Spectroscopy of Artificial Atoms and Molecules

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Abstract

Elementary experiments of atomic physics and quantum optics can be reproduced on a circuit board using elements built of superconducting materials. Such systems can show discrete energy levels similar to those of atoms. With respect to their natural cousins, the enhanced controllability of these 'artificial atoms' allows the testing of the laws of physics in a novel range of parameters. Also, the study of such systems is important for their proposed use as the quantum bits (qubits) of the foreseen quantum computer.

In this thesis, we have studied an artificial atom coupled with a harmonic oscillator formed by an $LC$-resonator. At the quantum limit, the interaction between the two can be shown to mimic that of ordinary matter and light. The properties of the system were studied by measuring the reflected signal in a capacitively coupled transmission line. In atomic physics, this has an analogy with the absorption spectrum of electromagnetic radiation. To simulate such measurements, we have derived the corresponding equations of motion using the quantum network theory and the semi-classical approximation. The calculated absorption spectrum shows a good agreement with the experimental data. By extracting the power consumption in different parts of the circuit, we have calculated the energy flow between the atom and the oscillator. It shows that, in a certain parameter range, the absorption spectrum obeys the Franck-Condon principle, and can be interpreted in terms of vibronic transitions of a diatomic molecule.

A coupling with a radiation field shifts the spectral lines of an atom. In our system, the interaction between the atom and the field is nonlinear, and we have shown that a strong monochromatic driving results in energy shifts unforeseen in natural or, even, other artificial atoms. We have used the Floquet method to calculate the quasienergies of the coupled system of atom and field. The oscillator was treated as a small perturbation probing the quasienergies, and the resulting absorption spectrum agrees with the reflection measurement.

Key words: Artificial atoms, Superconducting qubits, Landau-Zener tunneling, Floquet method, Bloch-Siegert shift, Franck-Condon principle
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During the last year of my studies, I have enormously enjoyed the collaboration with my fellow graduate student Matti Silveri. His help with the Floquet theory and long hours spent on the simulations were vital for the completion of this thesis and the reaching of its present level of quality. I have also had the honor to collaborate with Yuriy Makhlin from the Landau Institute for Theoretical Physics. I am indebted to the pre-examiners Professor Steven Girvin and Doctor Sorin Paraoanu for the invaluable comments and corrections on the manuscript of this thesis. Heartfelt thanks to Mira Leppiaho for the notes on the final manuscript.

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Finally, I would like to thank my beloved wife Sanna for the daily doses of love, support and devotion. I am forever grateful to her for reminding me of the importance of dreams. Nanna, Nea and Anton are my meaning of life.

Oulu, May 2010        Jani Tuorila
LIST OF ORIGINAL PAPERS

The present thesis contains an introductory part and the following papers which will be referred to in the text by their Roman numbers.


The author has had a central role in the theoretical analysis of all of the articles presented in this thesis. He has developed the theory and the computational tools around the phenomena. The author has done all of the numerical simulations in Papers I and II. In Paper IV he has done part of the simulations. Being a review article, Paper III is an exception and the author has made his contribution in the part discussing the results of Paper I. The author has written Paper III, the initial version of Paper IV, and contributed to the writing process of all the Papers.
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Prologue

Every story has its beginning. Traditionally, the narrator starts with an introduction of the setting and the main characters. In this sense, the story I am going to tell seems to be no different than any other. But soon, the reader will find out that it is not an ordinary tale. It has twists and catches that at first sight seem to be out of this world. The millions of years of natural selection have evolved our senses to comprehend only the things that occur in a very narrow range of length scale. We can understand when someone explains the working of a thing to which we are used in our every day lives, such as the movement of the cars or our rotation around the specific ball of hydrogen we call the Sun. But, it is nearly impossible for us to relate to the expansion of the Universe; we cannot see the strange dance of the electrons and we cannot watch as a neutron transforms into a proton in the reorganization of energy in beta decay. From this protoplasm filled with incomprehensibility rises up a new breed whose noble goal is to bring vision to the places that cannot be seen, to replace ignorance with understanding. This is the species of physicists that tries to conceive the world in the deepest way and to see things in places where the eyes are of no good. But, a physicist is not a character in this story. He is merely an explorer reporting his discoveries, a storyteller narrating the events.

This is a true story. The concept of truth, however, has a special meaning in physics and science, in general 1, 2. Physicists create theories in order to explain the wonders of the world. The theories are constructed from assumptions, which may or may not be intuitive but can never be substantiated. This means that physics (and any science) cannot answer to the question: Why? From the point of view of a theory, the phenomena of the world are implications deduced from the assumptions. The assumptions constitute the framework that explains the way things work. The object of science can then be summarized as a capability to answer to the question: How? The only test of any scientific theory is an experiment. We can justify the use of a theory only by its consistency with the real world phenomena. The more phenomena the theory is able to explain, the better it is. However, no matter how good the theory is, it does not contain any absolute truth. Any theory must be replaced as soon as some of its consequences contradict with a measurement. This means that theories can never be validated, only
proven wrong. Altogether, the story I am going to tell is true, but only in the scientific sense.

The story takes place in the realm of quantum physics. Quantum physics reveals itself usually in a very small length scale. In this limit inconceivable to the human eye, we have decided to call the objects particles. But, these particles are not the kind to which we are used. We cannot follow a quantum particle on its path through the space-time. We can only have probabilities for it to be located at a point at a given time. Simultaneously, as the size of the studied objects gets smaller, the more challenging it becomes to make and interpret measurements. Ultimately, the experimental apparatus is bound to have an influence on the state of the measured system which produces fundamental constraints on the measurable properties and makes the prediction of possible outcomes difficult. But, this entanglement of the apparatus and the particle is not the reason why we cannot trace the trajectories of the small. The probabilistic character of quantum objects is an inherent property of Nature. Nevertheless, from this counter-intuitive assumption for the common sense, we can deduce implications that can be tested with experiments. And the theory has passed so far these tests with an astounding accuracy. These are the reasons why quantum physics is both impossible to comprehend and intriguing to study.

A wise man once said that we do not do things because they are easy but because they are hard. This is the only way to obtain new and deeper understanding on the beautiful world in which we live. So, I invite the reader to sit back and enjoy as I will tell about my adventures in the land of the Quanta:

"In the beginning, everything seemed to be all mixed up. Soon after I entered this new realm, I realised that I would have to abandon all my previous views concerning the physical nature of things. But before I could do that, I had to find out what concepts I could carry with me into this strange world..."
Chapter 1

Introduction

Sometimes the things we think are close to us and feel we know the best, turn out to be nothing of the sort. This can happen in everyday life and it most certainly will in physics. As an example, let us consider the concept of energy. We all use the word on a daily basis. Our diets should consist of so and so many calories that, after digestion, are vital in sustaining our bodily functions. The electric gadgets surrounding us consume electricity that is produced by transformation of energy into its other forms. Altogether, when something happens, it is always related to energy either flowing or changing its appearance. But, no one knows exactly what energy is. We do not know of a single entity we can point out and call the elementary chunk of energy. All we can say about energy is that we can measure how much we need it in order to keep something on-going.

Energy can be seen as a measure of capability of change. For every event, there exists an energy threshold that needs to be exceeded in order for it to happen. This brings us to another carelessly used concept, which is time. Time is a tool to separate the "happenings" from the dead silence. One could say that if nothing were going on then time would lose its meaning. We need something to change in order to measure the time, and there is no change unless there is energy. So there exists a deep interconnection between energy and time that still remains a mystery to us, even though the words are firmly rooted into the use of everyday language.

But all this philosophy is useless, unless we can relate it with something real. Sometimes it is just better to take a pause from thinking and start doing. We know that things change because otherwise we would not have much to talk about. We know how to describe the changes in Nature by using the concepts of energy and time, even though we do not exactly know what they themselves are. However, it is important to keep in mind that things might not be so clear-cut as one would like them to be. We have to deal with our lack of knowledge until we get enlightened. Most of the current research in physics does not question the concepts of energy and time. This
is the case also with this thesis.

Cosmology tells us that only about 4% of the energy in the Universe consist of ordinary matter (atoms). Moreover, the contribution of radiation (photons) is only a fraction of that. The energy in the cosmos is mostly in forms of which we have no experience, or even a decent theory. One might then start to question: Why are we interested in such fractional issues as the atoms and the photons? The answer is simple: Because we can. The interaction between matter and radiation is described by quantum electrodynamics (QED) \[4\], which is the most accurately tested scientific theory to date. For example, according to QED the coupling strength between the atom and the field is determined by the so-called fine-structure constant. The magnitude of this interaction can be measured and it deviates from the predicted value only by parts per billion. But, even though the correspondence has been so far terrific, it might happen that there exists a situation where it is not. That is why we should test the theory in regimes where it has not been tested before and remember that we can only prove our theories wrong. This is what has been done in this thesis. At the same time, it is at least as important to concentrate our effort on finding out of what the Universe really is made and whether and how it interacts with everything else.

When the dimensions of the studied system get smaller and smaller, the classical view of the world starts to become less and less accurate. As we arrive at the length scale of the order of an Ångström \(10^{-10} \text{ m}\), we enter into the microscopic realm where the particles behave strangely for our macroscopic minds. In this limit, it is hard to use words that were designed for the common sense. Objects lose their point-like qualities and start to behave as fields with a probability amplitudes of being in different places at the same time. The particles become "spread-out" rather than localised, and we have to free our minds and start to speak about things with the words of quantum physics.

