ON THE SOLVABILITY OF GROUPS AND LOOPS

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Abstract

The dissertation consists of three articles in which the solvability of groups and the solvability of loops are considered. The first parts of the thesis survey some basic information and results on transversals and loops. The summarizing parts provide the three main results for the solvability of groups and loops.

Keywords: group, loop, solvability, transversal

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Oulu, January 2003 Kari Myllylä

List of the original articles

I Myllylä K & Niemenmaa M (1999) On the solvability of commutative loops and their multiplication groups. Comment Math Univ Carolinae 40,2: 209-213.

II Csörgö P, Myllylä K & Niemenmaa M (2000) On connected transversals to dihedral subgroups of order $2p^n$. Algebra Colloquium 7:1: 105-112.

III Myllylä K (2000) On connected transversals to dihedral subgroups. Acta Universitatis Ouluensis A 350, (23 pages).

Contents

1 Introduction

Let G be a group, H a subgroup of G and A, B two left transversals to H in G. Recall that a subset of elements of G is said to be a left transversal to H in G if it contains exactly one element of each left coset of H. If, further, the commutator subgroup $[A, B]$ is contained in H, then we say that the two transversals A and B are H -connected. Furthermore, a left transversal A is H -selfconnected if $[A, A] \leq H$. These definitions were introduced by Kepka and Niemenmaa in [15]. In this paper we are interested in investigating the relation between the solvability of G and the structure of H in the case that there exist H -connected transversals in G.

Kepka and Niemenmaa started to study this solvability problem in 1990 and they could first show in [11] and [15] that G is solvable in the case where H is cyclic. Next they showed in [16] that, if H is a finite abelian subgroup, then the solvability of G follows. After this Niemenmaa started to investigate the non-abelian case, and, in [17], he was able to show that G is solvable if $\mid H \mid = 6$. The more general problem, where the order of H is a product of two prime numbers p and q, was investigated by Niemenmaa in [19]. He could prove, by using the classification of finite simple groups, that G is solvable if $q = 2$ and $p \leq 61$, $q = 3$ and $p \leq 31$ or $q = 5$ and $p \leq 11$. In this paper we shall consider the case $\mid H \mid = 2p$, where $p = 4t + 3$ is a prime number, and there is an H -selfconnected transversal in G. After this Csörgö and Niemenmaa [5] have managed to show the solvability of G in the general case where $\mid H \mid = 2p$. They [6] have also studied the case where $\mid H \mid = pq$, where p and q are odd prime numbers and $p = 2q^m + 1$. Again the solvability of G was proved. Furthermore, Drapal has considered the general case $\mid H \mid = pq$ in [7]. The purpose of this paper is to investigate the problem, where H is a dihedral subgroup of order $2x$, where x is an odd number. We shall first consider the case $\mid H \mid = 2p^n$, where p is an odd prime number, and after this we shall prove our main result, the solvability of G in the case $\mid H \mid = 2x$. Niemenmaa has also considered the dihedral case in [18]. He was able to show that, if H is a dihedral 2 -group, then again the solvability of G follows.

Connected transversals are a very useful tool when studying the structure of loops. A groupoid Q is called a loop if it has a unique division on both sides and a neutral element 1, which means that loops can be thought of as nonassociative versions of groups. Now the permutations $L_x(a) = xa$ and $R_x(a) = ax$, where a is an element of Q and x goes through all the elements of Q, generate the multiplication group of Q, which is denoted by $M(Q)$. Furthermore, the stabilizer of the neutral element of Q, denoted by $I(Q)$, is the inner mapping group of Q. These two groups, which create a link between loop theory and group theory, were introduced by Bruck [1] and he used them to investigate the structure of loops. In loop theory the sets $\{L_a : a \in Q\}$ and $\{R_a : a \in Q\}$ are $I(Q)$ -connected transversals in $M(Q)$. For loops Bruck defined solvability as follows: a loop Q is solvable if it has a series $1 = Q_0 \subseteq ... \subseteq Q_n = Q$, where Q_{i-1} is a normal subloop of Q_i and the factor loop Q_i/Q_{i-1} is an abelian group. Of course, normal subloops are kernels of loop homomorphisms.

The structure of the multiplication group reflects some of the properties of its corresponding loop. In 1996 Vesanen [21] proved that the solvability of the multiplication group $M(Q)$ implies the solvability of Q in the case that Q is a finite loop. On the other hand Kepka and Niemenmaa [15] have proved the connection between connected transversals and multiplication groups of loops: a group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $L_G(H) = 1$ and H -connected transversals A and B satisfying $G = \langle A, B \rangle$. When we combine these two important results we can investigate the solvability of a loop Q by studying the properties of the inner mapping group $I(Q)$. Hence, by the results for the solvability of groups, a finite loop Q is solvable in the cases that the inner mapping group $I(Q)$ is a finite abelian group or a group of order $2p$ or pq, where p and q are odd prime numbers. Furthermore a loop Q is solvable, when the inner mapping group $I(Q)$ is a cyclic group or a dihedral 2-group. Similarly, from our main result it follows that a finite loop Q is solvable in the case that the inner mapping group $I(Q)$ is a dihedral group of order $2x$, where x is an odd number.

2 Connected transversals in groups

In this section we begin with the basic theory of transversals. After this we prove some preliminary lemmas, which are needed in the proofs of our main theorems concerning the solvability of groups.

As in the introduction, we assume that G is a group and H a subgroup of G. We denote the *core* of H in G by $L_G(H)$, i.e. $L_G(H) = \cap H^g$, where g goes through all the elements of G. It is clear that $L_G(H)$ is now the largest normal subgroup of G contained in the subgroup H. Let A be a subset of elements of G, which contains exactly one element from each left coset of H; then the subset A is said to be a *left transversal* to H in G. This definition means that $AH = G$ and $\mid A \cap gH \mid = 1$ for every element g of G. Furthermore $\mid G \mid = \mid A \mid \mid H \mid$. A *right transversal* is defined similarly. A left transversal A is called *stable*, if the set Ax is also a left transversal to H in G for every element x of the group G. Next we prove our first lemma for transversals.

Lemma 2.1. *The following conditions are equivalent:*

(1) A *is a stable transversal to* H *in* G,

(2) A^g *is a left (right) transversal to* H *in* G *for every* $g \in G$.

Proof. Assume first that A is a stable transversal. This means that $\mid Ag \cap gxH \mid = 1$ for every g and x from G and thus

$| A^g \cap xH |= 1$. On the other hand, $| Agy^{-1} \cap gH |= 1$ for every g and y from G, which means that $| A^g \cap Hy |= 1$. Hence A^g is a left and right transversal to H in G for every $g \in G$.

Assume now that A^g is a left transversal, i.e. $| A^g \cap xH |= 1$ for every g and x from G. This means that $| Ag \cap gxH |= 1$ and so A is a stable transversal to H in G.

Let A and B be left transversals to H in G. If, further, the commutator subgroup $[A, B]$ is a subgroup of H, then these two transversals are H *-connected*. This definition means that an element $a^{-1}b^{-1}ab$, where $a \in A$ and $b \in B$, is always an element of H. This useful definition was introduced by Kepka and Niemenmaa in [15]. Furthermore, a left transversal A is H *-selfconnected*, if $[A, A] \leq H$. In the following lemma [15, Lemma 2.2], we get the connection between stable transversals and connected transversals.

Lemma 2.2. *Let A and B be H -connected transversals in G. Then A and B are both stable transversals to H in G.*

Proof. Consider the set Ax, where $x \in G$. Now we can write $x = bh$, where $b \in B$ and $h \in H$. If $a, c \in A$ and $axH = cxH$, then $(ax)^{-1}(cx) = h^{-1}b^{-1}a^{-1}cbh \in H$ and so $b^{-1}a^{-1}cb \in H$. Now $a^{-1}c = (a^{-1}b^{-1}ab)(b^{-1}a^{-1}cb)(b^{-1}c^{-1}bc)$ and hence $a^{-1}c$ is an element of H. Thus $aH = cH$, which is possible if and only if $a = c$, and also $ax = cx$. This means that there can be at most one element of Ax in every left coset of H. Furthermore we must show that there is always this one element. Let y be an element of G and $b^{-1}y = ak$, where $a \in A$ and $k \in H$. Now $y = bak = abhh^{-1}b^{-1}a^{-1}bak = ax(h^{-1}b^{-1}a^{-1}bak) = (ax)l$, where $l \in H$. Thus $(Ax)H = G$ and so Ax is a left transversal and A is a stable transversal to H in G. We have a similar proof for the transversal B.

When we combine these two lemmas, we get the next result.

Lemma 2.3. *If A and B are H -connected transversals in G, then they are left and right transversals to H^g for every $g \in G$. Furthermore, if $L_G(H) = 1$, then $1 \in A \cap B$.*

Proof. Now A and B are stable transversals to H in G and so A^g and B^g are left and right transversals to H in G for every $g \in G$. This means that $\mid A^g \cap xH \mid = \mid A \cap x^g H^g \mid = 1$ for every x and g in G. Thus A is a left transversal to H^g in G. The proof is similar for B and the case of right transversals.

Assume now, in addition, that $L_G(H) = 1$. If x is an element of $A \cap H$, then $b^{-1}x^{-1}bx \in H$ for every $b \in B$. But then $x \in H^g$ for every $g \in G$, and so $x \in L_G(H) = 1$. In the same way we can prove that $1 \in B$ and so $1 \in A \cap B$.

