in } \Omega, \\ \partial_{\mathbf{n}}\Phi(\mathbf{x}) = -F(\mathbf{x},\Phi(\mathbf{x})) + g_{N\Gamma}(\mathbf{x}), & \text{on } \Gamma. \end{cases} \tag{1.2}$$

Another time dependent model problem is the homogeneous heat equation with the given Dirichlet type boundary condition and vanishing initial data (see [14], [29], and paper I)

$$\begin{cases} \partial_t \Phi - \Delta\Phi = 0, & \text{in } Q_T = \Omega \times (0,T), \\ \Phi(\mathbf{x}) = g_{D\Gamma}(\mathbf{x},t), & \text{on } \Sigma_T = \Gamma \times (0,T), \\ \Phi(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega. \end{cases} \tag{1.3}$$

The classical single layer and double layer heat potentials $\mathcal{V}$ and $\mathcal{W}$ are defined by the expressions

$$(\mathcal{V}\sigma)(\mathbf{x},t) = \int_0^t \int_\Gamma \sigma(\mathbf{y},\tau)\mathcal{E}(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{d}\Gamma_{\mathbf{y}}\,\mathrm{d}\tau,\ \left((\mathbf{x},t)\in Q_T\cup Q_T^c\right), \tag{1.4}$$

$$(\mathcal{W}\mu)(\mathbf{x},t) = \int_0^t \int_\Gamma \mu(\mathbf{y},\tau)\partial_{\mathbf{n}_y}\mathcal{E}(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{d}\Gamma_{\mathbf{y}}\,\mathrm{d}\tau,\ \left((\mathbf{x},t)\in Q_T\cup Q_T^c\right), \tag{1.5}$$

where $Q_T^c = \Omega^c \times (0,T)$, $\Omega^c = \mathbb{R}^2\backslash\bar{\Omega}$, $\partial_{\mathbf{n}_y}$ is the exterior normal derivative and

$$\mathcal{E}(\mathbf{x},t) = \begin{cases} \frac{1}{4\pi t}\mathrm{e}^{-\frac{|\mathbf{x}|^2}{4t}}, & t>0,\\ 0, & t\le 0,\end{cases} \tag{1.6}$$

denotes the fundamental solution of the two-dimensional heat equation. For sufficiently smooth boundary densities σ and μ, the heat potential

$$\Phi(\mathbf{x},t) = (\mathcal{V}\sigma)(\mathbf{x},t) - (\mathcal{W}\mu)(\mathbf{x},t), \quad ((\mathbf{x},t)\in Q_T), \tag{1.7}$$

combined with the boundary behaviour of the single layer and double layer potential and their normal derivatives $\partial_{\mathbf{n}}\mathcal{V}$, $\partial_{\mathbf{n}}\mathcal{W}$ yields

$$\Phi|_{\Sigma_T} = \mathcal{S}_\Gamma\sigma + (\tfrac{1}{2}I - \mathcal{D}_\Gamma)\mu \tag{1.8}$$
$$\partial_{\mathbf{n}}\Phi|_{\Sigma_T} = (\tfrac{1}{2}I + \mathcal{D}'_\Gamma)\sigma + \mathcal{H}_\Gamma\mu. \tag{1.9}$$

This is, in fact, a compact way to write the well known boundary behaviour

$$\mathcal{V}\sigma|_{\Sigma_T} = \mathcal{S}_\Gamma\sigma, \tag{1.10}$$
$$\mathcal{W}\mu|_{\Sigma_T} = (\mathcal{D}_\Gamma - \tfrac{1}{2}I)\mu, \quad \text{(interior limit)}. \tag{1.11}$$

Here, the boundary integral operators $\mathcal{S}_\Gamma$, $\mathcal{D}_\Gamma$, $\mathcal{D}'_\Gamma$ and $\mathcal{H}_\Gamma$ with corresponding representations are the single layer heat operator

$$(\mathcal{S}_\Gamma\sigma)(\mathbf{x},t) = \int_0^t \int_\Gamma \sigma(\mathbf{y},\tau)\mathcal{E}(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{d}\Gamma_{\mathbf{y}}\,\mathrm{d}\tau, \quad (\mathbf{x}\in\Gamma) \tag{1.12}$$

the double layer heat operator

$$(\mathcal{D}_\Gamma\mu)(\mathbf{x},t) = \int_0^t \int_\Gamma \mu(\mathbf{y},\tau)\partial_{\mathbf{n}_y}\mathcal{E}(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{d}\Gamma_{\mathbf{y}}\,\mathrm{d}\tau, \quad (\mathbf{x}\in\Gamma) \tag{1.13}$$

the spatial adjoint of the double layer heat operator

$$(\mathcal{D}'_\Gamma\sigma)(\mathbf{x},t) = \int_0^t \int_\Gamma \sigma(\mathbf{y},\tau)\partial_{\mathbf{n}_x}\mathcal{E}(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{d}\Gamma_{\mathbf{y}}\,\mathrm{d}\tau, \quad (\mathbf{x}\in\Gamma) \tag{1.14}$$

and the hypersingular heat operator

$$
\begin{aligned}
(\mathcal{H}_\Gamma \mu)(\mathbf{x},t) &= -\partial_{\mathbf{n}_x} \int_0^t \int_\Gamma \mu(\mathbf{y},\tau)\partial_{\mathbf{n}_y}\mathcal{E}(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{d}\Gamma_{\mathbf{y}}\,\mathrm{d}\tau, \qquad (1.15)\\
&= -\int_0^t \int_\Gamma \mu(\mathbf{y},\tau)\partial_{\mathbf{n}_x}\partial_{\mathbf{n}_y}\mathcal{E}(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{d}\Gamma_{\mathbf{y}}\,\mathrm{d}\tau, \quad (\mathbf{x}\in\Gamma).
\end{aligned}
$$

The direct boundary integral method is, in the case of the Dirichlet problem (1.3), based on the heat potential

$$\Phi = \mathcal{V}\sigma - \mathcal{W}g_{D\Gamma} \qquad (1.16)$$

combined with the boundary relation (1.8). This approach yields the single layer heat operator equation of the first kind (see papers I and II)

$$\mathcal{S}_\Gamma \sigma = (\tfrac{1}{2}I + \mathcal{D}_\Gamma)g_{D\Gamma}. \qquad (1.17)$$

For the linear Neumann problem (1.1) with $F = 0$, the heat potential

$$\Phi = \mathcal{V}g_{N\Gamma} - \mathcal{W}\mu \qquad (1.18)$$

together with the boundary relation (1.9) yields the hypersingular heat operator equation of the first kind (see [29] and paper III)

$$\mathcal{H}_\Gamma \mu = (\tfrac{1}{2}I - \mathcal{D}'_\Gamma)g_{N\Gamma}. \qquad (1.19)$$

In the following, the Nemitsky operator is defined by $\mathcal{F}(\mu)(\mathbf{x},t) = F(\mathbf{x},t,\mu(\mathbf{x},t))$. The properties of the nonlinearity are given later after fixing the spaces of functions. In the direct boundary integral method, the heat potential Φ in (1.1) is given by means of the representation

$$\Phi = \mathcal{V}(g_{N\Gamma} - \mathcal{F}(\mu)) - \mathcal{W}\mu, \qquad (1.20)$$

which by (1.10) and (1.11) yields the nonlinear boundary integral equation of the second kind, (see [30] and paper V)

$$(\tfrac{1}{2}I + \mathcal{D}_\Gamma)\mu + \mathcal{S}_\Gamma\mathcal{F}(\mu) = \mathcal{S}_\Gamma g_{N\Gamma}. \qquad (1.21)$$

We observe that the linear case $F = 0$ reduces to the linear boundary integral equation of the second kind

$$(\tfrac{1}{2}I + \mathcal{D}_\Gamma)\mu = \mathcal{S}_\Gamma g_{N\Gamma}, \qquad (1.22)$$

which has a well established theory [14], [16], [29], [40].

In certain anisotropic Sobolev spaces, equations (1.17), (1.19), (1.21) and (1.22) are uniquely solvable and the solutions admit the interpretations

$$
\begin{aligned}
\Phi\big|_{\Sigma_T} &= \mu, \qquad (1.23)\\
\partial_{\mathbf{n}}\Phi\big|_{\Sigma_T} &= \sigma. \qquad (1.24)
\end{aligned}
$$

On the other hand, the indirect single layer potential approach

$$\Phi = \mathcal{V}(\tilde{\sigma}), \tag{1.25}$$

yields for the boundary density the nonlinear boundary integral equation

$$(\tfrac{1}{2}I + \mathcal{D}'_\Gamma)\tilde{\sigma} + \mathcal{F}(\mathcal{S}_\Gamma\tilde{\sigma}) = g_{N\Gamma}. \tag{1.26}$$

This approach has been applied to the case of the boundary value problem of the Laplacian with a nonlinear boundary condition [48], [49], [50] as well as to the case of the boundary and initial value problem of the heat equation [52]. In these articles, unique solvability of the resulting nonlinear operator equation has been established, as well as stability and convergence of the chosen numerical method in appropriate spaces of functions.

1.3 Reduction to an integral equation

For linear problems, we perform the analysis in spaces of functions which are periodic with respect to the space variable. In the case of nonlinear problems, this reduction is not used.

We assume that the boundary curve has the smooth, regular, one-periodic parametric representation $\mathbf{x} : \mathbb{R} \to \Gamma$ such that the Jacobian $|\mathbf{x}'(\theta)|$ is strictly positive. Let $\mathbb{R}^2_T = \mathbb{R} \times (0, T)$. Then, the associated heat operators can be written as

$$\begin{aligned}
(\mathcal{S}u)(\theta,t) &= \int_0^t\int_0^1 u(\varphi,\tau)\mathcal{E}(\mathbf{x}(\theta)-\mathbf{x}(\varphi),t-\tau)|\mathbf{x}'(\varphi)|\,\mathrm{d}\varphi\,\mathrm{d}\tau,\\
(\mathcal{D}u)(\theta,t) &= \int_0^t\int_0^1 u(\varphi,\tau)\frac{\mathbf{n}_\varphi\cdot(\mathbf{x}(\theta)-\mathbf{x}(\varphi))}{2(t-\tau)}\mathcal{E}(\mathbf{x}(\theta)-\mathbf{x}(\varphi),t-\tau)|\mathbf{x}'(\varphi)|\,\mathrm{d}\varphi\,\mathrm{d}\tau,\\
(\mathcal{D}'u)(\theta,t) &= \int_0^t\int_0^1 u(\varphi,\tau)\frac{\mathbf{n}_\theta\cdot(\mathbf{x}(\theta)-\mathbf{x}(\varphi))}{2(t-\tau)}\mathcal{E}(\mathbf{x}(\theta)-\mathbf{x}(\varphi),t-\tau)|\mathbf{x}'(\varphi)|\,\mathrm{d}\varphi\,\mathrm{d}\tau,\\
(\mathcal{H}u)(\theta,t) &= -\int_0^t\int_0^1 u(\varphi,\tau)\partial_{\mathbf{n}(\theta)}\partial_{\mathbf{n}(\varphi)}\mathcal{E}(\mathbf{x}(\theta)-\mathbf{x}(\varphi),t-\tau)|\mathbf{x}'(\varphi)|\,\mathrm{d}\varphi\,\mathrm{d}\tau.
\end{aligned}$$

We emphasize that the kernel function corresponding to the hypersingular heat operator has strong singularity, and appropriate regularization methods are needed in actual numerical computations.

Using the notations $g_D(\theta,t) = g_{D\Gamma}(\mathbf{x}(\theta),t)$, $g_N(\theta,t) = g_{N\Gamma}(\mathbf{x}(\theta),t)$, we obtain $(\mathcal{S}_\Gamma\sigma)(\mathbf{x}(\theta),t) = (\mathcal{S}u)(\theta,t)$, with $u(\theta,t) = \sigma(\mathbf{x}(\theta),t)$ and $(\mathcal{H}_\Gamma\mu)(\mathbf{x}(\theta),t) = (\mathcal{H}u)(\theta,t)$ with $u(\theta,t) = \mu(\mathbf{x}(\theta),t)$. Thus, equations (1.17) and (1.19) on the boundary transform to ordinary integral equations

$$(\mathcal{S}u)(\theta,t) = \tfrac{1}{2}g_D(\theta,t) + (\mathcal{D}g_D)(\theta,t), \tag{1.27}$$

$$(\mathcal{H}u)(\theta,t) = \tfrac{1}{2}g_N(\theta,t) - (\mathcal{D}'g_N)(\theta,t), \tag{1.28}$$

over the domain $(\theta,t)\in\mathbb{R}\times[0,T]$.

We recall that sufficiently smooth functions, which are one-periodic with respect to variable θ, admit the Fourier representation

$$u(\theta,t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \hat{u}(n,\eta)\mathrm{e}^{\mathrm{i}n2\pi\theta+\mathrm{i}\eta t}\,\mathrm{d}\eta,$$

where the Fourier coefficient is defined by $\hat{u}(n,\eta) = \int_0^1\int_{\mathbb{R}} u(\theta,t)\mathrm{e}^{-\mathrm{i}n2\pi\theta-\mathrm{i}\eta t}\,\mathrm{d}t\,\mathrm{d}\theta$. In terms of the Heaviside step function H, the heat kernel corresponding to the circular boundary of radius ρ, defined by

$$\mathcal{E}_\rho(\theta-\varphi,t-\tau) = H(t-\tau)\,\frac{\exp\big(-\frac{\rho^2\sin^2\pi(\theta-\varphi)}{(t-\tau)}\big)}{4\pi(t-\tau)},$$

is a function of the difference of the coordinates. Therefore, the associated boundary integral operators $\mathcal{S}_\rho$, $\mathcal{D}_\rho$, $\mathcal{D}'_\rho$ and $\mathcal{H}_\rho$ are of a convolutional type

$$(\mathcal{S}_\rho u)(\theta,t) = \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} s(n,\eta)\hat{u}(n,\eta)\mathrm{e}^{\mathrm{i}n2\pi\theta+\mathrm{i}\eta t}\,\mathrm{d}\eta, \tag{1.29}$$

$$(\mathcal{D}_\rho u)(\theta,t) = \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} d(n,\eta)\hat{u}(n,\eta)\mathrm{e}^{\mathrm{i}n2\pi\theta+\mathrm{i}\eta t}\,\mathrm{d}\eta, \tag{1.30}$$

$$(\mathcal{D}'_\rho u)(\theta,t) = \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} d'(n,\eta)\hat{u}(n,\eta)\mathrm{e}^{\mathrm{i}n2\pi\theta+\mathrm{i}\eta t}\,\mathrm{d}\eta, \tag{1.31}$$

$$(\mathcal{H}_\rho u)(\theta,t) = \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} h(n,\eta)\hat{u}(n,\eta)\mathrm{e}^{\mathrm{i}n2\pi\theta+\mathrm{i}\eta t}\,\mathrm{d}\eta. \tag{1.32}$$

As an example, we give here the explicit representation formula of the single layer heat operator

$$\begin{aligned}(\mathcal{S}_\rho u)(\theta,t) &= \int_0^t\int_0^1 u(\varphi,\tau)\frac{\exp\big(-\frac{\rho^2\sin^2\pi(\theta-\varphi)}{(t-\tau)}\big)}{4\pi(t-\tau)}\rho\,\mathrm{d}\varphi\,\mathrm{d}\tau\\ &= \tfrac{\rho}{2\pi}\int_{-\infty}^{+\infty}\int_0^1 \hat{u}(\varphi,\eta)K_0(2\rho\sqrt{\mathrm{i}\eta}\,\sin\pi|\theta-\varphi|)\,\mathrm{d}\varphi\,\mathrm{d}\eta\\ &= \tfrac{\rho}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} I_n(\rho\sqrt{\mathrm{i}\eta})\,K_n(\rho\sqrt{\mathrm{i}\eta})\hat{u}(n,\eta)\mathrm{e}^{\mathrm{i}n2\pi\theta+\mathrm{i}\eta t}\,\mathrm{d}\eta.\end{aligned}$$

Here I_n and K_n are the modified Bessel functions, [1]. See also [39]. For the hypersingular heat operator, representation (1.32) follows from the fact that the kernel

$$-\partial_{\mathbf{n}(\theta)}\partial_{\mathbf{n}(\varphi)}\mathcal{E}_\rho(\theta-\varphi,t-\tau) = -\partial_{\mathbf{n}(\theta)}\partial_{\mathbf{n}(\varphi)}\Big(\frac{\exp\big(-\frac{\rho^2\sin^2\pi(\theta-\varphi)}{t-\tau}\big)}{4\pi(t-\tau)}\Big) = \tilde{h}(\theta-\varphi,t-\tau)$$

is a function of the difference of the coordinates. Similar considerations are true for $\mathcal{D}_\rho$ and $\mathcal{D}'_\rho$.

1.4 Spaces

Let $H^s(\Gamma)$, $s \in \mathbb{R}$ (see [2]) be the usual Sobolev space equipped with the norm $\|u\|_s = (u\,,\,u)_s^{\frac{1}{2}}$. In particular, $L^2(\Gamma) = H^0(\Gamma)$ and $(u\,,\,v)_{0,\Gamma} = \int_\Gamma u(\mathbf{y})v(\mathbf{y})\mathrm{d}\Gamma_{\mathbf{y}}$. For $s > 0$ the norm is given by

$$\|u\|_s^2 = \sum_{k=0}^{m} \|D^k u\|_0^2 + \int_\Gamma \int_\Gamma \frac{|D^m u(\mathbf{x}) - D^m u(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{1+2\delta}} \mathrm{d}\Gamma_{\mathbf{x}} \mathrm{d}\Gamma_{\mathbf{y}},$$

where $s = m + \delta$, $0 < \delta < 1$ and the derivative is taken with respect to the arc length. The negative order Sobolev spaces are defined by duality with respect to the L^2- inner product.

Theoretical analysis of the heat equation leads to the concept of anisotropic Sobolev spaces. Here, we concentrate on the anisotropic spaces on the space-time boundary. These are defined as intersection spaces ([33] p. 87, [34] p. 8).

$$H^{r,s}(\Sigma_T) = H^0((0,T);H^r(\Gamma)) \cap H^s((0,T);H^0(\Gamma)), \quad (r,s \geq 0).$$

In particular, we describe the vanishing initial condition in terms of the subspace

$$H^{r,s}_{00}(\Sigma_T) = \{u \mid u = U|_{\Sigma_T} \,:\, U \in H^{r,s}(\Sigma_T),\, U(\cdot,t) = 0,\, t < 0\}, \quad (0 < T < \infty).$$

The negative order spaces $H^{-r,-s}_{00}(\Sigma_T)$ for $0 < r < 1$, $0 < s < \frac{1}{2}$ are defined by duality $H^{-r,-s}_{00}(\Sigma_T) = (H^{r,s}_{00}(\Sigma_T))'$.