Quantum physics is not just the physics of the small, even though in large groups the non-classical effects are often washed out by the unavoidable couplings to the surrounding world. Sometimes, it can happen that even a macroscopic party of microscopic revellers start to dance with a common quantum beat. One such example is the phenomenon of superconductivity. In certain materials, electrons can form pairs when the disturbances due to the temperature have been cooled down. Due to their bosonic nature, the pairs do not obey the Pauli exclusion principle and can go into a common eigenstate. In this condensate state, the material loses its electrical resistivity, repels the external magnetic fields, and in other words, becomes superconductive. One can create systems with discrete energy levels by using superconducting materials, which means that we can discuss these macroscopic quantum objects in terms of atomic physics. This analogy has been extremely fruitful and productive and allowed us to call such constructs as
artificial atoms.

One of the most intriguing scientific challenges of the present day is the quest for a quantum computer [5]. Even though they have not yet been realised, it has been shown theoretically that these novel devices could solve certain tasks exponentially faster than their classical counterparts [6, 7]. The speed-up is based on the direct use of the quantum interference and entanglement, which allows massive parallelism in the algorithms. Quantum computers are made of quantum bits, or qubits, which are the equivalent of the classical bits in the quantum world. In addition to the zero and one states of their digital ancestors, qubits are quantum two-level systems and therefore can represent also any superposition of those. It is possible to build qubits out of superconducting materials. Due to their wide controllability and strong coupling with others, the superconducting qubits are good candidates for the building blocks of the quantum computers in the future.

This thesis involves basic research and introduces the reader to the theoretical results presented in the original Papers I-IV. We will show that superconducting circuits can behave as artificial atoms. Their large inherent tunability allows us to enter parameter regimes that are hard, or impossible, to achieve with natural atoms. The thesis is organised in the following way. We start with an introduction to the needed concepts in Chapter 2. Large part of the physics we use were first studied in the context of real atoms. It benefits to review it in the original language to lay the foundation on a more familiar ground. In Chapter 3 we introduce our superconducting devices and show that they are analogous to natural atoms interacting with light. The coupling between light and the energy levels of our atom is nonlinear which leads to effects that are unprecedented in natural, optical, and other solid-state systems. In a certain parameter region, the devices are shown to have similarities also with diatomic molecules. We review the novel results of the Papers in Chapter 4 and discuss some details that originally received only a little attention. In the end, the work is summarised.
Chapter 2

Physics of the Natural

What is called here natural is no more so than systems studied in the rest of this thesis. As a matter of fact, the artificial objects discussed later are made of the same physical units dealt with in this chapter. The whole notion is somewhat vague since we are merely dealing with mathematical abstractions created to describe Nature in a consistent way. In a sense, all physics in this thesis is completely, or not at all, natural, depending on the way you look at it. In either case, the present reasons for the term are historical.

The idea that everything consists of elementary indivisible particles has been around since the ancient Greeks. Ordinary matter around us consists of atoms that are made of a nucleus and a surrounding cloud of electrons. Although we have learned that even the nucleus has an internal structure, we will stop the division here. For our purposes, it will be enough to say that everything is made of atoms. The appointment as the building blocks of Nature intimates to call them natural. They are not artifacts but things that have been here before the epoch of the engineers.

The concepts we will use were first developed in the field of atomic physics. The interactions between the atoms and light are described by the most accurately tested scientific theory we have, namely quantum electrodynamics (QED) (for a nice layman oriented introduction, see [4]). Perhaps the simplest description of this interplay is obtained when atoms are placed in between reflecting mirrors, things we call cavities. In this field of cavity-QED, it is possible to see the coupling of light and matter at the quantum level.

We will go through the relevant details of atoms, photons and their interactions briefly in this chapter. Even though the discussion will be mostly repeated in the coming chapter, it is also beneficial to do it here. Namely, when the artificial objects are introduced, it is nice to have some analogy with concepts that are more familiar. After all, it will be quite surprising to see, that under the right conditions one can combine a myriad of atoms into a single macroscopic entity in which all parts work in unison and whose operation is indistinguishable from that of a single atom.
CHAPTER 2. PHYSICS OF THE NATURAL

2.1 Characters

We will start with the introduction of the two most important concepts in physics: the harmonic oscillator and the two-level system. A harmonic oscillator coupled with a two-level system can be used as a simple, but very powerful tool in the illustration of interactions between matter and light. On the other hand, a similar coupling can be used in the description of simultaneous transitions of the vibrational state of the nuclei and the electronic states in diatomic molecules.

2.1.1 Harmonic Oscillator as Light

Most of us are familiar with the swing. Once set into motion, there is a restoring force caused by Earth’s gravity trying to return the swing back to its equilibrium position. In the absence of a swinger, this leads to oscillating motion that eventually damps due to the friction and the drag of air. The swing belongs to a wide class of harmonic oscillators that are systems perturbed slightly from the equilibrium positions and then set into motion. The classical picture of a harmonic oscillator is known to those that have either swung themselves or know something about Newtonian mechanics.

Let us denote the deviation from the equilibrium position by \( x \). The rate of change of \( x \) is then characterised by the momentum \( p = m \dot{x} \) (time derivatives in this thesis are denoted by \( dx/dt \equiv \dot{x} \)). According to classical mechanics, the total energy of the driven oscillator (or the pushed swingee) is then described by the Hamiltonian function

\[
H = \frac{p^2}{2m} + \frac{k}{2} x^2 - F(t)x, \tag{2.1}
\]

where \( m \) is the mass of the oscillator and \( k \) is the spring constant, the proportionality factor of the linear force. The first term in the Hamiltonian is the kinetic energy and the second one is the potential energy. The last term incorporates a time-dependent driving \( F(t) \) of the oscillator. Hamiltonian equations give the classical equations of motion

\[
\dot{x} \equiv \frac{\partial H}{\partial p} = \frac{p}{m}, \tag{2.2}
\]
\[
\dot{p} \equiv -\frac{\partial H}{\partial x} = -kx + F(t). \tag{2.3}
\]

We can model the dissipation by inserting a phenomenological term \(-\gamma p\) into the equation of motion \(2.3\), leading to

\[
\ddot{x} + \gamma \dot{x} + \omega_n^2 x = f(t), \tag{2.4}
\]
where $\omega^2_L = k/m$ is the natural frequency of the oscillator and $f(t) = F(t)/m$. The damping coefficient $\gamma$ describes the rate at which the oscillator loses energy into the environment. Equation of motion (2.4) for the driven and damped harmonic oscillator is the starting point in numerous physical problems.

Let us then assume that the driving is monochromatic $f(t) = f_0 \sin(\omega_d t)$. The steady state (all the transients due to the initial conditions have died out) solution of the equation of motion is

$$x(t) = A \sin(\omega_d t + \phi),$$

where

$$A = \frac{f_0}{\sqrt{(\gamma \omega_d)^2 + (\omega^2_d - \omega^2_L)^2}}$$

$$\tan \phi = \frac{\gamma \omega_d}{\omega^2_d - \omega^2_L}.$$ (2.7)

The amplitude $A$ is related to the absorption from the drive and $\phi$ is the phase difference of the particle and the force, called dispersion (see Figure 2.1 for a sketch of the typical absorption and dispersion profiles). An important quantity is the quality factor

$$Q = \frac{\omega_L}{\gamma},$$

which determines the bandwidth $\Delta \omega$ of the oscillator. The higher the $Q$ is, the narrower is the range of frequencies with which the oscillator "swings".

**Quantum Oscillator**

We talk mostly about classical oscillators in this thesis. Nevertheless, it is worthwhile to notice that the electromagnetic field (i.e. light) can be modelled as an infinite set of small harmonic oscillators working at the quantum
level. Therefore, it serves as a good introduction and a mood-setter to discuss what happens to the swings in regions where the quantum physics dominates.

We follow the standard quantization procedure and replace the degrees of freedom with operators and assume that the corresponding Poisson bracket is replaced by the commutation relation

\[ x \rightarrow \hat{x}, \quad p \rightarrow \hat{p} \]

\[ \{x, p\} = 1 \quad \rightarrow \quad [\hat{x}, \hat{p}] = i\hbar. \]

We define the so-called annihilation and creation operators

\[ \hat{a} = \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega_L} \hat{x} + i \frac{\hat{p}}{\sqrt{m\omega_L}} \right) \]

\[ \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega_L} \hat{x} - i \frac{\hat{p}}{\sqrt{m\omega_L}} \right), \]

respectively. Finally, we obtain the Hamiltonian operator for the undriven and undamped harmonic oscillator

\[ \hat{H}_L = \hbar \omega_L (\hat{a}^\dagger \hat{a} + \frac{1}{2}). \]

It can be shown [8, 9, 10] that the electromagnetic field can be represented by an infinite sum of harmonic oscillators with different frequencies (modes). If only a single mode is excited we call the field monochromatic.

Let us denote the eigenstates of the harmonic oscillator with \(|n\rangle\) where \(n\) is a non-negative integer. These are often referred to as Fock states since the index \(n\) indicates the number of excitations in the oscillator. The reason for the names of the operators comes from their impact on the Fock states

\[ \hat{a} |n\rangle = \sqrt{n} |n-1\rangle, \]

\[ \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \]

\[ \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle. \]

A simple interpretation of this is that \(\hat{a}\) (\(\hat{a}^\dagger\)) annihilates (creates) one quantum and \(\hat{a}^\dagger \hat{a}\) gives the number of quanta. Fock states have been observed in superconducting artificial systems [11]. Just recently, there was also a report on reaching the ground state of a mechanical oscillator [12].