The following two lemmas for transversals are important in the proofs of Theorems 4.12, 4.13 and 4.14. Proofs of these lemmas can be found in [15, Proposition 2.7, Lemma 2.5].

Lemma 2.4. *Let $H \leq G$ and suppose that A and B are H -connected transversals in G. If $L_G(H) = 1$, then the normalizer $N_G(H)$ is equal to $H \times Z(G)$.*

Proof. If x is an element of $N_G(H)$, we can define a mapping f_x from the transversal A to itself by $x^{-1}ax \in f_x(a)H$. By Lemma 2.2 and Lemma 2.1, this mapping f_x is a permutation on A. If x and y are elements of $N_G(H)$, then $x^{-1}ax = f_x(a)u$, $y^{-1}f_x(a)y = f_y(f_x(a))v$ and $y^{-1}x^{-1}axy = f_{xy}(a)w$ for some $u, v, w \in H$. Then $f_{xy}(a)w = f_y(f_x(a))vy^{-1}uy$ and, since $vy^{-1}uy \in H$, it follows that $f_{xy}(a) = f_y f_x(a)$. Thus we have $f_{(xy)^{-1}}(a) = f_{x^{-1}} f_{y^{-1}}(a)$ and the mapping $F(x) = f_{x^{-1}}$ is a group homomorphism from $N_G(H)$ to the symmetric group S_A. Since A and B are H -connected transversals, we have that $B^{-1} \cap N_G(H) \subseteq Ker(F)$, which means that also $D = B \cap N_G(H)$ is contained in $Ker(F)$. If $x \in Ker(F) \cap H$, then $x \in H$ and $a^{-1}xa \in H$ for every $a \in A$. But this means that $x \in H^{a^{-1}}$ for every $a \in A$ and so $x \in L_G(H) = 1$.

Now $N_G(H) = DH$ and, since $Ker(F) \cap H = 1$ and F is a group homomorphism, we have that $Ker(F) = \langle D \rangle = D$. Hence D is a normal subgroup of $N_G(H)$ and so $N_G(H) = H \times D$.

It remains to show that $D = Z(G)$. If g is an element of G, then $g = ah$, where $a \in A$ and $h \in H$. If $d \in D$, then $d^{-1}g^{-1}dg = d^{-1}h^{-1}a^{-1}dah = h'd^{-1}a^{-1}dah$, where h' is an element of H. Hence $d^{-1}g^{-1}dg \in H$, which means that the commutator subgroup $E = [D, G]$ is contained in H. Now the subgroup E is normal in G, which means that $E = 1$. Therefore $D \leq Z(G)$ and, because we clearly have that $Z(G) \subseteq Ker(F)$, we have $N_G(H) = H \times Z(G)$.

Lemma 2.5. *Let $H \leq G$ and A and B be H-connected transversals. Furthermore, let $C \subseteq A \cup B$ and $K = \langle H, C \rangle$. Then $C \subseteq L_G(K)$.*

Proof. Let g be an element of G and c be an element of C. We can assume that $c \in A$ and $g = bh$, where $b \in B$ and $h \in H$. Now $g^{-1}c^{-1}g = h^{-1}b^{-1}c^{-1}bh = h^{-1}b^{-1}c^{-1}bcc^{-1}h$. Since $b^{-1}c^{-1}bc \in H$, it follows that $g^{-1}c^{-1}g \in \langle H, C \rangle = K$. So we have proved that $g^{-1}c^{-1}g \in K$ for every $c \in C$ and for every $g \in G$. This means that $c^{-1} \in K^g$ for every c and g and so $C \subseteq L_G(K)$.

In the following three lemmas, which are needed in the proofs of Theorems 4.13 and 4.14, we again assume that H is a subgroup of a group G and we have H-connected transversals A and B in G. Furthermore we presume that H is a dihedral group of order $2x$, where x is an odd number. Proofs of these lemmas can be found in [13, Lemma 2.8, Lemma 2.9 and Lemma 2.10].

Lemma 2.6. *If H is a maximal subgroup of G and $L_G(H) = 1$, then $\mid H \cap H^g \mid \leq 2$ for every $g \in G \setminus H$.*

Lemma 2.7. *Let H be a maximal subgroup of G and the core $L_G(H) = 1$. If $aH = bH$, where $a \in A$ and $b \in B$, then $b^{-1}a = a^{-1}b \in H \cap H^{a^{-1}} = H \cap H^{b^{-1}}$.*

Lemma 2.8. *Let G be a finite group, H a maximal subgroup of G and the core $L_G(H) = 1$. If $H \cap H^a = 1$ for some $1 \neq a \in A$, then $A = B$ and G is solvable.*

This is the first minor result for the solvability of groups with transversals. More information about solvability cases can be found in Section 4.

3 Some results of loop theory

As we mentioned in the introduction, loops can be thought of as nonassociative versions of groups. This means that a set Q is a *loop*, if it satisfies the following three conditions:

(1) To every ordered pair x and y of elements of Q, there corresponds a unique element xy of Q, called their product,
(2) If in the equation $xy = z$ any two of the elements x, y, z are assigned as elements of Q, the third is also uniquely determined as an element of Q,
(3) There exists a unique element 1, a neutral element of Q, with the property that $1x = x1 = x$ for every element x of Q.

An *associative loop* Q satisfies, in addition, the associative law $(xy)z = x(yz)$ for every element x, y, z of Q. Of course, associative loops are groups. If a loop Q satisfies the commutative law, $xy = yx$ for every x, y from Q, then it is called a *commutative loop*. Although loops usually do not satisfy the associative law, we can make same kinds of definitions as with groups. The following definitions and results are mainly based in Bruck's important paper *Contributions to the theory of loops* [1].

As in group theory, if a subset H of a loop Q obeys these three laws with respect to the same operation, this subset is called a *subloop*, and we denote this by $H \leq Q$. Similarly the *order* of a loop is by definition its cardinal number. If x is any fixed element of a loop Q, we may define two mappings L_x and R_x of Q into itself by $L_x(a) = xa$ and $R_x(a) = ax$, where a is an element of Q. Because Q is a loop, it is clear that mappings L_x and R_x are permutations on Q. In loop theory we can define normal subloops in three different ways. We shall later present these definitions and show that they really are equivalent.

A mapping f from a loop Q to a loop Q' is said to be a *homomorphism* if $f(xy) = f(x)f(y)$ for every element x and y of Q. Under this condition the loop Q is said to be *homomorphic* with the loop Q'. Let $f(Q)$ be the set of all elements of the form $f(x)$, where f is a homomorphism of the loop Q and x is an arbitrary element of Q. Now the set $f(Q)$ is called the *homomorphic image* of the loop Q with regard to mapping f. The subset K of the loop Q that includes all the elements x, which satisfy $f(x) = f(1)$, is called the *kernel* of the homomorphism f. Next it will be proved that the homomorphic image $f(Q)$ and the kernel K are also loops.

Lemma 3.1. *A homomorphic image $f(Q)$ of a loop Q is a loop.*

Proof. Now $f(x)f(y) = f(xy)$ and so the product of elements $f(x)$ and $f(y)$ is also an element of the set $f(Q)$. Because Q is a loop, any element of an equation $xy = z$ can be solved in Q, when two other elements have been given as elements of the loop Q. Let us assume (for example) that $f(x)u = f(z)$, where $f(x)$ and $f(z)$ are known elements and u an unknown element of the set $f(Q)$. Because u is an element of $f(Q)$, it is of form $f(y)$, where y is an unknown element of the loop Q. Thus $f(x)f(y) = f(z)$, which means that $f(xy) = f(z)$. The equality is valid when $xy = z$ and from this y can be solved as a unique element of the loop Q. Thus $u = f(y)$ can be solved as an unambiguous element of the set $f(Q)$. The proof is similar in the case $uf(y) = f(z)$. In addition $f(1)$ is the neutral element of this set $f(Q)$ and thus the homomorphic image $f(Q)$ is a loop.

Lemma 3.2. *The kernel K of a homomorphism f of a loop Q is a subloop of Q.*

Proof. Let x and y be two elements of the kernel K, which means that $f(x) = f(y) = f(1)$. Then, in addition, we have that $f(xy) = f(x)f(y) = f(1)f(1) = f(1)$. Therefore xy is an element of the kernel K. Because Q is a loop, any equation $xy = z$ can be solved for any of the three elements as an element of the loop Q, when the other two elements have been given as elements of Q. Let us prove that this is also valid when considering the kernel K. When two of the elements are given as elements of the kernel K, then the third one will be solved as an element of the loop Q. When x and y are elements of the kernel K, the result is clear. Let now x and z be elements of the kernel K and $xy = z$. Then $f(xy) = f(x)f(y) = f(1)f(y) = f(y) = f(z) = f(1)$. Thus y is also an element of the kernel K. Similarly it can be shown that if y and z are elements of K, is x also an element of the kernel K. In addition the neutral element 1 is an element of the kernel K and thus K is a subloop of Q.

A subloop H of a loop Q is said to be a *normal subloop* if and only if H is the kernel of some homomorphism of the loop Q. Then the expression $H \trianglelefteq Q$ is used. Now the loop $\{1\}$, which has only one element 1, is a normal subloop of Q; we often simply denote this subloop by 1. We say that a loop Q is *simple*, if it has only $\{1\}$ as a proper normal subloop.

As we mentioned earlier, there are two other criteria for the normality. We shall now consider these criteria and also prove that they are equivalent.

