ON CONNECTED TRANSVERSALS TO DIHEDRAL SUBGROUPS

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Abstract

Let $G$ be a group with a dihedral subgroup $H$ of order $2x$, where $x$ is an odd number. We show that, if there exist $H$-connected transversals $A$ and $B$ in $G$, then $G$ is a solvable group. We apply this result to loop theory and we show that, if the inner mapping group $I(Q)$ of a finite loop $Q$ is dihedral of order $2x$, then $Q$ is a solvable loop.

Keywords: group, loop, solvability
Contents

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1. Introduction

Let \( G \) be a group, \( H \) a subgroup of \( G \) and \( A, B \) two left transversals to \( H \) in \( G \). Now a subset of elements of \( G \) is said to be a left transversal to \( H \) in \( G \) if it contains exactly one element of each left coset of \( H \). A right transversal is defined similarly. If further \( [A, B] \leq H \), then we say that these transversals are \( H \) -connected. This definition was introduced by Kepka and Niemenmaa in [10]. This new concept presents the interesting possibility of studying structural properties of \( G \) by using the structure of \( H \). In this paper we are interested in investigating the relation between the solvability of \( G \) and the structure of \( H \).

Kepka and Niemenmaa started to study this solvability problem in 1990 and they could first show in [7], that \( G \) is solvable in the case \( H \) is cyclic. Next they showed in [11] that, if \( H \) is a finite abelian subgroup, then the solvability of \( G \) follows. After this Niemenmaa started to investigate the non-abelian case, and in [12] he was able to show that \( G \) is solvable if \( |H| = 6 \). The more general problem, where \( |H| \) is a product of two prime numbers \( p \) and \( q \), Niemenmaa investigated in [14]. He could prove, by using the classification of finite simple groups, that \( G \) is solvable if \( q = 2 \) and \( p \leq 61 \), \( q = 3 \) and \( p \leq 31 \) or \( q = 5 \) and \( p \leq 11 \). Furthermore Myllylä and Niemenmaa considered this situation in [8], where they showed that the solvability follows in the case \( |H| = 2p \), where \( p = 4t + 3 \) is a prime number. The examination of the solvability continued to
the dihedral case and in [13] Niemenmaa considered the situation that $H$ is a dihedral $2$-group and once again the solvability of $G$ followed. In [4] Csörgő, Myllylä and Niemenmaa were able to show that $G$ is solvable provided that $H$ is a dihedral group of order $2p^n$, where $p$ is an odd prime number. In the present paper we are able to show that $G$ is also solvable in the case that $H$ is a dihedral group of order $2x$, where $x$ is an odd number. Furthermore the case $|H|=pq$ in general form is now under investigation by Csörgő and Niemenmaa. They have already managed to show the solvability of $G$ in the situation, where $p$ and $q$ are odd prime numbers, $p>q$ and $p=2q^m+1$. In the future, the case where $H$ is a nilpotent subgroup and the general case where $H$ is dihedral, might also be very interesting.

Connected transversals are also a very useful tool when studying the structure of loops. A groupoid $Q$ is called a loop if $Q$ has a unique division on both sides and a neutral element $e$, which means that loops can be understood as nonassociative versions of groups. Now all permutations $L_a(x)=ax$ and $R_a(x)=xa$, where $a \in Q$, generate the multiplication group of $Q$, which is denoted by $M(Q)$. Furthermore the stabilizer of $e \in Q$, denoted by $I(Q)$, is the inner mapping group of $Q$. These two groups, which connect loop theory and group theory, were introduced by Bruck [1] and he used these groups to investigate the structure of loops. For loops Bruck defined solvability as follows: a loop $Q$ is solvable if it has a series $1=Q_0 \subseteq \ldots \subseteq Q_n = Q$, where $Q_{i-1}$ is a normal subloop of $Q_i$ and $Q_i/Q_{i-1}$ is an abelian group. Of course, normal subloops are kernels of loop homomorphisms.