Driving can be added by introducing a drive Hamiltonian

\[ \hat{H}_d = -F(t) \sqrt{\frac{\hbar}{2m\omega_L}} (\hat{a} + \hat{a}^\dagger). \]

The damping is more complicated due to the reversible nature of quantum mechanics due to the Hamiltonian formalism (there is no simple way of introducing damping phenomenologically, like in the classical case).
2.1. CHARACTERS

Master Equation

There is nothing ideal in the real world. Physical entities cannot be thought isolated as they are inevitably connected with systems nearby that can draw away energy/information for good. In quantum language this is called the loss of quantum coherence or decoherence for short. Formally, this can be modelled by introducing an environment consisting of a large number of harmonic oscillators (like the modes of the electromagnetic field) coupled linearly to the system. In general, the system becomes entangled with the environment meaning that the wave function formalism becomes inadequate and the quantum state must be described in terms of a density operator \( \hat{\rho} \) [13], defined as

\[
\hat{\rho} = \sum_i p_i |a_i\rangle\langle a_i|,
\]  

where \( p_i \) is the probability that the system is in state \( |a_i\rangle \). Due to the largeness of the environment, it easily becomes impossible to follow the time evolution of the whole system. It turns out, however, that under some reasonable assumptions the dynamics of the harmonic oscillator can be described by a simple equation of motion for the reduced density matrix \( \hat{\rho}_L = \text{Tr}_R(\hat{\rho}) \). By tracing out the environmental variables, making a second order perturbation theory with respect to the interaction (Born approximation), and assuming that the system has no long-term memory (Markov approximation) we end up with the master equation

\[
\frac{d\hat{\rho}_L}{dt} = -\frac{i}{\hbar} [\hat{H}_L + \hat{H}_d, \hat{\rho}_L] + \kappa \mathcal{D}[\hat{a}]\hat{\rho}_L,
\]  

where the superoperator \( \mathcal{D}[\hat{A}] \) is defined as \( \mathcal{D}[\hat{A}]\hat{\rho} = \frac{1}{2}(2\hat{A}\hat{\rho}\hat{A}^\dagger - \hat{A}^\dagger\hat{A}\hat{\rho} - \hat{\rho}\hat{A}^\dagger\hat{A}) \) for any operator \( \hat{A} \). This is the equation of motion for the driven harmonic oscillator density matrix in the Schrödinger picture and at zero temperature. The expectation value of any operator \( \hat{A} \) in the harmonic oscillator space can be calculated as

\[
\langle A \rangle = \text{Tr}(\hat{A}\hat{\rho}_L).
\]  

2.1.2 Two-Level System as Atom

Atoms are microscopic particles that are the building blocks of everything around us. They consist of a dense nucleus of protons and (possible) neutrons. The positively charged (due to protons) nucleus is surrounded by the negative electrons conducting their probabilistic quantum dance. Regardless of the apparent attraction of the electrons and the nucleus, each atomic system keeps its characteristic shape and does not collapse. Quantum physics explains this by stating that no particle can have definite position and momentum at the same instant. This prohibition is more commonly known as
the Heisenberg uncertainty principle and it is responsible for the limits of how much information we can gather from any given system. Even though we cannot pinpoint the electrons exactly, we know that their energies can only attain some discrete values. This has some profound implications on the way matter behaves when interacting with light [4].

The natural atoms are so small that we cannot see them in the way to which we are used. The size of an atom is measured in picometers ($10^{-12}$ meters) whereas the shortest wave length of light our eyes can render is of the order of hundreds of nanometers ($10^{-9}$ meters). This means that we have to use something like the so-called scanning tunneling microscopes, instead of optical ones, if we want to make at least some kind of image of an atom. This is, however, an indirect way to see an atom since it does not use light in the production. Another way of seeing atoms is to use radiation with shorter wavelengths.

Let us consider an atom interacting with radiation. For simplicity, let us talk about an atom that has only one proton and one electron, i.e. about hydrogen. The discussion can be generalized to atoms with larger atomic number. QED says that atoms interact with photons via electrons. This means that atoms can emit and absorb the photons whose energies coincide with the transitions between energy levels of the electron. The thing about energy is that we can measure only its differences. Therefore, we can denote the lower of the two energies involved in a transition by $-\hbar\omega_0/2$. The corresponding quantum state is named as $|\downarrow\rangle$. Similarly, the higher energy state $|\uparrow\rangle$ has the energy $+\hbar\omega_0/2$. So, the transition between the two states occurs when the photon energy is $\hbar\omega_0$. The Hamiltonian of the two states can be written compactly as

$$\hat{H}_A = \frac{\hbar\omega_0}{2} (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) = \frac{\hbar\omega_0}{2} \hat{\sigma}_z. \tag{2.21}$$

The operators in the two-dimensional Hilbert space can always be represented in terms of the identity matrix and the Pauli spin matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.22}$$

where the last two matrices are the creation and annihilation operators of the two-level system, respectively. We will use this notation throughout this thesis when talking about either natural or artificial atoms.

**Bloch Equations**

We can also write a master equation for the density matrix of a two-level system. We use a similar model as in the simple harmonic oscillator case,
but we include also a longitudinal coupling with the environment causing fluctuations of the level separation leading to dephasing. After a similar set of approximations as in the oscillator case, we obtain the master equation for the two-level atom

\[
\frac{d\hat{\rho}_A}{dt} = -\frac{i}{\hbar}[\hat{H}_A, \hat{\rho}_A] + \gamma D[\hat{\sigma}_-]\hat{\rho}_A + \frac{\gamma\phi}{2}D[\hat{\sigma}_z]\hat{\rho}_A. \tag{2.23}
\]

It is illustrative to write the density matrix in the fictitious spin representation as

\[
\hat{\rho}_A = \frac{1}{2}(\hat{I} + \mathbf{S} \cdot \hat{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + S_z & S_x - iS_y \\ S_x + iS_y & 1 - S_z \end{pmatrix}, \tag{2.24}
\]

where \(S_i = \langle \hat{\sigma}_i \rangle\). If we also write the two-level Hamiltonian as \(\hat{H} = \frac{1}{2}\mathbf{\Omega} \cdot \hat{\sigma}\), the master equation becomes

\[
\begin{align*}
\dot{S}_x &= (\mathbf{\Omega} \times \mathbf{S})_x - \frac{S_x}{T_2} \tag{2.25} \\
\dot{S}_y &= (\mathbf{\Omega} \times \mathbf{S})_y - \frac{S_y}{T_2} \tag{2.26} \\
\dot{S}_z &= (\mathbf{\Omega} \times \mathbf{S})_z - \frac{S_z - S_{z0}}{T_1}. \tag{2.27}
\end{align*}
\]

These are the Bloch equations that are used extensively in nuclear magnetic resonance, magnetic resonance imaging and optics, for example. The equations describe a (fictitious) spin-\(\frac{1}{2}\) particle (with spin vector \(\hbar\mathbf{S}/2\)) rotating around the magnetic field \(\mathbf{\Omega}\). The equilibrium value of the spin \(S_{z0}\) and the relaxation times \(T_1\) and \(T_2\) are defined in the zero temperature limit as

\[
\begin{align*}
S_{z0} &= -1 \tag{2.28} \\
T_1^{-1} &= \gamma \tag{2.29} \\
T_2^{-1} &= \frac{\gamma}{2} + \gamma\phi. \tag{2.30}
\end{align*}
\]

Usually, the phenomenological time constants \(T_1\) and \(T_2\) are called the relaxation and dephasing times, respectively.

### 2.2 Dance of Light and Matter

Now we are ready to combine the two subsystems in order to describe the interactions between matter and light. We start with an unperturbed Hamiltonian of a single-mode field (described by a harmonic oscillator) and a two-level atom

\[
\hat{H}_0 = \hat{H}_L + \hat{H}_A = \hbar\omega_L\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_0}{2}\hat{\sigma}_z. \tag{2.31}
\]
Eigenstates of $\hat{H}_0$ are of form $|\uparrow, \downarrow, n\rangle$, where $n$ is the number of quanta in the field. This set of unperturbed states serves as our basis set. The interaction between the two subsystems can be described elegantly in the dipole approximation by

$$\hat{H}_{\text{int}} = \hbar g (\hat{a}^\dagger + \hat{a})(\hat{\sigma}_+ + \hat{\sigma}_-).$$

(2.32)

Here the frequency $g$ is the coupling constant that determines the strength of the interaction. When the detuning between the light and atomic frequencies is small compared with the level separation of the atom, i.e. $\delta_L \ll \omega_0$, we say that the light is in resonance with the atom. In that case, we can make the so-called rotating wave approximation (RWA) and neglect the counter-rotating terms in $\hat{H}_{\text{int}}$. We end up with the so-called Jaynes-Cummings (JC) Hamiltonian

$$\hat{H}_{\text{JC}} = \hbar \omega_L \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \hbar g (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+),$$

(2.34)

which is used as a simple description of (nearly) resonant interactions between atom and light in quantum physics. It was introduced by Jaynes and Cummings in the context of matter-field coupling in free space [14].

**Fine-Structure Constant**

A strong coupling between atoms and photons is hard to achieve in free space. The weak interaction has its origin in the smallness of the so-called fine-structure constant $\alpha \simeq 1/137$, a fundamental constant of Nature that cannot be derived from any present theory (it is also responsible for the narrow atomic linewidths, see e.g. [15]). One can try to circumvent this by placing the atom between reflecting mirrors that allow the photon pass it many times leading to increased coupling with respect to the relaxation [15]. This field of physics is referred to as cavity-QED.