Theorem 3.3. *If H is a subloop of a loop Q, then the following condition is a necessary and sufficient one for the normality of the subloop H in the loop Q. Let x and y be arbitrary elements of the loop Q and let us consider the equation*

$$(h_1x)(h_2y) = h_3(xy).$$

When any two of the elements h_1, h_2 and h_3 have been given as elements of the subloop H, then the third one will also be uniquely determined as an element of H.

Proof. First shall we prove the sufficiency. Let us presume that a subloop H fullfills the condition of the theorem. Let us consider

Hx, the set of elements of the form hx, where h goes through all the elements of the subloop H. By setting $x = 1$ in the equation of the theorem, it will be seen that any element of the set Hy determines the same set entirely.

In addition every element x and y of the loop Q satisfies $HxHy = Hxy$. Thus the set Q/H, which includes as elements all the separable sets Hx, always includes products of elements Hx. Let x and z be given elements of Q and y be the unique solution of the equation $xy = z$. From the equation of the theorem it follows that $(Hx)u = Hz$, for some element u of the loop Q, if and only if u is an element of the set Hy. Similarly, if y and z are given elements and x can be solved from the equation $xy = z$, then a necessary and sufficient condition for $v(Hy) = Hz$ for some element v of the loop Q is that v is an element of the set Hx. Furthermore $H1 = H$ is the neutral element of Q/H and so the set Q/H is a loop.

According to the equation $HxHy = Hxy$, a mapping f that satisfies $f(x) = Hx$ is a homomorphism of the loop Q. In addition $f(x) = f(1)$ if and only if $Hx = H$, which means that x is an element of the subloop H. Thus H is the kernel of the homomorphism f, which means that H is a normal subloop of the loop Q.

Next shall we prove the necessity. Let H be a normal subloop of a loop Q. Thus H is the kernel of some homomorphism f of the loop Q. If $f(x) = f(z)$, we define $u = R_x^{-1}(z)$, which means that $ux = z$ and $f(u)f(x) = f(ux) = f(z) = f(x) = f(1)f(x)$. Hence $f(u) = f(1)$ and u is an element of the normal subloop H. Thus $z = ux$ is an element of the set Hx. On the other hand, if u is an element of the normal subloop H and $z = ux$ is an element of the set Hx, then $f(z) = f(ux) = f(u)f(x) = f(1)f(x) = f(x)$. Hence the set Hx includes precisely those elements z that satisfy $f(z) = f(x)$.

Let us now examine the equation $(h_1x)(h_2y) = h_3(xy)$, where h_1 is taken as an unknown. Now $f[(h_1x)(h_2y)] = f[h_3(xy)]$. Since h_2 and h_3 are elements of the normal subloop H, then we have that $f[(h_1x)(h_2y)] = f(h_1x)(f(h_2)f(y)) = f(h_1x)(f(1)f(y)) = f(h_1x)f(y)$ and $f[h_3(xy)] = f(h_3)f(xy) = f(1)f(xy) = f(xy) = f(x)f(y)$. As a result, we have that $f(h_1x) = f(x)$. Thus h_1x is an element of the set Hx, which means that h_1 is an element of the normal subloop H. Hence $h_1x = z$, where z is a unique element of the set Hx. This has the unique solution $h_1 = R_x^{-1}(z)$, which is now a unique element of the normal subloop H. Similar conclusion can be drawn if the unknown element is h_2 or h_3.

Now let H be a normal subloop of a loop Q. Then the *factor loop* Q/H is a loop, elements of which are separate *cosets* Hx. Of course, the neutral element of this loop is H.

Corollary 3.4. *Let H be a normal subloop of a loop Q. This means that $xH = Hx$ for every element x of the loop Q.*

Proof. Let x be an arbitrary element of the loop Q. In the equation of Theorem 3.3, we choose $h_1 = 1$, $y = 1$ and let h_2 be any element of the normal subloop H. According to Theorem 3.3 there exists such an element h_3 of the normal subloop H that $(1x)(h_21) = h_3(x1)$, which means $xh_2 = h_3x$. Thus $xH \subseteq Hx$.

Respectively, if h_3 has been given as an element of the normal subloop H, there exists such an element h_2 of the loop H that $h_3x = xh_2$, which means $Hx \subseteq xH$. Hence $xH = Hx$.

Two loops Q and Q' are said to be *isomorphic* if there exists a bijection f from Q to Q' such that $f(xy) = f(x)f(y)$ for every element x and y of the loop Q. This kind of mapping f is called an *isomorphism.* So an isomorphism is a bijective homomorphism.

Theorem 3.5. *Let K be the kernel of a homomorphism f of a loop Q. Hence the factor loop Q/K is isomorphic to the homomorphic image $f(Q)$ in terms of the mapping $Kx \mapsto f(x)$.*

Proof. Now the mapping that has been defined above is a homomorphism, because $(Kx)(Ky) = Kxy \mapsto f(xy) = f(x)f(y)$. Furthermore, according to the proof of Theorem 3.3, a coset Kx includes exactly those elements y of the loop Q, which have $f(y) = f(x)$. Hence the mapping $Kx \mapsto f(x)$ is a bijection. Thus the factor loop Q/K is isomorphic to the homomorphic image $f(Q)$.

Now define mappings $R_{x,y}$ and $M_{x,y}$ in the following way: $R_{x,y} = R_{xy}^{-1} R_y R_x$ and $M_{x,y} = R_{xy}^{-1} L_x R_y$, where x and y are elements of a loop Q. By setting $h_2 = 1$ in the equation of Theorem 3.3, it follows that $R_y R_x(h_1) = R_{xy}(h_3)$ or $R_{x,y}(h_1) = h_3$. Respectively, by setting $h_1 = 1$ it follows that $L_x R_y(h_2) = R_{xy}(h_3)$ or $M_{x,y}(h_2) = h_3$. With this result we get, from Theorem 3.3, the following lemma.

Lemma 3.6. *If a subloop H of a loop Q is normal, then permutations $R_{x,y}$ and $M_{x,y}$ of the loop Q are also permutations on the subloop H.*

Later it will be proved that this condition of Lemma 3.6 is also a sufficient one for the normality.

Again, let Q be a loop and $M(Q)$ be a permutation group, which is generated by all the permutations R_x and L_x, where x runs through all the elements of the loop Q. The group $M(Q)$ is called the *multiplication group* of the loop Q. Furthermore, let $I(Q)$ be the subgroup of the multiplication group $M(Q)$ generated by the permutations $R_{x,y}$ and $M_{x,y}$, where x and y run through all the elements of the loop Q. This subgroup $I(Q)$ is called the *inner mapping group* of the loop Q. Next we are going to demonstrate another definition for the inner mapping group $I(Q)$ of a loop Q.

Theorem 3.7. *Let Q be a loop and $I(Q)$ be its inner mapping group. Under this situation $U(1) = 1$ for every element U of the group $I(Q)$. In addition, with each and every element X of the multiplication group $M(Q)$, there exists a unique presentation of the form $X = R_x U$, where U is an element of the inner mapping group $I(Q)$ and $x = X(1)$.*

Proof. From the definition of mappings $R_{x,y}$ and $M_{x,y}$, we have that $R_{x,y}(1) = R_{xy}^{-1}(xy) = 1$ and so we also have that $R_{x,y}^{-1}(1) = 1$. Similarly $M_{x,y}(1) = R_{xy}^{-1}(xy) = 1$ and $M_{x,y}^{-1}(1) = 1$. Because the permutations $R_{x,y}$ and $M_{x,y}$ generate the group $I(Q)$, it is clear that $U(1) = 1$ for each and every element U of the inner mapping group $I(Q)$.

If $X = R_x U$, where U is an element of the group $I(Q)$, then $X(1) = R_x(1) = x$. This means that x has been uniquely defined.

In addition $U = R_x^{-1}X$ and so U is also unambiguous. What remains to be shown, is that, for every element X of the group $M(Q)$, there exists at least one presentation in the form $X = R_xU$, where U is an element of $I(Q)$. This follows by using the following seven identities:

(i) $R_yR_x = R_{xy}R_{x,y}$,
(ii) $L_yR_x = R_{yx}M_{y,x}$,
(iii) $R_y^{-1}R_x = R_pR_{p,y}^{-1}$, where $p = R_y^{-1}(x)$,
(iv) $L_y^{-1}R_x = R_qM_{y,q}^{-1}$, where $q = L_y^{-1}(x)$,
(v) $L_x = R_xM_{x,1}$,
(vi) $R_x^{-1} = R_uR_{u,x}^{-1}$, where $u = R_x^{-1}(1)$,
(vii) $L_x^{-1} = R_vM_{x,v}^{-1}$, where $v = L_x^{-1}(1)$.

Equations (i) and (ii) are trivial. Let us examine now equation (iii). If $p = R_y^{-1}(x)$, then $py = x$ and so $R_{p,y} = R_x^{-1}R_yR_p$ or $R_{p,y}^{-1} = R_p^{-1}R_y^{-1}R_x$. Hence $R_y^{-1}R_x = R_pR_{p,y}^{-1}$. Respectively, in equation (iv) $q = L_y^{-1}(x)$ or $yq = x$. Thus $M_{y,q} = R_x^{-1}L_yR_q$ and $M_{y,q}^{-1} = R_q^{-1}L_y^{-1}R_x$. This means that equation (iv) is valid. In addition three last equations are consequences of equations (ii), (iii) and (iv) by imposing $x = 1$.