Having introduced the parametric representation for the boundary curve, it is enough to consider functions which are one-periodic with respect to the spatial variable. First, for any $r \in \mathbb{R}$, let H^r be the Sobolev space of one-periodic functions on $\mathbb{R}$. The anisotropic spaces are defined by

$$H^{r,s} := H^0(\mathbb{R};H^r) \cap H^s(\mathbb{R};H^0), \quad (r,s \geq 0).$$

In terms of the Fourier transform with respect to the time variable

$$(\mathcal{F}_t u)(\theta,\eta) = \int_{-\infty}^{\infty} u(\theta,t)\mathrm{e}^{-\mathrm{i}\eta t}\,\mathrm{d}t,$$

the norm of the space $H^{r,s}$ is given by

$$\|u\|^2_{H^{r,s}} = \int_{-\infty}^{\infty} (\|(\mathcal{F}_t u)(\cdot,\eta)\|^2_{H^r} + |\eta|^{2s}\|(\mathcal{F}_t u)(\cdot,\eta)\|^2_{H^0})\,\mathrm{d}\eta. \tag{1.33}$$

The space $H^{r,s}(\mathbb{R}^2_T)$, $r,s \geq 0$, $0 < T \leq \infty$ is the space of restrictions to $\mathbb{R}^2_T$ of functions belonging to $H^{r,s}$ equipped with the infimum norm

$$H^{r,s}(\mathbb{R}^2_T) = \{\, u \mid u = U|_{\mathbb{R}^2_T} \,:\, U \in H^{r,s} \,\}, \tag{1.34}$$

$$\|u\|_{H^{r,s}(\mathbb{R}^2_T)} = \inf\{\, \|U\|_{H^{r,s}} \mid u = U|_{\mathbb{R}^2_T}\}. \tag{1.35}$$

Again, we have the Hilbert space

$$H^{r,s}(\mathbb{R}^2_T) = H^0((0,T);H^r) \cap H^s((0,T);H^0), \quad (0 < T \le \infty)$$

endowed with the norm

$$\|u\|^2_{H^{r,s}(\mathbb{R}^2_T)} = \int_0^T \|u(\cdot,t)\|^2_{H^r}\,\mathrm{d}t + \|u\|^2_{H^s((0,T);H^0)}, \quad (0 < T < \infty).$$

We also use equivalent norms defined by

$$\|u\|^2_{H^{r,s}(\mathbb{R}^2_T)} = \int_0^T \|u(\cdot,t)\|^2_{H^r}\,\mathrm{d}t + \int_0^T\int_0^T \frac{\|u(\cdot,t)-u(\cdot,\tau)\|^2_{H^0}}{|t-\tau|^{1+2s}}\,\mathrm{d}t\,\mathrm{d}\tau, \quad (0 < s < 1)$$

and

$$\|u\|^2_{H^{r,s}(\mathbb{R}^2_T)} = \int_0^T \|u(\cdot,t)\|^2_{H^r}\,\mathrm{d}t + \int_0^1 \|u(\theta,\cdot)\|^2_{H^s(0,T)}\,\mathrm{d}\theta, \quad (0 \le s \le 1). \tag{1.36}$$

Moreover, we introduce the subspace

$$H^{r,s}_{00}(\mathbb{R}^2_T) = \{u \mid u = U|_{\mathbb{R}^2_T} : U \in H^{r,s},\, U(\cdot,t) = 0,\, t < 0\}, \quad (0 < T < \infty).$$

Thus, for example $H^{r,s}_{00}(\mathbb{R}^2_\infty)$ is the space of those functions in $H^{r,s}(\mathbb{R}^2_\infty)$ for which the zero extension with respect to the time variable remains in $H^{r,s}$. Finally, we need the negative order space $H^{-r,-s}_{00}(\mathbb{R}^2_T)$ for $0 < r < 1$, $0 < s < \frac{1}{2}$, which is defined by duality $H^{-r,-s}_{00}(\mathbb{R}^2_T) = (H^{r,s}_{00}(\mathbb{R}^2_T))'$.

Next, we consider some sets of continuous (respectively smooth) functions, which are *one-periodic* with respect to the first argument. Let $\mathcal{C}_1(\mathbb{R}^2)$ (resp. $\mathcal{C}^\infty_1(\mathbb{R}^2)$) be the space of continuous (resp. infinitely smooth) functions. The space $\mathcal{C}_{10}(\mathbb{R}^2)$ (resp. $\mathcal{C}^\infty_{10}(\mathbb{R}^2)$) consists of continuous (resp. infinitely smooth) functions u, which are defined on $\mathbb{R}^2$, such that the support $\operatorname{supp} u(\theta,\cdot) \subset K$, where $K \subset \mathbb{R}$ is a compact set. The spaces of restrictions $\mathcal{C}_1(\mathbb{R}^2_T)$ (resp. $\mathcal{C}^\infty_1(\mathbb{R}^2_T)$) consist of continuous (resp. infinitely smooth) functions such that $u = U|_{\mathbb{R}^2_T}$. The space $\mathcal{C}_{10}(\mathbb{R}^2_T)$ (resp. $\mathcal{C}^\infty_{10}(\mathbb{R}^2_T)$) consists of continuous (resp. infinitely smooth) functions u, which are defined on $\mathbb{R}^2_T$, such that the support $\operatorname{supp} u(\theta,\cdot) \subset K$, where $K \subset \mathbb{R}$ is a compact set. In order to describe the initial condition, we introduce the subspace

$$\mathcal{C}^\infty_{00}(\mathbb{R}^2_T) = \{u \mid u = U|_{\mathbb{R}^2_T} : U \in \mathcal{C}^\infty_1(\mathbb{R}^2),\, U(\cdot,t) = 0,\, t < 0\}, \quad (0 < T \le \infty).$$

This subspace enjoys the property that the embedding $\mathcal{C}^\infty_{00}(\mathbb{R}^2_T) \subset H^{r,s}_{00}(\mathbb{R}^2_T)$ is dense for all $r,s \in \mathbb{R}$, if $s \ge 0$.

A comprehensive proof of the following Sobolev embedding theorem is given in [31] or [32]. For the convenience of the reader the proof is repeated here.

Theorem 1 *In the anisotropic spaces the embeddings*

$$H^{r,\frac{r}{2}}(\mathbb{R}^2) \quad \subset \quad \mathcal{C}(\mathbb{R}^2), \quad H^{r,\frac{r}{2}}(\mathbb{R}^2_T) \subset \mathcal{C}(\overline{\mathbb{R}}^2_T), \quad H^{r,\frac{r}{2}}_{00}(\mathbb{R}^2_T) \subset \mathcal{C}_{00}(\overline{\mathbb{R}}^2_T)$$

are continuous provided that $r > \frac{3}{2}$.

Proof. We present the proof for $H^{r,\frac{r}{2}} = H^{r,\frac{r}{2}}(\mathbb{R}^2)$. Note that the anisotropic norm (1.33) is given by

$$\|U\|^2_{H^{r,\frac{r}{2}}} = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} (1+|n|+|\eta|^{\frac{1}{2}})^{2r} |\hat{U}(n,\eta)|^2 \,\mathrm{d}\eta, \quad (U \in H^{r,\frac{r}{2}}).$$

Since $\mathcal{C}^\infty_{10}(\mathbb{R}^2)$ is dense in $H^{r,\frac{r}{2}}$, $r \geq 0$, it is enough to present the proof for $U \in \mathcal{C}^\infty_{10}(\mathbb{R}^2)$. The Cauchy-Schwarz inequality applied to the Fourier representation

$$U(\theta,t) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \hat{U}(n,\eta) \mathrm{e}^{\mathrm{i}n2\pi\theta + \mathrm{i}\eta t} \,\mathrm{d}\eta$$

gives

$$\begin{aligned} |U(\theta,t)| &\leq \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} |\hat{U}(n,\eta)| \,\mathrm{d}\eta \\ &= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} (1+|n|+|\eta|^{\frac{1}{2}})^{-r} (1+|n|+|\eta|^{\frac{1}{2}})^{r} |\hat{U}(n,\eta)| \,\mathrm{d}\eta \\ &\leq \frac{1}{2\pi} \sqrt{\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} (1+|n|+|\eta|^{\frac{1}{2}})^{-2r}} \sqrt{\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} (1+|n|+|\eta|^{\frac{1}{2}})^{2r} |\hat{U}(n,\eta)|^2 \,\mathrm{d}\eta}. \end{aligned}$$

The result follows, since the function $I(r) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} (1+|n|+|\eta|^{\frac{1}{2}})^{-2r}$ is bounded, if $r > \frac{3}{2}$. The other embedding results follow by applying the infimum characterization of the norm for the subspaces of restrictions. □

The anisotropic Sobolev spaces enjoy a similar compactness property as ordinary Sobolev spaces.

Theorem 2 *Assume that* $0 \leq r_1 \leq r$ *and* $0 \leq s_1 \leq s$. *Then the embedding* $H^{r,s}_{00}(\mathbb{R}^2_T) \subset H^{r_1,s_1}_{00}(\mathbb{R}^2_T)$ *is compact for any finite* $T > 0$.

The proof is given in article [29], Theorem 3.1 p. 93.

1.5 Mapping properties

The convergence analysis and error estimates are based on the mapping properties of the heat operators in the anisotropic Sobolev spaces. In this work, we just quote the results. The proofs are lengthy and technical, and can be found in the given articles. Throughout this work c, c', c_i etc. are generic constants, which may change in size, but are always independent of the mesh parameters. In the following, the duality pairing extends the $L^2(\mathbb{R}^2_T)$-inner product.

Theorem 3 *The single layer heat operator* $\mathcal{S} : H_{00}^{r,\frac{r}{2}}(\mathbb{R}_T^2) \to H_{00}^{r+1,\frac{r+1}{2}}(\mathbb{R}_T^2)$ *is an isomorphism for all* $r \geq -\frac{1}{2}$*. Furthermore, it is coercive such that*

$$(\mathcal{S}w\,,\,w) \geq c\,\|w\|^2_{-\frac{1}{2},-\frac{1}{4}} \quad \textit{for all } \; w \in H_{00}^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}_T^2). \tag{1.37}$$

For the proof see [3], [37], [14], [28]. The mapping properties for the double layer heat operator and its spatial adjoint are given in

Theorem 4 *The mapping properties of the operators* $\mathcal{D}$ *and* $\mathcal{D}'$ *are:*

(i) $\mathcal{D}, \mathcal{D}' : H^{r,\frac{r}{2}}(\mathbb{R}_T^2) \to H^{r+1,\frac{r+1}{2}}(\mathbb{R}_T^2)$ *are bounded for all* $r \geq -\frac{1}{2}$.

(ii) $\mathcal{D}, \mathcal{D}' : H^{r,\frac{r}{2}}(\mathbb{R}_T^2) \to H^{r,\frac{r}{2}}(\mathbb{R}_T^2)$ *are compact for all* $r \geq -\frac{1}{2}$.

(iii) For any $a \neq 0$, $r \geq -\frac{1}{2}$ *the operators* $aI+\mathcal{D}$, $aI+\mathcal{D}' : H^{r,\frac{r}{2}}(\mathbb{R}_T^2) \to H^{r,\frac{r}{2}}(\mathbb{R}_T^2)$ *are isomorphisms.*

This result is proved in paper [29], Theorem 4.2. See also [14], Corollary 3.14.

Theorem 5 *The hypersingular heat operator* $\mathcal{H} : H_{00}^{r,\frac{r}{2}}(\mathbb{R}_T^2) \to H_{00}^{r-1,\frac{r-1}{2}}(\mathbb{R}_T^2)$ *is an isomorphism for* $r \geq \frac{1}{2}$*. Furthermore, it is coercive such that*

$$\big(\mathcal{H}w\,,\,w\big) \quad \geq \quad c\,\|w\|^2_{\frac{1}{2},\frac{1}{4}} \quad \textit{for all } \; w \in H_{00}^{\frac{1}{2},\frac{1}{4}}(\mathbb{R}_T^2). \tag{1.38}$$

The proof is given in [29], Theorem 4.4 or [14], Corollary 3.13. The following coercivity estimate is used for the analysis of the nonlinear boundary integral equation. The proof can be found in [30], Theorem 4.1.

Theorem 6 *The operator* $\mathcal{S}_\Gamma^{-1}(\frac{1}{2}I + \mathcal{D}_\Gamma) : H_{00}^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \to H_{00}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ *is an isomorphism. Further, it is strongly coercive, i.e. there exists a positive constant* c *such that*

$$\big(\mathcal{S}_\Gamma^{-1}(\tfrac{1}{2}I + \mathcal{D}_\Gamma)w\,,\,w\big)_{\Sigma_T} \geq c\,\|w\|^2_{\frac{1}{2},\frac{1}{4}} \quad \textit{for all } w \in H_{00}^{\frac{1}{2},\frac{1}{4}}(\Sigma_T). \tag{1.39}$$

The kernel functions of the heat operators are, in the case of circular boundaries, functions of the difference of the coordinates. Therefore, the heat operators formally commute with the partial differentation operator. Also, the mean value functional $(J\,w)(t) := \int_0^1 w(\theta,t)\,\mathrm{d}\theta$ commutes with the heat operators.

Theorem 7 *Let* $w \in H_{00}^{r,\frac{r}{2}}(\mathbb{R}_T^2)$.

$$\textit{For } r \geq s - \tfrac{1}{2},\; \partial_\theta^s(\mathcal{S}_\rho w)(\theta,t) = (\mathcal{S}_\rho(\partial_\varphi^s w))(\theta,t),\; s = 1,2,\ldots \tag{1.40}$$

$$\textit{For } r \geq s + \tfrac{1}{2},\; \partial_\theta^s(\mathcal{H}_\rho w)(\theta,t) = (\mathcal{H}_\rho(\partial_\varphi^s w))(\theta,t),\; s = 1,2,\ldots \tag{1.41}$$

$$\textit{Moreover, } J\,(\mathcal{S}_\rho w)(t) = \mathcal{S}_\rho(J\,w)(t) \textit{ and } J\,(\mathcal{H}_\rho w)(t) = \mathcal{H}_\rho(J\,w)(t). \tag{1.42}$$

For the moment, we adopt some notations and results from [31] and [32]. Let $\mathcal{S}_\circ$ be the single layer heat operator for the circle with radius $\frac{\rho}{2\pi}$. We define

$$\begin{aligned} \underline{\partial}_\theta &:= \partial_\theta + J, \\ \mathcal{K}w &:= (\mathcal{S} - \mathcal{S}_\circ)w, \quad \underline{\mathcal{K}}w = \underline{\partial}_\theta \mathcal{K} \underline{\partial}_\theta^{-1}, \\ \underline{\mathcal{S}}w &:= \mathcal{S}_\circ w + \underline{\mathcal{K}}w. \end{aligned}$$

Theorem 8 *Assume that the one-periodic parametric representation of the boundary curve is chosen such that $|\mathbf{x}'(\theta)| = \rho =$ constant > 0. Let $0 < T < \infty$. Then the operator $\underline{\mathcal{S}}$ admits the representation $\underline{\mathcal{S}} = \underline{\partial_\theta} \mathcal{S} \underline{\partial_\theta}^{-1}$, and we have the mapping properties*

$$\underline{\mathcal{S}} : H_{00}^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}_T^2) \rightarrow H_{00}^{\frac{1}{2},\frac{1}{4}}(\mathbb{R}_T^2) \text{ is an isomorphism,}$$
$$\underline{\mathcal{K}} : H_{00}^{r,\frac{r}{2}}(\mathbb{R}_T^2) \rightarrow H_{00}^{r+2,\frac{r+2}{2}}(\mathbb{R}_T^2) \text{ continuous, if } r \geq -\tfrac{1}{2}.$$

The proof of these results can be found in [31], Section 3. It is based on explicit estimation of the symbol of the perturbation operator $\mathcal{K}$. The continuity properties of the operator $\underline{\mathcal{K}}$ combined with the compactness result given in Theorem 2, show that the operator $\underline{\mathcal{K}}$ can be interpreted as a compact perturbation of the principal part $\mathcal{S}_\circ$. Thus, a standard theory of compact perturbations is available.

We emphasize that this technique generalizes the analysis of our Petrov-Galerkin methods to general, smooth boundary curves with the restriction that the parametric representation is with respect to the arc length parameter. However, these aspects are not explicitly shown here.