In the next chapter we will see that we can control the coupling coefficient of the artificial atom+field system. Whereas one can achieve coupling strengths of tens of megahertz’s in the natural Rydberg atoms [16], we foresee the possibility of interaction of the order of a gigahertz in our superconducting atom. The drawback of this is that also the interaction of the atom with undesired modes increases leading to broadening of the atom linewidth. A major issue in such circuit-QED systems is then to minimize the environment induced dissipation in order to reach the strong coupling regime (see Section 2.2.3).
2.2. DANCE OF LIGHT AND MATTER

2.2.1 Dressed Atom

The Jaynes-Cummings coupling in Equation (2.34) is such that an excitation of the atom is always accompanied by a reduction of the photon number by one. Mathematically, this means that the state $|\downarrow, n+1\rangle$ is coupled only with the state $|\uparrow, n\rangle$, apart from itself. Accordingly, the state space splits into manifolds $\mathcal{E}(n) = \{|\downarrow, n+1\rangle, |\uparrow, n\rangle\}$ whose members are close to each other in energy. Within a given manifold, our total Hamiltonian reduces to

$$\hat{H}_{JC} = \hbar \omega_L \left(n + \frac{1}{2}\right) + \frac{\hbar}{2} \left( \frac{\delta_L}{\Omega_R} \Omega_R - \delta_L \right),$$

(2.35)

where we have defined the resonant Rabi frequency $\Omega_R = 2g \sqrt{n+1}$. We solve the eigenvalue equation for $\hat{H}_{JC}$ and obtain the JC-eigenenergies

$$E_{\pm,n} = \hbar \omega_L \left(n + \frac{1}{2}\right) \pm \frac{\hbar}{2} \sqrt{\delta_L^2 + \Omega_R^2},$$

(2.36)

and the corresponding eigenvectors

$$|1(n)\rangle = \sin \theta |\downarrow, n+1\rangle + \cos \theta |\uparrow, n\rangle$$

$$|2(n)\rangle = \cos \theta |\downarrow, n+1\rangle - \sin \theta |\uparrow, n\rangle.$$  

(2.37)

In the above, we have used the convention to denote the higher energy eigenstate with $|1(n)\rangle$ [8]. Also, we have defined

$$\tan 2\theta = \frac{\Omega_R}{\delta_L}. \tag{2.38}$$

In Figure 2.2 we have plotted on the left-hand side the unperturbed states from manifolds $\mathcal{E}(n)$ and $\mathcal{E}(n-1)$, where we have assumed that $\delta_L > 0$. When the JC-coupling is turned on, the eigenstates (2.37) of the Hamiltonian (2.35) become superpositions of the manifold states, depicted on the right in Figure 2.2. This entanglement of matter and light can be seen as "an atom dressed in field quanta", which has led to an intimating terming of the states (2.37) as dressed states. Also, at the resonance ($\delta_L = 0$), the degeneracy is lifted.

As an annotation, the state $|\downarrow, n+1\rangle$ is not an energy eigenstate. If we can initialize the system in that state, there will Rabi oscillations at rate $\Omega_R$ between the states $|\downarrow, n+1\rangle$ and $|\uparrow, n\rangle$. This will be seen also in the analysis of the classically driven two-level atom in a following section. In the quantum case, one can relate explicitly the excitation of the atom with a removal of a quantum from the field. The dancing quantum of energy swaps its partner in time similar to the Salsa Rueda.
Figure 2.2: Dressed states in the Jaynes-Cummings model. Arrows indicate transitions induced by a resonant probe. Rabi transitions $\Omega_R$ are not allowed in the dipole approximation. The allowed transitions $\omega_L$, $\omega_L \pm \Omega_R$ form the Mollow triplet in the absorption spectrum.

2.2.2 Dynamic Stark Effect

Time-dependent driving affects the spectral properties of an atom. The energy levels of the combined atom-field system are different than those of the uncoupled atom. This can be seen, for example, as a drastic change in the absorption spectrum of a weak probe. In an undriven atom, the probe absorption spectrum consists of a single line centered at the atomic level separation. When we apply the drive with angular frequency $\omega_L$, the atomic states become dressed. The absorption spectrum consists now of three peaks referred to as the Mollow triplet [17], a central one at $\omega_L$ and two side peaks at $\omega_L \pm \Omega_R$ (see Figure 2.2). The phenomenon that the dressed atomic energies differ from the bare ones is called the dynamic (ac) Stark effect [18].

In static electric fields the energy levels of an atom are also shifted and split, analogously to the Zeeman effect (induced by a static magnetic field). This is often referred to as the dc Stark shift. In this thesis, we consider only the dynamic Stark shift but a discussion of its difference with the dc Stark shift can be found in Reference [19].

**Bloch-Siegert Shift**

If the field amplitude becomes strong enough (roughly of the order of the atomic energy) one needs to include the effects produced by the counter-rotating term neglected in the RWA of the dressed atom model. This correction is referred to as the Bloch-Siegert (BS) shift [20] and leads to a deviation from the resonance condition $\delta_L = 0$. Real atoms couple weakly with electromagnetic fields due to the smallness of the fine-structure constant, which means that the BS shift is a negligible effect. Artificial atoms can be operated in the limit of high field intensity leading to the insufficiency of RWA. In Paper [LV] we present a longitudinally driven artificial atom that is simultaneously coupled with multiple (non-resonant) field modes resulting to energy...
2.2. DANCE OF LIGHT AND MATTER

corrections even beyond the BS shift.

2.2.3 Strong Coupling Regime

There are inevitable losses of quantum coherence also in the cavity-QED system. This is modelled by assuming that the atom and the cavity dissipate energy into the environment with characteristic rates $\gamma$ and $\kappa$, respectively. The time-evolution of the density operator $\hat{\rho}_{JC}$ of the Jaynes-Cummings system can then be determined from the master equation (in this discussion we have neglected the effects of dephasing $\gamma_\phi$)

$$\frac{d\hat{\rho}_{JC}}{dt} = -\frac{i}{\hbar} [\hat{H}_{JC}, \hat{\rho}_{JC}] + \kappa \mathcal{D}[\hat{a}]\hat{\rho}_{JC} + \gamma \mathcal{D}[\hat{\sigma}_-]\hat{\rho}_{JC}. \quad (2.39)$$

We see that the JC-master equation is a combination of the individual atom and oscillator equations (2.19) and (2.23).

Let us then recall that at the quantum level the atom and the cavity exchange energy with each other with the rate determined by the vacuum Rabi frequency $\Omega_{R0} = 2g$. When this rate of absorption or emission of a single photon is larger than any of the rates of loss ($\Omega_{R0} \gg \kappa, \gamma$) we say that the cavity-QED system is in the strong coupling regime. The atom and the oscillator can exchange a photon multiple times before it is lost in dissipation. These vacuum Rabi oscillations can be observed only in the strong coupling regime. In artificial Josephson atoms (which is the subject of this thesis), they were seen first in References [21] and [22].

Even though the Rabi splitting of the spectrum can be seen as an indication of coherent quantum interaction between the matter and the field, it can be explained classically as the normal mode splitting of two oscillators [23]. More profound evidence of the quantum nature of the coupling is the $\sqrt{n}$ scaling of the Jaynes-Cummings energy levels due to the separation of the JC-doublets by the Rabi frequency $\Omega_R = 2g\sqrt{n} + 1$. This nonlinear dependence on the number of excitations in the field has been seen in the superconducting cavity-QED systems in time-domain [24] and in direct spectroscopic measurements [25].

2.2.4 Classical Coupling of Light and Matter

Next we assume that there are so many quanta in the oscillator that it can be taken as classical. We first consider a longitudinal coupling ($\hbar \omega(t)/2$) of the atom and the field. The bare states of the atom are also coupled with a constant transverse component ($\hbar \Delta/2$). The resulting Hamiltonian $2 \times 2$-matrix can be written compactly as

$$\hat{H}_{LZ} = \frac{\hbar}{2} \left( \omega_0 + \omega(t) \right) \hat{\sigma}_z + \Delta \hat{\sigma}_x = \frac{\hbar}{2} \begin{pmatrix} \omega_0 + \omega(t) & \Delta \\ \Delta & -\omega_0 - \omega(t) \end{pmatrix}. \quad (2.40)$$
We diagonalize the Hamiltonian (2.40) and obtain the adiabatic eigenenergies 
\[ E_{\pm}(t) = \pm \frac{\hbar}{2} \sqrt{\left(\omega_0 + \omega(t)\right)^2 + \Delta^2}. \]
These are depicted in Figure 2.3 as a function of \( \omega_0 \) together with the diabatic energies \( \pm \hbar \omega_0/2 \) (black dashed lines). We see that at \( \omega_0 = 0 \) the degeneracy is lifted and we obtain the so-called avoided crossing of energy levels equal to \( \hbar \Delta \).

Let us then consider that at some moment in time we start to look at the system in the ground (diabatic) state far away from the degeneracy point. By turning on the field, one is capable of travelling through the avoided crossing to the other side of the degeneracy point. The calculation of this special initial value problem is called the Landau-Zener (LZ) problem [26, 27, 28, 29]. By knowing the initial state of the system, one is able to calculate the state vector far at the other side of the crossing by using a unitary rotation defined in the next subsection.

When the avoided crossing is traversed multiple times, the population of the excited state becomes dependent on the dynamical phase gathered in between the crossings. In addition to the LZ-picture, this can be discussed in a rotating frame where it is referred to as Rabi problem [30].

**Landau-Zener Picture**

Whenever a two-level system travels through an avoided crossing (like the one in Figure 2.3), one can use the methods that were developed independently by Landau, Zener, Stückelberg and Majorana [27, 26, 28, 29]. Equation (2.40)
2.2. **DANCE OF LIGHT AND MATTER**

is the usual prototype *Landau-Zener Hamiltonian* but in here the coupling of the field is not explicitly determined.

Landau-Zener (LZ) tunneling can occur when a two-level system travels through a minimum energy separation. Two successive LZ transitions can interfere with each other giving rise to the so-called St"uckelberg \[28\] oscillations of the transition probability which depend on the dynamical phase gathered in between the transitions. This kind of interferometer can be generalized by making the system cross the minimum energy separation periodically. Similar discussion can be found in \[31\] but with slightly different notation.