From the definition of the multiplication group $M(Q)$ it follows that its every element X has at least one presentation in the form $X = X_r...X_2X_1$, where $r \geq 1$ is a finite integer and every X_i is some of elements R_y, L_y, R_y^{-1} and L_y^{-1}, where y is an arbitrary element of Q. Next shall we prove our theorem by induction. In the case $r = 1$, the result follows immediately from equations $(v), (vi)$ and (vii). Let us now presume that theorem is valid, when there are less than r elements in the product $X_s...X_2X_1$. This means that $X = X_rR_xV$, where V is an element of the inner mapping group $I(Q)$, x an element of the loop Q and X_r one of four types of permutations R_y, L_y, R_y^{-1} and L_y^{-1}. In each case we can see, by using equations $(i) - (iv)$, that $X_rR_x = R_zW$, where W is an element of the group $I(Q)$ and z is an element of the loop Q. Hence we finally have that $X = R_zWV = R_zU$, where U is an element of the inner mapping group $I(Q)$ and thus our induction is finished. This proves Theorem 3.7.

Corollary 3.8. *An element U of the multiplication group $M(Q)$ of a loop Q is an element of the inner mapping group $I(Q)$ if and only if $U(1) = 1$.*

Corollary 3.9. *Every element X of the multiplication group $M(Q)$ has a unique presentation in the form $X = L_x V$, where V is an element of the inner mapping group $I(Q)$ and $x = X(1)$.*

Corollary 3.10. *Let a be an element of a loop Q and X an element of the multiplication group $M(Q)$ of the loop Q. Then $X(a) = U(a)x = xV(a)$, where U and V are elements of the inner mapping group $I(Q)$ and $x = X(1)$.*

Corollary 3.11. *If Q is a finite loop, then $\mid Q \mid= [M(Q) : I(Q)]$. In other words, the order of a loop Q is the index of the inner mapping group $I(Q)$ in the multiplication group $M(Q)$.*

Proof. The necessary condition is clear by Theorem 3.7. Assume now that $X(1) = 1$ for some element X of the multiplication group $M(Q)$. According to Theorem 3.7 $X(1) = R_x U(1) = R_x(1) = x$. This means that $x = 1$ and so $X = R_1 U = U$. Hence the permutation X is an element of the inner mapping group $I(Q)$.

If $X(1) = x$, let us take $V = L_x^{-1} X$. Hence we have that $V(1) = L_x^{-1}(x) = 1$ and so V is an element of the inner mapping group $I(Q)$, and in addition $X = L_x V$. Thus Corollary 3.9 is valid. Furthermore, Corollary 3.10 follows from Theorem 3.7 and from Corollary 3.9.

From Theorem 3.7 we get that each and every coset $XI(Q)$ of the subgroup $I(Q)$ in the group $M(Q)$ defines uniquely an element x of the loop Q so that $XI(Q) = R_x I(Q)$. Thus the number of the separate cosets equals the order of the loop Q.

By Corollary 3.8 we can now define that the inner mapping group $I(Q)$ of a loop Q is the stabilizer of the neutral element 1 of the loop Q.

In the following theorem we get the third criterion for the normality of a subloop H in Q by using the inner mapping group $I(Q)$.

Theorem 3.12. *Let Q be a loop and $I(Q)$ its inner mapping group. Then a necessary and sufficient condition for the normality of a subloop H in the loop Q is that all elements of the inner mapping group $I(Q)$ are permutations on the subloop H.*

Proof. The necessity follows from the definition of the group $I(Q)$ and Lemma 3.6. To prove the sufficiency we shall prove the validity of Theorem 3.3. This is done by applying Corollary 3.10 several times.

Assume now that elements of $I(Q)$ are permutations on a subloop H of the loop Q. Let x and y be arbitrary elements of the loop Q. In the case, where h_1 and h_2 are any two elements of the loop Q but not necessarily elements of the subloop H, we get that $(h_1x)(h_2y) = R_{h_2y}R_x(h_1) = U(h_1)R_{h_2y}R_x(1) = U(h_1)[x(h_2y)] = U(h_1)L_xR_y(h_2) = U(h_1)[V(h_2)L_xR_y(1)] = U(h_1)[V(h_2)(xy)] = [WU(h_1)V(h_2)](xy)$, where U and V are elements of $I(Q)$ and are independent of the element h_1. Furthermore, the permutation W is defined in the following way: $W = R^{-1}_{V(h_2)}R^{-1}_{xy}R_{V(h_2)(xy)}$, which means that $W^{-1} = R^{-1}_{V(h_2)(xy)}R_{xy}R_{V(h_2)} = R_{V(h_2),xy}$ and so W is also an element of $I(Q)$. Thus the equation of Theorem 3.3 can be expressed in the form $WU(h_1)V(h_2) = h_3$. If h_2 is an element of the subloop H, then $V(h_2)$ is also an element of H. Now $WU(h_1) = R^{-1}_{V(h_2)}(h_3)$ and so $WU(h_1)$, as well as h_1, is an element of the subloop H if and only if h_3 is an element of the subloop H.

Respectively we get $(h_1x)(h_2y) = L_{h_1x}R_y(h_2) = U(h_2)[(h_1x)y] = U(h_2)R_yR_x(h_1) = U(h_2)[V(h_1)(xy)] = [WU(h_2)V(h_1)](xy)$, with suitable elements U, V and W of the inner mapping group $I(Q)$. Furthermore, the more accurate definition for the element W is now $W = R^{-1}_{V(h_1)}R^{-1}_{xy}R_{V(h_1)(xy)}$ or $W^{-1} = R^{-1}_{V(h_1)(xy)}R_{xy}R_{V(h_1)} = R_{V(h_1),xy}$. Hence the equation of Theorem 3.3 is in the form $WU(h_2)V(h_1) = h_3$ or $WU(h_2) = R^{-1}_{V(h_1)}(h_3)$. If h_1 is an element of the subloop H, then h_2 is an element of H if and only if h_3 is an element of H. According to Theorem 3.3, H is now a normal subloop of the loop Q.

The *centre* $C = C(Q)$ of a loop Q is the subset of the loop Q that includes the elements a of Q such that $a(xy) = (ax)y$, $x(ay) = (xa)y$, $x(ya) = (xy)a$ and $ax = xa$ for all elements x and y of Q. In other words, the centre C includes all such elements of the loop Q that are commutative and associative with all elements of Q.

Theorem 3.13. *The centre C of a loop Q is a set of the elements a of the loop Q such that $U(a) = a$ with every element U of the inner mapping group $I(Q)$. Furthermore, if $U(a) = a$ for every element U of the inner mapping group $I(Q)$, x is an element of the loop Q and Y is an element of the multiplication group $M(Q)$, then $Y(ax) = aY(x)$.*

Proof. Let us first prove the second statement of the theorem. Let x and y be elements of the loop Q and a such an element of Q that $U(a) = a$ for every U of $I(Q)$. Thus according to Theorem 3.7 $R_y(ax) = (ax)y = U(a)(xy) = aR_y(x)$. According to this $R_y(aR_y^{-1}(x)) = aR_yR_y^{-1}(x) = ax$ and so $R_y^{-1}(ax) = aR_y^{-1}(x)$. Respectively $L_y(ax) = U(a)(yx) = aL_y(x)$. According to this we get again $L_y(aL_y^{-1}(x)) = aL_yL_y^{-1}(x) = ax$ and so we have that $L_y^{-1}(ax) = aL_y^{-1}(x)$. Because the multiplication group $M(Q)$ is generated by all permutations R_y and L_y, has the second statement been proved.

Let us now prove the first statement of the theorem. Assume first that $U(a) = a$ for every element U of $I(Q)$. Now with all elements x of the loop Q we have that $xa = L_x(a) = U(a)x = ax$. In addition, for all elements x and y of the loop Q, the following equations are valid: $(ax)y = a(xy)$, $(xa)y = x(ay)$ and $(xy)a = x(ya)$. This follows from the second statement of the theorem and the commutative property of the element a. Thus it follows from the equations that the element a is included in the centre C.

Conversely, if we assume that a is an element of the centre C of the loop Q, then $R_{x,y}(a) = R_{xy}^{-1}R_yR_x(a) = R_{xy}^{-1}((ax)y) = R_{xy}^{-1}(a(xy)) = R_{xy}^{-1}R_{xy}(a) = a$. Similarly $M_{x,y}(a) = R_{xy}^{-1}L_xR_y(a) = R_{xy}^{-1}(x(ay)) = R_{xy}^{-1}(a(xy)) = a$. Because $R_{x,y}$ and $M_{x,y}$ generate the group $I(Q)$, we get, as a result, that $U(a) = a$ with all elements a of the centre C and with all elements U of the inner mapping group $I(Q)$.

Corollary 3.14. *The centre C of a loop Q is a subloop of Q.*

Proof. Let a and b be two arbitrary elements of the centre C. Hence each and every element U of the inner mapping group $I(Q)$ satisfies $U(a) = a$ and $U(b) = b$. By Theorem 3.13 we have that $U(ab) = aU(b) = ab$ and thus the product ab is an element of the centre C. Of course, the neutral element 1 is an element of C. If $ax = b$, then $aU(x) = U(ax) = U(b) = b = ax$ and so $U(x) = x$ for every U of $I(Q)$. This means that x is an element of the centre C. Because $ax = xa$, then the element x of the equation $xa = b$ is also an element of the centre C. Hence the centre C is a subloop of the loop Q.

By the definition of the centre it is clear that the centre $C(Q)$ is an associative and commutative loop and thus an abelian group.