The structure of the multiplication group reflects some of the properties of its corresponding loop. In 1996 Vesamnen [16] proved that the solvability of $M(Q)$ implies the solvability of $Q$ in the case that $Q$ is a finite loop. On the other hand Kepka and Niemenmaa [10] have proved the connection between connected transversals and multiplication groups of loops. When we combine these two important results we can consider the solvability of a loop $Q$ by studying the properties of the inner mapping group $I(Q)$ of the loop $Q$. 
2. Connected transversals

In this section we begin with seven lemmas and theorems, which give basic information about connected transversals and consider some solvability cases. After this we prove three preliminary lemmas, which are needed later in the next section. Like in the introduction, we assume that $G$ is a group, $H \subseteq G$ and $A, B$ are $H$-connected transversals in $G$. By $L_G(H)$ we denote the core of $H$ in $G$, i.e. $L_G(H) = \cap H^g$, where $g \in G$. It is clear that $L_G(H)$ is now the largest normal subgroup of $G$ contained in the subgroup $H$.

**Lemma 2.1.** $A$ and $B$ are left and right transversals to $H^g$ for every $g \in G$. If $L_G(H) = 1$, then $1 \in A \cap B$.

**Lemma 2.2.** If $L_G(H) = 1$, then $N_G(H) = H \times Z(G)$.

**Lemma 2.3.** If $D \subseteq A \cup B$ and $K = \langle H, D \rangle$, then $D \subseteq L_G(K)$.

For the proofs of Lemmas 2.1 - 2.3, see [10, p.113, Proposition 2.7 and Lemma 2.5, respectively].

**Theorem 2.4.** If $H$ is a finite abelian group, then $G$ is solvable.
Theorem 2.5. If $H$ is cyclic, then $G$ is solvable.

Theorem 2.6. If $H$ is a dihedral subgroup of order $2p^n$, where $p$ is an odd prime, then $G$ is solvable.

Lemma 2.7. Suppose $G = QM$ where $G$ is a finite group. If $Q$ is an abelian group and $M$ has a nilpotent subgroup of index at most 2, then $G$ is solvable.

For the proofs of Theorems 2.4 - 2.6 and Lemma 2.7, see [11, 7, 4 and 2, respectively].

In the next three lemmas, $H \leq G$ is a dihedral group of order $2x$, where $x$ is an odd number, and we denote by $C$ the cyclic subgroup of order $x$ in $H$. Now $C$ is abelian and normal in $H$, which means that $H$ is solvable.

Lemma 2.8. If $H$ is a maximal subgroup of $G$ and $L_G(H) = 1$, then $|H \cap H^g| \leq 2$ for every $g \in G \setminus H$.

Proof. Clearly, $N_G(H) = H$ and $H \neq H^g$ whenever $g \in G \setminus H$. If $|H \cap H^g| > 2$, then $H \cap H^g$ contains a $p$-group $P$, where $p$ is an odd prime dividing $x$. Now $P \leq C$ and furthermore $P \lhd H$ and also $P \lhd H^g$. This means that $P$ is normal in $\langle H, H^g \rangle = G$, a contradiction. We conclude that $|H \cap H^g| \leq 2$.

Lemma 2.9. Let $H$ be a maximal subgroup of $G$ and $L_G(H) = 1$. If $aH = bH$, where $a \in A$ and $b \in B$, then $b^{-1}a = a^{-1}b \in H \cap H^a = H \cap H^b$.

Proof. Now $a^{-1}b^{-1}ab \in H$, $Ha^{-1} = Hb^{-1}$ and $b \in aH$. Hence $H^a = H^b$ and $b^{-1}a \in H \cap H^a$. From Lemma 2.8 it follows that either $b^{-1}a$ is an involution or $b^{-1}a = 1$. In both cases $b^{-1}a = a^{-1}b$. 

Lemma 2.10. Let $G$ be a finite group, $H$ a maximal subgroup of $G$ and the core $L_G(H) = 1$. If $H \cap H^a = 1$ for some $1 \neq a \in A$, then $A = B$ and $G$ is solvable.

Proof. Let $H \cap H^a = 1$ and $aH = bH$, where $a \in A$ and $b \in B$. This means that $H \cap H^{a^{-1}} = 1$ and so $a = b$ by Lemma 2.9.

Assume that $uH = wH$, where $u \in A$ and $w \in B$. Now $a \in A \cap B$ and so $a^{-1}u^{-1}au \in H$. Thus $a^{-1}u^{-1}aw \in H$ and finally $a^{-1}u^{-1}wa \in H$. This means that $u^{-1}w \in H \cap H^{a^{-1}}$, hence $u = w$. We conclude that $A = B$.