In the classic LZ-theory, two intersecting energy levels $\varepsilon_-$ and $\varepsilon_+$ are coupled, leading to an avoided crossing with a gap size $\hbar \Delta$. We assume that the uncoupled energies can be controlled with a time-dependent field, leading to energy separation $\varepsilon(t) = \hbar \omega_0 + \hbar \omega(t)$ where we have denoted $\varepsilon_+ - \varepsilon_- = \hbar \omega_0$. As the system is swept through the minimum energy separation it has an asymptotic probability $P_{LZ}$ to tunnel from one (adiabatic) level to another:

$$P_{LZ} = \exp\left(-2\pi \frac{\Delta^2}{4\nu}\right) = \exp\left(-2\pi \gamma\right).$$ \hspace{1cm} (2.41)

The parameter

$$\nu = \frac{1}{\hbar} \left| \frac{d\varepsilon}{dt} \right|$$ \hspace{1cm} (2.42)

is defined at $\varepsilon(t) = 0$ and describes the rate of change of the energy difference between the two uncoupled (diabatic) levels at the crossing point.

In Landau-Zener-St"uckelberg model, the time evolution between two successive level-crossings is discretised. The resulting characteristic interference patterns depend on the quantum phase gathered during the free evolution in between the two LZ-events. The following results are derived in Appendix A. According to References \[32\] \[33\] and Appendix A, one can now describe the tunneling process as a unitary rotation in the adiabatic basis as

$$U_{LZ} = \left( \begin{array}{cc} \cos \frac{\theta}{2} \exp(-i\phi_S) & \sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \exp(i\phi_S) \end{array} \right),$$ \hspace{1cm} (2.43)

where $\sin^2(\theta/2) = P_{LZ}$ and the Stokes phase $\phi_S = \pi/4 + \text{arg} [\Gamma(1 - i\gamma)] + \gamma(\ln \gamma - 1)$, which describes the difference between the dynamic phases gathered by the diabatic and adiabatic states. Furthermore, the time evolution of a level crossing problem can be divided into a tunneling part and into a free evolution part. In optics terms, the avoided crossing acts in a way as a beam splitter with an adjustable division ratio.

As promised earlier, we do not specify the physical realisation. It can be any two-level system that has the diabatic energies $\varepsilon_-$ and $\varepsilon_+$, adiabatic energies $E_+(t)$ and $E_-(t)$ and a coupling $\Delta$ between the diabatic states. It
turns out that there exists two separate regimes determined by the adiabatic condition \[ |\langle E^+(t) \frac{\partial}{\partial t} E^-(t) \rangle| \ll \frac{|E^+(t) - E^-(t)|}{\hbar}. \] (2.44)

Physically, this means that the system follows its adiabatic eigenstate if the coupling between the states at each given time is much smaller than the gap. We obtain the maximum of the left-hand part and the minimum of the right-hand part at the avoided crossing. After some algebra, the adiabatic condition becomes

\[ \nu \ll \Delta^2, \] (2.45)

where it should be kept in mind that \( \Delta \) has the unit of angular frequency in our notation.

Let us consider one cycle in the adiabatic basis. It can be produced by, for example, driving the system periodically at frequency \( \omega \). The amplitude of the drive has to be larger than the distance to the avoided crossing. During one drive period \( T = \frac{2\pi}{\omega} \), the diabatic levels accumulate a phase difference

\[ \varphi = \frac{1}{\hbar} \int_0^T \left[ E^+(t) - E^-(t) \right] dt. \] (2.46)

The free evolution is interrupted only by the two LZ-processes.

A St"uckelberg cycle goes as follows. Let us start from the right most point of the sweep (see Figure 2.3). At first, the system evolves freely gathering only dynamical phase \( \varphi_R/2 \). At the avoided crossing, a possible LZ-tunneling occurs, described with \( U_{LZ} \). On the left-hand side, we again have free evolution, and the accumulated phase is \( \varphi_L \). When the system again reaches the crossing point, it undergoes another LZ-transition \( U_{LZ} \). On the remaining of the cycle, the system acquires a phase shift \( \varphi_R/2 \). Notice that \( \varphi = \varphi_L + \varphi_R \). Now, the total time evolution operator of one such cycle is

\[ U_S = U_{\varphi_R/2} U_{LZ} U_{\varphi_L} U_{LZ} U_{\varphi_R/2} = \begin{pmatrix} e^{-i(\frac{\varphi}{2} + 2\varphi_S)} \cos^2 \frac{\theta}{2} & e^{-i\frac{\varphi_L + 2\varphi_S}{2}} \sin^2 \frac{\theta}{2} \\ -i \sin \frac{\varphi_L + 2\varphi_S}{2} \sin \theta & e^{i(\frac{\varphi}{2} + 2\varphi_S)} \cos^2 \frac{\theta}{2} + e^{i\frac{\varphi_L + 2\varphi_S}{2}} \sin^2 \frac{\theta}{2} \end{pmatrix}. \] (2.47)

Starting from the (adiabatic) ground state \( |0\rangle = (0, 1)^T \), the probability of being in the excited state \( |1\rangle = (1, 0)^T \) is

\[ P_S = |\langle 1|U_S|0\rangle|^2 = 2P_{LZ}(1 - P_{LZ})(1 - \cos(\varphi_L + 2\varphi_S)). \] (2.48)

We see that the condition for constructive interference is

\[ \varphi_L + 2\varphi_S - \pi = n \times 2\pi. \] (2.49)

This is the familiar Mach-Zehnder interferometer formula where the phase \( \varphi_L + 2\varphi_S - \pi \) corresponds to the phase gathered in between the beam splitters.
2.2. DANCE OF LIGHT AND MATTER

We are interested in a periodic driving. In that case we have multiple consecutive tunneling processes which persuades us to call the system as a multi-pass Stückelberg interferometer. We calculate the excited state population $P_N$ after $N$ drive periods, and by using the same methods as in Reference [33], we obtain

$$P_N = |\langle 1 | U^N_S | 0 \rangle|^2 = P_S \left( \frac{\sin N\xi}{\sin \xi} \right)^2,$$  \hspace{1cm} (2.50)

where

$$\cos \xi = \cos \frac{\varphi + 4\phi_S}{2} \cos^2 \frac{\theta}{2} + \cos \frac{\varphi_R - \varphi_L}{2} \sin^2 \frac{\theta}{2}.$$  \hspace{1cm} (2.51)

(It is interesting to notice that this result for $P_N$ looks very much similar to that of a diffraction grating with $N$ apertures.) Now, for having the maximum excited state population we have to have $\cos \xi = \pm 1$ which implies

$$(1 - P_{LZ}) \cos \frac{\varphi + 4\phi_S}{2} + P_L \cos \frac{\varphi_R - \varphi_L}{2} = \pm 1$$ \hspace{1cm} (2.52)

We also had the resonance condition for the prefactor $P_S$

$$\varphi_L + 2\phi_S - \pi = n \times 2\pi.$$ \hspace{1cm} (2.53)

Equations (2.52) and (2.53) determine the locations of the population maxima of the excited state. One can discuss this result in terms of quantum trajectories. At each crossing the "particle" has two possible ways to go which are denoted by the adiabatic states $E_-$ and $E_+$. The probability of choosing the path is determined by the LZ-formula. There are $2^{2N-1}$ ways to get to the excited state after $N$ drive periods. When these paths interfere constructively we see the population maxima. Correspondingly in the case of destructive interference, the system remains in the ground state. This is called the coherent destruction of tunneling (CDT) \[35, 33\]. We get the anti-resonance conditions

$$\left(1 - P_{LZ}\right) \cos \frac{\varphi + 4\phi_S}{2} + P_{LZ} \cos \frac{\varphi_R - \varphi_L}{2} = 0$$ \hspace{1cm} (2.54)

$$\varphi_L + 2\phi_S = n \times 2\pi.$$ \hspace{1cm} (2.55)

We can now restrict ourselves to two regions. First, let us assume that the adiabatic condition is fulfilled, i.e. we travel through the crossing slowly and, consequently, $P_{LZ} \ll 1$. In this region we have the strongest resonance when

$$\varphi_L + \varphi_R + 4\phi_S = m \times 2\pi$$ \hspace{1cm} (2.56)

This is the adiabatic resonance condition. The deeper we are operating in the adiabatic regime the better the latter equality holds. As we shall see in the
next chapter, the resonance lines lie on top of (2.56) and the intersecting lines (2.53) determine the locations of the maxima of the excited state occupation. On the other hand, if we assume the diabatic following, we have that $1 - P_{LZ} \ll 1$. In that case,

$$\varphi_R - \varphi_L = m \times 2\pi,$$

which is the diabatic resonance condition. The resonance lines lie now on top of (2.57) and the intersecting lines (2.53), again, determine the locations of the excited state occupation maxima. Condition (2.57) was used in the interpretation of [37] which had $\Delta = 4$ MHz and $\omega = 2\pi \times 1.2$ GHz which place the system comfortably into the diabatic regime. If they had also analyzed (2.53) they would have located the minimum and maximum populations without RWA analysis. We will return to this when we have introduced our artificial atom.

Rabi Picture

The above discussion deals with an atom whose uncoupled energy levels oscillate in time. The off-diagonal coupling in the Hamiltonian leads to population oscillations between the uncoupled states which are treated in terms of interference between successive transitions at the avoided crossing point. The results were derived assuming that the time evolution can be discretised leading to a countable set of unitary operations that rotate the initial state vector forward in time.

