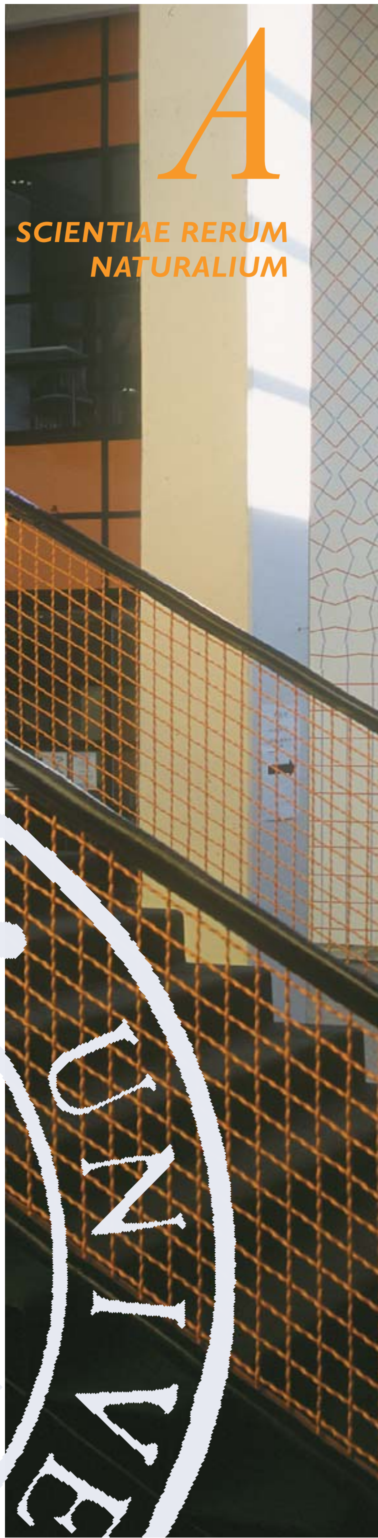
OULU 2010
A 548

ACTA UNIVERSITATIS OULUENSIS



Jukka Kemppainen

BEHAVIOUR OF THE BOUNDARY POTENTIALS AND BOUNDARY INTEGRAL SOLUTION OF THE TIME FRACTIONAL DIFFUSION EQUATION

FACULTY OF SCIENCE,
DEPARTMENT OF MATHEMATICAL SCIENCES,
UNIVERSITY OF OULU

ACTA UNIVERSITATIS OULUENSIS
A Scientiae Rerum Naturalium 548

JUKKA KEMPPAINEN

BEHAVIOUR OF THE BOUNDARY POTENTIALS AND BOUNDARY INTEGRAL SOLUTION OF THE TIME FRACTIONAL DIFFUSION EQUATION

Academic dissertation to be presented with the assent of the Faculty of Science of the University of Oulu for public defence in OP-sali (Auditorium L10), Linnanmaa, on 10 April 2010, at 12 noon

UNIVERSITY OF OULU, OULU 2010

Acta Univ. Oul. A 548, 2010

Supervised by
Professor Keijo Ruotsalainen

Reviewed by
Professor Martin Costabel
Professor Dorina Mitrea

ISBN 978-951-42-6131-2 (Paperback)
ISBN 978-951-42-6132-9 (PDF)
http://herkules.oulu.fi/isbn9789514261329/
ISSN 0355-3191 (Printed)
ISSN 1796-220X (Online)
http://herkules.oulu.fi/issn03553191/

Cover design
Raimo Ahonen

JUVENES PRINT
TAMPERE 2010

Kemppainen, Jukka, Behaviour of the boundary potentials and boundary integral solution of the time fractional diffusion equation
Faculty of Science, Department of Mathematical Sciences, University of Oulu, P.O.Box 3000, FI-90014 University of Oulu, Finland
Acta Univ. Oul. A 548, 2010
Oulu, Finland

Abstract

The dissertation considers the time fractional diffusion equation (TFDE) with the Dirichlet boundary condition in the sub-diffusion case, i.e. the order of the time derivative is $\alpha \in (0,1)$. In the thesis we have studied the solvability of TFDE by the method of layer potentials. We have shown that both the single layer potential and the double layer potential approaches lead to integral equations which are uniquely solvable.

The dissertation consists of four articles and a summary section. The first article presents the solution for the time fractional diffusion equation in terms of the single layer potential. In the second and third article we have studied the boundary behaviour of the layer potentials for TFDE. The fourth paper considers the spline collocation method to solve the boundary integral equation related to TFDE.

In the summary part we have proved that TFDE has a unique solution and the solution is given by the double layer potential when the lateral boundary of a bounded domain admits C^1 regularity. Also, we have proved that the solution depends continuously on the datum in the sense that a nontangential maximal function of the solution is norm bounded from above by the datum in $L^2(\Sigma_T)$. If the datum belongs to the space $H^{1,\alpha/2}(\Sigma_T)$, we have proved that the nontangential function of the gradient of the solution is norm bounded from above by the datum in $H^{1,\alpha/2}(\Sigma_T)$.

Keywords: boundary integral equation, double layer potential, single layer potential, spline collocation, time fractional diffusion

Acknowledgements

I wish to express my deepest gratitude to my supervisor, professor Keijo Ruotsalainen, for introducing to me this fascinating topic of my dissertation and for his constant support and rewarding collaboration and for his careful reading of my texts.

I am grateful to Professor Dorina Mitrea and Professor Martin Costabel for carefully reviewing the manuscript and giving valuable comments and suggestions.

I wish to express my thanks to professor emeritus Jukka Saranen for guiding me into "the world of boundary integral equations", Martti Hamina Ph.D for many valuable discussions and advice, and lecturer emerita Alli Huovinen for her enthusiasm for mathematics and offering me an opportunity to do tutorial work at the university in the first place which started my academic career (if there is any). I also thank professors Juha Kinnunen and Valery Serov. Attending their seminars and their advice have been a great help for me. Many others deserve thanks as well but there is no room to thank you all, you know who I mean.

The staff at both Mathematics Division and the Department of Mathematical Sciences of the University of Oulu deserve warm thanks for the working atmosphere that exists there. I am also indebted to them for financial support.

The English language was revised by Eva Braidwood Ph.D; to whom I express my thanks for her contribution.

Finally, I would like to thank my friends for many relaxing moments on the banks of some salmon rivers and elsewhere. My warmest thanks go to Anu and my parents for all their love, patience and encouragement.

Oulu, January 2010 Jukka Kemppainen

List of original articles

I Kemppainen J & Ruotsalainen K (2009) Boundary Integral Solution of the Time-Fractional Diffusion Equation. Integr Equ Oper Theory 64: 239–249.

II Kemppainen J (2009) Properties of the Single Layer Potential for the Time Fractional Diffusion Equation. Journal of Integral Equations and Applications. In Press.

III Kemppainen J (2010) Boundary Behaviour of the Layer Potentials for the Time Fractional Diffusion Equation in Lipschitz domains. Journal of Integral Equations and Applications. In Press.

IV Kemppainen J & Ruotsalainen K (2010) Boundary Element Collocation Method for Time-Fractional Diffusion Equations. In Constanda C & Pérez ME (eds) Integral Methods in Science and Engineering, Vol 2, Computational Methods. Birkhäuser Boston: 223–232.

Contents

1 Introduction

In the thesis we study the boundary integral solution of the time fractional diffusion equation (TFDE)

$$\begin{aligned} \partial_t^{\alpha}\Phi - \Delta\Phi &= 0, \text{ in } Q_T = \Omega \times (0,T), \\ \Phi &= g, \text{ on } \Sigma_T = \Gamma \times (0,T), \\ \Phi(x,0) &= 0, \ x \in \Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and ∂_t^{α} is the Caputo time derivative of the fractional order $0 < \alpha \leq 1$.

The concept of fractional differentiation and integration is nearly as old as the theory of calculus. Usually the fractional calculus is associated with the name Liouville, however, the history dates back to 1695 when Leibnitz wrote a letter to L'Hospital and discussed whether or not the meaning of derivatives with integer orders could be generalized to derivatives with non-integer orders. L'Hospital asked: "What if the order were $1/2$?" Leibnitz answered: "It would lead to a paradox from which one day useful consequences will be drawn." [22]

Recently, new applications have been found for the fractional derivatives in engineering, physics, finance and hydrology. In physics, fractional diffusion-type equations describe anomalous diffusion on heterogeneous media. They describe the non–Fickian transport phenomenon with long memory. What makes fractional diffusion–type equations useful on a wide range of applications is the strong relationship between these equations and fractional Brownian motion, the continuous–time random walk (CTRW) models, the Lévy stable probability distributions, and the generalized central limit theorem.