Suppose that $d \in a^2H \cap A$. Now $a^{-1}d^{-1}ad \in H$, which means that $adH = daH$. Hence $a^{-1}daH = a^{-1}adH = a^2H$ and thus $d^a \in a^2H$. It follows that $(d^{-1}a^2)^a = (d^{-1})^aa^2 \in H$ and so $d^{-1}a^2 \in H^{a^{-1}}$. On the other hand $d^{-1}a^2 \in H$ and we get $d^{-1}a^2 \in H^{a^{-1}} \cap H = 1$. Thus $d = a^2$ and $a^2 \in A$. Next assume that $e \in A$, hence $e^{-1}e^a$ and $e^{-1}e^a$ are elements of $H$. Because $(e^{-1}e^a)^aH = (e^{-1})^ae^aH = (e^{-1})^aeH = H$, we conclude that $e^{-1}e^a \in H^{a^{-1}} \cap H = 1$. Thus $e \in C_G(a)$ and since $e$ was an arbitrary element of $A$, we get that $A \subseteq C_G(a)$. Since $H \cap H^a = 1$, we have $C_G(a) \cap H = 1$. On the other hand $C_G(a)H = G$, which means that $A = C_G(a)$.

Thus $A$ is a subgroup of $G$ and since $[A, A] \leq A \cap H = 1$, it follows that $A$ is an abelian group. Since $G = AH$ and $C \leq H$ is abelian and so nilpotent, it follows from Lemma 2.7 that $G$ is solvable.
3. Solvability of $G$

In this section we prove our main theorem about the solvability of a group $G$ with a certain dihedral subgroup with connected transversals. We begin now by introducing some general group theoretic results that are needed later in the proof of our theorem.

**Lemma 3.1.** Let $G$ be a transitive permutation group on the finite set $X$ and for $g \in G$ denote $\text{fix}_X(g) = \{i \in X \mid g(i) = i\}$. Then $|G| = \sum_{g \in G} |\text{fix}_X(g)|$.

For the proof, see [9, Theorem 9.1].

**Definition.** Let $K$ be a subgroup of a group $G$. Then a subgroup $Q \leq G$ is a complement of $K$ in $G$ if $K \cap Q = 1$ and $KQ = G$. Furthermore, if $Q$ is normal in $G$, $Q$ is a normal complement of $K$ in $G$.

**Lemma 3.2.** Let $G$ be a finite group and let $S$ be an abelian Sylow subgroup contained in the center of its normalizer, i.e. $S \leq Z(N_G(S))$. Then $S$ has a normal complement in $G$.

For the proof, see [5, Theorem 7.4.3].
**Definition.** A nonidentity abelian subgroup \( T \) of a finite group \( G \) is said to be strongly self-centralizing if \( C_G(t) = T \) for every nonidentity element \( t \) of \( T \).

**Lemma 3.3.** Let \( T \) be a strongly self-centralizing subgroup of a finite group \( G \) and assume that \( [N_G(T) : T] = 2 \). If \( G \) is simple, then \( G \) contains exactly one conjugacy class of involutions.

For the proof, see [3, p.124-126].

**Definition.** A group \( G \) is a Frobenius group if and only if it contains a proper subgroup \( H \neq 1 \), called a Frobenius complement, such that \( H \cap H^g = 1 \) for all \( g \in G \setminus H \). Furthermore, the set \( N = G \setminus \cup_{g \in G} (H \setminus 1)^g \) is called the Frobenius kernel.

**Lemma 3.4.** Let \( G \) be a Frobenius group and \( H \) a Frobenius complement in \( G \). Then the Frobenius kernel \( N \) is a normal subgroup of \( G \) such that \( G = HN \) and \( H \cap N = 1 \). Furthermore the Frobenius kernel \( N \) is always nilpotent.

For the proof, see [6, p.499].

**Lemma 3.5.** Let \( G \) be a finite simple group having an elementary abelian Sylow \( 2 \) -subgroup \( T \) of order \( 2^n \), where \( n \geq 2 \). If \( C_G(t) = T \) for all nonidentity \( t \in T \), then \( G \cong SL(2,2^n) \).