On the other hand, if we want to treat time continuously we can do it by transforming the system into a different frame. Again, we consider the LZ-Hamiltonian

$$\hat{H}_{LZ} = \frac{\hbar}{2}(\omega_0 + \omega(t)) \hat{\sigma}_z + \frac{\hbar \Delta}{2} \hat{\sigma}_x.$$  \hspace{1cm} (2.58)

Let us then make a unitary rotation $\hat{U}^\dagger \hat{H} \hat{U} + i\hbar \frac{\partial \hat{U}^\dagger}{\partial t} \hat{U}$ into a non-uniformly rotating frame with

$$\hat{U} = \exp \left[ -i\hat{\sigma}_z \int_{-\infty}^t \frac{\omega(t')}{2} dt' \right].$$ \hspace{1cm} (2.59)

It is also worthwhile to notice that the transformation leaves the level occupations unchanged

$$\hat{U}^\dagger |\Psi\rangle = \hat{U}^\dagger \left( |\psi_{\uparrow}\rangle \uparrow + |\psi_{\downarrow}\rangle \downarrow \right)$$

$$= e^{i\int_{-\infty}^t \frac{\omega(t')}{2} dt'} |\psi_{\uparrow}\rangle \uparrow + e^{-i\int_{-\infty}^t \frac{\omega(t')}{2} dt'} |\psi_{\downarrow}\rangle \downarrow.$$ \hspace{1cm} (2.60)

Let us then make an assumption that we carry along throughout this thesis. We take that the oscillations of the level separation are periodic and
monochromatic. This is the case when an atom is coupled with a laser, and especially, the case in many experiments done with the artificial superconducting qubits that study quantum interference phenomena in macroscopic scale \[36, 37, 38, 39\]. We, thus, assume that \( \omega(t) = A_L \cos \omega_L t \). This way the LZ-Hamiltonian is represented in the rotating frame as

\[
\hat{H}_R = \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \frac{\hbar \Delta}{2} \left( e^{i \frac{A_L}{\omega_L} \sin \omega_L t} \hat{\sigma}_+ + e^{-i \frac{A_L}{\omega_L} \sin \omega_L t} \hat{\sigma}_- \right) = \frac{\hbar}{2} \left[ \omega_0 \hat{\sigma}_z + \sum_{k=-\infty}^{\infty} \left( \Omega_k \hat{\sigma}_+ + \Omega_{-k} \hat{\sigma}_- \right) e^{ik \omega_L t} \right],
\]

which we refer to as the Rabi Hamiltonian. We have defined the \( n \)-photon Rabi coupling coefficients \( \Omega_k = \Delta J_k \left( \frac{\Delta}{\omega_L} \right) \) where \( J_k(x) \) are first-order Bessel functions. In the last equality we have used the Jacobi-Anger relations \[40\]. We see that the time-dependence in the level separation can be transformed into a driving field containing the harmonics of the drive frequency. Accordingly, the basis states in this frame are called the longitudinal states.

It is instructive to make a so-called secular approximation where we neglect all but one of the frequencies which can be done when \( \omega_0 \approx n \omega_L \). This leads to

\[
\hat{H}_R = \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \Omega_n e^{i n \omega_L t} \\ \Omega_n e^{-i n \omega_L t} & \omega_0 \end{pmatrix}.
\]

The above (multi-photon) Rabi Hamiltonian describes the \( n \)-photon Rabi oscillations of the excited state population. The oscillations occur at the generalized Rabi frequency \( \Omega_R^{(n)} = \sqrt{\left( \omega_0 - n \omega_L \right)^2 + \Omega_n^2} \). The atom and the field are in resonance when they obey the RWA resonance condition

\[
\omega_0 = n \omega_L.
\]

We discussed earlier in the case of quantum field, that in the limit of strong Rabi driving, the inclusion of the counter-rotating term leads to Bloch-Siegert shift to this simple resonance condition. In natural atoms this effect is usually small, but we show in the following chapter that in solid-state analogs we have to go even beyond the BS-correction.

Let us consider resonant Rabi oscillations. We notice that whenever \( \Omega_n = 0 \) there are no oscillations and the system remains in the ground state. These zeroes of the resonant \( n \)-photon Rabi frequencies occur when the corresponding Bessel functions \( J_n \left( \frac{\Delta}{\omega_L} \right) \) vanish. In the discussion on the destructive interference in the LZ-picture, we called this phenomenon coherent destruction of tunneling. Now in the rotating Rabi-frame, we see that the interference oscillations of the excited state population can be alternatively seen as the Bessel-modulation of the Rabi frequency.
A typical situation in atomic physics deals with single-photon Rabi coupling \((n = 1)\), conventionally produced directly with an oscillating transverse \((\hat{\sigma}_x)\) drive. This leads to the celebrated Rabi oscillations of the excited state occupation \([30]\), which can be interpreted as an indication of a coherent quantum evolution. In superconducting artificial atoms, coherent oscillations have been observed first in Reference \([41]\). Multi-photon Rabi dynamics and the above indicated Bessel-dependence were seen in Reference \([36]\) and later in References \([37, 39]\).

2.3 Molecules

All life on Earth is made of molecules. They are electrically neutral groups of atoms that are held together with chemical bonds. The size of a molecule can range from Ångstrom-size \((10^{-10} \text{ meters})\) diatomic molecules (like \(\text{H}_2\)) up to macroscopic size of the human DNA \((10^{-6} \text{ meters} \text{ when folded; two meters when extended})\). In this context we will consider only diatomic molecules which consist of two atoms.

Like an atom, a molecule consists of a nucleus and a surrounding cloud of electrons. Because the nucleus consists of multiple nuclei, we have two new degrees of freedom. The nuclei can vibrate with respect to some equilibrium separation and there can also be rotational motion around some fixed axis. The nuclei are in general much heavier than the electrons. Accordingly, the electronic motion is much faster meaning that the nuclei occupy nearly fixed positions in the molecule. This allows us to separate the degrees of freedom in the wave function of the molecule as

\[
|\Psi\rangle = |\Psi_e\rangle|\Psi_n\rangle, \tag{2.65}
\]

which is the Born-Oppenheimer approximation \([42]\). Furtheron, the nuclear part can be written as

\[
|\Psi_n\rangle = |\Psi_v\rangle|\Psi_r\rangle, \tag{2.66}
\]

where \(|\Psi_v\rangle\) and \(|\Psi_r\rangle\) describe the vibrational and rotational states of the nucleus, respectively. Since \(|\Psi_v\rangle\) are the vibrational state vectors they can be taken in the first approximation to be those of the simple harmonic oscillator \([42]\).

2.3.1 Vibronic Transitions

We want to study the diatomic molecule spectroscopically. The electromagnetic field can induce transitions between states \((2.65)\) with different quantum numbers. We are interested in the intensities of such transitions where the rotational state does not change. We allow simultaneous transitions of the vibrational and electronic states which are called vibronic. Since the nuclei
are much heavier they change their relative position and momentum only after an electronic transition. This means that the electronic transition can be taken instantaneous in the time scale of the vibrations.

Let us consider a transition between two electronic states. We have depicted the situation in Figure 2.4. There are two nearly parabolic vibrational potentials, one for each electronic state. In general, the locations of the minima of the potentials are different allowing the vibronic transitions. We have denoted electronic transitions with vertical arrows in Figure 2.4. The most intense transitions occur when the overlap integral

$$f_{v',v} = \int \langle \Psi_{v'} | \Psi_v \rangle dR$$

between vibrational states $|\Psi_v\rangle$ and $|\Psi_{v'}\rangle$ in different electronic states has a maximum [42]. Classically this means that the most intense vibronic transitions occur between vibrational states that have coincident turning points. This is called the Franck-Condon principle [44, 45].
Chapter 3

Physics of the Artificial

There exists no more powerful tool in learning than a good analogy. Linking an unfamiliar phenomenon with something that one knows well often leads to better understanding and deeper knowledge. But analogies are double-edged swords. One has to be very careful about limitations on the depth of the identifications that can be made. No matter how good it is, if an analogy is drawn too far, it may send the negligent mind onto wrong tracks for a long time.

The prefix artificial has been used abundantly in this chapter. The meaning of the word could be rephrased as something that is produced as an imitation of something natural. But in this case, what is produced to imitate what? The physics in this chapter is the same as in the previous one. It is the physics of the quanta. The main characters are also the same: we will discuss on atoms and light. But contrary to the previous chapter, the number of atoms in the objects escalates from one to a macroscopic amount of ten to the power of ten, or so. It will be shown, that by a careful design and engineering we can produce macroscopic elements that are mathematically indistinguishable from the elements of which they were built. This is what is meant by the word artificial in this chapter. Artificial elements are such that when we describe them in terms of quantum physics their behaviour is analogous to the very elements of which they were made.

3.1 Preliminaries

3.1.1 Superconductivity

The artificial objects in this thesis are made out of superconducting materials. Superconductivity itself is an intriguing phenomenon of Nature. Many materials lose their electric resistivity below some material dependent critical temperature $T_c$. This was discovered in 1911 by Dutch physicist H. Kamerlingh Onnes [46, 47]. Another fundamental property of the superconducting
state is, that the magnetic field is excluded from the material. This is known as the *Meissner effect* [46, 48]. Even though it serves as the backbone of our artificial devices, superconductivity is discussed in this thesis only qualitatively without going into deep details. The discussion follows loosely Reference [49].

J. Bardeen, L. N. Cooper and J. R. Schrieffer presented in 1957 a microscopic theory of superconductivity, called the *BCS-theory* [50]. They showed that superconductivity can be explained if the electrons can somehow overcome the Coulomb repulsion and form pairs. This attraction has its origin in electron-phonon interactions between the electrons and the ionic lattice. These so-called *Cooper pairs* consist of two electrons which can, via weak attraction, be spread over considerable distances and several pairs can occupy the same region of space at the same time [49].

But how do these Cooper pairs explain superconductivity? The answer lies in the spin of a Cooper pair. A single electron is a fermion and, therefore, follows the *Pauli exclusion principle*, which says that no two identical fermions can be in the same physical state at the same time. On the other hand, a Cooper pair consists of two electrons and is, therefore, a boson. The bosons do not follow Pauli's principle. Quite the contrary, when there are many bosons in a given state there is an especially large probability for the others to go to the same state [49]. This way one can describe the state of a superconductor by a single wave function

$$\Psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r})} e^{i\theta(\mathbf{r})},$$  \hfill (3.1)

that describes the behaviour of the whole electron system as a unit. Here $\rho(\mathbf{r})$ is proportional to the Cooper-pair density and $\theta(\mathbf{r})$ is the phase of the wave function.