1.6 Spline spaces

Next, we discuss the space of approximations. Let $0 = \theta_0 < \theta_1 < \ldots < \theta_N = 1$ be a one-periodic mesh, with $h = \max\{\theta_{n+1} - \theta_n\}$. The mesh is called quasi-uniform if

$$\frac{\max\{\theta_{n+1} - \theta_n\}}{\min\{\theta_{n+1} - \theta_n\}} \leq C,$$

where $C \geq 1$ is a constant. We denote by $S_h^{d_\theta}([0,1])$ the space of one-periodic, smoothest splines of degree d_θ. Note that $S_h^{d_\theta}([0,1])$ is a N-dimensional linear space with basis functions $\{\psi_n^{d_\theta}\}_{n=1}^N$. Analogously, we define the quasi-uniform grid $0 = t_0 < t_1 < \ldots < t_M = T$, $k = \max\{t_{m+1} - t_m\}$. With respect to the time variable, we consider the lowest order spaces $S_{k,[0,T]}^{d_t}$. In practice, these are the spaces of piecewise constant splines generated by the basis functions ($m = 1, \ldots, M$)

$$\phi_m^0(t) = \begin{cases} 1, & \text{if } t_{m-1} < t < t_m, \\ 0, & \text{otherwise,} \end{cases}$$

or the spaces of piecewise linear continuous splines generated by the basis functions ($m = 1, \ldots, M$)

$$\phi_m^1(t) = \begin{cases} 1 + \frac{t - t_m}{t_m - t_{m-1}}, & \text{if } t_{m-1} < t < t_m, \\ 1 - \frac{t - t_m}{t_{m+1} - t_m}, & \text{if } t_m \leq t < t_{m+1}, \\ 0, & \text{otherwise.} \end{cases}$$

We draw the attention of the reader to the fact that the elements of the space $\varphi \in S_{k,[0,T]}^1$ satisfy the initial condition $\varphi(0) = 0$. The trial functions on the space-time boundary are tensor products

$$S_{h,k}^{d_\theta,d_t}(\mathbb{R}_T^2) := S_h^{d_\theta}([0,1]) \times S_{k,[0,T]}^{d_t} = \{v \,|\, v(\theta,t) = \sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_{m,n}\psi_n^{d_\theta}(\theta)\phi_m^{d_t}(t)\},$$

where $\{\psi_n^{d_\theta}\}_{n=1}^N$ and $\{\phi_m^{d_t}\}_{m=1}^M$ are the one-dimensional basis functions. We notice that the inclusion

$$S_{h,k}^{d_\theta,d_t}(\mathbb{R}_T^2) \subset H_{00}^{r,\frac{r}{2}}(\mathbb{R}_T^2), \quad (r < \min\{d_\theta + \tfrac{1}{2}, 2d_t + 1\}),$$

is generally valid for the tensor product spline spaces. Our methods use pointwise values. Therefore, it is essential that the images $\mathcal{S}u$ and $\mathcal{H}u$ are continuous. In the light of the mapping properties combined with the Sobolev embedding theorem, we obtain appropriate ranges for Sobolev indices

$$\begin{array}{ll} \frac{1}{2} < r < \min\{d_\theta + \frac{1}{2}, 2d_t + 1\} & \text{for the single layer heat operator,} \\ \frac{5}{2} < r < \min\{d_\theta + \frac{1}{2}, 2d_t + 1\} & \text{for the hypersingular heat operator.} \end{array}$$

1.6.1 Inverse properties

We now recall some basic inverse and approximation results. For quasi-uniform meshes $\{\theta_n\}$ and $\{t_m\}$, the one-dimensional inverse estimates

$$\begin{array}{llll} \|\psi\|_r & \le c\, h^{-(r-s)}\|\psi\|_s, & (\psi \in S_h^{d_\theta}, & s \le r < d_\theta + \frac{1}{2}), \\ \|\phi\|_r & \le c\, k^{-(r-s)}\|\phi\|_s, & (\phi \in S_{k,[0,T]}^0, & s \le r < \frac{1}{2}), \\ \|\phi\|_r & \le c\, k^{-(r-s)}\|\phi\|_s, & (\phi \in S_{k,[0,T]}^1, & s \le r \le 1), \end{array}$$

are available [21]. In the anisotropic spaces, the following inverse estimates are valid

$$\|v\|_{r,\frac{r}{2}} \le c \max\big(h^{-r}, k^{-r/2}\big)\|v\|_{0,0}, \qquad (v \in S_{h,k}^{0,0},\ 0 \le r < \tfrac{1}{2}), \quad (1.43)$$

$$\|v\|_{0,0} \le c \max\big(h^{-r}, k^{-r/2}\big)\|v\|_{-r,-\frac{r}{2}}, \qquad (v \in S_{h,k}^{0,0},\ 0 \le r \le \tfrac{1}{2}), \quad (1.44)$$

$$\|v\|_{r,\frac{r}{2}} \le c \max\big(h^{-r}, k^{-r/2}\big)\|v\|_{0,0}, \qquad (v \in S_{h,k}^{1,0},\ 0 \le r < 1), \quad (1.45)$$

$$\|v\|_{0,0} \le c \max\big(h^{-r}, k^{-r/2}\big)\|v\|_{-r,-\frac{r}{2}}, \qquad (v \in S_{h,k}^{1,0},\ 0 \le r < 1), \quad (1.46)$$

$$\|v\|_{r,\frac{r}{2}} \le c \max\big(h^{-r}, k^{-r/2}\big)\|v\|_{0,0}, \qquad (v \in S_{h,k}^{1,1},\ 0 \le r < 1), \quad (1.47)$$

$$\|v\|_{0,0} \le c \max\big(h^{-r}, k^{-r/2}\big)\|v\|_{-r,-\frac{r}{2}}. \qquad (v \in S_{h,k}^{1,1},\ 0 \le r < 1). \quad (1.48)$$

Note that estimates (1.43) and (1.44), as well as the other two pairs, can be combined to a single estimate. For the sake of simplicity, the use of two estimates is preferred. We recall here the proof given in paper I, Lemma 5.2 p. 56-57.

Proof. Let $v \in S^{1,1}_{h,k}$ and $0 \le r < 1$. The representation formula (1.36) of the anisotropic norm, together with one-dimensional inverse estimates yields (1.47)

$$\begin{aligned}\|v\|^2_{r,\frac{r}{2}} &\le c \int_0^T \|v(\cdot,t)\|^2_{H^r}\,\mathrm{d}t + c\int_0^1 \|v(\theta,\cdot)\|^2_{H^{\frac{r}{2}}(0,T)}\,\mathrm{d}\theta \\ &\le \int_0^T c\, h^{-2r}\|v(\cdot,t)\|^2_{H^0}\,\mathrm{d}t + \int_0^1 c\, k^{-r}\|v(\theta,\cdot)\|^2_{H^0(0,T)}\,\mathrm{d}\theta \\ &\le c\left(h^{-2r} + k^{-r}\right)\|v\|^2_{0,0} \le c\left(\max(h^{-r},\, k^{-\frac{r}{2}})\right)^2\|v\|^2_{0,0}.\end{aligned}$$

We use the duality $H^{-r,-\frac{r}{2}}_{00}(\mathbb{R}^2_T) = \left(H^{r,\frac{r}{2}}_{00}(\mathbb{R}^2_T)\right)'$ for $0 \le r < 1$. Then, the Schwarz inequality implies

$$\|v\|^2_{0,0} \le \|v\|_{r,\frac{r}{2}}\|v\|_{-r,-\frac{r}{2}} \le c\,\max(h^{-r},\, k^{-\frac{r}{2}})\|v\|_{-r,-\frac{r}{2}}\|v\|_{0,0},$$

which yields (1.48). The inverse estimates (1.45), (1.46) for $0 \le r < 1$ and (1.43), (1.44) for $0 \le r < \frac{1}{2}$ follow analogously. To see (1.44) for $r = \frac{1}{2}$ we use

$$\|v\|_{0,0} \le c\max(h^{-\frac{1}{4}},\, k^{-\frac{1}{8}})\|v\|_{-\frac{1}{4},-\frac{1}{8}}, \quad (v \in S^{0,0}_{h,k}), \tag{1.49}$$

which further gives

$$\begin{aligned}\|v\|^2_{-\frac{1}{4},-\frac{1}{8}} &\le \|v\|_{0,0}\|v\|_{-\frac{1}{2},-\frac{1}{4}} \\ &\le c\,\max(h^{-\frac{1}{4}},\, k^{-\frac{1}{8}})\|v\|_{-\frac{1}{4},-\frac{1}{8}}\|v\|_{-\frac{1}{2},-\frac{1}{4}},\end{aligned}$$

and therefore

$$\|v\|_{-\frac{1}{4},-\frac{1}{8}} \le c\,\max(h^{-\frac{1}{4}},\, k^{-\frac{1}{8}})\|v\|_{-\frac{1}{2},-\frac{1}{4}}.$$

Using this with (1.49) we obtain (1.44) with $r = \frac{1}{2}$. $\square$

1.6.2 Approximation properties

The associated one-dimensional orthogonal projection operator $P^{d_\theta}_h : H^0 \to S^{d_\theta}_h$ is defined by $(P^{d_\theta}_h u\,,\, v) = (u\,,\, v)$ for all $v \in S^{d_\theta}_h$. The following approximation result is true for periodic splines, [21],

$$\|u - P^{d_\theta}_h u\|_{H^r} \le c\, h^{s-r}\|u\|_{H^s}, \quad (u \in H^s, \quad 0 \le r \le s \le d_\theta + 1,\ r < d_\theta + \tfrac{1}{2}).$$

With respect to the time variable, the orthogonal projection $P^{d_t}_k : H^0(0,T) \to S^{d_t}_{k,[0,T]}$ is defined analogously. Now, we restrict ourselves to the approximation properties

$$\begin{aligned}\|u - P^0_k u\|_{H^r(0,T)} &\le c\, k^{s-r}\|u\|_{H^s(0,T)}, \quad (u \in H^s(0,T), 0 \le r \le s \le 2,\ r < \tfrac{1}{2}) \\ \|u - P^1_k u\|_{H^r(0,T)} &\le c\, k^{s-r}\|u\|_{H^s(0,T)}, \quad (u(0) = 0, u \in H^s(0,T), 0 \le r \le 1).\end{aligned}$$

In the piecewise linear case, u satisfies the initial condition $u(0) = 0$ and $s = 1$ or $s = 2$. For $r = 0$ or $r = 1$, the result is a standard error bound for the piecewise linear case. The interpolation inequality yields the given estimate for $0 < r < 1$. Note that in the piecewise constant case, no initial condition is required. In the following, we use the same notation for two-dimensional and one-dimensional interpolation and projection operators e.g. $P_h^{d_\theta} = P_h^{d_\theta} \otimes I$.

The two-dimensional L^2-projection $P_{h,\,k}^{d_\theta,d_t} : L^2(\mathbb{R}_T^2) \to S_{h,k}^{d_\theta,d_t}(\mathbb{R}_T^2)$ is defined by requiring $(P_{h,\,k}^{d_\theta,d_t} u\,,\,v) = (u\,,\,v)$ for all $v \in S_{h,k}^{d_\theta,d_t}(\mathbb{R}_T^2)$. We have also the representation $P_{h,\,k}^{d_\theta,d_t} = P_h^{d_\theta} \otimes P_k^{d_t} = P_k^{d_t} \otimes P_h^{d_\theta}$. The low order projections have the approximation property

$$\|u - P_{h,\,k}^{1,0} u\|_{r,\frac{r}{2}} \leq \quad c\left(h^{s-r} + k^{(s-r)/2}\right)\|u\|_{s,\frac{s}{2}},\ (u \in H_{00}^{s,\frac{s}{2}}(\mathbb{R}_T^2)), \tag{1.50}$$

where $0 \leq r \leq s \leq 2$, $r < 1$. For the higher order cases, we restrict ourselves to the following basic bounds

$$\|u - P_{h,\,k}^{1,1} u\|_{r,\frac{r}{2}} \leq c\left(h^{s-r} + k^{(s-r)/2}\right)\|u\|_{s,\frac{s}{2}},\ (0 \leq r \leq 2, s = 1 \text{ or } s = 2), \tag{1.51}$$

$$\|u - P_{h,\,k}^{3,1} u\|_{r,\frac{r}{2}} \leq c\left(h^{s-r} + k^{(s-r)/2}\right)\|u\|_{s,\frac{s}{2}},\ (0 \leq r \leq 2, s = 2 \text{ or } s = 4), \tag{1.52}$$

for $u \in H_{00}^{s,\frac{s}{2}}(\mathbb{R}_T^2)$. Compare also [14], and [37].

The spline interpolation makes sense only when the associated functions are continuous. According to the Sobolev embedding theorem this is guaranteed if $u \in H^r$, $r > \frac{1}{2}$. The interpolation operator $I_h^{d_\theta} : H^r \to S_h^{d_\theta}$, $r > \frac{1}{2}$, is defined as

$$\begin{cases} I_h^{d_\theta} u \in S_h^{d_\theta} \\ (I_h^{d_\theta} u)(\tilde{\theta}_i) = u(\tilde{\theta}_i),\ i = 1, 2, \ldots, N, \end{cases}$$

where $\tilde{\theta}_i$ are the usual interpolation points

$$\tilde{\theta}_i = \begin{cases} \theta_i & \text{for all } i, \text{ when } d_\theta \text{ is odd} \\ \frac{1}{2}(\theta_{i-1} + \theta_i) & \text{for all } i, \text{ when } d_\theta \text{ is even.} \end{cases}$$

In this work, we apply only odd degree interpolation. In the periodic case, we have for quasi-uniform meshes the approximation property [21]

$$\|u - I_h^{d_\theta} u\|_{H^r} \leq c\, h^{s-r}\|u\|_{H^s}, \quad (0 \leq r \leq s \leq 2,\ r < d_\theta + \tfrac{1}{2},\ s > \tfrac{1}{2}). \tag{1.53}$$

Our Petrov-Galerkin approach is to use interpolation with respect to the space variable and L^2-projection with respect to the time variable. For this, we define the two-dimensional projection operator $Q_{h,k}^{d_\theta,d_t} := I_h^{d_\theta} \otimes P_k^{d_t} = P_k^{d_t} \otimes I_h^{d_\theta}$. The following approximation property is used for the analysis of the nonlinear problems.

Lemma 1 *Let the meshes be quasi-uniform. Then we have the estimates*

$$\|u - Q_{h,k}^{1,0} u\|_{0,0} \quad \leq \quad c\left(h^s + k^{\frac{s}{2}}\right)\|u\|_{s,\frac{s}{2}}, \quad (u \in H_{00}^{s,\frac{s}{2}}(\mathbb{R}_T^2)), \tfrac{1}{2} < s \leq 2), \tag{1.54}$$

$$\|u - Q_{h,k}^{1,0} u\|_{\frac{1}{2},\frac{1}{4}} \quad \leq \quad c\left(h^{\frac{1}{2}} + k^{\frac{1}{4}}\right)\|u\|_{1,\frac{1}{2}}, \quad (u \in H_{00}^{1,\frac{1}{2}}(\mathbb{R}_T^2)). \tag{1.55}$$

Proof. We use the decomposition

$$(I - Q^{1,0}_{h,k})u = (I - P^0_k)u - P^0_k(I - I^1_h)u \tag{1.56}$$

and bound the $H^{0,0}$-norm of the terms separately. For the first term, we have

$$\|(I - P^0_k)u\|^2_{0,0} \le \int_0^1 c\, k^{2r} \|u(\theta, \cdot)\|^2_{H^r(0,T)}\, \mathrm{d}\theta = c\, k^{2r} \|u\|^2_{0,r}, \quad (0 \le r \le 2). \tag{1.57}$$

A similar estimate combined with the L^2-stability of the P^0_k-operator gives

$$\|P^0_k(I - I^1_h)u\|^2_{0,0} \le c \int_0^T \|(I - I^1_h)u(\cdot, t)\|^2_{H^0}\, \mathrm{d}t = c\, h^{2r} \|u\|^2_{r,0}, \ (\tfrac{1}{2} < r \le 2). \tag{1.58}$$

Estimate (1.54) for $\frac{1}{2} < s \le 2$ follows by putting together (1.56) with (1.57) and (1.58)

$$\|u - Q^{1,0}_{h,k}u\|_{0,0} \le c\, k^{\frac{s}{2}} \|u\|_{0,\frac{s}{2}} + c\, h^s \|u\|_{s,0} \le c\,(\, h^s + \, k^{\frac{s}{2}})\|u\|_{s,\frac{s}{2}}.$$

Another decomposition, namely

$$u - Q^{1,0}_{h,k}u = (u - P^{1,0}_{h,\,k}u) + (P^{1,0}_{h,\,k}u - Q^{1,0}_{h,k}u),$$

inverse estimate (1.45) and the approximation properties of the projection operators yield (1.55). □

1.7 Some Petrov-Galerkin approximations

For the single layer heat operator equation $\mathcal{S}u = f$, we have analysed several methods. These include collocation methods, as well as Petrov-Galerkin type methods, where collocation is used with respect to the space variable, and the Galerkin method with respect to the time variable. In the case of the hypersingular heat operator equation $\mathcal{H}u = f$, only the latter approach is used. The purpose of this chapter is to describe a unified approach which covers the treatment given in articles I, II, and III in the light of Theorem 11.

An essential device is the equivalence of the proposed projection method to a certain Galerkin method. In order to establish this equivalence, we need the mean value functional $(J\,w)(t) := \int_0^1 w(\theta, t)\, \mathrm{d}\theta$, as well as the corresponding trapezoidal rule approximation

$$(J_\Delta w)(t) := \sum_{n=1}^{N} \frac{\theta_{n+1} - \theta_{n-1}}{2} w(\theta_n, t). \tag{1.59}$$

We define the bilinear forms

$$B^{j_\theta,j_t}(w,v) = \big(\partial_t^{j_t}(\partial_\theta^{j_\theta} + J\,)w\,,\,\partial_t^{j_t}(\partial_\theta^{j_\theta} + J\,)v\big) \tag{1.60}$$

$$B_\Delta^{j_\theta,j_t}(w,v) = \big(\partial_t^{j_t}(\partial_\theta^{j_\theta} + J_\Delta)w\,,\,\partial_t^{j_t}(\partial_\theta^{j_\theta} + J\,)v\big). \tag{1.61}$$

In particular, we interpret here $\partial_t^0 u = u$. Throughout this work, the index j_θ is related to the degree of the spline approximation by the formula $d_\theta = 2j_\theta - 1$. With respect to the time index, this convention is not applicable. In practice, j_t takes the values $j_t = 0$ or $j_t = 1$. Using the orthogonality conditions

$$(\partial_\theta^s u\,,\,J\,u) = 0,\ \ (\partial_\theta^s u\,,\,J_\Delta u) = 0,\ \ s = 1, 2, \ldots,$$

we may rewrite the bilinear forms

$$B^{j_\theta,j_t}(w,v) = \big(\partial_t^{j_t}\partial_\theta^{j_\theta} w\,,\,\partial_t^{j_t}\partial_\theta^{j_\theta} v\big) + \big(\partial_t^{j_t} J\,w\,,\,\partial_t^{j_t} J\,v\big)$$

$$B_\Delta^{j_\theta,j_t}(w,v) = \big(\partial_t^{j_t}\partial_\theta^{j_\theta} w\,,\,\partial_t^{j_t}\partial_\theta^{j_\theta} v\big) + \big(\partial_t^{j_t} J_\Delta w\,,\,\partial_t^{j_t} J\,v\big).$$

The next step is to define a family of seminorms. For brevity, we introduce the notation

$$|\!|\!|v|\!|\!|_{j_\theta,j_t;r} = \|\partial_t^{j_t}\partial_\theta^{j_\theta} v\|_{r,\frac{r}{2}} + \|\partial_t^{j_t} J\,v\|_{r,\frac{r}{2}}. \tag{1.62}$$

The indices j_θ and j_t depend on the degree of the approximating tensor product splines. The third index r depends on the order of the operator under consideration. In the cases of the single layer heat operator equation and the hypersingular heat operator equation, the index r takes the values $r = -\frac{1}{2}$, $r = +\frac{1}{2}$, respectively. In any space of functions, where the expression $|\!|\!|\cdot|\!|\!|_{j_\theta,j_t;r}$ is well defined and finite, it gives a seminorm. However, in the subspace of the approximating functions it defines a norm. This is proved in the following Lemma.