For the proof, see [3, p.99-103].

Finally, we need the following result of Vesalanen [15, Theorem 4.2].

**Lemma 3.6.** Consider now the projective special linear group \( PSL(2,2^n) \), where \( n \geq 2 \), and let \( H \) be a maximal subgroup of order \( 2(2^n + 1) \). Then there exist no \( H \) -connected transversals.

After these preparations we are ready to prove our main theorem.
**Theorem 3.7.** Let $G$ be a group and let $H$ be a dihedral subgroup of order $2x$, where $x$ is an odd number. If there exist $H$-connected transversals $A$ and $B$ in $G$, then $G$ is solvable.

**Proof.** Again we denote by $C = \langle c \rangle$ the cyclic subgroup of order $x$ in $H$. We already know that $C$ is normal in $H$ and so $H$ is solvable.

First, let $G$ be a finite group. We assume that $G$ is a minimal counter example. If $L_G(H) > 1$, then $H/L_G(H)$ is cyclic of order $x_1 \leq x$ or dihedral of order $2x_1 < 2x$, where $x_1$ is an odd number. In both cases we have $H/L_G(H)$-connected transversals in $G/L_G(H)$. Hence $G/L_G(H)$ is solvable by Theorem 2.5 or by minimal counter example. Now $L_G(H)$ is solvable because $H$ is solvable. This means that also $G$ is solvable and thus we may assume that $L_G(H) = 1$. Furthermore, $1 \in A \cap B$ by Lemma 2.1.

If $H$ is not a maximal subgroup of $G$, then $G$ has a proper subgroup $K$ such that $H < K < G$. We may assume that $K = \langle H, D \rangle$, where $D \subset A$ and $D \neq 1$, and so $L_G(K) > 1$ by Lemma 2.3. Now $GL_G(K)/L_G(K)$ is cyclic of order $x_1$ or dihedral of order $2x_1$, where $x_1 \leq x$ is an odd number. Again we have $HL_G(K)/L_G(K)$-connected transversals in $G/L_G(K)$ and hence $G/L_G(K)$ is solvable by Theorem 2.5 or by minimal counter example. Furthermore $A \cap K$ and $B \cap K$ are $H$-connected transversals in $K$ and so $K$ and also $L_G(K)$ are solvable, which means that $G$ is solvable. Thus we may assume that $H$ is a maximal subgroup of $G$. Furthermore, $N_G(H) = H$.

Next we shall show that $G$ is simple. If $N$ is a nontrivial normal subgroup of $G$, then $G = NH$. We can write $H = CQ$, where $|C| = x$ and $|Q| = 2$. It is immediate that $N \cap H \triangleleft H$. If $|N \cap H| = 2$, then $Q \triangleleft H$ and so $H$ is abelian, which is not possible. On the other hand, if $|N \cap H| = 2x_1 < 2x$, where $x_1$ is an odd number, then $N \cap H$ is dihedral. We have now an involution $q$ in $N \cap H$ and $c \in C = \langle c \rangle$ is not an element of $N \cap H$. Because $N \cap H \triangleleft H$, the element $q^c \in N \cap H$, which is not possible. This means that $N \cap H \leq C$ and so $q$ is not an element of $N$. If we write $E = NC$, then $E$ is a proper subgroup of $G$, because we don’t have involution $q$ in $E$. Clearly, $N_E(C) = C$ and if $n \in E \setminus C$, then $C \cap C^n = 1$ by Lemma 2.8. Hence $E$ is
a Frobenius group with a Frobenius complement \( C \). By using the properties of Frobenius groups, mentioned in Lemma 3.4, it follows that the Frobenius kernel of the Frobenius group \( E \) is solvable, because it is nilpotent and finite. Hence \( E \) is solvable, because it is a product of the subgroup \( C \) and the Frobenius kernel. From this we conclude that \( G = NH \) is also solvable. Thus we may assume that \( G \) is simple.