As can be noted from above, all the Cooper pairs in a superconductor are in a common physical state and therefore have the same energy. This common state is referred to the *ground state* of the superconductor and the Cooper pairs are said to form a *condensate*. If one wants to excite this state, a finite energy must be expended. The finite energy difference between the ground state and the excited state is called the *energy gap* and denoted by $\Delta_g$. The excitation of the system can be described as a breakup of a Cooper pair and the energy needed for such an operation is $2\Delta_g$, which is the so-called *pair binding energy*. The critical temperature can now be explained by the existence of the energy gap. If the temperature $T \lesssim \Delta_g/k_B$ ($k_B$ is the Boltzmann constant), the electrons form Cooper pairs and the metal is in the superconducting state.

For example, the absence of the electrical resistance of the superconductors can be explained by the energy gap. That is, because it is known that the electrical resistance of a metal is due to the interaction between the moving electron system (that is, the electric current) and the vibrations of the lattice
or with impurities. Due to the energy gap, however, the needed quantum transitions will not always be possible. This implies, that the electron system will not be excited when it is moving slowly. It means, that the charge transfer is frictionless and no electrical resistance occurs.\textsuperscript{51}

An astonishing thing in the BCS-theory is the sole existence of the superconducting gap. It means that the energy spectrum of the common wave function is discrete. A superconductor described by billions of Cooper pairs forms a single entity and can, thus, be called a \emph{macroscopic quantum object}.

### 3.1.2 Josephson Junction

In 1962 B. D. Josephson\textsuperscript{52} was analyzing a situation where two superconducting regions have been connected by a weak insulating layer. He predicted that even when there is no potential difference between the two regions there exists a direct supercurrent through the junction. He also noted that if the two regions are connected to the two terminals of a battery, the current starts to oscillate at high frequency. The Josephson tunneling effect was shown experimentally already in 1963 by P. W. Anderson and J. M. Rowell\textsuperscript{53} and the oscillations were seen in 1965 by I. K. Yanson et al.\textsuperscript{54}.

The Josephson junctions are very interesting because with them one can display quantum effects on a macroscopic scale. Due to their discrete and nonlinear energy structure, one can call them \emph{artificial atoms}. These junctions have numerous applications. For example, two such junctions, when connected in series, can be used in very sensitive charge measurements\textsuperscript{49}. With many different compositions, one can build quantum two-level systems, or qubits, which are the building blocks of the quantum computer\textsuperscript{55}.

**Josephson Effect**

When the insulating layer between the two superconductors is thin enough, the probability for the Cooper pairs to tunnel through it becomes sufficiently large. If the coupling between the superconductors is weak, one can write the resulting Cooper pair current as

\begin{equation}
I = I_c \sin \varphi,
\end{equation}

where the material dependent $I_c$ is the \emph{critical current}, the maximum current the junction can carry without dissipation. This is the so-called \emph{dc Josephson effect} that relates the supercurrent through the junction with the quantum mechanical phase difference $\varphi$ between the condensates. The periodic dependence on the phase is a result from the discreteness of the tunneling Cooper pairs\textsuperscript{57}. 

If one applies a potential difference $V$ across the junction, the phase difference evolves according to

$$\dot{\phi} = \frac{2eV}{\hbar}$$  \hfill (3.3)

which is referred to as the *ac Josephson relation*. Now, we can write the phase in terms of the branch flux $\Phi_J$ (see the definition of the branch flux in Appendix B) as

$$\phi = \frac{2\pi \Phi_J}{\Phi_0} \mod 2\pi,$$  \hfill (3.4)

where $\Phi_0 = h/2e$ is the superconducting flux quantum. When the fluctuations of flux are small ($|\Phi_J| \ll \Phi_0$) one can linearize the current-flux relation. We recall that inductance is defined as the linear proportionality constant between the flux and the current ($\Phi = LI$). Thus, we can identify the *linear Josephson inductance*

$$L_{J0}^{-1} = \frac{2\pi I_c}{\Phi_0},$$  \hfill (3.5)

which depends only on the circuit materials and fundamental constants.

**Josephson Energy**

We show in Appendix B that Josephson junction can store a (inductive) potential energy

$$U_J = -E_J \cos \frac{2\pi \Phi_J}{\Phi_0}$$  \hfill (3.6)

which is referred to as the *Josephson energy*. In the above, we have defined the *Josephson coupling* $E_J = I_c \Phi_0/2\pi$. A potential difference $V$ builds up charges with opposite signs at the interfaces between the condensates and the insulator. This means that the junction has also capacitive properties characterised by the capacitance $C_J$. Accordingly, the junction has the *charging energy*

$$T_J = \frac{Q_J^2}{2C_J},$$  \hfill (3.7)

where the charge at the "capacitor plates" of the junction is $Q_J = C_J \dot{\Phi}_J$. The circuit equivalent of this is depicted in Figure 3.1.
3.1. PRELIMINARIES

Figure 3.2: Inductance $L_J$ and the potential energy $U_J$ of a Josephson junction (in arbitrary units).

The charging energy can be interpreted as the kinetic energy of the junction leading to the classical Hamiltonian function

$$H_J = \frac{Q_J^2}{2C_J} - E_J \cos \frac{2\pi \Phi_J}{\Phi_0}. \quad (3.8)$$

If we choose $\Phi_J$ as our canonical coordinate (see Appendix B) we identify the charge $Q_J$ as the corresponding canonical momentum. It is instructive to write down the Hamilton equations of motion

$$V_J \equiv \dot{\Phi}_J = \frac{dH_J}{dQ_J} = \frac{Q_J}{C_J}, \quad (3.9)$$

$$I_J \equiv -\dot{Q}_J = \frac{dH_J}{d\Phi_J} = I_c \sin \frac{2\pi \Phi_J}{\Phi_0}, \quad (3.10)$$

where we see that the voltage $V_J$ across the junction and the supercurrent $I_J$ through it can be obtained straight from the Hamiltonian function.

Again, if we assume that the fluctuations in the flux $\Phi_J$ are small we obtain (we neglect the constant energy $-E_J$)

$$H_J \approx \frac{Q_J^2}{2C_J} + \frac{\Phi_J^2}{2L_{J0}}. \quad (3.11)$$

We notice that if we identify $\Phi_J$ as the coordinate and $Q_J$ the corresponding canonical momentum, the linearized Hamiltonian describes a harmonic oscillator with natural frequency

$$\omega_J = 1/\sqrt{L_{J0}C_J} = \sqrt{8E_cE_J/\hbar}. \quad (3.12)$$

The charging energy of a single electron is defined as $E_c = e^2/2C_J$. As the fluctuations in the flux start to grow, the anharmonic terms in the cosine-potential become noticeable. So, we see that the Josephson junction can be
used as a nonlinear superconductive circuit element. Next we will show how this nonlinearity can be exploited in mimicking atomic physics on a circuit board.

We saw earlier in Equation (3.5) that in the limit of small flux, the current-flux relationship is linear which can be used as a definition of the linear inductance. In general, the fluctuations may not be centered around \( \Phi_J = 0 \). This means that we have to take into account the curvature \( \frac{d^2 H_J}{d\Phi^2} \) of the total energy when determining the inductance. We generalize \( \Phi = LI \) by the definition of the nonlinear Josephson inductance

\[
L_J^{-1} = \frac{d^2 H_J}{d\Phi^2} = L_{J0}^{-1} \cos \frac{2\pi \Phi_J}{\Phi_0}.
\]

(3.13)

We see in Figure 3.2 that under proper choices of the flux \( \Phi_J \) the Josephson inductance can become infinite or negative.

**Macroscopic Quantum Effects**

The discussion above shows that it is possible to describe the macroscopic state of the Cooper pairs with a single collective state vector. This is often called the semi-classical picture of the Josephson junction, which is somewhat misleading since we are talking in terms of the quantum mechanical phase difference. Essentially what has been done is, that we have made the so-called first quantization of the Josephson junction. What we do next is that we quantize also the phase difference \( \varphi \) (i.e. the flux \( \Phi \)) which is analogous to the quantization of the state vector in the quantum field theory. Accordingly, we call this procedure the second quantization of the Josephson junction.

The quantization procedure follows the guidelines of that of the harmonic oscillator (2.2), i.e.

\[
\Phi_J \rightarrow \hat{\Phi}_J, \quad Q_J \rightarrow \hat{Q}_J
\]

(3.14)

\[
\{\Phi_J, Q_J\} = 1 \rightarrow [\hat{\Phi}_J, \hat{Q}_J] = i\hbar.
\]

(3.15)

We choose to write down the resulting Hamiltonian operator in the eigenbasis \( \{|N\}, N \in \mathbb{Z}\) of the operator \( \hat{Q}_J = 2e\hat{N} \) that describes the charge that is tunneled through the junction. Since the charge is in the form of Cooper pairs, we can describe the charge states by the number of tunneled Cooper pairs \( N \). We end up with the Hamiltonian operator

\[
\hat{H}_J = \sum_N \left[ 4E_c(N - N_0)^2|N\rangle\langle N| - \frac{E_J}{2}\left( |N + 1\rangle + |N - 1\rangle \right)\langle N| \right].
\]

(3.16)

We have included the residual offset charge \( Q_0 = 2eN_0 \) into the Hamiltonian. It has its origin in the charged impurities, and one can show that by coupling the junction capacitively to the external circuitry, one can control \( Q_0 \)
3.1. PRELIMINARIES

Figure 3.3: Energy spectrum of the Josephson junction with $E_J/E_c = 0.5$ (left) and $E_J/E_c = 3$ (right). The dashed lines are the charge state energies with $N = -2, -1, 0, 1, 2, 3$ (0 and 1 indicated in the plots).

continuously \[56\] (this the so-called Cooper-pair box setup). This should be contrasted with the charge $Q_J$ of the tunneled Cooper pairs which can have only values that are integer multiples of $2e$. The offset charge $Q_0$ has a similar meaning as the vector potential in the case of an electron in a magnetic field \[57\]. One should also notice, that the Josephson term couples charge states that differ by just one Cooper pair.