Lemma 2 *Let $r \in \mathbb{R}$, $d_\theta = 2j_\theta - 1$, and $j_t = 0$ or $j_t = 1$. Then the mapping $v \mapsto |\!|\!|v|\!|\!|_{j_\theta,j_t;r}$ defines a norm in the space $S_{h,k}^{d_\theta,d_t}(\mathbb{R}_T^2)$.*

Proof. It suffices to show that the condition

$$|\!|\!|v|\!|\!|_{j_\theta,j_t;r} = 0, \quad \text{for} \quad v \in S_{h,k}^{d_\theta,d_t}(\mathbb{R}_T^2) \tag{1.63}$$

implies $v(\theta,t) \equiv 0$. We consider only the case $j_t = 1$ ($d_t = 1$), since the other case $j_t = 0$ ($d_t = 0$) is a simplified version of the proof. Assuming (1.63), we obtain from the basis representation

$$(\partial_t\partial_\theta^{j_\theta} v)(\theta,t) = \partial_t\Bigg(\sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_{m,n}(\partial_\theta^{j_\theta}\psi_n^{d_\theta})(\theta)\phi_m^{d_t}(t)\Bigg) \tag{1.64}$$

and

$$(\partial_t J\,v)(t) = \partial_t\Bigg(\sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_{m,n}(J\,\psi_n^{d_\theta})\phi_m^{d_t}(t)\Bigg). \tag{1.65}$$

According to (1.63) and (1.64) we have $\sum_{m=1}^{M}\left(\sum_{n=1}^{N}\alpha_{m,n}(\partial_\theta^{j_\theta}\psi_n^{d_\theta})(\theta)\right)\phi_m^{d_t}(t)\equiv C(\theta)$, which, because of the initial condition $\phi_m^1(t)=0$ for all $m=1,\dots,M$, yields

$$\sum_{n=1}^{N}\alpha_{m,n}(\partial_\theta^{j_\theta}\psi_n^{d_\theta})(\theta)\equiv 0, \quad (m=1,\dots,M). \tag{1.66}$$

Analogously, from (1.63) and (1.65), we deduce that

$$\sum_{n=1}^{N}\alpha_{m,n}(J\,\psi_n^{d_\theta})=0, \quad (m=1,\dots,M). \tag{1.67}$$

Adding (1.66) and (1.67) together gives

$$\sum_{n=1}^{N}\alpha_{m,n}(\partial_\theta^{j_\theta}+J\,)\psi_n^{d_\theta}=(\partial_\theta^{j_\theta}+J\,)(\sum_{n=1}^{N}\alpha_{m,n}\psi_n^{d_\theta})\equiv 0, \quad (m=1,\dots,M).$$

The operator $(\partial_\theta^{j_\theta}+J\,)$ is an isomorphism from $S_h^{d_\theta}$ to the space $S_h^{(d_\theta-1)/2}$. Thus

$$\sum_{n=1}^{N}\alpha_{m,n}\psi_n^{d_\theta}=0, \quad (m=1,\dots,M),$$

which implies that all the coefficients $\alpha_{m,n}$ vanish, proving our statement. □

After these preparations, we are able to reformulate the proposed projection methods as certain Galerkin methods. This equivalence is carried by the approximating bilinear form defined in (1.61). Our next aim is to establish the coercivity of the approximating bilinear form with respect to the associated energy norm $\|\!|\cdot|\!\|_{j_\theta,j_t;r}$. For our method, it is enough that this property is valid in the subspace $S_{h,k}^{d_\theta,d_t}(\mathbb{R}_T^2)$.

1.7.1 Reformulation of the collocation problem

The lowest order spline collocation equations corresponding to the single layer heat operator equation are the following:

$$\text{find } u_\Delta\in S_{h,k}^{d_\theta,1} \text{ such that } I_{h,k}^{d_\theta,1}\mathcal{S}u_\Delta=I_{h,k}^{d_\theta,1}f, \quad d_\theta=1 \text{ or } d_\theta=3. \tag{1.68}$$

For a sufficiently smooth solution to the equation $\mathcal{S}u=f$, the collocation problem is equivalent to the Galerkin problem:

$$\text{find } u_\Delta\in S_{h,k}^{d_\theta,1} \text{ such that } B_\Delta^{j_\theta,1}(\mathcal{S}u_\Delta,v)=B_\Delta^{j_\theta,1}(\mathcal{S}u,v). \tag{1.69}$$

The case $d_\theta=1$ was analysed by Hamina & Saranen in paper I for a circular boundary. Their regularity requirements were described in terms of the spaces of continuous functions. Later Hämäläinen & Saranen [32] and Hämäläinen [31] reduced the regularity requirements by introducing appropriate spaces of the Sobolev type. They also generalized the analysis to cover the case of a general smooth boundary curve, provided that the parametric representation is with respect to the arc length parameter. The case $d_\theta = 3$ has not been explicitly treated in the articles.

1.7.2 Reformulation of the Petrov-Galerkin method

Another possibility is to use the projection operator $Q_{h,k}^{d_\theta,d_t}$. Then the lowest order approximations to the single layer heat operator equation are the following.

$$\text{Find } u_\Delta \in S_{h,k}^{1,0}(\mathbb{R}_T^2) \quad \text{such that} \quad Q_{h,k}^{1,0}\mathcal{S}u_\Delta = Q_{h,k}^{1,0}f. \tag{1.70}$$

$$\text{Find } u_\Delta \in S_{h,k}^{1,1}(\mathbb{R}_T^2) \quad \text{such that} \quad Q_{h,k}^{1,1}\mathcal{S}u_\Delta = Q_{h,k}^{1,1}f. \tag{1.71}$$

$$\text{Find } u_\Delta \in S_{h,k}^{3,1}(\mathbb{R}_T^2) \quad \text{such that} \quad Q_{h,k}^{3,1}\mathcal{S}u_\Delta = Q_{h,k}^{3,1}f. \tag{1.72}$$

The equivalent Galerkin method is now available.

Theorem 9 *Let $u \in H_{00}^{s,\frac{s}{2}}(\mathbb{R}_T^2)$, $s > \frac{1}{2}$ be the solution of the equation $\mathcal{S}u = f$. The function $u_\Delta \in S_{h,k}^{1,0}(\mathbb{R}_T^2)$ is a solution of problem (1.70) if and only if*

$$B_\Delta^{1,0}(\mathcal{S}u_\Delta, v) = B_\Delta^{1,0}(\mathcal{S}u, v), \quad (v \in S_{h,k}^{1,0}(\mathbb{R}_T^2)). \tag{1.73}$$

The function $u_\Delta \in S_{h,k}^{1,1}(\mathbb{R}_T^2)$ is a solution of problem (1.71) if and only if

$$B_\Delta^{1,0}(\mathcal{S}u_\Delta, v) = B_\Delta^{1,0}(\mathcal{S}u, v), \quad (v \in S_{h,k}^{1,1}(\mathbb{R}_T^2)). \tag{1.74}$$

Let $u \in H_{00}^{s,\frac{s}{2}}(\mathbb{R}_T^2)$, $s > \frac{3}{2}$ be the solution of the equation $\mathcal{S}u = f$. Then the function $u_\Delta \in S_{h,k}^{3,1}(\mathbb{R}_T^2)$ is a solution of problem (1.72) if and only if

$$B_\Delta^{2,0}(\mathcal{S}u_\Delta, v) = B_\Delta^{2,0}(\mathcal{S}u, v), \quad (v \in S_{h,k}^{3,1}(\mathbb{R}_T^2)). \tag{1.75}$$

The proof is similar to that of Theorem 10, and is omitted. It is remarkable here that the required regularity for the equivalence is described in terms of the anisotropic Sobolev spaces contrary to article II, where the spaces of essentially bounded functions were used. See also Theorem 18, formula (3.4).

The hypersingular heat operator decreases smoothness. Therefore, we have considered only the lowest order method corresponding to the projection operator $Q_{h,k}^{d_\theta,d_t}$. The discrete approximation to the hypersingular heat operator equation is:

$$\text{find } u_\Delta \in S_{h,k}^{3,1}(\mathbb{R}_T^2) \text{ such that } Q_{h,k}^{3,1}\mathcal{H}u_\Delta = Q_{h,k}^{3,1}f. \tag{1.76}$$

This operator equation is equivalent to the following system of equations:

find $u_\Delta \in S^{3,1}_{h,k}(\mathbb{R}^2_T)$ such that

$$\int_0^T \phi^1_m(t)(\mathcal{H}u_\Delta)(\theta_n,t)\,\mathrm{d}t = \int_0^T \phi^1_m(t) f(\theta_n,t)\,\mathrm{d}t,\ (n=1,\dots,N,\ m=1,\dots,M). \tag{1.77}$$

As an example, we prove that problem setting (1.76) is equivalent to a certain Galerkin problem. In this particular case, the bilinear forms are

$$B^{2,0}(w,v) = \big(\partial^2_\theta w\,,\,\partial^2_\theta v\big) + \big(J\,w\,,\,J\,v\big) \tag{1.78}$$

$$B^{2,0}_\Delta(w,v) = \big(\partial^2_\theta w\,,\,\partial^2_\theta v\big) + \big(J_\Delta w\,,\,J\,v\big). \tag{1.79}$$

The proof is quoted from paper III.

Theorem 10 *Let $u \in H^{s,\frac{s}{2}}_{00}(\mathbb{R}^2_T)$, $s > \frac{5}{2}$ be the solution of the equation $\mathcal{H}u = f$. Then the function $u_\Delta \in S^{3,1}_{h,k}(\mathbb{R}^2_T)$ is a solution of problem (1.76) if and only if*

$$B^{2,0}_\Delta(\mathcal{H}u_\Delta, v) = B^{2,0}_\Delta(\mathcal{H}u, v), \quad (v \in S^{3,1}_{h,k}(\mathbb{R}^2_T)). \tag{1.80}$$

Proof. We denote $\tilde{w} = \mathcal{H}u_\Delta - f$, and consider the condition

$$B^{2,0}_\Delta(\tilde{w}, v) = 0, \quad \text{for all } v \in S^{3,1}_{h,k}(\mathbb{R}^2_T). \tag{1.81}$$

It is known that the operator $\partial^2_\theta + J \,:\, S^3_h \to S^1_h$ is an isomorphism between the one-dimensional periodic spline spaces. Therefore, we can choose the basis functions $v_{m,n}$ of the trial space $S^{3,1}_{h,k}(\mathbb{R}^2_T)$ such that

$$(\partial^2_\theta + J\,)v_{m,n}(\theta,t) = \phi^1_m(t)\psi^1_n(\theta),$$

where ψ^1_n is the 1-periodic Courant basis function, such that for $\theta \in [0,1]$

$$\psi^1_n(\theta) = \begin{cases} 1 + (\theta-\theta_n)/(\theta_n-\theta_{n-1}), & \theta_{n-1} < \theta < \theta_n, \\ 1 - (\theta-\theta_n)/(\theta_{n+1}-\theta_n), & \theta_n < \theta < \theta_{n+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then we obtain by partial integration

$$\begin{aligned} B^{2,0}_\Delta(\tilde{w}, v_{m,n}) &= \big((\partial^2_\theta + J_\Delta)\tilde{w}\,,\,(\partial^2_\theta + J\,)v_{m,n}\big) = \big((\partial^2_\theta + J_\Delta)\tilde{w}\,,\,\phi^1_m\psi^1_n\big) \\ &= -\big(\partial_\theta\tilde{w}\,,\,\phi^1_m\partial_\theta\psi^1_n\big) + \big(J_\Delta\tilde{w}\,,\,\phi^1_m\psi^1_n\big) \\ &= -\int_0^T \phi^1_m(t)\left[\frac{\tilde{w}(\theta_n,t)-\tilde{w}(\theta_{n-1},t)}{\theta_n-\theta_{n-1}} - \frac{\tilde{w}(\theta_{n+1},t)-\tilde{w}(\theta_n,t)}{\theta_{n+1}-\theta_n}\right]\mathrm{d}t \\ &\qquad + J\,\psi^1_n \textstyle\sum_{n=1}^N \frac{\theta_{n+1}-\theta_{n-1}}{2} \displaystyle\int_0^T \phi^1_m(t)\tilde{w}(\theta_n,t)\,\mathrm{d}t. \end{aligned} \tag{1.82}$$

Suppose first that u_Δ satisfies the discrete equations, which means that

$$\int_0^T \phi_m^1(t)\tilde{w}(\theta_n,t)\,\mathrm{d}t = 0, \qquad \text{for all } n=1,\dots,N, m=1,\dots,M. \tag{1.83}$$

According to (1.82), the bilinear form $B_\Delta^{2,0}(\tilde{w},v)$ reduces to a linear combination of integrals of type (1.83) and consequently $B_\Delta^{2,0}(\tilde{w},v)=0$ for all $v\in S_{h,k}^{3,1}(\mathbb{R}_T^2)$.

Conversely, assume that (1.81) is valid. For all $v\in S_{h,k}^{3,1}(\mathbb{R}_T^2)$ also $J\,v\in S_{h,k}^{3,1}(\mathbb{R}_T^2)$. The identity $B_\Delta^{2,0}(\tilde{w},Jv)=\int_0^T(J_\Delta\tilde{w})(t)(J\,v)(t)\,\mathrm{d}t$ together with (1.79) implies

$$\int_0^T (J_\Delta\tilde{w})(t)(J\,v)(t)\,\mathrm{d}t \;= 0, \quad (v\in S_{h,k}^{3,1}(\mathbb{R}_T^2)), \tag{1.84}$$

$$(\partial_\theta^2\tilde{w}\,,\,\partial_\theta^2 v) \;= 0, \quad (v\in S_{h,k}^{3,1}(\mathbb{R}_T^2)). \tag{1.85}$$

As in (1.82), formula (1.85) yields

$$\int_0^T \phi_m^1(t)\left[\frac{\tilde{w}(\theta_{n+1},t)-\tilde{w}(\theta_n,t)}{\theta_{n+1}-\theta_n}-\frac{\tilde{w}(\theta_n,t)-\tilde{w}(\theta_{n-1},t)}{\theta_n-\theta_{n-1}}\right]\mathrm{d}t=0 \tag{1.86}$$

for $1\le n\le N$, $1\le m\le M$. Let m be fixed. Then we have for all $n=1,...,N$

$$\int_0^T \phi_m^1(t)\frac{\tilde{w}(\theta_{n+1},t)-\tilde{w}(\theta_n,t)}{\theta_{n+1}-\theta_n}\,\mathrm{d}t=\int_0^T \phi_m^1(t)\frac{\tilde{w}(\theta_n,t)-\tilde{w}(\theta_{n-1},t)}{\theta_n-\theta_{n-1}}\,\mathrm{d}t=C_m. \tag{1.87}$$

Using the periodicity of $\tilde{w}$ with respect to θ, we obtain

$$\sum_{n=1}^N(\tilde{w}(\theta_n,t)-\tilde{w}(\theta_{n-1},t))=\tilde{w}(\theta_N,t)-\tilde{w}(\theta_0,t)=0. \tag{1.88}$$

On the other hand, (1.87) implies

$$\begin{aligned} C_m &= C_m\sum_{n=1}^N(\theta_n-\theta_{n-1})=\sum_{n=1}^N\int_0^T \phi_m^1(t)(\tilde{w}(\theta_n,t)-\tilde{w}(\theta_{n-1},t))\,\mathrm{d}t \\ &= \int_0^T \phi_m^1(t)\sum_{n=1}^N(\tilde{w}(\theta_n,t)-\tilde{w}(\theta_{n-1},t))\,\mathrm{d}t, \end{aligned}$$

which, with (1.88), gives $C_m=0$. Inserting this back to (1.87), we have

$$\int_0^T \phi_m^1(t)\tilde{w}(\theta_n,t)\,\mathrm{d}t=\int_0^T \phi_m^1(t)\tilde{w}(\theta_0,t)\,\mathrm{d}t, \quad (1\le n\le N). \tag{1.89}$$

Finally, according to (1.84) and (1.89)

$$0 = J\,\psi_n^3 \sum_{n=1}^{N} \frac{\theta_{n+1} - \theta_{n-1}}{2} \int_0^T \phi_m^1(t)\tilde{w}(\theta_n, t)\,\mathrm{d}t = J\,\psi_n^3 \int_0^T \phi_m^1(t)\tilde{w}(\theta_0, t)\,\mathrm{d}t. \quad (1.90)$$

Since $J\,\psi_n^3$ is nonzero, the value of the integral $\int_0^T \phi_m^1(t)\tilde{w}(\theta_0, t)\,\mathrm{d}t$ vanishes and, due to formula (1.89), all integrals of type (1.83) vanish. Thus, u_Δ satisfies Petrov-Galerkin equations (1.76). □

1.8 Description of the nonlinearity

Now, we specify the assumptions on the nonlinearity. The real valued nonlinear function $F(\mathbf{x}, t, \xi) : \Sigma_T \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the *Carathéodory conditions*:

The mapping $(\mathbf{x}, t) \mapsto F(\mathbf{x}, t, \xi) : \Sigma_T \to \mathbb{R}$ is measurable for all fixed $\xi \in \mathbb{R}$(1.91)

The mapping $\xi \mapsto F(\mathbf{x}, t, \xi) : \mathbb{R} \to \mathbb{R}$ is continuous for almost all $(\mathbf{x}, t) \in \Sigma_T$(1.92)

The associated Nemitsky operator or superposition operator

$$u \mapsto \mathcal{F}(u) : L^2(\Sigma_T) \to L^2(\Sigma_T); \quad \mathcal{F}(u)(\mathbf{x}, t) = F(\mathbf{x}, t, u(\mathbf{x}, t))$$

is well defined, provided that the Carathéodory conditions and the following growth condition

$$|F(\mathbf{x}, t, \xi)| \le b(\mathbf{x}, t) + c\,|\xi|, \quad ((\mathbf{x}, t, \xi) \in \Sigma_T \times \mathbb{R}) \quad (1.93)$$

are valid (see [38]). Here, c is a constant and $b(\mathbf{x}, t) \in L^2(\Sigma_T)$. For the analysis of the numerical approximation scheme, we assume that the mapping $\xi \mapsto F(\mathbf{x}, t, \xi)$ is nondecreasing for each $(\mathbf{x}, t) \in \Sigma_T$

$$(F(\mathbf{x}, t, \xi) - F(\mathbf{x}, t, \xi'))(\xi - \xi') \ge 0 \quad \text{for all} \quad \xi, \xi' \in \mathbb{R}. \quad (1.94)$$

Note that the Carathéodory conditions, together with (1.93) and (1.94) imply that the Nemitsky operator $\mathcal{F} : L^2(\Sigma_T) \to L^2(\Sigma_T)$ is monotone, which by definition means that

$$(\mathcal{F}(u) - \mathcal{F}(w)\,,\, u - w) \ge 0.$$

In order to obtain regularity results and error estimates, we need the Lipschitz and Hölder conditions.
There exists a constant $L > 0$ such that

$$|F(\mathbf{x}, t, \xi) - F(\mathbf{x}, t, \xi')| \leq L|\xi - \xi'| \quad \text{for all } \xi, \xi' \in \mathbb{R}, (\mathbf{x}, t) \in \Sigma_T. \tag{1.95}$$

There are constants $N, M > 0$ and $0 < \lambda \leq 1$ such that

$$|F(\mathbf{x}, t, \xi) - F(\mathbf{y}, \tau, \xi)| \leq N|\mathbf{x} - \mathbf{y}|^{\lambda} + M|t - \tau|^{\frac{\lambda}{2}} \tag{1.96}$$

for all $\xi \in \mathbb{R}$, $(\mathbf{x}, t), (\mathbf{y}, \tau) \in \Sigma_T$.

We recall some basic properties concerning the Nemitsky operator in anisotropic Sobolev spaces. The first one is a special case of the results given for example in book [38]. The latter is presented in [30]. For the convenience of the reader, the proofs are repeated here.