Corollary 3.15. *The centre C of a loop Q is a normal subloop of the loop Q.*

Corollary 3.16. *If a is an element of the centre C of a loop Q, x an element of the loop Q and Y an element of the multiplication group $M(Q)$, then $Y(ax) = aY(x)$.*

Proof. Corollary 3.16 is an immediate consequence of Theorem 3.13. Similarly Corollary 3.15 follows from Theorem 3.12 and Theorem 3.13.

Next we shall prove an interesting result about the relationship between the centre $C(Q)$ of a loop Q and the centre $Z(M(Q))$ of the multiplication group of the loop Q.

Lemma 3.17. *The centre $C(Q)$ of a loop Q and the centre $Z(M(Q))$ of the multiplication group $M(Q)$ are isomorphic.*

Proof. Let us first prove that the permutations R_c, where c runs through all the elements of the centre C, generate the centre of the multiplication group $M(Q)$. If x is an element of the loop Q and c, d are elements of the centre C, then $L_c(x) = cx = xc = R_c(x)$ and $R_cR_d(x) = (xd)c = x(cd) = R_{cd}(x)$. Furthermore, the centre C is an abelian group and thus $R_c^{-1} = R_{c^{-1}}$. Hence $\langle\ R_c\ \rangle = \{\ R_c\ \}$, where c runs through all the elements of C.

Let x and a be elements of the loop Q and c be an element of the centre C. Now $R_cR_x(a) = (ax)c = (ac)x = R_xR_c(a)$ and $R_cL_x(a) = (xa)c = x(ac) = L_xR_c(a)$. According to this we have that $R_cR_x = R_xR_c$ and $R_cL_x = L_xR_c$, which means that we also have that $R_cR_x^{-1} = R_x^{-1}R_c$ and $R_cL_x^{-1} = L_x^{-1}R_c$. Hence the set $\{R_c : c \in C\}$ is included in the centre $Z(M(Q))$.

Conversely, assume that Z is an element of the centre $Z(M(Q))$ of the multiplication group $M(Q)$; then $ZX(a) = XZ(a)$ for every element a of the loop Q and for every element X of the multiplication group $M(Q)$. Hence we have $ZL_x(1) = L_xZ(1)$, which means that $Z(x) = xZ(1) = R_{Z(1)}(x)$ for every element x of the loop Q. Let us now examine more carefully the element $Z(1)$. If U is an element of the inner mapping group $I(Q)$, then $U(Z(1)) = UZ(1) = ZU(1) = Z(1)$ and thus $Z(1)$ is an element of the centre C by Theorem 3.13. Hence we have proved that the centre $Z(M(Q))$ equals $\{R_c : c \in C\}$.

Next we shall examine a mapping f, $f(c) = R_c$, from the centre C of the loop Q to the centre $Z(M(Q))$ of the multiplication group $M(Q)$. Now it is clear that this mapping f is a bijection and $f(c_1c_2) = R_{c_1c_2} = R_{c_1}R_{c_2} = f(c_1)f(c_2)$. Thus the mapping f is an isomorphism from the centre $C(Q)$ of Q to the centre $Z(M(Q))$ of $M(Q)$. The proof is complete.

The following theorem is often seen as a basic result in loop theory and it will help us to understand the definitions for nilpotency and solvability of loops.

Theorem 3.18. *Let H be a normal subloop of a loop Q. Hence every subloop S of the factor loop Q/H can be represented in the form K/H, where K is a uniquely determined subloop of the loop Q. In addition H is a normal subloop of the loop K. Now K is the set of all such elements of the loop Q that are elements of the cosets Hx, which are included in the subloop S. Furthermore, K/H is a normal subloop of the factor loop Q/H if and only if K is a normal subloop of the loop Q. When this holds, the factor loop Q/K is isomorphic to the factor loop $(Q/H)/(K/H)$.*

Proof. Let S be a subloop of the factor loop Q/H. Thus S includes some elements Hx of the factor loop Q/H. Because S is a loop, it includes the neutral element H. Let K be the set of elements of the loop Q that are included in the cosets Hx in the subloop S. The defined set K is unambiguous and includes the subloop H. It remains to show that the set K, which has been defined above, is a loop.

Let x and y be elements of the set K, so that Hx and Hy are elements of the loop S. Thus $HxHy = Hxy$ is also an element of the loop S and thus xy is an element of the set K. Let $xy = z$, where x and z are elements of the set K and y is unknown. Then y can be solved uniquely as an element of the loop Q. Now Hx and Hz are elements of the subloop S and $Hz = Hxy = HxHy$. Because S is a loop, Hy can be unambiguously solved as an element of the loop S and thus y is an element of the set K. Similarly, in the situation where $xy = z$, y and z are elements of the set K and x is unknown, x can be solved uniquely as an element of the set K. Because the loop H includes the neutral element 1, then the set K includes it as well. Thus K is a loop. Given the definitions of the factor loop and the loop K, it is clear that $S = K/H$.

To continue our proof, we need the following minor result. Now it is clear that $R_{Hv}(Hu) = H[R_v(u)]$ and $L_{Hv}(Hu) = H[L_v(u)]$ for every element u and v of the loop Q. Given this, the coset $Hu = R^{-1}_{Hv}(H[R_v(u)])$ and $Hu = L^{-1}_{Hv}(H[L_v(u)])$. If we define $x = R_v(u)$ and $y = L_v(u)$ or $u = R_v^{-1}(x)$ and $u = L_v^{-1}(y)$, we get $H[R_v^{-1}(x)] = R^{-1}_{Hv}(Hx)$ and $H[L_v^{-1}(y)] = L^{-1}_{Hv}(Hy)$. Because R_v and L_v are permutations on the loop Q, we see that $H[R_v^{-1}(u)] = R^{-1}_{Hv}(Hu)$ and $H[L_v^{-1}(u)] = L^{-1}_{Hv}(Hu)$ for every element u and v of the loop Q.

Let us now assume that K/H is a normal subloop of the factor loop Q/H. Thus $U_{Q/H}(K/H) \subseteq K/H$, where $U_{Q/H}$ is an arbitrary element of the inner mapping group $I(Q/H)$ of the factor loop Q/H. Let z be an element of the subloop K and thus Hz is an element of the factor loop K/H. Then $R_{Hx,Hy}(Hz) = R^{-1}_{Hxy}R_{Hy}R_{Hx}(Hz) = R^{-1}_{Hxy}[(HzHx)Hy] = H[R^{-1}_{xy}((zx)y)] = H[R_{x,y}(z)]$. Similarly $M_{Hx,Hy}(Hz) = R^{-1}_{Hxy}L_{Hx}R_{Hy}(Hz) = R^{-1}_{Hxy}[Hx(HzHy)] = H[R^{-1}_{xy}(x(zy))] = H[M_{x,y}(z)]$. Since K/H is a normal subloop, then $R_{Hx,Hy}(Hz)$ and $M_{Hx,Hy}(Hz)$ are elements of the loop K/H and so $R_{x,y}(z)$ and $M_{x,y}(z)$ are elements of the loop K for every element x and y of the loop Q. Hence K is a normal subloop of the loop Q.

Conversely, let K be a normal subloop of the loop Q. Thus $U(K) \subseteq K$ for every element U of the inner mapping group $I(Q)$ of the loop Q. If z is an element of the subloop K, then $R_{x,y}(z) = R_{xy}^{-1}R_yR_x(z) = R_{xy}^{-1}[(zx)y] \in K$ and $M_{x,y}(z) = R_{xy}^{-1}L_xR_y(z) = R_{xy}^{-1}[x(zy)] \in K$. Thus the coset $H[R_{x,y}(z)] = H[R_{xy}^{-1}((zx)y)] = R_{Hxy}^{-1}[(HzHx)Hy] = R_{Hx,Hy}(Hz) \in K/H$. Similarly $H[M_{x,y}(z)] = H[R_{xy}^{-1}(x(zy))] = R_{Hxy}^{-1}[Hx(HzHy)] = M_{Hx,Hy}(Hz) \in K/H$. This means that the factor loop K/H is a normal subloop of the factor loop Q/H.

Assume now that K is a normal subloop of the loop Q and that $H \leq K$. We shall define a mapping f from the factor loop Q/H to the factor loop Q/K by $f(Hx) = Kx$.

Let us first prove that the mapping f is well defined. Assume that $Hx_1 = Hx_2$ and thus that $H = R_{Hx_2}^{-1}(Hx_1) = H[R_{x_2}^{-1}(x_1)]$. Hence $R_{x_2}^{-1}(x_1)$ is an element of the subloop H and thus an element of the subloop K. According to this there exists an element k of K such that $R_{x_2}^{-1}(x_1) = k$ and so $x_1 = R_{x_2}(k) = kx_2$. This means that $Kx_1 = Kkx_2 = KkKx_2 = Kx_2$ and the mapping f is well defined. Clearly the mapping f is a surjection, because, for each element Kx of the factor loop Q/K, there exists an element Hx in the factor loop Q/H that $f(Hx) = Kx$. Furthermore $f(Hx_1Hx_2) = f(Hx_1x_2) = Kx_1x_2 = Kx_1Kx_2 = f(Hx_1)f(Hx_2)$ and thus f is also a homomorphism. Hence the factor loop Q/K is the homomorphic image of the factor loop Q/H in terms of the mapping f.

Let us now examine the kernel of this mapping f. If the kernel includes an element Hx of the factor loop Q/H, then we have $f(Hx) = Kx = K$. This means that the element x must be an element of the subloop K and so Hx is also an element of the factor loop K/H. On the other hand, if Hx is an element of the factor loop K/H, then $f(Hx) = Kx = K$. Hence the kernel of the homomorphism f is the factor loop K/H. By Theorem 3.5 the factor loops $(Q/H)/(K/H)$ and Q/K are isomorphic.