Our model problem is a special case of an anomalous diffusion. One way to characterize diffusion processes is the mean squared displacement (MSD). The evolution of MSD in standard diffusion is linear in time. However, many practical experiments indicate a power-law type evolution, i.e. MSD $\sim t^{\alpha}$ with $\alpha \neq 1$. We say that the diffusion process is anomalous if the evolution of MSD is not linear. For example, it has been observed that the stress $\sigma(t)$ in some complex viscoelastic materials decays like $t^{-\alpha}$ with $0 < \alpha < 1$. Therefore Giona et al. have proposed a (one-dimensional) TFDE model for the relaxation in complex viscoelastic materials [13].

From the mathematical point of view, however, the initial value problems have been considered mainly (see [25] and references therein). An expression for the fundamental solution of the corresponding Cauchy problem was constructed in [20] and [33] by means of a Fox H-function. In [20] it was proved that the classical solution of the Cauchy problem, with exponentially bounded initial data that is locally Hölder continuous, is unique. In [25] the uniqueness of the classical solution of nonhomogeneous TFDE, with the source term $F \in \mathscr{C}(Q_T)$, the boundary data $g \in \mathscr{C}(\Sigma_T)$, nonzero initial data $u_0 \in \mathscr{C}(\overline{\Omega})$ and Δ replaced by a linear elliptic second order differential operator $L = \sum_{k=1}^{n} \left(p(x)\partial_{x_k}^2 + \partial_{x_k} p(x)\partial_{x_k}\right) - q(x)$, was proved. According to the author's knowledge, only the existence and uniqueness of the classical solution of TFDE in bounded domains have been considered until now. The weak or the variational formulation of the problem is a new approach in this setting. As far as I am aware, TFDE has not so far been investigated in the classical function spaces, such as $L^2(\Sigma_T)$ or in the scale of anisotropic Sobolev spaces.

The purpose of the thesis is to determine the solvability theory for TFDE in the aforementioned classical function spaces by using the method of layer potentials. Since this approach is new in this setting, we first had to investigate the behaviour of the layer potentials in order to determine an integral equation corresponding TFDE. Then we were able to derive the corresponding boundary integral equation and study its solvability. We have shown that the use of both the single layer potential and the double layer potential leads to an integral equation, which is uniquely solvable. Then the solution of the original problem may be expressed in terms of the boundary data.

We have made different regularity assumptions on the domain in different papers. At the moment, we have been able to deduce the unique solvability for TFDE only in $\mathscr{C}^1$ domains. In the future, it would be useful to generalize our results for Lipschitz domains.

2 Summary of the original articles

The thesis consists of four articles. The first article presents the solution for TFDE in terms of the single layer potential. Using the double layer ansatz for the Dirichlet problem is more common. The reason for this is that the single layer approach leads to an integral equation of the first kind, which is usually ill-posed. Indeed, it follows from the proof of Theorem 1 in II that the single layer potential is an integral operator with a weakly singular kernel. Hence, it is a compact operator and therefore its inverse cannot be bounded, e.g., in $L^p(\Sigma_T)$ with a Lipschitz smooth lateral boundary Γ. However, note that the well-posedness of a problem depends not only on the operator but on the spaces and the norms of the spaces [21, Chapter 15].

In paper I we have shown that the single layer approach for the Dirichlet problem of TFDE leads to a well-posed integral equation in a scale of anisotropic Sobolev spaces. Hence, the single layer approach makes sense for solving TFDE. A drawback is that there seems to be no clear physical meaning for the quantity σ in the boundary integral equation $V\sigma = g$. If we would like to have a direct physical meaning of the solution of the integral equation, we should construct it with the direct method. Instead, we have used the indirect method because it is easier in the sense that the right hand side g is the boundary data. Still, there is one drawback. The solvability of the integral equation is achieved in a space that is weaker than $L^2(\Sigma_T)$. One could naturally ask whether is it possible to obtain a corresponding integral equation which is solvable in a smoother space such as $L^2(\Sigma_T)$ or $H^{1,\frac{\alpha}{2}}(\Sigma_T)$. It turns out that this is indeed the case when we use the double layer approach for solving TFDE.

In the second and third article we have studied the boundary behaviour of the layer potentials for TFDE. Since our technique is based on the well-known technique used in elliptic and parabolic case, let us recall some landmarks on the history. The jump relation of the gradient of the single layer potential for the parabolic equation in nondivergence form was shown in [29] when the lateral boundary has Lyapunov regularity. Then solvability for the Dirichlet and Neumann problems follows from the classical theory, found e.g. in [12] and [21]. In [11] Fabes and Riviére were able to construct solutions to the Dirichlet and Neumann problems for the heat equation in $\mathscr{C}^1$-cylinders by the method of layer potentials. They used the Fourier transform in the time variable to reduce the study to an elliptic problem and then using the results in [10]. Their proof of the invertibility of the boundary integral operator $-\frac{1}{2}I+J$ relies on their result which states that the norm of J is small with T as an operator on $L^2(\Sigma_T)$. Then a well-known iteration argument [11, proof of Theorem 3.3] gives the inverse on $L^2(\Sigma_T)$ for arbitrary T as long as $T<\infty$. The same conclusion is also true in the space $H^{1,\frac{1}{2}}(\Sigma_T)$.

On Lipschitz domains, however, the norm of J remains bounded away from zero as $T \to 0$. In [5] R. M. Brown managed to solve the Dirichlet and Neumann problems for the heat equation in Lipschitz cylinders by using the technique developed in [35] and [36].

We started to study the behaviour of the single layer potential near the boundary of a $\mathscr{C}^{1+\lambda}$, $0<\lambda<1$, smooth bounded domain. We have proved the continuity of the single layer potential across the boundary that the normal derivative of the single layer potential satisfies the usual jump relation known for the heat equation and that the single layer potential is Hölder continuous. The results are presented in paper II.

Paper III is based on the results in [11] and [18]. In the proofs we use the same technique as in [11]. We also need to know the behaviour of the Fox H-functions which can be found from [4], [18] and [19]. Our main result in paper III states that the maximal function $\widetilde{D\psi}$, associated with the double layer potential D is bounded in $L^2(\Sigma_T)$ and that D has the nontangential limit $-\frac{1}{2}I+J$ almost everywhere.

The fourth paper considers the spline collocation method to solve the boundary integral equation related to TFDE. In this paper we study TFDE in a smooth, bounded domain $\Omega \subset \mathbb{R}^2$. We use the single layer approach which leads to a boundary integral equation of the first kind. The corresponding integral equation is solved by the spline collocation method. As trial functions we use the tensor products of continuous piecewise linear splines and the collocation points are the nodal points. In this paper we show that the spline collocation method is stable in a suitable anisotropic Sobolev space and it furnishes quasi-optimal error estimates. The paper is mainly based on the work [9], [15] and [16].

2.1 Notation and preliminaries

In this section we introduce the notation and some basic concepts needed in this work.

The fractional Caputo derivative appearing in TFDE is defined as follows. Let us first introduce the fractional integral operator J^α of order $\alpha>0$,

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad t>0.$$

The Caputo derivative ∂_t^α of order $0 < \alpha < 1$ is defined as $\partial_t^\alpha f(t) = J^{1-\alpha} D^1 f(t)$, where D^1 denotes the usual derivative operator. It can be written in a form

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau.$$

The fractional integral operators have the semigroup property $J^\alpha J^\beta = J^{\alpha+\beta}$ which is needed in the proof of positive defineteness of the Caputo derivative in paper I [32, Formula 2.21].

The analysis of the Fox H-functions play an important role in this work. We shall not represent the general definition. The Fox H-functions needed in this work are defined as follows. Let m, p, q be nonnegative integers such that $m \leq q$. For $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \alpha_j \in \mathbb{R}_+ = (0, \infty)$ with $i = 1, \ldots, p$ and $j = 1, \ldots, q$, the Fox H-function $H_{p,q}^{m,0}(z)$ is defined as the Mellin-Barnes integal

$$H_{p,q}^{m,0}(z) = H_{p,q}^{m,0}\left(z \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix}\right) = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}} \mathscr{H}_{p,q}^{m,0}(s) z^{-s} \mathrm{d}s \tag{2}$$

with

$$\begin{aligned} \mathscr{H}_{p,q}^{m,0}(s) =& \mathscr{H}_{p,q}^{m,0}\left(s \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix}\right) \\ =& \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j)}. \end{aligned} \tag{3}$$

If an empty product occurs in (3), it is defined to be one. The contour $\mathscr{C}$ in (2) is a loop starting at the point $-\infty + \mathrm{i}\varphi_1$ and terminating at the point $-\infty + \mathrm{i}\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < \infty$ such that the poles of the Gamma function $\Gamma(b_j + \beta_j s)$ lie on the left of $\mathscr{C}$. For the properties of the Fox H-functions we refer to [19].