Lemma 3 *Assume that the Carathéodory conditions are valid.*

(*i*) *If (1.93) is valid, then* $\mathcal{F} : L^2(\Sigma_T) \to L^2(\Sigma_T)$ *is bounded and continuous.*

(*ii*) *If, in addition (1.94) is valid, then the Nemitsky operator* $\mathcal{F} : L^2(\Sigma_T) \to L^2(\Sigma_T)$ *is monotone.*

Proof. Assuming growth condition (1.93), we obtain

$$\begin{aligned}
\|\mathcal{F}(u)\|^2_{L^2(\Sigma_T)} &= \int_0^T \int_\Gamma |(\mathcal{F}(u))(\mathbf{x}, t)|^2 \mathrm{d}\Gamma_\mathbf{x}\, \mathrm{d}t = \int_0^T \int_\Gamma |F(\mathbf{x}, t, u(\mathbf{x}, t))|^2 \mathrm{d}\Gamma_\mathbf{x}\, \mathrm{d}t \\
&\leq \int_0^T \int_\Gamma (b(\mathbf{x}, t) + c|u(\mathbf{x}, t)|)^2 \mathrm{d}\Gamma_\mathbf{x}\, \mathrm{d}t \\
&\leq \|b\|^2_{L^2(\Sigma_T)} + c\, \|u\|^2_{L^2(\Sigma_T)}.
\end{aligned}$$

The monotonicity of the Nemitsky operator follows because

$$(\mathcal{F}(u) - \mathcal{F}(w)\, ,\, u - w) = \int_0^T \int_\Gamma (F(\mathbf{x}, t, u(\mathbf{x}, t)) - F(\mathbf{x}, t, w(\mathbf{x}, t)))(u(\mathbf{x}, t) - w(\mathbf{x}, t)) \mathrm{d}\Gamma_\mathbf{x}\, \mathrm{d}t$$

is non-negative. □

Lemma 4 *Assume that Carathéodory conditions are valid.*

(*i*) *If (1.93) and (1.95) are valid, then* $\mathcal{F} : L^2(\Sigma_T) \to L^2(\Sigma_T)$ *is Lipschitz continuous.*

(*ii*) *If (1.93),(1.95) and (1.96) are valid, then* $\mathcal{F} : H^{s,\frac{s}{2}}_{00}(\Sigma_T) \to H^{s,\frac{s}{2}}_{00}(\Sigma_T)$ *is bounded for all* $0 \leq s < \lambda$.

Proof. The norm of the space $H_{00}^{r,\frac{r}{2}}(\Sigma_T)$ is defined by

$$\|u\|^2_{s,\frac{s}{2}} = \|u\|^2_{L^2(\Sigma_T)} + |u|^2_{s,0} + |u|^2_{0,\frac{s}{2}},$$

where the seminorms are defined by

$$\begin{aligned} |u|^2_{s,0} &= \int_0^T \int_\Gamma \int_\Gamma \frac{|u(\mathbf{x},t)-u(\mathbf{y},t)|^2}{|\mathbf{x}-\mathbf{y}|^{1+2s}} \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}\Gamma_{\mathbf{y}}\, \mathrm{d}t, \\ |u|^2_{0,\frac{s}{2}} &= \int_0^T \int_0^T \int_\Gamma \frac{|u(\mathbf{x},t)-u(\mathbf{x},\tau)|^2}{|t-\tau|^{1+s}} \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}t\, \mathrm{d}\tau. \end{aligned}$$

(i) The Lipschitz continuity of the Nemitsky operator follows from condition (1.95)

$$\begin{aligned} \|\mathcal{F}(u)-\mathcal{F}(w)\|^2_{L^2(\Sigma_T)} &= \int_0^T \int_\Gamma |(\mathcal{F}(u))(\mathbf{x},t)-(\mathcal{F}(w))(\mathbf{x},t)|^2 \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}t \\ &= \int_0^T \int_\Gamma |F(\mathbf{x},t,u(\mathbf{x},t))-F(\mathbf{x},t,w(\mathbf{x},t))|^2 \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}t \\ &\le \int_0^T \int_\Gamma L^2|u(\mathbf{x},t)-w(\mathbf{x},t)|^2 \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}t \\ &= L^2\|u-w\|^2_{L^2(\Sigma_T)}. \end{aligned}$$

(ii) The Lipschitz and Hölder conditions for the function $F = F(\mathbf{x},t,\xi)$ yield for the seminorms the estimates

$$\begin{aligned} |\mathcal{F}(u)|^2_{s,0} &= \int_0^T \int_\Gamma \int_\Gamma \tfrac{|(\mathcal{F}(u))(\mathbf{x},t)-(\mathcal{F}(u))(\mathbf{y},t)|^2}{|\mathbf{x}-\mathbf{y}|^{1+2s}} \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}\Gamma_{\mathbf{y}}\, \mathrm{d}t \\ &\le \int_0^T \int_\Gamma \int_\Gamma \Big[\tfrac{|F(\mathbf{x},t,u(\mathbf{x},t))-F(\mathbf{y},t,u(\mathbf{x},t))|^2}{|\mathbf{x}-\mathbf{y}|^{1+2s}} + \tfrac{|F(\mathbf{y},t,u(\mathbf{x},t))-F(\mathbf{y},t,u(\mathbf{y},t))|^2}{|\mathbf{x}-\mathbf{y}|^{1+2s}}\Big] \mathrm{d}\Gamma_{\mathbf{x}} \mathrm{d}\Gamma_{\mathbf{y}}\, \mathrm{d}t \\ &\le \int_0^T \int_\Gamma \int_\Gamma \tfrac{N^2|\mathbf{x}-\mathbf{y}|^{2\lambda}}{|\mathbf{x}-\mathbf{y}|^{1+2s}}\, \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}\Gamma_{\mathbf{y}}\, \mathrm{d}t + L^2|u|^2_{s,0} \\ &\le N^2 T \int_\Gamma \int_\Gamma |\mathbf{x}-\mathbf{y}|^{2\lambda-2s-1} \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}\Gamma_{\mathbf{y}} + L^2|u|^2_{s,0} \le c_1 + L^2|u|^2_{s,0} \end{aligned}$$

$$\begin{aligned} |\mathcal{F}(u)|^2_{0,\frac{s}{2}} &= \int_0^T \int_0^T \int_\Gamma \tfrac{|(\mathcal{F}(u))(\mathbf{x},t)-(\mathcal{F}(u))(\mathbf{x},\tau)|^2}{|t-\tau|^{1+s}} \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}t\ \mathrm{d}\tau \\ &\le \int_0^T \int_0^T \int_\Gamma \Big[\tfrac{|F(\mathbf{x},t,u(\mathbf{x},t))-F(\mathbf{x},\tau,u(\mathbf{x},t))|^2}{|t-\tau|^{1+s}} + \tfrac{L^2|u(\mathbf{x},t)-u(\mathbf{x},\tau)|^2}{|t-\tau|^{1+s}}\Big] \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}t\, \mathrm{d}\tau \\ &= \int_0^T \int_0^T \int_\Gamma \tfrac{M^2|t-\tau|^{\lambda}}{|t-\tau|^{1+s}}\, \mathrm{d}\Gamma_{\mathbf{x}}\, \mathrm{d}t\, \mathrm{d}\tau + L^2|u|^2_{0,\frac{s}{2}} \\ &= M^2 m(\Gamma) \int_0^T \int_0^T |t-\tau|^{\lambda-s-1}\, \mathrm{d}t\, \mathrm{d}\tau + L^2|u|^2_{0,\frac{s}{2}} = c_2 + L^2|u|^2_{0,\frac{s}{2}}. \end{aligned}$$

The assertion follows since

$$\begin{aligned} \|\mathcal{F}(u)\|^2_{s,\frac{s}{2}} &= \|\mathcal{F}(u)\|^2_{L^2(\Sigma_T)} + |\mathcal{F}(u)|^2_{s,0} + |\mathcal{F}(u)|^2_{0,\frac{s}{2}} \\ &\le \|b\|^2_{L^2(\Sigma_T)} + c_1 + c_2 + c\,\|u\|^2_{L^2(\Sigma_T)} + L^2|u|^2_{s,0} + L^2|u|^2_{0,\frac{s}{2}} \\ &\le c_3 + c_4\,\|u\|^2_{s,\frac{s}{2}}. \quad \square \end{aligned}$$

2 Abstract theory

The theory of the Finite Element Method, as well as that of the Boundary Element Method, is closely related to the abstract functional analysis. The so-called Lax-Milgram theorem, together with the underlying Riesz representation theorem, play crucial roles in this context. In this chapter, we introduce some fundamental elements of the linear and nonlinear theory.

In the nonlinear case, the essential references are Petryshyn's articles [41], [42] and [43], Minty's work [36], and Browder's results [11], [12]. The reader may find the textbooks by Pascali & Sburlan [38] and Deimling [19] useful.

2.1 Linear equations

Let X be a Hilbert space. Let $\ell : X \to \mathbb{R}$ be a bounded linear functional. Let $B(\cdot\,,\,\cdot) : X \times X \to \mathbb{R}$ be a bilinear form. The bilinear form is called

$$\text{bounded, if } |B(w,v)| \leq C_B\,\|w\|_X \|v\|_X, \text{ for all } \ w,v \in X,$$
$$\text{strongly coercive, if } B(w,w) \geq c_0\,\|w\|_X^2, \text{ for all } \ w \in X, \text{ with } c_0 > 0.$$

We consider the variational problem of finding $u \in X$, such that

$$B(u,v) = \ell(v) \qquad \text{for all } \ v \in X. \tag{2.1}$$

The solvability is given in the following so-called Lax-Milgram Theorem. For the proof, see [10] or [13].

Theorem 11 *Let $B(\cdot,\cdot)$ be a real valued, strongly coercive and bounded bilinear form in a Hilbert space X. Let ℓ be a bounded real valued linear form on X. Then the equation*

$$B(u,v) = \ell(v) \qquad \text{for all } \ v \in X$$

is uniquely solvable, and

$$\|u\|_X \leq \frac{1}{c_0}\,\|\ell\|.$$

In order to approximate the solution of the variational problem (2.1), we consider another variational problem of finding $u_n \in X_n$ such that

$$B(u_n, v_n) = \ell(v_n) \quad \text{for all} \;\; v_n \in X_n, \tag{2.2}$$

where $X_n \subset X$ is a closed subspace. We recall the so-called Cea Lemma, which describes the solvability of problem (2.2).

Theorem 12 *Let $B(\cdot,\cdot)$ be a real valued and bounded bilinear form in a Hilbert space X. Let X_n be a closed subspace of X. Let $B(\cdot,\cdot)$ be strongly coercive in X_n. Let ℓ be a bounded real valued linear form on X. Then equation (2.2) admits a unique solution such that*

$$\|u_n\|_X \leq \frac{c_B}{c_0}\,\|u\|_X.$$

For the approximation error in u_n, we have the quasi-optimal bound

$$\|u - u_n\|_X \leq \big(1 + \frac{c_B}{c_0}\big) \inf_{\psi \in X_n} \|u - \psi\|_X.$$

Proof. The solvability of the variational problem (2.2) follows from the Lax-Milgram theorem, since closed subspaces of Hilbert spaces are themselves Hilbert spaces. The coercivity and boundedness of the bilinear form combined to the equations

$$\begin{aligned} B(u, v_n) &= \ell(v_n) \qquad \text{for all} \;\; v_n \in X_n \\ B(u_n, v_n) &= \ell(v_n) \qquad \text{for all} \;\; v_n \in X_n \end{aligned}$$

give

$$c_0\,\|u_n\|^2_{X_0} \quad \leq \quad B(u_n, u_n) = B(u, u_n) \leq c_B\,\|u\|_{X_0}\|u_n\|_X.$$

This yields the stability

$$\|u_n\|_X \leq \frac{c_B}{c_0}\,\|u\|_X.$$

In particular, we have for all $\psi \in X_n$

$$\begin{aligned} c_0\,\|u_n - \psi\|^2_X &\leq B(u_n - \psi, u_n - \psi) \\ &= B(u - \psi, u_n - \psi) + B(u_n - u, u_n - \psi) \\ &= B(u - \psi, u_n - \psi) \\ &\leq c_B\,\|u - \psi\|_X\|u_n - \psi\|_X \end{aligned}$$

implying

$$\|u_n - \psi\|_X \leq \frac{c_B}{c_0} \inf_{\psi \in X_n} \|u - \psi\|_X.$$

Now, the triangle inequality gives for all $\psi \in X_n$

$$\|u - u_n\|_X \leq \|u - \psi\|_X + \|\psi - u_n\|_X \leq \big(1 + \frac{c_B}{c_0}\big) \inf_{\psi \in X_n} \|u - \psi\|_X$$

proving the quasi-optimality. $\square$

In practice, spaces X_n are finite dimensional. The elements $u_n \in X_n$ are called shape functions. The importance of the quasi-optimal error estimate is that the asymptotic convergence depends only on the approximation properties of the shape functions.

In this study we apply these fundamental tools to the case of small perturbations of coercive operators. The formulation of Theorems 11 and 12 can be generalized in several ways. It is possible to replace the coercivity by the Babuska stability condition [10]. The theorems can also be modified to cover the case of compact perturbations of isomorphic operators [13], Theorem 10.1.3.

2.2 Nonlinear equations

The purpose of this section is to give the general theoretical framework which has been applied to articles IV and V. The approach is based on Saranen's article [55] concerning projection methods applied to the equation

$$A(u) = g,$$

where $A : X \to X$ is a nonlinear Hammerstein operator

$$A(u) = Iu + Du + S(F(u)).$$

Here, S and D are given linear operators and F describes the nonlinearity. Let X_0 be a Hilbert space equipped with the norm $\|\cdot\|_{X_0}$ such that the embedding $X \subset X_0$ is dense and continuous. We denote the dual space of X by X^*. By identifying X_0 with its dual, we have $X \subset X_0 \subset X^*$. Let $(\cdot\,,\,\cdot)$ be the continuous bilinear form defined in $X \times X^*$ or $X^* \times X$ such that it extends the inner product of the pivot space X_0. We use another Hilbert space Z such that $Z \subset X$ with continuous injection.

We recall some concepts concerning nonlinear behaviour. The nonlinearity $F : X_0 \to X_0$ is called

monotone if $(F(u) - F(w)\,,\, u - w) \geq 0,$
strongly monotone if $(F(u) - F(w)\,,\, u - w) \geq c_F \|u - w\|^2_{X_0}.$

Let $(S^{-1})^* : X \to X^*$ be the adjoint of the operator $S^{-1} : X^* \to X$ defined by the relation $(S^{-1}u^*\,,\, u) = (u^*\,,\, (S^{-1})^* u),\ u \in X$. We assume that the operator A is strongly $(S^{-1})^*$-monotone in the sense that the inequality

$$(A(u) - A(w)\,,\, (S^{-1})^*(u - w)) \geq c_A \|u - w\|^2_X \tag{2.3}$$

is valid for all $u, v \in X$ with a positive constant c_A. The monotonicity (2.3) implies that the operator $A : X \to X$ is a homeomorphism.

Let $X_n \subset X$, $n \in \mathbb{N}$ be a sequence of finite dimensional subspaces with given projection $Q_n : X \to X_n$ for each $n \in \mathbb{N}$. We look for an approximation u_n for the solution u such that

$$Q_n A(u_n) = Q_n A(u). \tag{2.4}$$

The following abstract result describes the solvability of the approximate equation.

Theorem 13 *We assume that the following conditions are fulfilled*

$$D: \quad X \to Z \text{ is bounded,} \tag{2.5}$$
$$S: \quad X^* \to X \text{ is an isomorphism and } S: X_0 \to Z \text{ is bounded,} \tag{2.6}$$
$$F: \quad X_0 \to X_0 \text{ is Lipschitz continuous,} \tag{2.7}$$
$$A: \quad X \to X \text{ is strongly } (S^{-1})^* - \text{ monotone.} \tag{2.8}$$

Assume that the projection operators Q_n have the asymptotic approximation property

$$\|u - Q_n u\|_X \le \varepsilon(n)\|u\|_Z, \ n \in \mathbb{N} \tag{2.9}$$

such that $\varepsilon(n) \to 0$ when $n \to \infty$. Then there exists an integer $n_0 \in \mathbb{N}$ and a positive constant c_1 such that

$$\|Q_n A(v) - Q_n A(v')\|_X \ge c_1 \|v - v'\|_X \text{ for all } v, v' \in X_n, \tag{2.10}$$

when $n \ge n_0$. Moreover, equation (2.4) is uniquely solvable for $n \ge n_0$.