Next we give definitions for the nilpotency and the solvability of a loop Q. If we set $C_0 = 1$, $C_1 = C(Q)$ and the factor loop $C_i/C_{i-1} = C(Q/C_{i-1})$, then we obtain a series of normal subloops of the loop Q. If C_{n-1} is a proper subloop of Q but $C_n = Q$, then we say that the loop Q is *nilpotent of class* n.

In 1946 Bruck [1] showed that, if Q is a nilpotent loop, then the multiplication group $M(Q)$ is a solvable group. He also showed that if $M(Q)$ is a nilpotent group, then Q is a nilpotent loop.

Bruck defined the solvability of loops as follows: A loop Q is *solvable* if it has a series $1 = Q_0 \subseteq Q_1 \subseteq ... \subseteq Q_n = Q$, where Q_{i-1} is a normal subloop of Q_i and Q_i/Q_{i-1} is an abelian group. Of course, nilpotent loops are solvable.

In 1996, Vesanen [21, Theorem 1] investigated the relationship between solvable loops and solvable groups and he was able to prove the next theorem.

Theorem 3.19. *Let* Q *be a finite loop. If the multiplication group* $M(Q)$ *is a solvable group, then* Q *is a solvable loop.*

This result is very fundamental and deep and opens a large variety of possibilities for creating solvability criteria for finite loops in terms of the properties of the inner mapping group. Hence we are interested in those properties of the inner mapping group $I(Q)$, which guarantee the solvability of the multiplication group $M(Q)$ and this subject will be examined in Section 4.

At the end of this section we shall once more take a close look at the multiplication group $M(Q)$ and the inner mapping group $I(Q)$ of a loop Q. As we know, the multiplication group $M(Q)$ is generated by the permutations L_x and R_x of the loop Q, where x goes through all the elements of the loop Q. Since Q is a loop, the permutation group $M(Q)$ is transitive on Q and thus the stabilizers of the elements of the loop Q are conjugate in the multiplication group $M(Q)$.

Next we prove two minor results concerning the inner mapping group $I(Q)$.

Lemma 3.20. *If a loop* Q *is a group, then the inner mapping group* $I(Q)$ *is a group of inner automorphisms.*

Proof. As we know, $I(Q) = \langle\, R_{x,y}, M_{x,y} \,\rangle$. When Q is a group, then we have $R_x^{-1} = R_{x^{-1}}$ and $L_x^{-1} = L_{x^{-1}}$. This means that $R_{x,y}(a) = R_{xy}^{-1} R_y R_x(a) = R_{xy}^{-1}((ax)y) = R_{xy}^{-1}(a(xy)) = a$ and $M_{x,y}(a) = R_{xy}^{-1} L_x R_y(a) = R_{(xy)^{-1}}(x(ay)) = x(ay)y^{-1}x^{-1} = xax^{-1}$ for any element a of the loop Q. Thus the permutation $R_{x,y}$ is an identity mapping and the permutation $M_{x,y}$ is an inner automorphism.

Lemma 3.21. *A loop* Q *is a commutative group if and only if its inner mapping group* $I(Q)$ *includes only the identity mapping.*

Proof. Let us first assume that the loop Q is an abelian group. By Lemma 3.20 the inner mapping group $I(Q)$ is a group of inner automorphisms. Since the group Q is abelian, then inner automorphisms are identity mappings.

Conversely, assume that the inner mapping group $I(Q)$ includes only the identity mapping. Hence $U(a) = a$ for every element a of the loop Q and for every element U of the inner mapping group $I(Q)$. By Theorem 3.13 $Q = C(Q)$ and thus the loop Q is an abelian group.

Loops appear in a natural way as algebraic structures on transversals of a subgroup in a group. With the following theorem we get a link between loop theory and group theory with transversals. Subsets A and B of the multiplication group $M(Q)$ of a loop Q will be defined in the following way, $A = \{\, L_a \,\}$ and $B = \{\, R_a \,\}$, where a goes through all the elements of the loop Q. In the following theorem it will be proved that A and B are $I(Q)$ -connected transversals in $M(Q)$.

Theorem 3.22. *Let* Q *be a loop and* $M(Q)$ *the multiplication group and* $I(Q)$ *the inner mapping group of the loop* Q*. Hence subsets* $A = \{L_a : a \in Q\}$ *and* $B = \{R_a : a \in Q\}$ *of the multiplication group* $M(Q)$ *are* $I(Q)$ *-connected transversals in the group* $M(Q)$*.*

Proof. By Theorem 3.7 and Corollary 3.9 every element X of the multiplication group $M(Q)$ has a unique representation in the form $X = R_xU$ and the form $X = L_xV$, where U and V are elements of the inner mapping group $I(Q)$ and $x = X(1)$. This means that the sets $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ are left transversals to the inner mapping group $I(Q)$ in the multiplication group $M(Q)$.

Next we prove that transversals A and B are also $I(Q)$ -connected in $M(Q)$. Now $L_a^{-1}R_b^{-1}L_aR_b(1) = L_a^{-1}R_b^{-1}(ab) = L_a^{-1}R_b^{-1}R_b(a) = L_a^{-1}(a) = 1$. This means that the commutator subgroup $[A, B]$ is a subgroup of the inner mapping group $I(Q)$.

In addition, subgroups of the inner mapping group $I(Q)$ have the following property.

Corollary 3.23. *If* $1 < H \leq I(Q)$, *then the subgroup* H *is not a normal subgroup of the multiplication group* $M(Q)$.

Proof. Let us presume that a subgroup H of the inner mapping group $I(Q)$ is normal in the multiplication group $M(Q)$. Then $H^{L_a} = H$ for every element a of the loop Q. Let U be an element of the subgroup H so that $U(1) = 1$. Because H is normal in $M(Q)$, we also have that $U^{L_a}(1) = L_a^{-1}UL_a(1) = L_a^{-1}U(a) = 1$. From this it follows that $U(a) = L_a(1) = a$ for every element a of the loop Q. This means that U is an identity mapping. Hence, if H is a normal subgroup of the multiplication group $M(Q)$, then $H = 1$.

Furthermore, it is now clear that the sets $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ generate the multiplication group $M(Q)$. To summarize, if Q is a loop, then there exist $I(Q)$ -connected transversals A and B in $M(Q)$ such that $M(Q) = \langle A, B\rangle$. In addition, the core $L_{M(Q)}[I(Q)]$ is equal to 1.

Given this, a question arises as to which kinds of groups can be multiplication groups of a loop. The following theorem, which was proved by Kepka and Niemenmaa [15, Theorem 4.1], describes the relationship between multiplication groups of loops and connected transversals.

Theorem 3.24. *A group G is isomorphic to the multiplication group of a loop Q if and only if there exist a subgroup H of G satisfying $L_G(H) = 1$ and H -connected transversals A and B such that $G = \langle A, B\rangle$.*

Proof. The necessity follows from the definition of the multiplication group, Theorem 3.22 and Corollary 3.23.

Assume now that a group G has a subgroup H and H -connected transversals A and B satisfying the conditions of the theorem. For every element x of G there is a unique element $f(x)$ in the transversal A such that $f(x)H = xH$. Let K be the set of left cosets of H in G. We shall define a binary operation $(*)$ on the set K in the following way: $(xH) * (yH) = f(x)yH$.

First we shall show that this operation $(*)$ is well defined. If $xH = uH$ and $yH = vH$, then $f(x) = f(u)$ and $f(x)yH = f(u)yH = f(u)vH$. This means that $(xH) * (yH) = (uH) * (vH)$ and the operation $(*)$ is well defined.

Next we shall show that the set $(K, *)$ is a loop. Now it is clear that a product of two elements of K is always included in the set K. Furthermore, by Lemma 2.3, we have that $1 \in A$. Hence $(1H) * (yH) = f(1)yH = yH$ and $(xH) * (1H) = f(x)H = xH$, which means that the set $(K, *)$ has the neutral element $1H$. If $(xH) * (yH) = zH$, where xH and zH are known elements and yH an unknown element of the set K, then $yH = f(x)^{-1}zH$ is a uniquely determined element of K. Respectively, let us have $(xH) * (yH) = zH$, where yH and zH are known elements and xH an unknown element of K. Now Ay is a left transversal to H in G by Lemma 2.2. This means that $f(x)$ can be solved uniquely as an element of A in the equation $zH = f(x)yH$. Hence $xH = f(x)H$ can be solved uniquely as an element of K, and so the set $(K, *)$ is a loop.

For every element y of G there exists exactly one element $g(y)$ in the transversal B such that $yH = g(y)H$. This means that $(xH) * (yH) = f(x)yH = f(x)g(y)H$. Since transversals A and B are H -connected, we have $f(x)g(y)H = g(y)f(x)H$. Hence we have that $L_{yH}(xH) = (yH) * (xH) = f(y)xH$ and respectively $R_{yH}(xH) = (xH) * (yH) = f(x)g(y)H = g(y)f(x)H = g(y)xH$. Because $G = \langle A, B\rangle$, we have a group homomorphism from the group G to the multiplication group $M(K)$ of the loop K. The kernel of this homomorphism is the core $L_G(H)$. Now we have $L_G(H) = 1$ and so the group G is isomorphic to the multiplication group $M(K)$.