We recall the definition of the regularity of the domain and introduce the spaces we have used [23], [24], [38].

First, we recall the definition of the Lipschitz domain.

Definition 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. We say that $\Omega \in \mathrm{Lip}$ or $\Gamma \in \mathrm{Lip}$ if for each point $x \in \Gamma$ there exists a neighborhood U_x, a rectangular coordinate system $(y', y_n) := (y_1, \ldots, y_n)$ and a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with the following properties.*

1. $\varphi \in \mathrm{Lip}(\mathbb{R}^{n-1})$;
2. $U_x \cap \Omega = \{(y', y_n) : y_n > \varphi(y')\} \cap U_x$;

3. $U_x \cap \Gamma = \{(y', y_n) : y_n = \varphi(y')\} \cap U_x$.

Similarly, we say that $\Omega \in \mathscr{C}^1$ ($\Gamma \in \mathscr{C}^1$) or $\Omega \in \mathscr{C}^{1+\lambda}$ ($\Gamma \in \mathscr{C}^{1+\lambda}$), $0 < \lambda < 1$, if the function φ appearing in the definition has $\mathscr{C}^1$ or $\mathscr{C}^{1+\lambda}$ smoothness.

For given $r, s \geq 0$ the anisotropic Sobolev space $H^{r,s}(Q_T)$ is defined by

$$H^{r,s}(Q_T) = H^0(0,T;H^r(\Omega)) \cap H^s(0,T;H^0(\Omega)), \tag{4}$$

which is a Hilbert space with the norm

$$\|u\|^2_{H^{r,s}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{H^r(\Omega)} \mathrm{d}t + \|u\|^2_{H^s(0,T;H^0(\Omega))}. \tag{5}$$

If Ω and $(0,T)$ are replaced by $\mathbb{R}^n$ and $\mathbb{R}$, we denote $H^r := H^r(\mathbb{R}^n)$ and $H^{r,s} := H^{r,s}(\mathbb{R}^n \times \mathbb{R})$. The norm in H^r may be defined via Fourier transform,

$$\|u\|^2_{H^r} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^r |\hat{u}(\xi)|^2 \mathrm{d}\xi,$$

where $\hat{u}(\xi) = (\mathscr{F}u)(\xi)$ denotes the Fourier transform of $u(x)$, and the norm in $H^{r,s}$ is given by

$$\|u\|^2_{H^{r,s}} = \int_{\mathbb{R}^{n+1}} \left\{(1 + |\xi|^2)^r + (1 + |\eta|^2)^s\right\} |\hat{u}(\xi,\eta)|^2 \mathrm{d}\xi \mathrm{d}\eta, \tag{6}$$

where $\hat{u}(\xi,\eta) = (\mathscr{F}u)(\xi,\eta)$ denotes the Fourier transform of $u(x,t)$.

Since our domains possess at least Lipschitz regularity, the norm $\|\cdot\|_{H^r(\Omega)}$ may be defined by the usual infimum norm [38, Theorems 5.3 and 5.4]

$$\|u\|_{H^r(\Omega)} = \inf\{\|U\|_{H^r} : u = U|_\Omega, U \in H^r\}.$$

We also need the Sobolev spaces defined on the boundary. Let $\{U_j\}_{j=1}^M$ be an open cover for $\Gamma \in$ Lip and $\{\beta_j\}_{j=1}^M$ a subordinate partition of unity. It follows from the definition of the Lipschitz domain that there exists a Lipschitz continuous diffeomorphism ϕ_j from U_j onto the central plane $x_n = 0$ of the cube $W = \{x \in \mathbb{R}^n : |x_i| < 1, i = 1, \dots, n\}$ [38, Theorem 2.5]. Since β_j has compact support in U_j, $(\beta_j u) \circ \phi_j^{-1}$ has compact support in $W \cap \{x_n = 0\}$ for any function u defined on Γ. Therefore we may extend $(\beta_j u) \circ \phi_j^{-1}$ by zero out of $W \cap \{x_n = 0\}$. The Sobolev space $H^r(\Gamma)$ consists of all functions u for which $(\beta_j u) \circ \phi_j^{-1} \in H^r(\mathbb{R}^{n-1})$ holds for all $j = 1, \dots, M$. The norm in the space $H^r(\Gamma)$ is given by

$$\|u\|^2_{H^r(\Gamma)} = \sum_{j=1}^M \|(\beta_j u) \circ \phi_j^{-1}\|^2_{H^r(\mathbb{R}^{n-1})}. \tag{7}$$

The anisotropic spaces $H^{r,s}(\Sigma_T)$ are defined by setting

$$H^{r,s}(\Sigma_T) = H^0(0,T;H^r(\Gamma)) \cap H^s(0,T;H^0(\Gamma)) \tag{8}$$

and the norm is given by (5) with Ω replaced by Γ.

The anisotropic Sobolev spaces $H^{-r,-s}(Q_T)$ are defined by duality as follows. Let $H_0^{r,s}(Q_T)$ be the closure of $\mathscr{C}_0^\infty(Q_T)$ in $H^{r,s}(Q_T)$. We define $H^{-r,-s}(Q_T) = (H_0^{r,s}(Q_T))'$. The spaces $H^{-r,-s}(\Sigma_T)$ are defined analogously.

Furthermore, we need the spaces which take the zero initial condition into account. We denote $\Sigma := \Gamma \times \mathbb{R}$ and define the space $H^{r,s}(\Sigma)$ similarly as $H^{r,s}(\Sigma_T)$. Then the space $\widetilde{H}^{r,s}(\Sigma_T)$ is the space of restrictions of those functions in $H^{r,s}(\Sigma)$ that vanish on the negative time axis, i.e.

$$\widetilde{H}^{r,s}(\Sigma_T) = \{u = U|_{\Sigma_T} : U \in H^{r,s}(\Sigma), U(\cdot,t) = 0, t < 0\}. \tag{9}$$

In paper IV we need Sobolev spaces consisting of distributions which are periodic in the spatial variable. These spaces are defined in IV and in order to avoid a confusion with the notations used above, we refer to article IV for the details.

Let us now turn back to our problem. The boundary relation is interpreted in paper I in the sense of the trace mapping. This means that the mapping $\gamma : u \mapsto u|_{\Sigma_T}$, originally defined locally by $\gamma u(x,t) = u(x', \varphi(x'), t)$ as a mapping $\mathscr{C}_0^\infty(\overline{\Omega}) \to \mathscr{C}_0^\infty(\Gamma)$, is extended as the mapping acting on the anisotropic Sobolev spaces, $\gamma : H^{1,\frac{\alpha}{2}}(Q_T) \to H^{\frac{1}{2},\frac{\alpha}{4}}(\Sigma_T)$.

In papers II and III the boundary relation is interpreted in the sense of nontangential limits

$$\lim_{\substack{x \to x_0 \\ x \in K}} u(x,t) = g(x_0,t), \quad (x_0,t) \in \Sigma_T.$$

The set K is defined by

$$K = \{(x,t) \in Q_T : x \in \Omega, |x - x_0| < \delta, \langle x_0 - x, n(x_0)\rangle > \beta |x - x_0|\},$$

where $n(x_0)$ is the outward unit normal at x_0 and δ is a positive constant depending on Ω and $0 < \beta < 1$ (here we assume that the domain is at least $\mathscr{C}^1$).

We recall the definitions of the boundary potentials. The single layer potential is defined by

$$(S\varphi)(x,t) = \int_0^t \int_\Gamma G(x-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau, \quad (x,t) \in (\mathbb{R}^n \setminus \Gamma) \times \mathbb{R}_+, \tag{10}$$

where G is the fundamental solution of the fractional diffusion equation. It is known that

$$G(x,t) = \begin{cases} \pi^{-n/2} t^{\alpha-1} |x|^{-n} H^{20}_{12}\left(\frac{1}{4}|x|^2 t^{-\alpha} \Big|^{(\alpha,\alpha)}_{(n/2,1),(1,1)}\right) & , \quad x \in \mathbb{R}^n, t > 0, \\ 0 & , \quad x \in \mathbb{R}^n, t < 0, \end{cases} \tag{11}$$

where H is the Fox H-function (see [19],[28],[30]). The double layer potential is defined by

$$(D\varphi)(x,t) = \int_0^t \int_\Gamma \partial_{n(y)} G(x-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau, \quad (x,t) \in (\mathbb{R}^n \setminus \Gamma) \times \mathbb{R}_+. \quad (12)$$

As it was mentioned before, the single layer potential is continuous across the lateral boundary. In paper III we have shown that the double layer potential has the nontangential limiting value $-\frac{1}{2}\varphi + J\varphi$ a.e., where the operator J is defined by

$$(J\varphi)(x,t) = \lim_{\varepsilon \downarrow 0} \int_0^{t-\varepsilon} \int_\Gamma \partial_{n(y)} G(x-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau. \quad (13)$$

Therefore, solving TFDE by the single layer potential leads us to the boundary integral equation $V\varphi = g$, where $V\varphi$ denotes the trace of $S\varphi$ on the boundary. By using the double layer approach we end up in the boundary integral equation $(-\frac{1}{2}I + J)\varphi = g$.