Proof. Since Q_n is a projection $Q_n v = v$ for all $v \in X_n$, and we can write

$$(I - Q_n)(A(v) - A(v')) = (I - Q_n)D(v - v') + (I - Q_n)S(F(v) - F(v')),$$

for all $v, v' \in X_n$. A lower bound is obtained by decomposing and using $(S^{-1})^*$ -monotonicity

$$\begin{aligned}&\big(Q_n A(v) - Q_n A(v'), (S^{-1})^*(v - v')\big)\\ &\quad = \big(A(v) - A(v'), (S^{-1})^*(v - v')\big) - \big((I - Q_n)\big(A(v) - A(v')\big), (S^{-1})^*(v - v')\big)\\ &\quad \ge c\,\|v - v'\|_X^2 - c'\|(I - Q_n)\big(A(v) - A(v')\big)\|_X\|v - v'\|_X.\end{aligned}$$

On the other hand, the Schwarz inequality yields the upper bound

$$\begin{aligned}\big(Q_n A(v) - Q_n A(v'), (S^{-1})^*(v - v')\big) &\le c\,\|Q_n A(v) - Q_n A(v')\|_X\|(S^{-1})^*(v - v')\|_{X^*}\\ &\le c\,\|Q_n A(v) - Q_n A(v')\|_X\|v - v'\|_X.\end{aligned}$$

Therefore, we obtain the estimate

$$\|Q_n A(v) - Q_n A(v')\|_X \ge c\,\|v - v'\|_X - c'\,\|(I - Q_n)\big(A(v) - A(v')\big)\|_X. \tag{2.11}$$

According to approximation property (2.9) combined with assumptions (2.5) - (2.8), we find

$$\begin{aligned}\|(I - Q_n)\big(A(v) - A(v')\big)\|_X &\le \|(I - Q_n)D(v - v')\|_X + \|(I - Q_n)S(F(v) - F(v'))\|_X\\ &\le c\,\varepsilon(n)\|D(v - v')\|_Z + c\,\varepsilon(n)\|S(F(v) - F(v'))\|_Z\\ &\le c\,\varepsilon(n)\|v - v'\|_X + c\,\varepsilon(n)\|F(v) - F(v')\|_{X_0}\\ &\le c\,\varepsilon(n)\|v - v'\|_X + c\,\varepsilon(n)\|v - v'\|_{X_0}\\ &\le c\,\varepsilon(n)\|v - v'\|_X.\end{aligned}$$

Thus, (2.11) yields the stability

$$\|Q_n A(v) - Q_n A(v')\|_X \ge c_1 \|v - v'\|_X, \text{ for all } v, v' \text{ and } n \ge n_0.$$

Due to stability, the operator $Q_nA : X_n \to X_n$ is an injection. According to Brouwer's theorem on invariance of the domain, the range $R(Q_nA)$ is open. Because of the stability and continuity of Q_nA in X_n, the range $R(Q_nA)$ is also closed in X_n. Therefore, $R(Q_nA) = X_n$, and Q_nA is a homeomorphism. □

Theorem 14 *We assume that conditions (2.5) - (2.8) and approximation property (2.9) are valid. Then, we have the asymptotic error estimate*

$$\|u - u_n\| \le c\,\|u - Q_nu\|_X. \tag{2.12}$$

Furthermore, if the mapping $D : X_0 \to Z$ is bounded, then

$$\|u - u_n\| \le c\,\|u - Q_nu\|_{X_0}. \tag{2.13}$$

Proof. For all $v \in X_n$, because of equation (2.4) it holds that

$$\begin{aligned} Q_nA(u_n) - Q_nA(v) &= Q_nA(u) - Q_nA(v) \\ &= Q_n(u-v) + Q_nD(u-v) + Q_nV(F(u)-F(v)). \end{aligned} \tag{2.14}$$

Since Q_n is a projection operator in X_n, the choice $v = Q_nu$ gives

$$Q_nA(u_n) - Q_nA(Q_nu) = Q_nD(u - Q_nu) + Q_nV(F(u) - F(Q_nu)). \tag{2.15}$$

Now, it follows by using (2.5) and approximation property (2.9)

$$\begin{aligned} \|Q_nD(u-Q_nu)\|_X &\le \|D(u-Q_nu)\|_X + \|(I-Q_n)D(u-Q_nu)\|_X \\ &\le c\,\|u-Q_nu\|_X + c\,\varepsilon(n)\|D(u-Q_nu)\|_Z \\ &\le c\,\|u-Q_nu\|_X + c\,\varepsilon(n)\|u-Q_nu)\|_X \\ &\le c\,\|u-Q_nu\|_X. \end{aligned} \tag{2.16}$$

An analogous estimate follows by using (2.6) and (2.7)

$$\begin{aligned} \|Q_nV(F(u) - F(Q_nu))\|_X &\le \|V(F(u)-F(Q_nu))\|_X \\ &\quad + \|(I-Q_n)V(F(u)-F(Q_nu))\|_X \\ &\le c\,\|F(u)-F(Q_nu)\|_{X_0} + c\,\varepsilon(n)\|V(F(u)-F(Q_nu))\|_Z \\ &\le c\,\|u-Q_nu\|_{X_0} + c\,\varepsilon(n)\|F(u)-F(Q_nu)\|_{X_0} \\ &\le c\,\|u-Q_nu\|_{X_0}. \end{aligned} \tag{2.17}$$

Combining stability (2.10) with (2.15), (2.16) and (2.17) we have

$$\begin{aligned} c_1\,\|u_n - Q_nu\|_X &\le \|Q_nA(u_n) - Q_nA(Q_nu)\|_X \\ &\le c\,\|u-Q_nu\|_X + c\,\|Q_nu - u_n\|_X. \end{aligned}$$

Finally, the triangle inequality yields the convergence

$$\|u - u_n\|_X \le \|u - Q_nu\|_X + \|Q_nu - u_n\|_X \le c\,\|u - Q_nu\|_X.$$

If mapping property (2.13) is valid then we may replace (2.16) by the stronger estimate

$$\|Q_n D(u - Q_n u)\|_X \leq \|u - Q_n u\|_{X_0},$$

which together with (2.10), (2.15) and (2.17) implies

$$\|u_n - Q_n u\|_X \leq c\,\|u - Q_n u\|_{X_0}.$$

The theorem is proved. □

3 Summary of the original articles

The thesis consists of two lines of research. The first deals with questions of the stability and convergence of the spline collocation method applied to the single layer heat operator equation. These results are generalized to some spline Petrov-Galerkin methods applied to the single layer heat operator equation, as well as to the case of the hypersingular heat operator equation. The analysis is based on the assumption that the spatial domain is a disk.

The rest of the thesis consists of the analysis of the boundary element approximation of some problems, where the boundary condition is nonlinear. The effect of the numerical integration is analysed in the stationary case. Also some numerical results are given. Later, the theory is extended to the nonlinear time-dependent case.

3.1 Article I: On the spline collocation method for the single layer heat operator equation

As discussed in the introduction, the direct boundary element method yields the single layer heat operator equation for the unknown boundary density. We construct and analyse a spline collocation scheme for solution of the single layer heat operator equation, assuming that the spatial domain is two-dimensional and has a smooth boundary. We apply piecewise linear approximation both in space and time together with nodal point collocation. For the proof of the stability and the convergence we restrict ourselves here to the case of the circle, but the method can be applied to all smooth closed curves, as was shown in [31], [32].

The collocation equations corresponding to the single layer heat operator equation $\mathcal{S}u = f$ are: find $u_\Delta \in S^{1,1}_{h,k}$ such that

$$(\mathcal{S}u_\Delta)(\theta_n, t_m) = f(\theta_n, t_m), \quad (1 \leq n \leq N,\, 1 \leq m \leq M). \tag{3.1}$$

For the equivalent characterization of the collocation equations, we need some regularity assumptions. Let $\mathcal{C}_1(\overline{\mathbb{R}}^2_T)$ be the space of continuous functions $f(\theta, t)$

on the closure $\overline{\mathbb{R}}^2_T$ of $\mathbb{R}^2_T$ such that f is 1-periodic with respect to the variable θ. Moreover, we define the spaces

$$\begin{aligned} \mathcal{C}_{00}(\overline{\mathbb{R}}^2_T) &= \{f \in \mathcal{C}_1(\overline{\mathbb{R}}^2_T) \mid f(\theta,0) \equiv 0\}, \\ \mathcal{H}^c_{00}(\overline{\mathbb{R}}^2_T) &= \{f \in \mathcal{C}_{00}(\overline{\mathbb{R}}^2_T) \mid \partial_\theta f,\, \partial_t f,\, \partial_t\partial_\theta f = \partial_\theta\partial_t f \in \mathcal{C}_1(\overline{\mathbb{R}}^2_T)\}. \end{aligned}$$

Next, we give sufficient conditions on the function u to guarantee the property $\mathcal{S}u \in \mathcal{H}^c_{00}(\overline{\mathbb{R}}^2_T)$. Recall that the measurable function u is essentially bounded on $\overline{\mathbb{R}}^2_T$ if there exists $C > 0$ such that $|u(\theta,t)| \leq C$ for almost all $(\theta,t) \in \mathbb{R}^2_T$. We introduce the space

$$\mathcal{C}^t_{00}(\mathbb{R}^2_T) = \{f \mid f(\theta,\cdot) \text{ is continuous for almost all } \theta \text{ and } f(\theta,0) = 0\}.$$

The space $\mathcal{H}_{00}(\mathbb{R}^2_T)$ consists of functions $f \in \mathcal{C}^t_{00}(\mathbb{R}^2_T)$ such that $\partial_\theta f \in \mathcal{C}^t_{00}(\mathbb{R}^2_T)$ and the partial derivatives $\partial_t f, \partial_t\partial_\theta f = \partial_\theta\partial_t f$ are measurable, essentially bounded functions on $\overline{\mathbb{R}}^2_T$. We emphasize that $S^{1,1}_{h,k}$ is a subspace of $\mathcal{H}_{00}(\mathbb{R}^2_T)$.

Now, we are ready to establish the equivalence between the collocation problem and a certain Galerkin problem. For this, we define the bilinear form

$$\begin{aligned} B^{1,1}_\Delta(w,v) &= \big(\partial_t\partial_\theta w\,,\, \partial_t\partial_\theta v\big) + \big((\partial_t \otimes J_\Delta)w\,,\,(\partial_t \otimes J)v\big) \\ &= \int_0^1\int_0^T \partial_t\partial_\theta w(\theta,t)\partial_t\partial_\theta v(\theta,t)\,\mathrm{d}t\,\mathrm{d}\theta + \int_0^T ((\partial_t \otimes J_\Delta)w)(t)((\partial_t \otimes J)v)(t)\,\mathrm{d}t. \end{aligned}$$

Theorem 15 (I, Theorem 3.1) *Let $u \in \mathcal{H}_{00}(\mathbb{R}^2_T)$ be the solution of the equation $\mathcal{S}u = f$. Then the function $u_\Delta \in S^{1,1}_{h,k}$ is a solution of the collocation problem (3.1) if and only if*

$$B^{1,1}_\Delta(\mathcal{S}u_\Delta, v) = B^{1,1}_\Delta(\mathcal{S}u, v), \quad (v \in S^{1,1}_{h,k}).$$

The unique solvability of collocation equations (3.1) is a consequence of the coercivity and continuity of the bilinear form $B^{1,1}_\Delta(\mathcal{S}u,v)$. This yields stability and convergence of the method. To prove the coercivity, we consider the form $B^{1,1}_\Delta(w,v)$ as a small perturbation of the bilinear form

$$B^{1,1}(w,v) = \big(\partial_t\partial_\theta w\,,\,\partial_t\partial_\theta v\big) + \big((\partial_t \otimes J)w\,,\,(\partial_t \otimes J)v\big).$$

The energy norm induced by the bilinear form is

$$|\!|\!| v |\!|\!|_{-\frac{1}{2},-\frac{1}{4}} = \|\partial_t\partial_\theta v\|_{-\frac{1}{2},-\frac{1}{4}} + \|(\partial_t \otimes J)v\|_{-\frac{1}{2},-\frac{1}{4}}.$$

For the analysis, it is essential that $|\!|\!| \cdot |\!|\!|_{-\frac{1}{2},-\frac{1}{4}}$ is a norm in the subspace $S^{1,1}_{h,k}$. The proof of continuity and coercivity of the bilinear form $B^{1,1}(\mathcal{S}\cdot,\cdot)$ is based on certain commutation properties of the single layer heat operator. The bilinear form $B^{1,1}_\Delta(\mathcal{S}\cdot,\cdot)$ is a small perturbation of the form $B^{1,1}(\mathcal{S}\cdot,\cdot)$. Therefore, we obtain the continuity estimate

$$|B^{1,1}_\Delta(\mathcal{S}u,v)| \leq c_1\, |\!|\!| u |\!|\!|_{-\frac{1}{2},-\frac{1}{4}} |\!|\!| v |\!|\!|_{-\frac{1}{2},-\frac{1}{4}}, \quad (u,v \in \mathcal{H}_{00}(\mathbb{R}^2_T)),$$

as well as the coercivity

$$B_{\Delta}^{1,1}(\mathcal{S}v, v) \geq c_2 \, \|v\|_{-\frac{1}{2},-\frac{1}{4}}^2, \quad (v \in \mathcal{H}_{00}(\mathbb{R}_T^2)),$$

for a sufficiently small mesh parameter h. Now, the general theory gives the existence of the unique solution of collocation equations (3.1), as well as the stability

$$\|u_{\Delta}\|_{-\frac{1}{2},-\frac{1}{4}} \leq \frac{c_1}{c_2} \, \|u\|_{-\frac{1}{2},-\frac{1}{4}}$$

and the quasi-optimal approximation result

$$\|u - u_{\Delta}\|_{-\frac{1}{2},-\frac{1}{4}} \leq \big(1 + \frac{c_1}{c_2}\big) \inf_{v \in S_{h,k}^{1,1}} \|u - v\|_{-\frac{1}{2},-\frac{1}{4}}. \tag{3.2}$$

The convergence analysis of the collocation approximation defined by (3.1) is based on the quasi-optimal error estimate. It is natural first to discuss the asymptotic accuracy of the approximation when the error is measured by means of the energy norm $\|\cdot\|_{-\frac{1}{2},-\frac{1}{4}}$. For this, we need error estimates for the L^2-orthogonal projection $P_{h,k}^{1,1} : L^2(\mathbb{R}_T^2) \to S_{h,k}^{1,1}$ as well as some additional regularity assumptions on the solution of the single layer heat operator equation.

Lemma 5 *Assume that* $u \in \mathcal{H}_{00}(\mathbb{R}_T^2)$ *such that* $u(\theta,\cdot)$, $(\partial_\theta u)(\theta,\cdot) \in H^2(0,T)$ *for almost all* θ, *and* $(\partial_t u)(\cdot,t) \in H^2$ *for almost all* t. *Then we have the approximation results*

$$\begin{aligned} \|u - P_{h,k}^{1,1}u\|_{-\frac{1}{2},-\frac{1}{4}} &\leq c\, h^{\frac{3}{2}} \|\partial_t u\|_{2,0} + c\, k\big(h^{\frac{1}{2}} + k^{\frac{1}{4}}\big)\big(\|u\|_{0,2} + \|\partial_\theta u\|_{0,2}\big), \\ \|u - P_{h,k}^{1,1}u\|_{0,0} &\leq c\, h \|\partial_t u\|_{2,0} + c\, k\big(\|u\|_{0,2} + \|\partial_\theta u\|_{0,2}\big). \end{aligned}$$

Theorem 16 (I, Theorem 5.1) *Assume that the solution of the equation* $\mathcal{S}u = f$ *satisfies* $u \in \mathcal{H}_{00}(\mathbb{R}_T^2)$ *such that* $u(\theta,\cdot)$, $(\partial_\theta u)(\theta,\cdot) \in H^2(0,T)$ *for almost all* θ, *and* $(\partial_t u)(\cdot,t) \in H^2$ *for almost all* t. *Then the collocation approximation* $u_\Delta \in S_{h,k}^{1,1}$ *defined by (3.1) furnishes the asymptotic error estimate*

$$\|u - u_{\Delta}\|_{-\frac{1}{2},-\frac{1}{4}} \leq c\, h^{\frac{3}{2}} \|\partial_t u\|_{2,0} + c\, k\big(h^{\frac{1}{2}} + k^{\frac{1}{4}}\big)\big(\|u\|_{0,2} + \|\partial_\theta u\|_{0,2}\big).$$

In the estimate the time step dominates the order of the convergence. This effect can be compensated for by letting the time-discretization be finer than the discretization in the space variable. We also consider the convergence by using the norm $\|v\|_{0,0} = \|\partial_t \partial_\theta v\|_{0,0} + \|(\partial_t \otimes J)v\|_{0,0}$. For this we need the inverse estimate

$$\|\partial_t \partial_\theta v\|_{0,0} + \|(\partial_t \otimes J)v\|_{0,0} \leq c \max(h^{-\frac{1}{2}}, k^{-\frac{1}{4}}) \|v\|_{-\frac{1}{2},-\frac{1}{4}}, \quad (v \in S_{h,k}^{1,1}).$$

These aspects are summarized in the following theorem.

Theorem 17 (I, Theorems 5.2, 5.3 and 5.4) *Let the assumptions of Theorem 16 be valid. Moreover, suppose that* $k \leq c\, h^\nu$, *where* $h \leq h_0$ *is sufficiently small. Then the collocation approximation* u_Δ *satisfies*

$$\|u - u_{\Delta}\|_{-\frac{1}{2},-\frac{1}{4}} \leq c\, h^{\frac{3}{2}} \|\partial_t u\|_{2,0} + c\, h^{\min(\frac{5}{4}\nu, \nu + \frac{1}{2})}\big(\|u\|_{0,2} + \|\partial_\theta u\|_{0,2}\big).$$

In particular, for $\nu \geq \frac{6}{5}$

$$\|u - u_\Delta\|_{-\frac{1}{2},-\frac{1}{4}} \leq c\, h^{\frac{3}{2}} \big(\|u\|_{0,2} + \|\partial_\theta u\|_{0,2} + \|\partial_t u\|_{2,0}\big).$$

If in addition, $c_0\, \underline{h} \leq h \leq \underline{h}$ and $c_0\, h^2 \leq k \leq c\, h^2$, where $\underline{h} \leq h_0$ is sufficiently small. Then, we have for $\nu \geq \frac{6}{5}$

$$\|u - u_\Delta\|_{0,0} \leq c\, h\big(\|u\|_{0,2} + \|\partial_\theta u\|_{0,2} + \|\partial_t u\|_{2,0}\big).$$

The final results are the pointwise and L^2-convergence

$$\max_{\theta\in[0,1]} |(u - u_\Delta)(\theta, t)| \leq c\, t^{\frac{1}{2}} h\big(\|u\|_{0,2} + \|\partial_\theta u\|_{0,2} + \|\partial_t u\|_{2,0}\big),\ (0 \leq t \leq T),$$

$$\|u - u_\Delta\|_{0,\mathbb{R}^2_T} \leq c\, h\big(\|u\|_{0,2} + \|\partial_\theta u\|_{0,2} + \|\partial_t u\|_{2,0}\big).$$

We have carried out some numerical experiments which confirm the convergence indicating, even the quadratic rate of the convergence. The numerical implementation of the scheme, covering also the general case, has been explained in some detail in article I.

3.2 Article II: A collocation type projection method for the single layer heat operator equation

In order to simplify convergence analysis, and reduce regularity requirements, we propose another projection method for the approximation. Our approach is to use interpolation with respect to the space variable and L^2-projection with respect to the time variable. With this aim in view, we define $Q^{1,0}_{h,k} := I^1_h \otimes P^0_k = P^0_k \otimes I^1_h$, where $I^1_h : H^r \to S^1_h$, $r > \frac{1}{2}$, is the interpolation operator. Discrete equations corresponding to the equation $\mathcal{S}u = f$ are: find $u_\Delta \in S^{1,0}_{h,k}(\mathbb{R}^2_T)$ such that

$$Q^{1,0}_{h,k} \mathcal{S} u_\Delta = Q^{1,0}_{h,k} f. \tag{3.3}$$

In order to describe the required regularity, we define some spaces of functions. Let $\mathcal{C}_1(\overline{\mathbb{R}}^2_T)$ be the space of continuous functions $f(\theta, t)$ on the closure $\overline{\mathbb{R}}^2_T$ of $\mathbb{R}^2_T$ such that f is 1-periodic with respect to the variable θ. The measurable function u is essentially bounded on $\overline{\mathbb{R}}^2_T$ if there exists a constant $C > 0$ such that $|u(\theta, t)| \leq C$ for almost all $(\theta, t) \in \mathbb{R}^2_T$. The space of essentially bounded functions on $\mathbb{R}^2_T$ is denoted by $\mathcal{L}^\infty(\mathbb{R}^2_T)$. Moreover, we define the spaces

$$\begin{aligned}
\mathcal{C}_{00}(\overline{\mathbb{R}}^2_T) &= \{f \in \mathcal{C}_1(\overline{\mathbb{R}}^2_T) \mid f(\theta, 0) \equiv 0\},\\
\mathcal{H}^c_{00}(\overline{\mathbb{R}}^2_T) &= \{f \in \mathcal{C}_{00}(\overline{\mathbb{R}}^2_T) \mid \partial_\theta f \in \mathcal{C}_1(\overline{\mathbb{R}}^2_T)\},\\
\mathcal{H}_{00}(\mathbb{R}^2_T) &= \{f \in \mathcal{L}^\infty(\mathbb{R}^2_T) \mid \partial_\theta f \in \mathcal{L}^\infty(\mathbb{R}^2_T)\}.
\end{aligned}$$

We notice that the inclusions $S^{1,0}_{h,k}(\mathbb{R}^2_T) \subset H^{r,\frac{r}{2}}(\mathbb{R}^2_T)$, $r < 1$, $S^{1,0}_{h,k}(\mathbb{R}^2_T) \subset \mathcal{L}^{\infty}(\mathbb{R}^2_T)$ are valid for the trial spaces.