If there exists \( a \in A \) such that \( H \cap H^a = 1 \), then \( G \) is solvable by Lemma 2.10. Thus we may assume that \( H \cap H^a \geq 1 \), whenever \( 1 \neq a \in A \). Let \( P \) be a subgroup in \( H \) of order an odd prime with the biggest power. Now \( P \leq C \) and \( P \) is normal in \( H \). Assume now that \( |P| = p^r \), where \( p \) is an odd prime and \( r \) the biggest power of \( p \). Now \( N_G(P) = H \) and \( P \) is a Sylow \( p \)-subgroup of \( H \). Because \( N_G(P) = H \), it is not possible that there is a \( p \)-group \( P_p \leq G \) such that \( P \triangleleft P_p \), which means that \( P \) is also a Sylow \( p \)-subgroup of \( G \). Since by Lemma 2.8 \( |C \cap C^g| = 1 \), whenever \( g \in G \setminus H \), it follows that \( |P : P \cap P^g| = p^r \), whenever \( g \in G \setminus H \). We also know that all Sylow \( p \)-subgroups of \( G \) are conjugate and so \( |G : H| = 1 + kp^r \) ([6, p.36]). This calculation is the same for all Sylow \( p_i \)-subgroups of \( H \), where \( p_i \) is an odd prime number, and we conclude that \( |G : H| = 1 + kx \). If \( 1 + kx \) is an odd number, then \( Q \) is a Sylow \( 2 \)-subgroup of \( G \) and \( Q \) has a normal complement in \( G \) by Lemma 3.2. Because \( G \) is simple, we must assume that \( k \) is an odd number.

We can now consider \( G \) as a permutation group acting on the set with \( 1 + kx \) points and because \( H \) is not normal in \( G \), \( H \) is a one point stabilizer. Since \( |H \cap H^g| = 2 \), whenever \( g \in G \setminus H \), we conclude that in the action of \( H \) on the remaining \( kx \) points, the order of the stabilizer of any point is \( 2 \), which means that the length of any orbit is \( x \). If we consider one orbit, it is clear that \( H \) acts transitively on the \( x \) points. Because \( x \) is an odd number, every involution of \( H \) fixes at least one point in the orbit. Furthermore we know that the neutral element of \( H \) fixes every element in the orbit and we have \( x \) involution in \( H \), which means by Lemma 3.1 that every involution of \( H \) must fix one and only one point in the orbit. Just the same it is clear that other elements of \( H \), except involutions and \( 1 \), can’t fix any point in the orbit. This means that, if \( q \in H \) is an involution, \( q \) is a product of \( k(x - 1)/2 \)
distinct transpositions and fixes $k+1$ points. Thus $q$, and all conjugates of $q$ in $G$, fix $k+1$ points in the set with $1+\mathit{kk}x$ points. Furthermore we notice that, if $g \in G \setminus H$ fixes a point, then $g \in H^{g_1}$ for some $g_1 \in G \setminus H$. Now we remember that the number of conjugates of $H$ in $G$ is $1+bkx$. If we consider elements of $H$, except involutions and $1$, we know that they fix only one point and the same is true for elements of $H^{g_1}$. Furthermore $1 \in G$ fixes all points. We again apply the counting argument from Lemma 3.1 and we get

$$|G| = 1+bkx + (1+bkx)(x-1) + N(k+1),$$

where $N$ is the number of conjugates of $q$ in $G$, including the number of involutions in $H$ and in conjugates of $H$. From this result we get that $N = (1+bkx)x/(k+1)$, because $|G| = (1+bkx)2x$. On the other hand, $N = [G : C_G(q)]$, which means that $|C_G(q)| = 2(k+1)$.

Let $p$ be a prime, which divides $x$, and we shall now study if it is possible that $p$ divides $k+1$, which means that there is a $p$-group $P \leq C_G(q)$. If $P \leq H$, then $\langle q \rangle P \leq H$ is abelian, which is not possible. Thus $P$ is not a subgroup of $H$, but it is a subgroup of $H^g$, a conjugate of $H$. Now we have two possibilities for $q$. If $q \in H^g$, then $\langle q \rangle P \leq H^g$ is abelian, which is not possible, and we must suppose that $q$ is not in $H^g$. But this means that $P \triangleleft \langle H^g, q \rangle = G$, which is not possible. Our conclusion is that $p$ does not divide $k+1$ and so $x$ and $k+1$ don’t have common divisors. Now $k+1 > 1$ and we conclude that $k+1$ divides $1+bkx = (k+1) + k(x-1)$ and so $k+1$ divides $x-1$.