2.2 Solvability of TFDE

In paper I we have proved the solvability of TFDE when the datum belongs to $\widetilde{H}^{\frac{1}{2},\frac{\alpha}{4}}(\Sigma_T)$. However, the solvability of the corresponding integral equation was achieved in a space $\widetilde{H}^{-\frac{1}{2},-\frac{\alpha}{4}}(\Sigma_T)$ which is weaker than $L^2(\Sigma_T)$. Now we concentrate on the case where the datum belongs to $L^2(\Sigma_T)$. In addition, we use the double layer potential ansatz in contrast to the results in paper I where we used the single layer approach.

Combining the results in paper III and [11] implies that instead of TFDE we may consider the corresponding boundary integral equation of the second kind, $(-\frac{1}{2}I + J)\varphi = g$, where g is the Dirichlet boundary value. Although we considered the boundary behaviour of the layer potentials on Lipschitz domains in paper III, we are not able to prove existence and uniqueness of the solution in general Lipschitz domains at the moment. Therefore, in what follows, we assume that Ω is a bounded domain with $\mathscr{C}^1$ boundary.

We start with showing that the operator $-\frac{1}{2}I + J$ is invertible on $L^2(\Sigma_T)$. The proof follows from the proofs of Theorem 1 in paper III and [11, Theorem 1.3]. Indeed, by our assumption on Ω, the boundary Γ may be covered with finitely many neighborhoods $U_{x^{(l)}}$ and we may assume that the neighborhoods are balls, i.e. $U_{x^{(l)}} = B(x^{(l)}, r_l)$ for some $r_l > 0$. It is enough to consider one of these balls, say $B_l := B(x^{(l)}, r_l)$, and show that

$$(J_l\psi)(x,t) = \lim_{\varepsilon \to 0+} \int_0^{t-\varepsilon} \int_{\Gamma \cap B_l} \partial_{n(y)} G(x-y,t-\tau)\psi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau$$

is a continuous operator in $L^2(\Sigma_T)$ with the properties

1. for all a, $0 < a \leq T$, $J_l \chi_{(a,\infty)} = \chi_{(a,\infty)} J_l \chi_{(a,\infty)}$, where $\chi_{(a,\infty)}$ is the characteristic function of (a,∞);
2. if $(a,b) \subset (0,T)$, $\|J_l(\chi_{(a,b)}\psi)\|_{L^2(\Gamma\times(a,b))} \leq \omega(b-a)\|\psi\|_{L^2(\Gamma\times(a,b))}$, where $\omega(\delta) \to 0$ as $\delta \to 0+$.

The first condition is often called *the Volterra property*. The operator satisfying these properties is said to belong to the class $\mathscr{J}(\Sigma_T)$.

Then we divide the operator J_l into three parts; J_{l1}, J_{l2} and J_{l3} and all the arguments are the same as in [11], except in our case we have, using the same notations as in [11],

$$\|J_{l3}(\psi\chi_{(a,b)})\|_{L^2(\Gamma\cap B_l\times(a,b))} \leq C_k(b-a)^{\alpha/2}\|\psi\|_{L^2(\Gamma\cap B_l\times(a,b))}, \tag{14}$$

where

$$(J_{l3}\psi)(x,t) = \lim_{\varepsilon\to 0+}\int_0^{t-\varepsilon}\int_{B_l\cap\Gamma} K_3(x,y,t,\tau)\psi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau$$

with the kernel

$$K_3(x,y,t,\tau) = \langle\psi_k(x)-\psi_k(y), n_k(y)\rangle\widetilde{G}(x-y,t-\tau).$$

Here we have approximated the domain by a $\mathscr{C}^2$ domain, ψ_k denotes the graph of the local parametric representation and n_k denotes the normal vector of the corresponding domain. This regularization is needed because of the strong singularity of the kernel of the operator J_l. Recall that $\widetilde{G}$ was defined in paper III by

$$\partial_{n(y)}G(x-y,t-\tau) = \langle x-y, n(y)\rangle\widetilde{G}(x-y,t-\tau).$$

To prove (14), we use the asymptotic behaviour of the kernel $K_3(x,y,t,\tau)$. Denote $z = \frac{1}{4}(t-\tau)^{-\alpha}|x-y|^2$. It follows from the asymptotic behaviour of the Fox H-functions that, for example, in the case $n=3$ (the other cases of n's may be treated similarly)

$$K_3(x,y,t,\tau) \leq C_k\begin{cases}(t-\tau)^{-\frac{3\alpha}{2}-1}|x-y|^2\exp\{-\sigma(t-\tau)^{-\frac{\alpha}{2-\alpha}}|x-y|^{\frac{2}{2-\alpha}}\} & \text{, if } z\geq 1,\\ (t-\tau)^{-\alpha-1}|x-y| & \text{, if } z<1,\end{cases}$$

where we used the inequality $|\langle\psi_k(x)-\psi_k(y), n_k(y)\rangle| \leq C_k|x-y|^2$ and chose $\lambda = 0$ in Lemma 3 of paper II. Now, we choose an integrable majorant

$$\varphi(x) = \begin{cases}C_1|x|^2\exp\{-\sigma|x|^{\frac{2}{2-\alpha}}\} & \text{, if } |x|\geq 1,\\ C_2|x| & \text{, if } |x|<1\end{cases}$$

and use [34, Theorem 2(a) pp. 62-63]. Then we have for $t \in (a,b)$

$$|J_{l3}(\psi\chi_{(a,b)})(x,t)| \leq C_k \int_a^t (t-\tau)^{\frac{\alpha}{2}-1} M_\Gamma(\psi\chi_{(a,b)}(x,\tau))\mathrm{d}\tau,$$

where M_Γ denotes the Hardy-Littlewood maximal operator on Γ, i.e.,

$$M_\Gamma(g)(x) = \sup_{r>0} r^{-(n-1)} \int_{|x-y|<r} |g(y)|\mathrm{d}\sigma(y).$$

From this estimate it is easy to see that

$$\|J_{l3}(\psi\chi_{(a,b)})\|_{L^2(\Gamma\cap B_l\times(a,b))} \leq C_k(b-a)^{\alpha/2}\|\psi\|_{L^2(\Gamma\cap B_l\times(0,T))}.$$

Therefore we see that the operator J is of Volterra type and the norm of J is small with T (belongs to the class $\mathscr{J}(\Sigma_T)$). Hence

Theorem 2.2.1. *The operator $-\frac{1}{2}I+J$ is invertible on $L^2(\Sigma_T)$ for each $0<T<\infty$.*

Following the same lines as in [11] we may prove the following existence and uniqueness result.

Theorem 2.2.2. *Let $g \in L^2(\Sigma_T)$ in TFDE. Then TFDE admits a unique solution u and the solution is given by*

$$u(x,t) = D((-\frac{1}{2}I+J)^{-1}g)(x,t).$$

Moreover, the solution depends continuously on the data in the sense that

$$\|N(u)\|_{L^2(\Sigma_T)} \leq C\|g\|_{L^2(\Sigma_T)}.$$

Above N denotes a nontangential maximal function, which is defined by

$$N(u)(x,t) = \sup\{|u(y,t)| : y \in \Omega\cap B(x,\delta) \text{ such that } \langle y-x, n(x)\rangle > \beta|x-y|\}$$

for some $0<\beta<1$ and a positive constant δ depending on Ω and β. Note that, when $\Gamma \in \mathscr{C}^1$, such constants β and δ always exist.

2.3 Regularity in the initial–Dirichlet problem

In this section we study the regularity of the solution of the initial–Dirichlet problem when the lateral datum possesses some regularity. We will prove that if the lateral datum belongs to $H^{1,\frac{\alpha}{2}}(\Sigma_T)$ then also the nontangential maximal function of the gradient of the solution belongs to $L^2(\Sigma_T)$. Note that we cannot expect this to be true in general if the datum belongs to $L^2(\Sigma_T)$.

We first study the mapping properties of J. We have the following result. See [11, Theorem 3.1].