Problem setting (3.3) is also equivalent to a certain Galerkin problem. For this, we define the bilinear form

$$\begin{aligned} B^{1,0}_{\Delta}(w,v) &= (\partial_\theta w\,,\,\partial_\theta v) + (J_\Delta w\,,\,J\,v) \\ &= \int_0^1\int_0^T \partial_\theta w(\theta,t)\partial_\theta v(\theta,t)\,\mathrm{d}t\,\mathrm{d}\theta + \int_0^T (J_\Delta w)(t)(J\,v)(t)\,\mathrm{d}t, \end{aligned}$$

which is a small perturbation of the bilinear form

$$B^{1,0}(w,v) = (\partial_\theta w\,,\,\partial_\theta v) + (J\,w\,,\,J\,v).$$

Theorem 18 (II, Theorem 2) *Let $u \in \mathcal{H}_{00}(\mathbb{R}^2_T)$ be the solution of the equation $\mathcal{S}u = f$. Then the function $u_\Delta \in S^{1,0}_{h,k}(\mathbb{R}^2_T)$ is a solution of discrete problem (3.3) if and only if*

$$B^{1,0}_{\Delta}(\mathcal{S}u_\Delta, v) = B^{1,0}_{\Delta}(\mathcal{S}u, v), \quad (v \in S^{1,0}_{h,k}(\mathbb{R}^2_T)). \tag{3.4}$$

Here, the energy norm $|||v|||_{-\frac{1}{2},-\frac{1}{4}} = \|\partial_\theta v\|_{-\frac{1}{2},-\frac{1}{4}} + \|J\,v\|_{-\frac{1}{2},-\frac{1}{4}}$ is a norm in the subspace $S^{1,0}_{h,k}(\mathbb{R}^2_T)$. The continuity and coercivity of the form $B^{1,0}(\mathcal{S}\cdot,\cdot)$ follows from certain commutation properties of the single layer heat operator. The bilinear form $B^{1,0}_{\Delta}(\mathcal{S}\,\cdot,\cdot)$ is a small perturbation. Therefore, we have, for sufficiently small h, the continuity

$$|B^{1,0}_{\Delta}(\mathcal{S}u,v)| \le c_1\,|||u|||_{-\frac{1}{2},-\frac{1}{4}}|||v|||_{-\frac{1}{2},-\frac{1}{4}}, \quad (u,v \in \mathcal{H}_{00}(\mathbb{R}^2_T)). \tag{3.5}$$

and the coercivity

$$B^{1,0}_{\Delta}(\mathcal{S}v,v) \ge c_2\,|||v|||^2_{-\frac{1}{2},-\frac{1}{4}}, \quad (v \in \mathcal{H}_{00}(\mathbb{R}^2_T)). \tag{3.6}$$

We assume that the solution of the equation $\mathcal{S}u = f$ satisfies $u \in \mathcal{H}_{00}(\mathbb{R}^2_T)$. Then for all $0 < h \le h_0$ there exists a unique solution u_Δ of equations (3.3). Moreover, we have the stability

$$|||u_\Delta|||_{-\frac{1}{2},-\frac{1}{4}} \le \frac{c_1}{c_2}\,|||u|||_{-\frac{1}{2},-\frac{1}{4}} \tag{3.7}$$

and the quasi-optimal error estimate

$$|||u-u_\Delta|||_{-\frac{1}{2},-\frac{1}{4}} \le \big(1+\frac{c_1}{c_2}\big) \inf_{v\in S^{1,0}_{h,k}(\mathbb{R}^2_T)} |||u-v|||_{-\frac{1}{2},-\frac{1}{4}}. \tag{3.8}$$

The quasi-optimal error estimate (3.8) yields the asymptotic accuracy of the approximation when the error is measured by means of the norm $|||\cdot|||_{-\frac{1}{2},-\frac{1}{4}}$.

For the proof of our convergence results, we need additional regularity assumptions on the solution of the equation $\mathcal{S}u = f$. We also need approximation properties with respect to the norm $|||v|||_{0,0} = \|\partial_\theta v\|_{0,0} + \|J\,v\|_{0,0}$. These are described in the following lemma.

Lemma 6 *Assume that* $u \in \mathcal{H}_{00}(\mathbb{R}^2_T)$ *such that* $u \in H^{s,\frac{s}{2}}_{00}(\mathbb{R}^2_T)$, $1 \le s \le 2$. *Then we have the approximation results*

$$\begin{aligned} \|u - P^{1,0}_{h,k}u\|_{-\frac{1}{2},-\frac{1}{4}} &\le c\,(h^{s-\frac{1}{2}} + k^{(s-\frac{1}{2})/2})\|u\|_{s,\frac{s}{2}}, \quad (1 \le s \le 2), \\ \|u - P^{1,0}_{h,k}u\|_{0,0} &\le c\left(h^{s-1} + k^{(s-1)/2}\right)\|u\|_{s,\frac{s}{2}}, \quad (1 \le s \le 2). \end{aligned}$$

¿From the quasi-optimal error estimate and Lemma 6, we obtain

Theorem 19 (II, Theorem 4) *Let* $\mathcal{S}u = f$, $u \in \mathcal{H}_{00}(\mathbb{R}^2_T)$ *and* $u \in H^{s,\frac{s}{2}}_{00}(\mathbb{R}^2_T)$, $1 \le s \le 2$. *Then the approximation* $u_\Delta \in S^{1,0}_{h,k}(\mathbb{R}^2_T)$ *defined by (3.3) furnishes the asymptotic error estimate*

$$\|u - u_\Delta\|_{-\frac{1}{2},-\frac{1}{4}} \le c\,(h^{s-\frac{1}{2}} + k^{(s-\frac{1}{2})/2})\|u\|_{s,\frac{s}{2}}. \tag{3.9}$$

In Theorem 19, the time step dominates the order of the convergence. This effect can be compensated for by letting the time-discretization be finer than the discretization in the space variable.

Theorem 20 (II, Theorem 5 and Theorem 6) *Let* $\mathcal{S}u = f$, $u \in \mathcal{H}_{00}(\mathbb{R}^2_T)$ *and* $u \in H^{s,\frac{s}{2}}_{00}(\mathbb{R}^2_T)$, $1 \le s \le 2$. *Moreover, suppose that* $h \le \underline{h}$ *and* $k \le c\,h^2$, *where* $\underline{h} \le h_0$ *is sufficiently small. Then the collocation type approximation* u_Δ *satisfies*

$$\|u - u_\Delta\|_{-\frac{1}{2},-\frac{1}{4}} \le c\; h^{\frac{3}{2}}\|u\|_{2,1}. \tag{3.10}$$

Further, let $c_0\,\underline{h} \le h \le \underline{h}$ *and* $c_0\,h^2 \le k \le c\,h^2$, *where* $\underline{h} \le h_0$ *is sufficiently small. Then we have for the collocation type approximation* u_Δ

$$\|u - u_\Delta\|_{0,0} \le c\,h\|u\|_{s,\frac{s}{2}}. \tag{3.11}$$

In particular, we obtain the L^2*-convergence* $\|u - u_\Delta\|_{0,0} \le c\,h\|u\|_{2,1}$.

We have proved that the Petrov-Galerkin approximation is stable and convergent. Compared to the collocation method, the regularity requirements are significantly reduced.

3.3 Article III: An approximation method for the hypersingular heat operator equation

Here, we consider the case of a pseudodifferential operator of positive order. The model problem is the hypersingular heat operator equation. Our Petrov-Galerkin approach uses interpolation with respect to the space variable and L^2-projection with respect to the time variable. For this, we define $Q^{3,1}_{h,k} := I^3_h \otimes P^1_k = P^1_k \otimes I^3_h$, where $I^3_h : H^r \to S^3_h$, $r > \frac{1}{2}$, is the interpolation operator. Since our method uses pointwise values, it is essential that the image $\mathcal{H}u$ is continuous. According to the Sobolev embedding theorem this condition is true if $u \in H^{r,\frac{r}{2}}_{00}(\mathbb{R}^2_T)$, $r > \frac{5}{2}$.

The discrete equation corresponding to the hypersingular heat operator equation $\mathcal{H}u = f$ is: find $u_\Delta \in S^{3,1}_{h,k}(\mathbb{R}^2_T)$ such that

$$Q^{3,1}_{h,k}\mathcal{H}u_\Delta = Q^{3,1}_{h,k}f. \tag{3.12}$$

This operator equation is equivalent to the following system of equations:
find $u_\Delta \in S^{3,1}_{h,k}(\mathbb{R}^2_T)$ such that

$$\int_0^T \phi^1_m(t)(\mathcal{H}u_\Delta)(\theta_n, t)\,\mathrm{d}t = \int_0^T \phi^1_m(t) f(\theta_n, t)\,\mathrm{d}t,\ (n = 1, \dots, N, m = 1, \dots, M).$$

Again, problem setting (3.12) is equivalent to a certain Galerkin problem. For this, we define the bilinear forms

$$\begin{aligned} B^{2,0}(w, v) &= \big(\partial^2_\theta w\,,\, \partial^2_\theta v\big) + \big(J\,w\,,\, J\,v\big) \\ B^{2,0}_\Delta(w, v) &= \big(\partial^2_\theta w\,,\, \partial^2_\theta v\big) + \big(J_\Delta w\,,\, J\,v\big). \end{aligned}$$

Theorem 21 (III, Theorem 2) *Let $u \in H^{s,\frac{s}{2}}_{00}(\mathbb{R}^2_T)$, $s > \frac{5}{2}$ be the solution of the equation $\mathcal{H}u = f$. Then the function $u_\Delta \in S^{3,1}_{h,k}(\mathbb{R}^2_T)$ is a solution of problem (3.12) if and only if*

$$B^{2,0}_\Delta(\mathcal{H}u_\Delta, v) = B^{2,0}_\Delta(\mathcal{H}u, v), \quad (v \in S^{3,1}_{h,k}(\mathbb{R}^2_T)). \tag{3.13}$$

Discrete equations (3.12) are uniquely solvable if the spatial discretization parameter is small enough. This result is a consequence of the coercivity estimate for the bilinear form $B^{2,0}_\Delta(\mathcal{H}u, v)$, which yields stability and convergence of the method. For the sake of brevity, we introduce the notation $|\!|\!| v |\!|\!|_{r,s} = \|\partial^2_\theta v\|_{r,s} + \|J\,v\|_{r,s}$. In any space of functions, where this expression is well defined and finite, it gives a seminorm. In particular, in the subspace of the approximating functions $|\!|\!| \cdot |\!|\!|_{s,\frac{s}{2}}$ defines a norm. Our next aim is to establish the required coercivity of the approximating bilinear form with respect to the norm $|\!|\!| \cdot |\!|\!|_{\frac{1}{2},\frac{1}{4}}$. For our method, it is enough that this property is valid in the subspace $S^{3,1}_{h,k}(\mathbb{R}^2_T)$.

Lemma 7 *Let $u, w \in H^{s,\frac{s}{2}}_{00}(\mathbb{R}^2_T)$, $s > \frac{5}{2}$. Then we have the continuity*

$$|B^{2,0}(\mathcal{H}u, w)| \le c_1\, |\!|\!| u |\!|\!|_{\frac{1}{2},\frac{1}{4}} |\!|\!| w |\!|\!|_{\frac{1}{2},\frac{1}{4}}, \quad (u, w \in H^{s,\frac{s}{2}}_{00}(\mathbb{R}^2_T)) \tag{3.14}$$

and the coercivity

$$B^{2,0}(\mathcal{H}w, w) \ge c_2\, |\!|\!| w |\!|\!|^2_{\frac{1}{2},\frac{1}{4}}, \quad (w \in H^{s,\frac{s}{2}}_{00}(\mathbb{R}^2_T)). \tag{3.15}$$

The perturbation due to the trapezoidal rule approximation is small in the sense that

$$|B^{2,0}(\mathcal{H}u, w) - B^{2,0}_\Delta(\mathcal{H}u, w)| \le c\, h\big(|\!|\!| u |\!|\!|_{0,0} + |\!|\!| u |\!|\!|_{0,\frac{1}{2}}\big) |\!|\!| w |\!|\!|_{0,0}, \tag{3.16}$$

for all $u, w \in H_{00}^{s,\frac{s}{2}}(\mathbb{R}_T^2)$. *In particular,*

$$|B^{2,0}(\mathcal{H}v, v) - B_\Delta^{2,0}(\mathcal{H}v, v)| \leq c\, h\big(1 + k^{-\frac{1}{4}}\big)\|\!|v|\!\|^2_{\frac{1}{2},\frac{1}{4}}, \quad (v \in S_{h,k}^{3,1}(\mathbb{R}_T^2)). \tag{3.17}$$

Therefore, also the approximating bilinear form is coercive

$$B_\Delta^{2,0}(\mathcal{H}v, v) \geq c_3\, \|\!|v|\!\|^2_{\frac{1}{2},\frac{1}{4}}, \quad (v \in S_{h,k}^{3,1}(\mathbb{R}_T^2)) \tag{3.18}$$

for $0 < h \leq h_0$, $h^{4-\varepsilon} \leq c_0\, k$, $\varepsilon > 0$ *and* h_0 *is sufficiently small.*

The main result is the following error bound.

Theorem 22 (III, Theorem 3) *Assume that* $u \in H_{00}^{s,\frac{s}{2}}(\mathbb{R}_T^2)$, $s > \frac{5}{2}$ *and* $\mathcal{H}u = f$. *Then, for all* $0 < h \leq h_0$, $h^{4-\varepsilon} \leq c_0\, k$, $\varepsilon > 0$ *and* h_0 *is sufficiently small, there exists a unique solution* u_Δ *of Petrov-Galerkin equations (3.12). Moreover, we have the approximation result*

$$\|\!|u - u_\Delta|\!\|_{\frac{1}{2},\frac{1}{4}} \leq C \inf_{v \in S_{h,k}^{3,1}(\mathbb{R}_T^2)} \big(\|\!|u - v|\!\|_{\frac{1}{2},\frac{1}{4}} + h\|\!|u - v|\!\|_{0,\frac{1}{2}}\big). \tag{3.19}$$

The final aim is to establish L^2-convergence for the discrete solution. Based on the error estimate, we first discuss the asymptotic accuracy of the approximation when the error is measured by means of the norm $\|\!|\cdot|\!\|_{\frac{1}{2},\frac{1}{4}}$. For the proof of the convergence results, we make additional regularity assumptions on the solution of the equation $\mathcal{H}u = f$.

Lemma 8 *Assume that* $u \in H_{00}^{4,2}(\mathbb{R}_T^2)$. *Then we have the approximation results*

$$\begin{aligned}
\|\!|u - P_{h,\,k}^{3,1}u|\!\|_{0,0} &\leq c\,\big(h^2 + k\big)\|u\|_{4,2},\\
\|\!|u - P_{h,\,k}^{3,1}u|\!\|_{\frac{1}{2},\frac{1}{4}} &\leq c\,(h^{\frac{3}{2}} + k^{\frac{3}{4}})\|u\|_{4,2}.
\end{aligned}$$

¿From Theorem 22 and Lemma 8, we obtain

Theorem 23 (III, Theorem 4 and Theorem 5) *Let* $\mathcal{H}u = f$, *such that* $u \in H_{00}^{4,2}(\mathbb{R}_T^2)$. *Then the Petrov-Galerkin approximation* $u_\Delta \in S_{h,k}^{3,1}(\mathbb{R}_T^2)$ *defined by (3.12) furnishes the asymptotic error estimate*

$$\|\!|u - u_\Delta|\!\|_{\frac{1}{2},\frac{1}{4}} \leq c\,\big(h^{\frac{3}{2}} + k^{\frac{3}{4}}\big)\|u\|_{4,2}. \tag{3.20}$$

Moreover, suppose that $h^{4-\varepsilon} \leq c_0\, k \leq c\, h^2$, *where* $h \leq h_0$, *and* h_0 *is sufficiently small. Then we have the asymptotic estimates*

$$\|\!|u - u_\Delta|\!\|_{\frac{1}{2},\frac{1}{4}} \leq c\, h^{\frac{3}{2}}\|u\|_{4,2}, \tag{3.21}$$

$$\|\!|u - u_\Delta|\!\|_{0,0} = \|\partial_\theta^2(u - u_\Delta)\|_{0,0} + \|J\,(u - u_\Delta)\|_{0,0} \leq c\, h^{\frac{3}{2}}\|u\|_{4,2}. \tag{3.22}$$

In particular, we have the L^2*-convergence* $\|u - u_\Delta\|_{0,0} \leq c\, h^{\frac{3}{2}}\|u\|_{4,2}$.

Note that the asymptotic order of convergence with respect to the $\|\!|\cdot|\!\|_{0,0}$ remains the same. It is remarkable here that the required regularity of the solution is described in terms of the anisotropic Sobolev spaces. This technique can be applied also to the case of the single layer heat operator equation.

3.4 Article IV: The numerical approximation of the solution of a nonlinear boundary integral equation with the collocation method

In the stationary case, the nonlinear heat equation reduces to the potential equation (1.2) with a given nonlinear Neumann type boundary condition. The direct boundary integral approach yields the nonlinear boundary integral equation, [53]

$$(\tfrac{1}{2}I + D_\Gamma)\mu + S_\Gamma \mathcal{F}(\mu) = S_\Gamma g_{N\Gamma}. \tag{3.23}$$

We denote the associated nonlinear integral operator $A(u) = (\frac{1}{2}I + D_\Gamma)u + S_\Gamma \mathcal{F}(u)$. Here, $\mathcal{F}$ describes the nonlinearity, and the linear boundary integral operators are the single layer and double layer operators, respectively

$$S_\Gamma u(\mathbf{x}) = -\frac{1}{2\pi}\int\limits_\Gamma u(\mathbf{y}) \ln|\mathbf{x}-\mathbf{y}| d\Gamma_{\mathbf{y}}, \qquad D_\Gamma u(\mathbf{x}) = \frac{1}{2\pi}\int\limits_\Gamma u(\mathbf{y}) \frac{\partial}{\partial n_y} \ln|\mathbf{x}-\mathbf{y}| d\Gamma_{\mathbf{y}}.$$

The purpose of our paper is to introduce an approximation scheme for (3.23) by using an easily computable L^2-orthogonal projection of the nonlinear function. This approach applies to general projection methods, but for simplicity we discuss only collocation.