Furthermore, $1H = gH$ if and only if g is an element of the subgroup H. Hence the subgroup H is isomorphic with the inner mapping group $I(K)$, which is the stabilizer of the neutral element $1H$ of the loop K.

By the proof of Theorem 3.24, if A is an H-selfconnected transversal in G satisfying $G = \langle A \rangle$ and $L_G(H) = 1$, then $(xH) * (yH) = f(x)yH = f(x)f(y)H = f(y)f(x)H = (yH) * (xH)$ and K is a commutative loop. On the other hand, if a loop Q is commutative, then always $R_q = L_q$ for every element $q \in Q$. Thus the following corollary is clear.

Corollary 3.25. *A group G is isomorphic to the multiplication group of a commutative loop Q if and only if there exists a subgroup H of G satisfying $L_G(H) = 1$ and an H-selfconnected transversal A satisfying $G = \langle A \rangle$.*

4 Summary of the results of the original articles for the solvability of groups

By Theorem 3.19 a finite loop Q is solvable, if its multiplication group $M(Q)$ is a solvable group. Hence we have the question: when is the multiplication group solvable? Furthermore, by Theorem 3.24, a group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $L_G(H) = 1$ and H-connected transversals A and B such that $G = \langle A, B \rangle$. Given these results it is reasonable to examine the solvability of such a group G which has a subgroup H with H-connected transversals.

In this section we assume now that the group G has a subgroup H and H-connected transversals A and B. We shall study the solvability in three different cases depending on the order of the subgroup H. As a reminder, a group G is *solvable* if it has a series $1 = G_n \leq ... \leq G_1 \leq G_0 = G$ of subgroups G_i, where each G_{i+1} is a normal subgroup of G_i and the factor groups G_i/G_{i+1} are abelian groups.

We begin by introducing some solvability results, which are needed for our main theorems. As we mentioned in the introduction, Kepka and Niemenmaa [11, Theorem 2.2 and Corollary 2.3] have proved the following result, which is very important for the proofs of our forthcoming theorems.

Theorem 4.1. *Let H be a cyclic subgroup of a group G. If there exist H-connected transversals A and B in G, then G is a solvable group.*

Kepka and Niemenmaa [16, Theorem 4.1] have also studied the situation, where H is a finite abelian subgroup, and they proved the following theorem.

Theorem 4.2. *Let H be a finite abelian subgroup of a group G such that there exist H-connected transversals A and B in G. Then G is a solvable group.*

In the following two results for solvability, we do not assume that we have connected transversals, but these lemmas have a quite important role in proofs of our main theorems. The first one has been proved by Herstein in [9, Theorem 1].

Lemma 4.3. *Let G be a finite group and M an abelian subgroup of G. If M is maximal in G, then G is a solvable group.*

The second lemma for the solvability in the case of finite groups has been proved by Carr [2, Main Theorem].

Lemma 4.4. *Suppose that $G = MN$ is a finite group. If M is an abelian subgroup and N has a nilpotent subgroup of index at most 2, then G is a solvable group.*

As we have already mentioned, our purpose in this section is to introduce our three main theorems for the solvability of a group G with a subgroup H of a certain given order and H-connected transversals. We begin now by introducing some group theoretic results, which also have an important role in the proofs of our main theorems. The first result, with its corollary, is from the theory of permutation groups and it was proved in [14, Theorem 9.1].

Lemma 4.5. *Let G be a permutation group on a set X. Furthermore, let $fix_X(g) = \{i \in X \mid g(i) = i\}$, where $g \in G$. Then the number of orbits of G on X is $\frac{1}{|G|}\sum_{g\in G} \mid fix_X(g) \mid$.*

Corollary 4.6. *Let G be a transitive permutation group on the finite set X and for $g \in G$ let $fix_X(g) = \{i \in X \mid g(i) = i\}$. Then $\mid G \mid = \sum_{g\in G} \mid fix_X(g) \mid$.*

In our next lemma we need the definition for a complement and a normal complement. Let K be a subgroup of a group G. Then a subgroup $H \leq G$ is a *complement* of the subgroup K in G if $K \cap H = 1$ and $KH = G$. Furthermore, if H is normal in G, then H is a *normal complement* of K in G. The following result for complements is based on [8, Theorem 7.4.3].

Lemma 4.7. *Let G be a finite group and let S be an abelian Sylow subgroup of G contained in the centre of its normalizer, i.e. $S \leq Z(N_G(S))$. Then S has a normal complement in G.*

We shall also need following two results of Collins [3, Corollary 3.18 and Theorem 2.56] for finite groups. By the way, a nonidentity abelian subgroup T of a finite group G is said to be *strongly self-centralizing* if $C_G(t) = T$ for every nonidentity element t of the subgroup T.

Lemma 4.8. *Let T be a strongly self-centralizing subgroup of a finite group G and assume that $[N_G(T) : T] = 2$. If G is simple, then G contains exactly one conjugacy class of involutions.*

Lemma 4.9. *Let G be a finite simple group having an elementary abelian Sylow 2-subgroup T of order 2^n, where $n \geq 2$. If $C_G(t) = T$ for all nonidentity $t \in T$, then $G \cong SL(2, 2^n)$.*

A group G is said to be a *Frobenius group* if and only if it contains a proper subgroup $H \neq 1$, called a *Frobenius complement*, such that the intersection $H \cap H^g$ is trivial for all elements $g \in G \setminus H$.

Furthermore, the set $N = G \setminus \cup_{g\in G}(H \setminus 1)^g$ is called the *Frobenius kernel.* The following lemma for Frobenius groups has been proved by Huppert [10, Theorem 5.7.6 and Theorem 5.8.7].

Lemma 4.10. *Let G be a Frobenius group and H a Frobenius complement in G. Then the Frobenius kernel N is a normal subgroup of G such that $G = HN$ and $H \cap N = 1$. Furthermore the Frobenius kernel N is always nilpotent.*

Finally, we need the following result of Vesanen [20, Theorem 4.2] for connected transversals in a projective special linear group.

Lemma 4.11. *Consider the projective special linear group $PSL(2, 2^n)$, where $n \geq 2$, and let H be a maximal subgroup of order $2(2^n + 1)$. Then there exist no H -connected transversals.*

After these preparations we are ready to concentrate to our three main theorems. As before, let G be a group and H be a subgroup of G. First we shall investigate the situation that there is an H -selfconnected transversal A in G. We also assume that $\mid H \mid= 2p$, where $p = 4t + 3$ is an odd prime number. By using the results for transversals in Section 2, Theorem 4.1 and the lemmas of this section, we derive the first of our main theorems. The proof of the following theorem can be found in the original article [12, Theorem 2.1]. Our method, which is based on elementary permutation group theory, can not be generalized to the case $p = 4t + 1$.

Theorem 4.12. *Let G be a group and let H be a subgroup of order $2p$, where $p = 4t+3$ is an odd prime number. If there exists an H -selfconnected transversal A in G, then G is a solvable group.*

The next question is, of course, about the solvability in the general case $\mid H \mid= 2p$, where p is an odd prime number. This matter has been studied by Csörgö and Niemenmaa and they managed to show, in [5, Theorem 2.4], that the solvability follows also in this general case provided that there are H -connected transversals A and B in the group G.

Csörgö and Niemenmaa have also studied the more general case, where $|H|$ is a product of two prime numbers p and q and there are H -connected transversals in G. Niemenmaa [19, Theorem 3.1] could first prove, by using the classification of finite simple groups, that, if G is finite, then G is solvable at least in the following cases: $p = q$, q is not a factor of $p - 1$, $q = 2$ and $p \leq 61$, $q = 3$ and $p \leq 31$ or $q = 5$ and $p \leq 11$. Together with Csörgö, Niemenmaa [6, Theorem 3.1] has studied the situation, where $p > q$ are odd prime numbers and $p = 2q^m + 1$. Again the solvability of G was proved.

After this the examination of the solvability continued to the case that a group G has a dihedral subgroup H with H -connected transversals. First, in [18, Theorem 3.3], Niemenmaa considered the situation where H is a dihedral 2 -subgroup with H -connected transversals and, once again, the solvability of G followed. Next we shall study the case, where H is a dihedral subgroup of order $2p^n$, where p is an odd prime number.

In the proof of our second main theorem we use solvability criteria for finite groups and for factorized groups, some permutation group theory and finally a detailed analysis on how involutions are distributed among the cosets of H. We first show that the result is true for finite groups, and from this we easily get that it holds for infinite groups, too. The proof of the following theorem can be found in the original article [4, Theorem 3.6].

Theorem 4.13. *Let G be a group and let H be a dihedral subgroup of order $2p^n$, where p is an odd prime number. If there exist H -connected transversals A and B in G, then G is a solvable group.*

After this result it is quite natural to be interested in the more general case, where H is a dihedral subgroup of order $2x$, where x is an odd number. That is the case, which we consider in the third main theorem. The basic structure of the proof is similar to the proof of Theorem 4.13, but, of course, is more complicated. The proof of the following theorem can be found in the original article [13, Theorem 3.7].

Theorem 4.14. *Let* G *be a group and let* H *be a dihedral subgroup of order* $2x$, *where* x *is an odd number. If there exist* H *-connected transversals* A *and* B *in* G, *then* G *is a solvable group.*

Naturally, the next question is: What about the situation, where the order of a dihedral subgroup is $2n$ with no constraints on the number n ? We hope to be able to answer this question in the future.