Theorem 2.3.1. *The operator J maps $H^{1,\frac{\alpha}{2}}(\Sigma_T)$ into itself.*

Proof. It is enough to show that the Euclidian operator

$$(\widetilde{J_0}\psi)(x,t) = \int_0^t \int_{\mathbb{R}^{n-1}} \langle \nabla G((x-y,\varphi(x)-\varphi(y)),t-\tau),(\nabla\varphi(y),-1)\rangle \psi(y,\tau)\mathrm{d}y\mathrm{d}\tau$$

maps $H^{1,\frac{\alpha}{2}}(\mathbb{R}^{n-1}_T)$ into itself when $\varphi \in \mathscr{C}_0^1(\mathbb{R}^{n-1})$. Since we already have established the continuity in $L^p(\Sigma_T)$ in Theorem 1 of paper III, we need to investigate the derivatives. We begin with the spatial derivative. Let $K(x,y,t-\tau)$ denote the kernel of $\widetilde{J_0}$. We write $\partial_{x_j}\widetilde{J_0}\psi(x,t)$ as follows

$$\begin{aligned}\partial_{x_j}\widetilde{J_0}\psi(x,t) = &\int_0^t \int_{\mathbb{R}^{n-1}} \partial_{x_j}K(x,y,t-\tau)(\psi(y,\tau)-\psi(x,\tau))\mathrm{d}y\mathrm{d}\tau \\ &+\int_0^t \psi(x,\tau)\partial_{x_j}\big(\int_{\mathbb{R}^{n-1}} K(x,y,t-\tau)\mathrm{d}y\big)\mathrm{d}\tau = A\psi(x,t)+B\psi(x,t).\end{aligned}$$

The Laplace transform of the function $B\psi$ is

$$\begin{aligned}&(\mathscr{L}(B\psi))(x,\mathrm{i}\eta)\\ =&(\mathscr{L}\psi)(x,\mathrm{i}\eta)\partial_{x_j}\int_{\mathbb{R}^{n-1}} K(x,y)H(\eta(|x-y|^2+(\varphi(x)-\varphi(y))^2)^{1/\alpha})\mathrm{d}y\\ =&(\mathscr{L}\psi)(x,\mathrm{i}\eta)\{\int_{\mathbb{R}^{n-1}} \partial_{x_j}K(x,y)\big(H(\eta(|x-y|^2+(\varphi(x)-\varphi(y))^2)^{1/\alpha})-H(0)\big)\mathrm{d}y\\ &+\int_{\mathbb{R}^{n-1}} K(x,y)\partial_{x_j}H(\eta(|x-y|^2+(\varphi(x)-\varphi(y))^2)^{1/\alpha})\mathrm{d}y\}\\ =&m(x,\eta)(\mathscr{L}\psi)(x,\mathrm{i}\eta),\end{aligned}$$

where

$$K(x,y) = \frac{\varphi(y)-\varphi(x)-\nabla\varphi(y)\cdot(y-x)}{(|x-y|^2+(\varphi(x)-\varphi(y))^2)^{\frac{n}{2}}}$$

and

$$H(\eta) = \int_0^\infty \exp(-\mathrm{i}\eta\tau)\tau^{\alpha-1}\{nH^{20}_{12}(\frac{1}{4}\tau^{-\alpha})+2H^{30}_{23}(\frac{1}{4}\tau^{-\alpha})\}\mathrm{d}\tau.$$

We will prove that for each x, the function $|\eta|^{-\frac{\alpha}{2}}m(x,\eta)$ is a Fourier multiplier on $L^2(\mathbb{R})$. By the Parseval theorem we need to prove the uniform estimate

$$||\eta|^{-\frac{\alpha}{2}}m(x,\eta)| \le C.$$

For the proof we need explicit dependence on η of the symbol m. There seems to be no formula for $H(\eta)$. But, we may use the Mellin convolution property

$$\mathscr{K}(x) = \int_0^\infty \mathscr{K}_1(\frac{x}{t})\mathscr{K}_2(t)\frac{\mathrm{d}t}{t} \overset{\mathscr{M}}{\leftrightarrow} \mathscr{K}^*(s) = \mathscr{K}_1^*(s)\mathscr{K}_2^*(s), \tag{15}$$

where we have denoted $f^*(s) = \int_0^\infty f(\tau)\tau^{s-1}\mathrm{d}\tau$ the Mellin transform to determine a formula for H. Substituting $\tau = u^{-1}$ in the definition of H we get

$$H(\eta) = \int_0^\infty \exp(-\mathrm{i}\eta/u)u^{-\alpha}\{nH_{12}^{20}(\frac{1}{4}u^\alpha) + 2H_{23}^{30}(\frac{1}{4}u^\alpha)\}\frac{\mathrm{d}u}{u}. \tag{16}$$

This may be interpreted as a Mellin convolution. We investigate the real and imaginary parts separately. Since the asymptotic behaviour is the same, it is enough to deal with the real part. We may also assume that $\eta > 0$. We use the following formulae

$$\begin{aligned}
(\mathscr{M}\cos 2\sqrt{(\cdot)})(s) &= \sqrt{\pi}\frac{\Gamma(s)}{\Gamma(\frac{1}{2}-s)}, \\
(\mathscr{M}\psi(\cdot^2))(s) &= \frac{1}{2}(\mathscr{M}\psi)(\frac{s}{2}), \\
(\mathscr{M}\psi(\frac{\cdot}{2}))(s) &= 2^s(\mathscr{M}\psi)(s)
\end{aligned}$$

to determine $\mathscr{K}_1^*(s)$. Substituting $\frac{1}{4}u^\alpha = t$ in the second function $\mathscr{K}_2$ we have (up to a multiplicative constant)

$$\mathscr{K}_2^*(s) = n(\mathscr{M}H_{12}^{20}))(\frac{s-\alpha}{\alpha}) + 2(\mathscr{M}H_{23}^{30})(\frac{s-\alpha}{\alpha}).$$

Since $(\mathscr{M}H_{pq}^{mn})(s) = \mathscr{H}_{pq}^{mn}(s)$, we have

$$\begin{aligned}
\mathscr{K}_2^*(s) &= n\mathscr{H}_{12}^{20}\left[\frac{s-\alpha}{\alpha}\Big|\,{}^{(\alpha,\alpha)}_{(n/2,1),\ (1,1)}\right] + 2\mathscr{H}_{12}^{20}\left[\frac{s-\alpha}{\alpha}\Big|\,{}^{(\alpha,\alpha),\ \ (0,1)}_{(n/2,1),\ (1,1),\ (1,1)}\right] \\
&= n\frac{\Gamma(\frac{n}{2}-1+\frac{1}{\alpha}s)\Gamma(\frac{1}{\alpha}s)}{\Gamma(s)} + 2\frac{\Gamma(\frac{n}{2}-1+\frac{1}{\alpha}s)\Gamma(\frac{1}{\alpha}s)\Gamma(\frac{1}{\alpha}s)}{\Gamma(s)\Gamma(-1+\frac{1}{\alpha}s)}.
\end{aligned}$$

Therefore by the Mellin convolution property we get (up to a multiplicative constant)

$$\begin{aligned}
&(\mathscr{M}\mathrm{Re}H)(s) \\
=&n\mathscr{H}_{14}^{30}\left[s\Big|\,{}^{(0,1)}_{(\frac{n}{2}-1,\frac{1}{\alpha}),\ (0,\frac{1}{\alpha}),\ (0,\frac{1}{2});\ (\frac{1}{2},\frac{1}{2})}\right] + 2\mathscr{H}_{25}^{40}\left[s\Big|\,{}^{(0,1),\ \ (-1,\frac{1}{\alpha})}_{(\frac{n}{2}-1,\frac{1}{\alpha}),\ (0,\frac{1}{\alpha}),\ (0,\frac{1}{\alpha}),\ (0,\frac{1}{2});\ (\frac{1}{2},\frac{1}{2})}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&(\mathrm{Re}H)(\eta) \\
=&nH_{14}^{30}\left[\eta\Big|\,{}^{(0,1),}_{(\frac{n}{2}-1,\frac{1}{\alpha}),\ (0,\frac{1}{\alpha}),\ (0,\frac{1}{2});\ (\frac{1}{2},\frac{1}{2})}\right] + 2H_{25}^{40}\left[\eta\Big|\,{}^{(0,1),\ \ (-1,\frac{1}{\alpha})}_{(\frac{n}{2}-1,\frac{1}{\alpha}),\ (0,\frac{1}{\alpha}),\ (0,\frac{1}{\alpha}),\ (0,\frac{1}{2});\ (\frac{1}{2},\frac{1}{2})}\right].
\end{aligned}$$

We start with the first integral in the definition of m. We split the integration into two parts depending on whether $\rho := \rho(x,y) = \eta(|x-y|^2 + (\varphi(x) - \varphi(y))^2)^{1/\alpha}$ is less than 1 or greater than 1. In the first part, i.e. $\rho < 1$, we use the fact that H is Hölder-continuous with exponent $0 < \beta < \alpha$. Because $|\partial_{x_j} K(x,y)| \le C|x-y|^{-n}$, we get for the first term of the real part an estimate

$$\begin{aligned} &\mathrm{Re}\Big\{ \int_{\rho<1} \partial_{x_j} K(x,y)\big(H(\rho) - H(0)\big)\mathrm{d}y \Big\} \\ \le & C\eta^\beta \int_{|x-y|<\eta^{-\alpha/2}} |x-y|^{-n}(|x-y|^2 + (\varphi(x) - \varphi(y))^2)^{\frac{\beta}{\alpha}} \mathrm{d}y \\ \le & C\eta^\beta \int_0^{\eta^{-\alpha/2}} r^{\frac{2\beta}{\alpha}-2} \mathrm{d}r \le C\eta^{\alpha/2}, \end{aligned}$$

where we have chosen $\alpha/2 < \beta < \alpha$ to guarantee the convergence of the integral. In the second part we may estimate H by constant and we get a similar bound.