Now, we specify the assumptions on the nonlinearity. The real valued nonlinear function $F(\mathbf{x}, \xi) : \Gamma \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the Carathéodory conditions:

The mapping $\mathbf{x} \mapsto F(\mathbf{x}, \xi) : \Gamma \to \mathbb{R}$ is measurable for all fixed $\xi \in \mathbb{R}$.
The mapping $\xi \mapsto F(\mathbf{x}, \xi) : \mathbb{R} \to \mathbb{R}$ is continuous for almost all $\mathbf{x} \in \Gamma$.

The Nemitsky operator $u \mapsto \mathcal{F}(u) : L^2(\Gamma) \to L^2(\Gamma)$; $\mathcal{F}(u)(\mathbf{x}) = F(\mathbf{x}, u(\mathbf{x}))$ is well defined, provided that the Carathéodory conditions and the growth condition $|F(\mathbf{x}, \xi)| \le b(\mathbf{x}) + c\,|\xi|$, $(\mathbf{x}, \xi) \in \Gamma \times \mathbb{R}$ are valid (see [38]). Here, c is a constant and $b(\mathbf{x}) \in L^2(\Gamma)$. For the analysis of the numerical approximation scheme, we assume that
(A1) the Nemitsky operator $\mathcal{F} : L^2(\Gamma) \to L^2(\Gamma)$ is strongly monotone,
(A2) the Nemitsky operator $\mathcal{F} : L^2(\Gamma) \to L^2(\Gamma)$ is Lipschitz continuous,
(A3) the Nemitsky operator $\mathcal{F} : H^s(\Gamma) \to H^s(\Gamma)$ is bounded for $0 \le s < 1$.
We point out that the Lipschitz continuity of the Carathéodory function F guarantees (A2) and (A3). These properties are necessary for regularity results and error estimates.

Theorem 24 *Let the capacity of the boundary curve differ from unity.*

(*i*) *For every $g \in H^{-\frac{1}{2}}(\Gamma)$ integral equation (3.23) has a unique solution $u \in H^{\frac{1}{2}}(\Gamma)$.*

(*ii*) *For the solution u, the following regularity result is true: If $g_{N\Gamma} \in H^{s-1}(\Gamma)$, $\frac{1}{2} \le s < 2$, and assumptions (A1), (A2) and (A3) are valid, then the solution satisfies $u \in H^s(\Gamma)$.*

The proof is presented in [53]. It is based on the fact that integral operator A is strongly S_Γ^{-1} monotone.

Next, we consider the collocation method for finding an approximate solution of equation (3.23). We require that $g_{N\Gamma} \in H^{s-1}(\Gamma),\ s > \frac{1}{2}$. Then the function $g_{N\Gamma}$ is continuous, and the collocation equations are given by: find $u_h \in S_h^d$ such that

$$(A(u_h))(\tilde{x}_i) = (S_\Gamma g_{N\Gamma})(\tilde{x}_i), \quad (i = 0, \ldots, N-1),$$

Nodal point collocation is applied for odd degree splines, and midpoint collocation is applied for even degree splines. An equivalent formulation of the collocation problem, given in terms of the interpolation operator is: find $u_h \in S_h^d$ such that

$$I_h^d A(u_h) = I_h^d S_\Gamma g_{N\Gamma}, \tag{3.24}$$

For numerical purposes, we define an approximate collocation equation as follows: find $\tilde{u}_h \in S_h^d$ such that

$$\tilde{A}_h(\tilde{u}_h) := \tfrac{1}{2}\tilde{u}_h - I_h^d D_\Gamma \tilde{u}_h + I_h^d S_\Gamma P_h^d \mathcal{F}(\tilde{u}_h) = I_h^d S_\Gamma g_{N\Gamma}. \tag{3.25}$$

The following theorem describes the convergence properties.

Theorem 25 (IV, Theorem 3.1 and Theorem 3.3) *Assume $d > 0$. Let $u \in H^s(\Gamma)$, $\frac{1}{2} < s \le d+1$, be the solution of (3.23) and suppose that (A1) and (A2) are valid. Then, for sufficiently small h, the collocation problem (3.24) as well as the approximate collocation problem (3.25) admit unique solutions u_h, and $\tilde{u}_h$, respectively. Furthermore, we have the asymptotic error estimates*

$$\|u - u_h\|_t \ \le\ c\, h^{s-t}\|u\|_s, \tag{3.26}$$

$$\|\tilde{u}_h - u_h\|_{\frac{1}{2}} \ \le\ c\, h^{s+\frac{1}{2}}\|u\|_s + c\, h^{\tau+\frac{1}{2}}\|\mathcal{F}(u)\|_\tau, \tag{3.27}$$

$$\|u - \tilde{u}_h\|_t \ \le\ c\, h^{s-t}\|u\|_s + c\, h^{\tau+1-\max(t,\frac{1}{2})}\|\mathcal{F}(u)\|_\tau, \tag{3.28}$$

for $0 \le t \le s$, $t < d + \frac{1}{2}$, provided that $\mathcal{F}(u) \in H^\tau(\Gamma), 0 \le \tau \le d+1$.

The proof for (3.26) presented in [53] covers indices $\frac{1}{2} \le t \le s$, and the results in [55] give (3.26) when $0 \le t < \frac{1}{2}$. Here, the crucial idea is to choose an appropriate Hilbert space, and show that the assumptions of Theorem 13 are fulfilled.

Another source of error is the effect of the numerical integration. We decompose the single layer operator to a singular part with logarithmic singularity, and to a smooth part. The singular part is integrated exactly. The smooth part as well as integrals concerning the double layer operator are computed using numerical quadratures. We suppose that the quadrature error can be bounded above by $c\,h^\sigma$. For the right hand side of the equation, we use the orthogonal projection approximation. The resulting nonlinear system defines the mapping $\hat{A} : S_h^d \to S_h^d$ such that

$$\hat{A}(\hat{v}_h) = I_h P_h g_{N\Gamma}. \tag{3.29}$$

The solution $\hat{v}_h$ is related to the approximate solution of the original problem by the formula $\hat{v}_h = |\mathbf{x}'|\hat{u}_h$. We summarize the final result in the following theorem. The details are written in the original article.

Theorem 26 (IV, Theorem 5.2) *Assume $d > 0$. Let $g \in H^{s-1}(\Gamma)$ and let $v \in H^s(\Gamma), \frac{1}{2} < s \le d+1$ be the solution of (3.29) and suppose that assumptions (A1), (A2) are valid. Then we have the estimate*

$$\|u - \hat{u}_h\|_t \quad \le \quad c\,h^{s-t}(\|v\|_s + \|g\|_{s-1}) + c\,h^{\tau+1-\max(t,\frac{1}{2})} + c\,h^{\sigma-1-\max(t,\frac{1}{2})},$$

for $0 \le t \le s$, $t < d + \frac{1}{2}$, provided that the data is sufficiently smooth.

It evolves that our method retains the optimal convergence order of the collocation method. Numerical experiments confirm our theoretical results.

3.5 Article V: On the numerical solution of a non-linear heat conduction problem

In this article, we analyse a projection method scheme for solution of the second kind non-linear heat operator equation (1.1), assuming that the spatial domain Ω is two-dimensional and has a smooth boundary Γ. The direct boundary integral approach yields the nonlinear boundary integral equation (1.21). We denote the associated nonlinear integral operator with

$$\mathcal{A}(u) = (\tfrac{1}{2}I + \mathcal{D}_\Gamma)u + \mathcal{S}_\Gamma\mathcal{F}(u) \tag{3.30}$$

and consider the nonlinear operator equation $\mathcal{A}(u) = \mathcal{S}_\Gamma g_{N\Gamma}$. Concerning the nonlinearity, we assume that Carathéodory conditions (1.91) and (1.92), growth condition (1.93) and Lipschitz condition (1.95) are true. These imply that the Nemitsky operator $\mathcal{F} : L^2(\Sigma_T) \to L^2(\Sigma_T)$ is a well-defined, bounded and Lipschitz continuous operator. If, in addition, the mapping $\xi \mapsto F(\mathbf{x}, t, \xi)$ is non-decreasing for each $(\mathbf{x}, t) \in \Sigma_T$ then the Nemitsky operator is monotone. See Lemma 3 and Lemma 4. According to the following theorem, equation (3.30) is uniquely solvable in $H_{00}^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$, [30].

Theorem 27 *The operator $\mathcal{A} : H_{00}^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \to H_{00}^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ is a homeomorphism. Furthermore, we have*

$$\big(\mathcal{A}(u) - \mathcal{A}(w)\,,\, \mathcal{S}_\Gamma^{-1}(u-w)\big)_{\Sigma_T} \ge c\,\|u-w\|^2_{\frac{1}{2},\frac{1}{4}}, \quad (u, w \in H_{00}^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)).$$

We propose a method where we use interpolation with respect to the space variable, and orthogonal projection with respect to the time variable. The trial functions are tensor products of piecewise linear and piecewise constant splines. We define the projection operator $Q_{h,k}^{1,0} := I_h^1 \otimes P_k^0 = P_k^0 \otimes I_h^1$. The discrete problem asks us to find $u_\Delta \in S_{h,k}^{1,0}$ such that

$$Q_{h,k}^{1,0}\mathcal{A}(u_\Delta) = Q_{h,k}^{1,0}\mathcal{S}_\Gamma g_{N\Gamma}. \tag{3.31}$$

The method requires evaluation of pointwise values. Thus, we need a certain regularity concerning $\mathcal{A}(u_\Delta)$ and $\mathcal{S}_\Gamma g_{N\Gamma}$. For the right-hand side, it is enough to assume that $g_{N\Gamma} \in H_{00}^{s,\frac{s}{2}}(\Sigma_T)$, $s > \frac{1}{2}$, since, according to the mapping property of the single layer operator, $\mathcal{S}_\Gamma g_{N\Gamma} \in H_{00}^{s,\frac{s}{2}}(\Sigma_T)$, $s > \frac{3}{2}$. Furthermore, it is known that the embedding $H_{00}^{r,\frac{r}{2}}(\Sigma_T) \subset \mathcal{C}(\overline{\Sigma_T})$ is continuous for $r > \frac{3}{2}$. Sufficient regularity for $\mathcal{A}(u_\Delta)$ follows easily from the special structure of the operator $\mathcal{A}$. The identity operator preserves splines and therefore the required pointwise values exist. On the other hand, the trial function space satisfies $S_{h,k}^{1,0} \subset H_{00}^{r,\frac{r}{2}}(\Sigma_T)$, $r < 1$, and the mapping properties of the associated operators yield $\mathcal{D}_\Gamma u_\Delta + \mathcal{S}_\Gamma \mathcal{F}(u_\Delta) \in \mathcal{C}(\overline{\Sigma_T})$.

Lemma 9 *If the meshes are quasi-uniform and $v, v' \in S_{h,k}^{1,0}$, then*

$$\|(I - Q_{h,k}^{1,0})\mathcal{D}_\Gamma(v - v')\|_{\frac{1}{2},\frac{1}{4}} \leq c\left(h^{\frac{1}{2}} + k^{\frac{1}{4}}\right)\|v - v'\|_{\frac{1}{2},\frac{1}{4}}, \qquad (3.32)$$

$$\|(I - Q_{h,k}^{1,0})\mathcal{S}_\Gamma(\mathcal{F}(v) - \mathcal{F}(v'))\|_{\frac{1}{2},\frac{1}{4}} \leq c\left(h + k^{\frac{1}{2}}\right)\|v - v'\|_{\frac{1}{2},\frac{1}{4}}. \qquad (3.33)$$

Proof. Estimate (3.33) follows from Lemma 1, with $s = 1$ by means of the mapping property of the single layer heat operator and the Lipschitz continuity of the Nemitsky operator

$$\begin{aligned}\|(I - Q_{h,k}^{1,0})\mathcal{S}_\Gamma(\mathcal{F}(v) - \mathcal{F}(v'))\|_{\frac{1}{2},\frac{1}{4}} &\leq c\left(h^{\frac{1}{2}} + k^{\frac{1}{4}}\right)\|\mathcal{S}_\Gamma(\mathcal{F}(v) - \mathcal{F}(v'))\|_{1,\frac{1}{2}} \\ &\leq c\left(h^{\frac{1}{2}} + k^{\frac{1}{4}}\right)\|\mathcal{F}(v) - \mathcal{F}(v')\|_{0,0} \\ &\leq c\left(h^{\frac{1}{2}} + k^{\frac{1}{4}}\right)\|v - v'\|_{0,0} \\ &\leq c\left(h^{\frac{1}{2}} + k^{\frac{1}{4}}\right)\|v - v'\|_{\frac{1}{2},\frac{1}{4}}.\end{aligned}$$

Estimate (3.33) is proved analogously. □

Theorem 28 (V, Theorem 3) *For sufficiently small h and k, we have the stability inequality*

$$\|Q_{h,k}^{1,0}\mathcal{A}(v) - Q_{h,k}^{1,0}\mathcal{A}(v')\|_{\frac{1}{2},\frac{1}{4}} \geq c_0\,\|v - v'\|_{\frac{1}{2},\frac{1}{4}}, \quad (v, v' \in S_{h,k}^{1,0}).$$

In addition, equation (3.31) is uniquely solvable and we have the error estimates

$$\|u - u_\Delta\|_{s,\frac{s}{2}} \leq c \inf_{v \in S_{h,k}^{1,0}} \|u - v\|_{s,\frac{s}{2}} \leq c\left(h^{2-s} + k^{1-\frac{s}{2}}\right)\|u\|_{2,1},\ (s = 0, s = \tfrac{1}{2}).$$

Proof. Since $Q_{h,k}^{1,0}$ is a projection, we can write

$$\begin{aligned}(I - Q_{h,k}^{1,0})(\mathcal{A}(v) - \mathcal{A}(v')) &= (I - Q_{h,k}^{1,0})\mathcal{D}_\Gamma(v - v') \\ &\quad + (I - Q_{h,k}^{1,0})\mathcal{S}_\Gamma(\mathcal{F}(v) - \mathcal{F}(v')).\end{aligned} \qquad (3.34)$$

The Schwarz inequality

$$\left(Q_{h,k}^{1,0}\mathcal{A}(v) - Q_{h,k}^{1,0}\mathcal{A}(v')\,,\,\mathcal{S}_\Gamma^{-1}(v - v')\right)_{\Sigma_T} \leq c\,\|Q_{h,k}^{1,0}\mathcal{A}(v) - Q_{h,k}^{1,0}\mathcal{A}(v')\|_{\frac{1}{2},\frac{1}{4}}\|v - v'\|_{\frac{1}{2},\frac{1}{4}}$$

combined with the decomposition

$$\begin{aligned}&\left(Q_{h,k}^{1,0}\mathcal{A}(v) - Q_{h,k}^{1,0}\mathcal{A}(v')\,,\,\mathcal{S}_\Gamma^{-1}(v - v')\right)_{\Sigma_T} \\ &= \left(\mathcal{A}(v) - \mathcal{A}(v')\,,\,\mathcal{S}_\Gamma^{-1}(v - v')\right)_{\Sigma_T} - \left((I - Q_{h,k}^{1,0})\left(\mathcal{A}(v) - \mathcal{A}(v')\right),\,\mathcal{S}_\Gamma^{-1}(v - v')\right)_{\Sigma_T} \\ &\geq c\,\|v - v'\|_{\frac{1}{2},\frac{1}{4}}^2 - c'\,\|(I - Q_{h,k}^{1,0})\left(\mathcal{A}(v) - \mathcal{A}(v')\right)\|_{\frac{1}{2},\frac{1}{4}}\|v - v'\|_{\frac{1}{2},\frac{1}{4}}\end{aligned}$$

leads to the estimate

$$\|Q_{h,k}^{1,0}\mathcal{A}(v)-Q_{h,k}^{1,0}\mathcal{A}(v')\|_{\frac{1}{2},\frac{1}{4}} \geq c\,\|v-v'\|_{\frac{1}{2},\frac{1}{4}} - c'\,\|(I-Q_{h,k}^{1,0})\big(\mathcal{A}(v)-\mathcal{A}(v')\big)\|_{\frac{1}{2},\frac{1}{4}}.$$

This implies stability for $0 < h \leq h_0$ and $0 < k \leq k_0$, since Lemma 9 together with (3.34) yields

$$\|(I-Q_{h,k}^{1,0})\big(\mathcal{A}(v)-\mathcal{A}(v')\big)\|_{\frac{1}{2},\frac{1}{4}} \leq c\,(\,h^{\frac{1}{2}} + k^{\frac{1}{4}})\|v-v'\|_{\frac{1}{2},\frac{1}{4}}.$$

Due to stability, the operator $Q_{h,k}^{1,0}\mathcal{A} : S_{h,k}^{1,0} \to S_{h,k}^{1,0}$ is an injection. According to Brouwer's theorem on domain invariance, the range $R(Q_{h,k}^{1,0}\mathcal{A})$ is open. Because of the stability and continuity of $Q_{h,k}^{1,0}\mathcal{A}$ in $S_{h,k}^{1,0}$, the range $R(Q_{h,k}^{1,0}\mathcal{A})$ is also closed in $S_{h,k}^{1,0}$. Therefore, $R(Q_{h,k}^{1,0}\mathcal{A}) = S_{h,k}^{1,0}$, and $Q_{h,k}^{1,0}\mathcal{A}$ is a homeomorphism. The quasi-optimality with respect to $\|\cdot\|_{\frac{1}{2},\frac{1}{4}}$ is a direct consequence of stability.

The $L^2(\Sigma_T)$-estimate is proved by means of the technique in [55]. In fact, the result follows from Theorem 14, since conditions (2.5) - (2.8), and approximation property (2.9) are true when we choose the spaces $X_0 = L^2(\Sigma_T)$, $X = H_{00}^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$, $Z = H_{00}^{1,\frac{1}{2}}(\Sigma_T)$ and the operators $S = \mathcal{S}_\Gamma$, $D = \mathcal{D}_\Gamma$, $F = \mathcal{F}$, $Q_n = Q_{h,k}^{1,0}$ and $X_n = S_{h,k}^{1,0}$. Also, the mapping property $D : X_0 \to Z$ is bounded, is fulfilled. $\square$

In the numerical implementation, the resulting time-integrals can be calculated exactly. Compared to the Galerkin method, this approach simplifies the realization of the matrix equations. It seems that straightforward application of the preceeding analysis does not extend to the collocation method.

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