Next we shall consider cosets $aH$. We first study if it is possible that a coset $aH$, where $1 \neq a \in A$, contains more than two conjugates of $q$. Let $q^{g_1}$, $q^{g_2}$ and $q^{g_3}$ be three different conjugates of $q$ in $aH$. Now $q^{g_1} = ah_1$, $q^{g_2} = ah_2$ and $q^{g_3} = ah_3$, where $h_1, h_2, h_3 \in H$. This means that $q^{g_1} = q^{g_1} = h_1^{-1}a^{-1}$ and so $q^{g_1}q^{g_2} = h_1^{-1}h_2 \in H$. In the same way $q^{g_1}q^{g_3}, q^{g_2}q^{g_3} \in H$. Clearly, $q^{g_1}a^{-1}, q^{g_2}a^{-1}$ and $q^{g_3}a^{-1}$ are elements of $H^{a^{-1}}$ and thus $q^{g_1}q^{g_2} = q^{g_1}a^{-1}aq^{g_2} \in H \cap H^{a^{-1}}$. By Lemma 2.8 $q^{g_1}q^{g_2}$ is an involution and three elements $q^{g_1}$, $q^{g_2}$ and $q^{g_1}q^{g_2}$ commute with each other. Likewise $q^{g_1}q^{g_3}$ and $q^{g_2}q^{g_3}$ are involutions in $H \cap H^{a^{-1}}$. Hence we have three distinct involutions $q^{g_1}q^{g_2}$, $q^{g_1}q^{g_3}$.
and \( q^{a_2}q^{a_3} \) in \( H \cap H^{a_1} \), a contradiction. We conclude that every coset \( aH, \quad 1 \neq a \in A, \) has at most two conjugates of \( q \).

The number of those conjugates of the involution \( q \), which are not contained in \( H \), is \( N - x = [2x(1 + kx)/2(k + 1)] - x = kx(x-1)/(k+1) \). Now every coset \( aH \), where \( 1 \neq a \in A \), contains at most two conjugates of \( q \), hence \( kx(x-1)/(k+1) \leq 2kx \) and so \( x-1 \leq 2(k+1) \). Because \( k+1 \) divides \( x-1 \), we get \( x-1 = k+1 \) or \( x-1 = 2(k+1) \).

If \( x-1 = k+1 \), then \( k = x-2 \) and \(|G| = [1+(x-2)x]2x = (x-1)^22x \). Now \( x-1 \) is an even number and we can assume that \(|G| = 2^s \cdot r \), where \( r \) is an odd number. If \( S \) is a Sylow 2 -subgroup of \( G \), then \(|S| = 2^s \). Since \( Z(S) > 1 \), \( S \) contains an involution \( z \) such that \( S \leq C_G(z) \). Now the group \( C \) is cyclic and so \( C \leq C_G(c) \) for every \( c \in C \backslash 1 \). If we have \( c_1 \in C_G(c) \backslash C \) and \( c_1 \notin H \), then \( \langle c \rangle \triangleleft \langle H, c_1 \rangle = G \), which is not possible. So we must have \( c_1 \in H \), but this means that \( c_1 \) is an involution and \( \langle c_1 \rangle \leq H \) is abelian, not possible. Thus \( C_G(c) = C \) for every \( c \in C \backslash 1 \), which means that \( C \) is strongly self-centralizing. Furthermore \( N_C(C) = H \) and so by Lemma 3.3 \( q \) and \( z \) are conjugate. It follows that \(|C_G(q)| = |C_G(z)| \) and so \( 2^s \) divides \(|C_G(q)| \) and \( 2^{s-1} \) divides \( k+1 = x-1 \). But then \((2^{s-1})^2 2 = 2^{2s-1} \) divides \(|G| = 2^s \cdot r \), where \( r \) is an odd number. But this is possible only if \( 2^{2s-1} = 2^s \) or \( s = 1 \). Hence \(|G| = 2^r \), where \( r \) is an odd number. But this is not possible, because \( G \) is simple and cannot have a Sylow 2 -subgroup of order \( 2 \).