In the second integral we consider, for example, the case $n = 3$. The other cases of n's may be treated similarly. If $\rho \le 1$, we use the asymptotic behaviour and logarithmic power series representation of the Fox H-functions [19, Theorem 1.5, p. 8] to get

$$\mathrm{Re}\Big\{ \int_{\rho<1} K(x,y)\partial_{x_j} H(\rho)\mathrm{d}y \Big\} = \int_{\rho<1} K(x,y)\partial_{x_j}(c_0 + c_1\rho^{\frac{\alpha}{2}} + c_2\rho^\alpha + o(\rho^\alpha))\mathrm{d}y.$$

However, an easy calculation shows that the residue for $(\mathscr{M}\mathrm{Re}H)(s)\rho^{-s}$ at $s = -\frac{\alpha}{2}$ is zero. Therefore $c_1 = 0$ and the second lowest order term in the series appearing on the right hand side is ρ^α. We stress that this better behaviour is needed in the following estimates. After differentiation we obtain

$$\mathrm{Re}\Big\{ \int_{\rho<1} K(x,y)\partial_{x_j} H(\rho)\mathrm{d}y \Big\} \le C\eta^\alpha \int_{|x-y|<\eta^{-\alpha/2}} |x-y|^{2-n}\mathrm{d}y \le C\eta^{\alpha/2}.$$

On the other hand, if $\rho > 1$, we may estimate $|\frac{\mathrm{d}}{\mathrm{d}\rho}H(\rho)|$ from above by a constant, from which it follows

$$\begin{aligned} &\mathrm{Re}\Big\{ \int_{\rho>1} K(x,y)\partial_{x_j} H(\rho)\mathrm{d}y \Big\} \\ \le & C\eta \int_{|x-y|>\eta^{-\alpha/2}} |x-y|^{1-n}(|x-y|^2 + (\varphi(x) - \varphi(y))^2)^{\frac{1}{\alpha}-\frac{1}{2}}\mathrm{d}y \le C\eta^{\alpha/2}. \end{aligned}$$

This finally finishes the proof of the uniform boundedness of $||\eta|^{-\alpha/2}m(x,\eta)|$.

Since $|\tau|^{-\alpha/2}m(x,\tau)$ is a Fourier multiplier on $L^2(\mathbb{R})$, we have

$$\begin{aligned} \|B\psi\|_{L^2(\mathbb{R}^{n-1}_T)} &\le C\|\mathscr{F}_\tau^{-1}(|\tau|^{\alpha/2}\mathscr{F}_t(\psi)(x,\tau))\|_{L^2(\mathbb{R}^n)} \\ &\le C\|\psi\|_{H^{1,\frac{\alpha}{2}}(\Sigma_T)}, \end{aligned}$$

where, for simplicity, we have denoted the extension of ψ in $\widetilde{H}^{1,\frac{\alpha}{2}}(\mathbb{R}^{n-1}\times\mathbb{R})$ by ψ.

For the operator

$$(A\psi)(x,t)=\int_0^t\int_{\mathbb{R}^{n-1}}\partial_{x_j}K(x-y,t-\tau)(\psi(y,\tau)-\psi(x,\tau))\mathrm{d}y\mathrm{d}\tau$$

we proceed as in [11, proof of Theorem 3.1]. To start with, we compute the Laplace transform of $A\psi$ in the time variable. We get

$$\mathscr{L}(A\psi)(x,\mathrm{i}\eta)=\int_{\mathbb{R}^{n-1}}\partial_{x_j}\{K(x,z)H(\rho)\}(g(y)-g(x))\mathrm{d}y,$$

where we have, for simplicity, denoted $g(x):=(\mathscr{L}\psi(x,\cdot))(\mathrm{i}\eta)$. Now, choose a cut-off function $\chi\in\mathscr{C}_0^\infty(\mathbb{R})$ such that $\chi(r)\equiv 1$ for $|r|\leq 1$ and $\chi(r)\equiv 0$ for $|r|>2$. We split the integral into three parts as follows

$$\begin{aligned}\mathscr{L}(A\psi)(x,\mathrm{i}\eta)=&\int_{\mathbb{R}^{n-1}}\partial_{x_j}[K(x,y)(H(\rho)-H(0)\chi(\rho)](g(y)-g(x))\mathrm{d}y\\&+H(0)\int_{\mathbb{R}^{n-1}}K(x,y)\partial_{x_j}(\chi(\rho))(g(y)-g(x))\mathrm{d}y\\&+H(0)\int_{\mathbb{R}^{n-1}}\partial_{x_j}K(x,y)\chi(\rho)(g(y)-g(x))\mathrm{d}y\\=&I_1+I_2+I_3.\end{aligned}$$

The last integral I_3 can be estimated as follows

$$I_3\leq\Big|\int_{\mathbb{R}^{n-1}}\partial_{x_j}K(x,y)(\chi(\rho)-1)(g(y)-g((x))\mathrm{d}y\Big|+\Big|\int_{\mathbb{R}^{n-1}}\partial_{x_j}K(x,y)(g(y)-g(x))\mathrm{d}y\Big|.$$

The second integral above is bounded in $L^2(\mathbb{R}^{n-1})$ by [10, Theorem 1.5]. The first integral above is divided into two parts depending on whether $\rho>2$ or $1<\rho\leq 2$ (note that the kernel is zero for $\rho\leq 1$). If $\rho>2$, then $1-\chi(\rho)=-1$ and we obtain an upper bound

$$\sup_{\varepsilon>0}\Big|\int_{|x-y|>\varepsilon}\partial_{x_j}K(x,y)(g(y)-g(x))\mathrm{d}y\Big|,$$

which is bounded in $L^2(\mathbb{R}^{n-1})$. If $1<\rho\leq 2$, we use the following inequality, found, e.g., in [14],

$$|u(x)-u(y)|\leq C|x-y|(M(\nabla u)(x)+M(\nabla u)(y))\quad\text{a.e.,}\tag{17}$$

which is valid for $u \in H^1(\mathbb{R}^{n-1})$. Using this result we have an upper bound

$$
\begin{aligned}
&C\int_{\frac{c_1}{\eta^{\alpha/2}}<|x-y|\le\frac{c_2}{\eta^{\alpha/2}}} |x-y|^{1-n}(M(\nabla g)(x)+M(\nabla g)(y))\mathrm{d}y\\
\le& CM(\nabla g)(x)\int_{\frac{c_1}{\eta^{\alpha/2}}<|x-y|\le\frac{c_2}{\eta^{\alpha/2}}} |x-y|^{1-n}\mathrm{d}y+\int_{\frac{c_1}{\eta^{\alpha/2}}<|x-y|\le\frac{c_2}{\eta^{\alpha/2}}} |x-y|^{1-n}M(\nabla g)(y)\mathrm{d}y\\
\le& CM(\nabla f(\cdot,\tau))(x)\log\frac{c_2}{c_1}+\frac{1}{|B(x,\frac{c_2}{\eta^{\alpha/2}})|}\int_{B(x,\frac{c_2}{\eta^{\alpha/2}})} M(\nabla g)(y)\mathrm{d}y
\end{aligned}
$$

for some constants c_1, c_2. The last upper bound is bounded in $L^2(\mathbb{R}^{n-1})$ due to the boundedness of the Hardy-Littlewood maximal function.

For the second integral I_2 we use (17) again. Note that now $\partial_{x_j}\chi(\rho)$ is nonzero only if $1<\rho<2$. Therefore we get,as above,

$$
\begin{aligned}
I_2 \le& C\eta M(\nabla g)(x)\int_{\frac{c_1}{\eta^{\alpha/2}}<r<\frac{c_2}{\eta^{\alpha/2}}} r^{\frac{2}{\alpha}-1}\mathrm{d}r+C\frac{1}{|B(x,\frac{c_2}{\eta^{\alpha/2}})|}\int_{B(x,\frac{c_2}{\eta^{\alpha/2}})} M(\nabla g)(y)\mathrm{d}y\\
\le& C(M(\nabla g)(x)+M(M(\nabla g))(x)),
\end{aligned}
$$

which is again bounded in $L^2(\mathbb{R}^{n-1})$.