5 Summary of the results of the original articles for the solvability of loops

As we have mentioned earlier, loops can be understood as nonassociative versions of groups, and the permutation group generated by all left and right translations of a loop Q is called the multiplication group $M(Q)$; so $M(Q) = \langle L_x, R_x \rangle$, where x runs through all the elements of the loop Q. Furthermore, the stabilizer of the neutral element 1 of the loop Q is called the inner mapping group $I(Q)$. By Theorem 3.22, the subsets $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ of the multiplication group $M(Q)$ are $I(Q)$ - connected transversals in $M(Q)$. Also it is clear that these subsets A and B generate the multiplication group $M(Q)$. In addition, by Corollary 3.23, it is not possible to have a nontrivial normal subgroup of the multiplication group $M(Q)$ in the inner mapping group $I(Q)$. This means that the core of $I(Q)$ in $M(Q)$ is trivial.

In this last section we shall consider the solvability of loops by studying the properties of inner mapping groups. As we have already defined in Section 3, a loop Q is a solvable loop if it has a series $1 = Q_0 \subseteq Q_1 \subseteq ... \subseteq Q_n = Q$, where Q_{i-1} is a normal subloop of Q_i and the factor loops Q_i/Q_{i-1} are abelian groups. By Theorem 3.19 we get the important connection between the solvability of a finite loop and the solvability of the multiplication group. If the multiplication group $M(Q)$ of a finite loop Q is solvable, then the loop Q is also a solvable loop.

To be able to use the solvability results of Section 4, we need the important result of Theorem 3.24. A group G is isomorphic with the multiplication group of a loop Q if and only if there exist a subgroup H satisfying $L_G(H) = 1$ and H -connected transversals A and B satisfying $G = \langle A, B\rangle$. Furthermore, by Corollary 3.25 a group G is isomorphic with the multiplication group of a commutative loop Q if and only if there exists a subgroup H of G satisfying $L_G(H) = 1$ and an H -selfconnected transversal A satisfying $G = \langle A\rangle$.

By Theorem 4.1 and Theorem 4.2, the multiplication group $M(Q)$ of a loop Q is a solvable group, if the inner mapping group $I(Q)$ is a cyclic group or a finite abelian group. This means that, under these conditions, the loop Q is also solvable provided that it is a finite loop [16, Corollary 6.4]. Furthermore, Kepka and Niemenmaa [11, Theorem 2.4] have proved the solvability of a loop Q with the cyclic inner mapping group in general, not only in the finite case.

By our first main theorem concerning the solvability of groups, Theorem 4.12, we get the following result.

Theorem 5.1. *If Q is a commutative loop such that $\mid I(Q) \mid= 2p$, where $p = 4t + 3$ is a prime number, then $M(Q)$ is a solvable group. Furthermore, if Q is finite, then it is a solvable loop.*

After this Csörgö and Niemenmaa continued to study the general case $\mid I(Q) \mid= 2p$ and they managed to show that the multiplication group $M(Q)$ of a loop Q is a solvable group if the inner mapping group $I(Q)$ is of order $2p$, where p is an odd prime number [5, Theorem 3.2]. Furthermore, if Q is a finite loop, then it is also solvable. Csörgö and Niemenmaa have also studied the case, where the order of the inner mapping group $I(Q)$ is a product of two odd prime numbers. They proved, in [6, Theorem 4.3], that, if $\mid I(Q) \mid= pq$, where $p > q$ are odd prime numbers such that $p = 2q^m + 1$, then the solvability of the multiplication group $M(Q)$ and the loop Q follows. Of course, the solvability of the loop Q provides that the loop Q is finite.

Furthermore, Drapal has also studied this case $\mid I(Q) \mid= pq$. He assumed that $q < p$ are prime numbers and that the inner mapping group $I(Q)$ is nonabelian. In addition he used slightly different methods in his proof. When we started with the solvability of groups with transversals, he started to consider multiplication groups and inner mapping groups at once. First Drapal proved that the factor loop $Q/C(Q)$ is finite with a trivial centre and that its inner mapping group is of order pq as well. After this he showed that, if the inner mapping group of a loop is a nonabelian group of order pq and the loop has a trivial centre, then the multiplication group of this loop is a solvable group. From this, with the result for finite abelian inner mapping groups, it follows that the multiplication group $M(Q/C(Q))$ is solvable. By using the result that the multiplication group $M(Q/C(Q))$ is isomorphic with the factor group $M(Q)/Z(M(Q))$, Drapal also proved that the multiplication group $M(Q)$ is a solvable group [7, Corollary 4.7]. Thus we have the following:

Theorem 5.2. *Let $I(Q)$ be the inner mapping group of a loop Q. If $I(Q)$ is a nonabelian group of order pq, where $q < p$ are prime numbers, then the multiplication group $M(Q)$ is a solvable group.*

From Theorem 5.2 we get again by Theorem 3.19 the following result concerning the solvability of finite loops.

Theorem 5.3. *If the inner mapping group $I(Q)$ of a finite loop Q is a nonabelian group of order pq, where $q < p$ are prime numbers, then the loop Q is a solvable loop.*

Now we go on to the dihedral cases, which were considered earlier with the solvability criteria for groups in Section 4. Niemenmaa managed to show in [18, Corollary 3.4, Theorem 4.2] that, if the inner mapping group $I(Q)$ of a loop Q is a dihedral 2-group, then, once again, the solvability of the multiplication group $M(Q)$ and the corresponding (finite) loop Q follows. Next we study the case, where the inner mapping group $I(Q)$ is a dihedral group of order $2p^n$, where p is an odd prime number. Given our second main theorem for the solvability of groups, Theorem 4.13, there follows the next result for loops.

Theorem 5.4. *If* Q *is a loop, whose inner mapping group* $I(Q)$ *is dihedral of order* $2p^n$, *where* p *is an odd prime number, then the multiplication group* $M(Q)$ *is a solvable group. If we further assume that* Q *is finite, then* Q *is a solvable loop.*

After this it is natural to consider the more general dihedral case where the order of the inner mapping group $I(Q)$ of a loop Q is $2x$, where x is an odd number. Of course, the solvability result of Theorem 5.4 is included in the following theorem, which follows from Theorem 4.14.

Theorem 5.5. *If* Q *is a loop, whose inner mapping group* $I(Q)$ *is dihedral of order* $2x$, *where* x *is an odd number, then the multiplication group* $M(Q)$ *is a solvable group. If we further assume that* Q *is finite, then* Q *is a solvable loop.*

After these theorems we know that, if the inner mapping group $I(Q)$ of a loop Q is a cyclic group, a finite abelian group, a group of order pq, where $q < p$ are prime numbers, or a dihedral group of order $2x$, where x is an odd number, then the multiplication group $M(Q)$ is a solvable group. Furthermore, if the loop Q is finite and the inner mapping group $I(Q)$ satisfies one of these conditions above, then Q is a solvable loop.

The next interesting situation is the case, where the order of the dihedral inner mapping group $I(Q)$ of a loop Q is $2n$ with no constraints on the number n. This means that we are considering the general case where the inner mapping group $I(Q)$ of a loop Q is dihedral. This matter we shall study more carefully in the future.

References

[1] Bruck RH (1946) Contributions to the theory of loops. Trans Amer Math Soc 60: 245-354.

[2] Carr JG (1976) A solubility criterion for factorised groups. Arch Math 27: 225-231.

[3] Collins MJ (1990) Representations and characters of finite groups. Cambridge University Press.

[4] Csörgö P, Myllylä K & Niemenmaa M (2000) On connected transversals to dihedral subgroups of order $2p^n$. Algebra Colloquium 7:1: 105-112.

[5] Csörgö P & Niemenmaa M (2000) Solvability conditions for loops and groups. J Algebra 232: 336-342.

[6] Csörgö P & Niemenmaa M (2002) On connected transversals to nonabelian subgroups. Europ J Comb 23: 179-185.

[7] Drápal A (2002) Orbits of inner mapping groups. Monatsh Math 134: 191-206.

[8] Gorenstein D (1968) Finite groups. Harper & Row, New York.

[9] Herstein IN (1958) A remark on finite groups. Proc Amer Math Soc 9: 255-257.

[10] Huppert B (1967) Endliche Gruppen I. Springer-Verlag, Berlin-Heidelberg.

[11] Kepka T & Niemenmaa M (1993) On loops with cyclic inner mapping groups. Arch Math 60: 233-236.

[12] Myllylä K & Niemenmaa M (1999) On the solvability of commutative loops and their multiplication groups. Comment Math Univ Carolinae 40,2: 209-213.

[13] Myllylä K (2000) On connected transversals to dihedral subgroups. Acta Universitatis Ouluensis A 350.

[14] Neumann PM, Stoy GA & Thompson EC (1994) Groups and geometry. Oxford University Press.

[15] Niemenmaa M & Kepka T (1990) On multiplication groups of loops. J Algebra 135: 112-122.

[16] Niemenmaa M & Kepka T (1994) On connected transversals to abelian subgroups. Bull Austr Math Soc 49: 121-128.

[17] Niemenmaa M (1995) Transversals, commutators and solvability in finite groups. Bollettino U M I (7) 9-A: 203-208.

[18] Niemenmaa M (1995) On loops which have dihedral 2-groups as inner mapping groups. Bull Austr Math Soc 52: 153-160.

[19] Niemenmaa M (1997) On connected transversals to subgroups whose order is a product of two primes. Europ J Comb 18: 915-919.

[20] Vesanen A (1994) The group PSL(2,q) is not the multiplication group of a loop. Comm Algebra 22(4): 1177-1195.

[21] Vesanen A (1996) Solvable groups and loops. J Algebra 180: 862-876.