Then assume that \( 2(k+1) = x-1 \) and so the number of those conjugates of the involution \( q \), which are not elements of \( H \), is \( kx(x-1)/(k+1) = 2kx \). We already know that every coset \( aH \neq H \) has at most two conjugates of \( q \), which means that every coset has exactly two conjugates of \( q \). Again \( C \) is strongly self-centralizing and so all involutions of \( G \) are conjugate. Thus we can assume that \( S \) is a Sylow 2 -subgroup of \( G \), which contains \( q \) and \( q \in Z(S) \). Hence \( S \leq C_G(q) \) and so the set \( C_G(q)H - H \) is not empty. Let now \( d = q^{y_1} \) and \( f = q^{y_2} \) be two involutions of a coset \( aH \), where \( a \in C_G(q)H - H \) and \( y_1, y_2 \in G \backslash H \). Now \( q^d \) and \( q^f \) are elements of \( H \) and so \( q \in H^d \) and also \( q \in H^f \). This means that \( q \) is an element of \( H \cap H^d \) and \( H \cap H^f \). By Lemma 2.8 \( q = q^d = q^f \), which means that \( d, f \in C_G(q) \). Now
$H$ is dihedral, not abelian, and so \(|C_G(q) \cap H| = 2\). Hence we have \(|C_G(q)H| = 2(k + 1) |H|/2 = (x - 1) |H|/2\). By this result \(|C_G(q)(H - H| = (x - 3) |H|/2\), which means that we have \((x - 3)/2\) cosets \(aH \neq H\), where \(a \in C_G(q)H - H\). We know that each coset \(aH \neq H\) contains exactly two conjugates of \(q\), hence we get always two elements into \(C_G(q)\) when we take two conjugates of \(q\) in every coset \(aH\). Thus we have at least \((x - 3)\) involutions in \(C_G(q)\), which are not elements of \(H\). Furthermore \(q \in C_G(q)\) and so we have at least \((x - 2)\) involutions in \(C_G(q)\). Since \(|C_G(q)| = x - 1\), we conclude that every element of \(C_G(q) \setminus 1\) is an involution, which means that \(C_G(q)\) is an elementary abelian group.

Now \(S \leq C_G(q)\) and we get that \(S = C_G(q)\) is an elementary abelian Sylow 2-subgroup of order \(2^s\), where \(s \geq 2\). On the other hand \(|C_G(q)| = |S| = x - 1\) and so \(x = 2^s + 1\). It is clear that \(S \leq C_G(y)\) for every nonidentity \(y \in S\), because \(S\) is abelian. Now all involutions of \(G\) are conjugate and so \(C_G(y)\) is also same kind of elementary abelian group like \(C_G(q)\). Hence we conclude that \(C_G(y) = S\) for every nonidentity \(y \in S\) and so \(G \approx SL(2, 2^s) = PSL(2, 2^s)\) by Lemma 3.5. Now \(H\) is a maximal subgroup of \(G\) and \(|H| = 2(2^s + 1)\), which is a contradiction by Lemma 3.6. We have now proved that \(G\) is solvable in the case that \(G\) is a finite group.

Next we prove that our theorem also holds when \(G\) is an infinite group. First assume that \(G = \langle A, B \rangle\) and \(L_G(H) = 1\). Let \(a\) be a fixed element of \(A\) and \(h\) a fixed element of \(H\) and write \(F(a, h) = \{b \in B : a^{-1}b^{-1}ab = h\}\). If \(b\) and \(c\) are elements of \(F(a, h)\), then \(bc^{-1} \in C_G(a)\) and \(b \in C_G(a)c\). Thus \(F(a, h) \subseteq C_G(a)b_h\), where \(b_h\) is a fixed element from \(F(a, h)\), and \(B = \bigcup F(a, h)\), where \(h\) goes through all the elements of \(H\). Hence \(G = BH \subseteq C_G(a)\{b_h : h \in H\}\) and thus \([G : C_G(a)] \leq |H|^{2}. Similarly \([G : C_G(b)] \leq |H|^{2}\) for every \(b \in B\). Since \(H \leq \langle A, B \rangle\) is finite, we conclude that \([G : C_G(H)] \leq [G : C_G(a_1) \cap \cdots C_G(a_n) \cap C_G(b_1) \cap \cdots C_G(b_m)] \leq [G : C_G(a_1)] \cdot \cdots [G : C_G(a_n)] \cdot [G : C_G(b_1)] \cdot \cdots [G : C_G(b_m)]\), where \(a_i\) and \(b_j\) are elements of transversals \(A\) and \(B\) with finite numbers \(m\) and \(n\). This means that the index \([G : C_G(H)]\) is finite, whence \([G : N_G(H)]\) is also finite. By Lemma 2.2, \(N_G(H) = H \times Z(G)\) and thus \(G/Z(G)\) is a finite group. Now
$Z(G)\cap H = 1$, which means that $HZ(G)/Z(G)$ is a dihedral group of order $2x$. By the first part of our proof $G/Z(G)$ is solvable and the solvability of $G$ easily follows.