We have to split the first integral I_1 into three parts depending on whether $\rho\le 1$ or $1<\rho\le 2$ or $\rho>2$. If $\rho\le 1$, then the integration is divided into two parts with kernels $\partial_{x_j}K(x,y)(H(\rho)-H(0)\chi(\rho))$ and $K(x,y)\partial_{x_j}H(\rho)$. The corresponding integrals are denoted by I_{11} and I_{12}. For the first part we use the fact that H is Hölder continuous with any $0<\beta<\alpha$ (see proof of Theorem 1 in paper III). Then, using (17), we get

$$
\begin{aligned}
I_{11} \le& C\int_{|x-y|\le\frac{c_2}{\eta^{\alpha/2}}} |x-y|^{-n}(\eta|x-y|^{2/\alpha})^{\beta}|g(y)-g(x)|\mathrm{d}y\\
\le& C\eta^{\beta}\int_{|x-y|\le\frac{c_2}{\eta^{\alpha/2}}} |x-y|^{1-n+\frac{2\beta}{\alpha}}(M(\nabla g)(x)+M(\nabla g)(y))\mathrm{d}y\\
\le& CM(\nabla g)(x)\eta^{\beta}\int_0^{c_2/\eta^{\alpha/2}} r^{\frac{2\beta}{\alpha}-1}\mathrm{d}r\\
&+C\eta^{\beta}\sum_{j=0}^{\infty}\int_{2^{-j-1}\frac{c_2}{\eta^{\alpha/2}}<|x-y|\le 2^{-j}\frac{c_2}{\eta^{\alpha/2}}} |x-y|^{1-n+\frac{2\beta}{\alpha}}M(\nabla g)(y)\mathrm{d}y\\
\le& CM(\nabla g)(x)+\sum_{j=0}^{\infty}2^{-\frac{2j\beta}{\alpha}}\frac{1}{|B(x,2^{-j}\frac{c_2}{\eta^{\alpha/2}})|}\int_{B(x,2^{-j}\frac{c_2}{\eta^{\alpha/2}})} M(\nabla g)(y)\mathrm{d}y\\
\le& C(M(\nabla g)(x)+M(M(\nabla g))(x)),
\end{aligned}
$$

which is bounded in $L^2(\mathbb{R}^{n-1})$.

For the second part I_{12} we need an explicit formula for H. Then we may use the estimate $|\partial_{x_j} H(\rho)| \le C\eta^{\alpha}|x-y|$ and proceed as above. We omit the details.

If $\rho > 2$, we use asymptotic behaviour of H at infinity. Again, we divide the integration into two parts with kernels $\partial_{x_j} K(x,y)H(\rho)$ and $K(x,y)\partial_{x_j} H(\rho)$. It is enough to consider the real part ReH since the asymptotic behaviour for the imaginary part is essentially the same. Further, it is enough to determine the asymptotic behaviour for the function

$$H_{25}^{40}(z) := H_{25}^{40}\left[z\Big|\begin{smallmatrix} (0,1), & (-1,\frac{1}{\alpha}) & & & \\ (\frac{n}{2}-1,\frac{1}{\alpha}), & (0,\frac{1}{\alpha}), & (0,\frac{1}{\alpha}), & (0,\frac{1}{2}); & (\frac{1}{2},\frac{1}{2}) \end{smallmatrix}\right].$$

Applying [19, Corollary 1.10.2], we have

$$H_{25}^{40}(z) = \mathscr{O}(z^{\frac{\alpha(n-1)}{4}} \exp\{2^{\frac{\alpha+2}{2}} \cos(\frac{4-\alpha}{4}\pi) z^{\frac{\alpha}{2}}\})$$

for $\mathbb{R} \ni z \to \infty$. Since $z \mapsto z^{\gamma}\exp(-cz^{\delta})$ is uniformly bounded in $\mathbb{R}_+$ for any $\gamma, \delta, c > 0$, we obtain, using the previous estimate and (17), the following upper bound for I_1 corresponding the kernel $\partial_{x_j} K(x,y)H(\rho)$

$$\begin{aligned} &C\eta^{\frac{\alpha(n-1)}{4}-\gamma} \int_{|x-y|>\frac{c}{\eta^{\alpha/2}}} |x-y|^{\frac{1}{2}-\frac{n}{2}-\frac{2\gamma}{\alpha}} (M(\nabla g)(x) + M(\nabla g)(y))\mathrm{d}y \\ \le & C\eta^{\frac{\alpha(n-1)}{4}-\gamma}\Big\{M(\nabla g)(x)\int_{\frac{c}{\eta^{\alpha/2}}} r^{\frac{n-3}{2}-\frac{2\gamma}{\alpha}}\mathrm{d}r + \sum_{j=0}^{\infty} \int_{2^j\frac{c}{\eta^{\alpha/2}}<|x-y|<2^{j+1}\frac{c}{\eta^{\alpha/2}}} |x-y|^{\frac{1-n}{2}-\frac{2\gamma}{\alpha}} M(\nabla g)(y)\mathrm{d}y\Big\} \\ \le & C(M(\nabla g)(x) + M(M(\nabla g))(x)), \end{aligned}$$

when we choose $\gamma > \frac{\alpha(n-1)}{4}$. The final upper bound is bounded in $L^2(\mathbb{R}^{n-1})$. The other terms may be estimated similarly and therefore the details are omitted.

Finally, it remains to show that $\mathscr{F}_{\tau}^{-1}((\mathrm{i}\tau)^{\alpha/2}\mathscr{F}_t(\widetilde{J_0}\psi)(\tau)) \in L^2(\mathbb{R}_T^{n-1})$. Because

$$\begin{aligned} &(\mathrm{i}\tau)^{\alpha/2}\mathscr{F}_t(\widetilde{J_0}\psi)(x,\tau) \\ =&\int_{\mathbb{R}^{n-1}} K(x,z)H(\tau(|x-y|^2 + (\varphi(x)-\varphi(y))^2)^{1/\alpha})(\mathrm{i}\tau)^{\alpha/2}\mathscr{F}_t(\psi)(y,\tau)\mathrm{d}y, \end{aligned}$$

we have

$$\mathscr{F}_{\tau}^{-1}((\mathrm{i}\tau)^{\alpha/2}\mathscr{F}(\widetilde{J_0}\psi)(\tau))(x,t) = \widetilde{J_0}(\mathscr{F}_{\tau}^{-1}((\mathrm{i}\tau)^{\alpha/2}\mathscr{F}_t(\psi))(x,t).$$

Since $\widetilde{J_0}$ is bounded in $L^2(\mathbb{R}_T^{n-1})$ and $\psi \in H^{1,\frac{\alpha}{2}}(\mathbb{R}_T^{n-1})$, we may finally conclude the claim. □

Now, we may proceed as in [11] to conclude that the norm of $\widetilde{J_0}$ is small with T on $H^{1,\frac{\alpha}{2}}(\mathbb{R}_T^{n-1})$, which implies

Lemma 2.3.2. *The norm of J is small with T as an operator on $H^{1,\frac{\alpha}{2}}(\Sigma_T)$.*

Then, applying the usual iteration argument, we get (see [11])

Theorem 2.3.3. *The operator $-\frac{1}{2}I + J$ is invertible on $H^{1,\frac{\alpha}{2}}(\Sigma_T)$.*

In addition, we have the following regularity result.