Then let $G = \langle A, B \rangle$ and assume that $L_G(H) > 1$. We write $\bar{G} = G/L_G(H)$ and $\bar{H} = H/L_G(H)$. Now $AL_G(H)/L_G(H)$ and $BL_G(H)/L_G(H)$ are $\bar{H}$-connected transversals in $\bar{G}$ and furthermore $\bar{G} = \langle AL_G(H)/L_G(H), BL_G(H)/L_G(H) \rangle$. If $H$ is cyclic, then $\bar{G}$ is solvable by Theorem 2.5. In the case that $\bar{H}$ is dihedral, we get that $L_{\bar{G}}(\bar{H}) = \{L_G(H)\}$ and so is trivial. Then proceed as in the previous part of the proof of the infinite case, it is immediate that $\bar{G}$ is solvable. But then it is clear that $G$ is solvable, too.

Finally, let $K = \langle A, B \rangle$ be a proper subgroup of $G$. Now $A$ and $B$ are $K\cap H$-connected transversals in $K$. If $K\cap H$ is cyclic then $K$ is solvable by Theorem 2.5. On the other hand the solvability of $K$, in the case that $K\cap H$ is dihedral, follows from the proof of the case $G = \langle A, B \rangle$. Now $[G : K] \leq |G| / |A|$ is finite, which also means that $[G : N_G(K)]$ and so the number of conjugates of $K$ in $G$ is finite. Since $[G : L_G(K)] \leq \Pi[G : K^g]$, we conclude that $G/L_G(K)$ is a finite group. Now $HL_G(K)/L_G(K)$ is cyclic or dihedral and the solvability of $G/L_G(K)$ follows. Since $L_G(K)$ is solvable, it follows that $G$ is solvable. The proof is complete.
4. Solvability of loops

As we mentioned in the introduction, loops can be understood as nonassociative versions of groups, and the permutation group generated by all left and right translations of a loop \( Q \) is called the multiplication group \( M(Q) \), so \( M(Q) = \langle L_a(x), R_a(x) \rangle \). Furthermore the stabilizer of the neutral element \( e \in Q \) is called the inner mapping group \( I(Q) \). Now \( \langle L_a(x) \rangle \) and \( \langle R_a(x) \rangle \) are \( I(Q) \)-connected transversals in \( M(Q) \) and the core of \( I(Q) \) in \( M(Q) \) is trivial [10, p.118]. In this section we shall consider the solvability of a loop, defined in the introduction, by studying the properties of the inner mapping group.

In 1996 Vesänen [16] proved the following connection between the solvability of a loop \( Q \) and the solvability of the multiplication group \( M(Q) \).

**Lemma 4.1.** Let \( Q \) be a finite loop. If \( M(Q) \) is solvable, then \( Q \) is solvable.

On the other hand Kepka and Niemenmaa [10, Theorem 4.1] have showed the connection between connected transversals and multiplication groups of loops.
**Lemma 4.2.** A group $G$ is isomorphic to the multiplication group of a loop $Q$ if and only if there exists a subgroup $H$ satisfying $L_G(H) = 1$ and $H$-connected transversals $A$ and $B$ satisfying $G = \langle A, B \rangle$.

We combine now these results with our Theorem 3.7 and we get an interesting theorem about the solvability of a loop $Q$.

**Theorem 4.3.** If $Q$ is a loop whose inner mapping group is dihedral of order $2x$, where $x$ is an odd number, then the multiplication group $M(Q)$ is a solvable group. If we further assume that $Q$ is finite, then $Q$ is a solvable loop.
References