Theorem 2.3.4. *Let $g \in H^{1,\frac{\alpha}{2}}(\Sigma_T)$ in TFDE. Then the solution*

$$u(x,t) = D((-\frac{1}{2}I + J)^{-1}g)(x,t)$$

of TFDE satisfies

$$\|N(\nabla u)\|_{L^p(\Sigma_T)} \leq C\|g\|_{H^{1,\frac{\alpha}{2}}(\Sigma_T)}.$$

2.4 The collocation method

We remark that the notations used in this section are taken from paper IV and the spaces used in this part are different from those defined before. The results in this part of the thesis are generalizations of those in [15] and [16]. Following the same lines as in the above-mentioned papers we have formulated an equivalent Galerkin problem in paper IV. Then, using the standard techniques of Galerkin methods we are able to prove the stability of the method and the quasioptimal error estimate in the approximation space M_1. The error estimate for the collocation method then follows from the well-known results from approximation theory. We have (IV, Theorem 6)

Theorem 2.4.1. *Assume that $0 < h_\theta, h_t < h_0$. Let $u_\Delta \in M_1$ be the solution of the collocation problem and $u \in \widetilde{H}^{2,1}(\mathbb{R}_T^2) \cap \widetilde{H}^{1,2}(\mathbb{R}_T^2)$ be the solution of the single layer integral equation $Vu = f$. Then*

$$|||u - u_\Delta|||_{-\frac{1}{2},-\frac{\alpha}{4}} \leq C_1 h_\theta^{\frac{3}{2}} \|u\|_{2,1} + C_2 h_t^{1+\frac{\alpha}{4}} \|u\|_{1,2}.$$

References

1. Adolfsson V, Jawerth B, Torres R (1994) A Boundary Integral Method for Parabolic Equations in Non-Smooth Domains. Commun Pure Appl Math 47(6): 861–892.
2. Abramovitz M & Stegun IA (eds) (1971) Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. Washington DC, US Government Printing Office.
3. Arendt W, Batty C, Hieber M & Neubrander F (2001) Vector-valued Laplace Transforms and Cauchy Problems. Basel, Birkhäuser Verlag.
4. Braaksma BLJ (1964) Asymptotic Expansions and Analytic Continuations for a Class of Barnes Integrals. Compos Math 15: 239–341.
5. Brown RM (1989) The Method of Layer Potentials for the Heat Equation in Lipschitz Cylinders. Amer J Math 111: 339–379.
6. Coifman RR, MacIntosh A & Meyer Y (1982) L'Intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes. Ann Math 116: 361–387.
7. Colton DL & Kress R (1983) Integral Equation Methods in Scattering Theory. John Wiley & Sons Inc.
8. Colton DL & Kress R (1998) Inverse Acoustic and Electromagnetic Scattering Theory. Berlin, Springer-Verlag.
9. Costabel M & Saranen J (2000) Spline Collocation for Convolutional Parabolic Boundary Integral Equations. Numer Math 84: 417–449.
10. Fabes EB, Jodeit M Jr, Riviére NM (1978) Potential Techniques for Boundary Value Problems on C^1-domains. Acta Math 141(3-4): 165–186.
11. Fabes EB & Riviére NM (1979) Dirichlet and Neumann Problems for the Heat Equation in C^1-cylinders. Proceedings of Symposia in Pure Mathematics, Vol. XXXV, Part 2: 179–196.
12. Friedman A (1983) Partial Differential Equations of Parabolic Type. Malabar USA, Robert E Krieger Publishing Company Inc.
13. Ginoa M, Cerbelli S, Roman HE (1992) Fractional Diffusion Equation and Relaxation in Complex Viscoelastic Materials. Physica A 191: 449–453.
14. Hajłasz P (2003) Sobolev Spaces on Metric Measure Spaces. Contemp Math 338: 173–218.
15. Hamina M & Saranen J (1994) On the spline collocation method for the single layer heat operator equation. Math Comp 62: 41–64.
16. Hämäläinen J (1998) Spline collocation for the single layer heat equation. Ann Acad Sci Fenn, mathematica dissertationes 113.
17. Jones BF (1964) A Class of Singular Integrals. Amer J Math 86: 441–462.
18. Eidelman SD, Kochubei AN (2004) Cauchy problem for fractional diffusion equations. J Differ Equ 199: 211–255.
19. Kilbas AA & Saigo M (2004) H-Transforms: Theory and Applications. CRC Press LLC.
20. Kochubei AN (1990) Fractional Order Diffusion. Differ Equ 26(4): 485–492.
21. Kress R (1999) Linear Integral Equations. New York, Springer-Verlag Inc.

22. Li C & Deng W (2007) Remarks on Fractional Derivatives. Appl Math Comput 187: 777–784.
23. Lions JL & Magenes E (1972) Non-homogeneous boundary value problems and applications I. Berlin, Springer.
24. Lions JL & Magenes E (1972) Non-homogeneous boundary value problems and applications II. Berlin, Springer.
25. Luchko Y (2009) Maximum Principle for the Generalized Time–Fractional Diffusion Equation. J Math Anal Appl 351: 218–223.
26. Marichev OI (1983) Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables. Chichester, Ellis Horwood.
27. McLean W (2000) Strongly Elliptic Systems and Boundary Integral Equations. New York, Cambridge University Press.
28. Podlubny I (1999) Fractional Differential Equations. San Diego USA, Academic Press.
29. Pogorzelski W (1957) Propriétés des intégrales de l'équation parabolique normale. Ann Polon Math 4: 61–92.
30. Prudnikov AP, Brychkov YuA & Marichev OI (1990) Integrals and Series, vol 3: More Special Functions. Amsterdam, Overseas Publishers Association.
31. Riviére NM (1971) Singular Integrals and Multiplier Operators. Ark Mat 9: 243–278.
32. Samko SG, Kilbas AA & Marichev OI (1993) Fractional Integrals and Derivatives, Theory and Applications. Amsterdam, Gordon and Breach Science Publishers.
33. Schneider WR & Wyss W (1989) Fractional Diffusion and Wave Equations. J Math Phys 30: 134–144.
34. Stein EM (1970) Singular Integrals and Differentiability Properties of Functions. Princeton, Princeton University Press.
35. Verchota G (1982) Layer Potentials and Boundary Value Problems for Laplace's Equations on Lipschitz Domains. PhD Thesis, University of Minnesota.
36. Verchota G (1984) Layer Potentials and Regularity for the Dirichlet problem for Laplace's Equation in Lipschitz Domains. J Funct Anal 59: 572–611.
37. Watson GN (1992) A Treatise on the Theory of Bessel Functions. UK, Cambridge University Press.
38. Wloka J (1987) Partial Differential Equations. UK, Cambridge University Press.

Original articles

I Kemppainen J & Ruotsalainen K (2009) Boundary Integral Solution of the Time-Fractional Diffusion Equation. Integr Equ Oper Theory 64: 239–249.

II Kemppainen J (2009) Properties of the Single Layer Potential for the Time Fractional Diffusion Equation. Journal of Integral Equations and Applications. In Press.

III Kemppainen J (2010) Boundary Behaviour of the Layer Potentials for the Time Fractional Diffusion Equation in Lipschitz domains. Journal of Integral Equations and Applications. In Press.

IV Kemppainen J & Ruotsalainen K (2010) Boundary Element Collocation Method for Time-Fractional Diffusion Equations. In Constanda C & Pérez ME (eds) Integral Methods in Science and Engineering, Vol 2, Computational Methods. Birkhäuser Boston: 223–232.

Original publications are not included in the electronic version of the dissertation.

530. Sørensen, Louise Ilum (2009) Grazing, disturbance and plant soil interactions in northern grasslands

531. Salo, Antti (2009) Expression of lysyl hydroxylases and characterization of a novel disorder caused by mutations in the lysyl hydroxylase 3 gene

532. Vaismaa, Matti (2009) Development of benign synthesis of some terminal ?-hydroxy ketones and aldehydes

533. Meriläinen, Gitte (2009) Structural and enzymological studies of the thiolase enzymes

534. Astorga, Anna (2009) Diversity patterns in marine and freshwater environments. The role of environmental and spatial factors across multiple scales

535. Rivinoja, Antti (2009) Golgi pH and glycosylation

536. Salmi, Tuukka (2009) Very small families generated by bounded and unbounded context-free languages

537. Kokkonen, Nina (2009) Role of altered pH homeostasis and hypoxia in the phenotypic changes of cancer cells

538. Kangas, Katja (2009) Recreation and tourism induced changes in northern boreal environments

539. Laari-Salmela, Sari (2009) The process of strategy formation in software business: three cases from Kainuu region, Finland

540. Väyrynen, Karin (2009) Evolution of software business in industrial companies: Resources, capabilities and strategy

541. Eskelinen, Anu (2009) Plant community dynamics in tundra: propagule availability, biotic and environmental control

542. Still, Kaisa (2009) Mobile technology for interest-based communities: concept design with a knowledge-based approach

543. Tarvainen, Oili (2009) Scots pine and its ectomycorrhizal symbionts under chronic low-level urban pollution—responses and restoration

544. Saari, Eija (2009) Towards minimizing measurement uncertainty in total petroleum hydrocarbon determination by GC-FID

545. Pihlajaniemi, Henna (2009) Success of micropropagated woody landscape plants under northern growing conditions and changing environment

UNIVERSITY OF OULU P.O.B. 7500 FI-90014 UNIVERSITY OF OULU FINLAND

ACTA UNIVERSITATIS OULUENSIS

ISBN 978-951-42-6131-2 (Paperback)
ISBN 978-951-42-6132-9 (PDF)
ISSN 0355-3191 (Print)
ISSN 1796-220X (Online)

UNIVERSITY of OULU
OULUN YLIOPISTO