We can solve the eigenenergies of (3.16) by diagonalising the corresponding truncated Hamiltonian matrix (the eigenenergies can be solved also in the flux basis, where it becomes a matter of solving the so-called Mathieu equation \[58\]). We can study the resulting energy spectrum in two limiting cases of the ratio $E_J/E_c$. When $E_J/E_c \ll 1$, we see that the Josephson coupling is small and the eigenvalues are close those of the charge states $|N\rangle$. We refer to this limit, accordingly, as the charge limit. In the opposite limit, $E_J/E_c \gg 1$, the Josephson coupling is dominant which means that the level separations become equidistant (with splitting $\hbar \omega_J$, see Equation (3.12)) and the junction resembles a harmonic oscillator. We call this the flux limit.

These features have been plotted in Figure 3.3, where we show the first four energies as a function of the offset charge. On the left-hand side plot, we have $E_J/E_c = 0.5$ and the energies follow closely the charge energies (black dashed lines). As the ratio becomes bigger, $E_J/E_c = 3$, the deviations from the charge states start to grow. This is depicted on the right.

We comment on two important facts. Firstly, the allowed energies of the Josephson junction are discrete. In this sense they are similar to the atoms in the ”real world” and it is proper to call them artificial atoms. Later, we will show that with small adjustments to the circuitry, we can obtain control over the offset charge $Q_0$ and the Josephson coupling energy $E_J$. This means that we can adjust the energy structure of these atoms, a property that is not
available, or at least very limited, in the natural atoms. Secondly, especially close to the charge limit, the energy level spacings are nonlinear which allows their use as qubits (see Chapter 1).

### 3.2 Artificial Atoms

We are interested in creating atomic physics experiments on a circuit board. We would also some day like to build a working quantum computer whose computational power in certain tasks exponentially exceeds its classical counterparts. There are many advantages in building the components and wiring out of superconducting materials. We have already discussed that the superconducting state is dissipationless which plays an important role in the preservation of quantum coherence. Also, we have shown that Josephson junctions have discrete energy levels. The parameters of the junctions are not fundamental constants of Nature, as in natural atoms, but can be changed in lithographic process of fabrication. There is also the possibility to control externally the parameters during an experiment, as we will soon see. Josephson junctions are inherently nonlinear allowing the energy structure to be non-equidistant. In fact, the Josephson junction is the only circuit element that is both nonlinear and non-dissipative at arbitrarily low temperatures.

There are a couple of important requirements that have to be fulfilled before superconducting atoms can display quantum effects. First, the temperature has to be smaller than the relevant level separation in the atom. This prevents the thermal fluctuations from disturbing the atomic operations. Second, the atomic energy has to be smaller than energy gap of the used superconducting material. This way the use of control signals (that are in the neighborhood of the atomic energy splitting) do not lead to quasiparticle creation or the destruction of the state.

We concentrate here on one particular realisation of an artificial atom, namely the single-Cooper-pair transistor. We show that it works as a single Josephson junction, but whose offset charge and Josephson coupling energy can be adjusted with electric and magnetic fields, respectively. This gives control over the energy spectrum of the atom, which is a useful feature in quantum computing and allows the testing of atomic physics and quantum optics in parameter regions prohibited from natural atoms. We also couple the atom with an $LC$-oscillator and show that one can mimic matter-light interactions with such a construct. The linear atom-field coupling strength in our system can be among the strongest ever realised. We discuss also the downside of the strong interaction leading to coupling with unwanted electromagnetic modes and resulting in decoherence effects ultimately preventing us from reaching the strong coupling regime of cavity QED in our
3.2. ARTIFICIAL ATOMS

Figure 3.4: Single-Cooper-pair transistor.

3.2.1 Charge-Flux Atom

We saw earlier that the eigenenergies of the Josephson junction are dependent on the offset charge $Q_0$ and the ratio $E_J/E_c$. Here we will present slight modifications to the circuitry which allow us to control these parameters with electric and magnetic fields (notice that the charging energy $E_c$ is difficult to control since it depends only on the capacitance of the junction).

In Figure 3.4, one can see the circuit diagram of our modified Josephson junction. It consists of a superconducting island that is coupled to the rest of the circuit via two Josephson junctions. We couple a voltage source $V_g$ to the island via the gate capacitance $C_g$. It can be used in controlling the charge to and from the island. Additionally, we will show later that if we can control the flux $\Phi = \Phi_1 + \Phi_2$ through the device we have means to adjust the effective Josephson energy of the "junction". This kind of setup is traditionally called the single-Cooper-pair transistor (SCPT).

We can use the quantum network theory \[60, 61\] and derive the Hamiltonian operator of the circuit. In Appendix B.1 we have gone through the intermediate steps for the SCPT for two reasons. First, it is a good test for the procedure. Second, a two-level system built using the SCPT is the main work-horse in this thesis so it is appropriate that the foundation is laid properly.

According to Appendix B.1 the Hamiltonian operator of the SCPT in charge basis is

$$\hat{H}_{\text{scpt}} = \sum_N \left[8E_c(N - N_0)^2|N\rangle - \frac{E_J(\Phi)}{2} \left(e^{i\phi}|N + 1\rangle + e^{-i\phi}|N - 1\rangle\right)\right] \langle N|,$$

(3.17)

where the charging energy $E_c = e^2/2C$; and the effective Josephson energy $E_J(\Phi) = E_{J0} \sqrt{\cos^2 \frac{\pi \Phi}{\Phi_0} + d^2 \sin^2 \frac{\pi \Phi}{\Phi_0}}$. Also, the offset charge $Q_0 = C_g V_g =$
2eN_0 and \( \tan \phi = -d \tan \frac{\Phi_0}{\Phi_0} \). We have defined \( E_{J0} = E_{J1} + E_{J2} \) and the asymmetry \( d = (E_{J2} - E_{J1})/E_{J0} \). We see that when the two junctions are symmetric (\( d = 0 \)) the Hamiltonian has the same form as (3.16). What we have achieved with our adjustments, is that we have control over the offset charge \( Q_0 \) and the Josephson energy \( E_J \). This means that the energy spectrum has a wide range of tunability that can be easily exploited during the measurements, which is a common feature of the artificial atoms made of superconducting tunnel junctions. We refer to our SCPT as the charge-flux atom because of the nature of the control parameters.

In Figure 3.5, we have plotted the three lowest eigenenergies of the SCPT with \( E_{J0}/E_c = 4 \) and \( d = 0.13 \). We have varied the flux bias from \( \Phi = 0.5\Phi_0 \) to \( \Phi = \Phi_0 \) resulting in bands of allowed energies. From the Figure, we see that the Josephson coupling lifts the degeneracies of the charge states leading to avoided crossings in the energy diagram at locations \( N_0 = k/2 \) where \( k \) is an integer, similarly to the single Josephson junction. We refer to these locations as charge degeneracy points. We observe that when \( \Phi \approx \Phi_0/2 \) the eigenenergies deviate from the charge states only in narrow regions near the crossing points. Exactly at flux degeneracy, \( \Phi = \Phi_0/2 \), the coupling between the two charge states is the smallest which leads to an avoided crossing of magnitude \( dE_{J0} \).

Let us then concentrate on the case, where \( dE_{J0} \ll E_c \). Similarly as in the case of a single junction, one can assume that the Josephson term couples only two (nearly) degenerate charge states. Accordingly, one can truncate
the Hamiltonian as
\[
\hat{H}_q \approx \frac{1}{2} \left[ E_{el} \hat{\sigma}_z - E_J(\Phi) \left( \cos \phi \hat{\sigma}_x + \sin \phi \hat{\sigma}_y \right) \right],
\]
where we have shifted the zero of energy into the middle of the gap at the avoided crossing by defining \( E_{el} = 4E_C(1 - 2N_0) \). We can calculate analytically the eigenenergies for the ground (-) and excited (+) states and obtain \( E_{\pm} = \frac{1}{2} \sqrt{E_{el}^2 + E_J(\Phi)^2} \). According to the preceding nomenclature, we will refer to this two-level atom as charge-flux qubit.

3.3 Modern Dance of Artificial Light and Matter

3.3.1 Dressed Artificial Atom

Now that we have introduced our artificial atom we would like to couple it with light. But what is light in electric circuits? We recall that light was modelled as a harmonic oscillator in the previous chapter. In circuit theory, we have a straightforward analog that is formed by an inductor and a capacitor. This so-called \( LC \)-resonator behaves like a source of monochromatic photons that can be easily coupled to the atom simply with leads. We present in Figure 3.6 the circuit schematic of the coupling. In addition to the oscillator, we have added an external coil \( L_{ext} \) that can be used in controlling the flux through the qubit. This turns out to have dramatic effects on the qubit properties, as we will point out later.

Again, we have gone through the necessary circuit theory in Appendix B.2 in order to see that we can imitate light-matter interactions. In the Appendix, we have shown that the branch flux of the qubit is determined by the oscillator flux \( \Phi \) and the external flux \( \Phi_{ext} \) through the inductance loop. The
Hamiltonian operator for the coupled system (in the two-state approximation (3.18)) is

\[
\hat{H}_{qo} = \frac{\hat{q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} \\
+ \frac{1}{2} \left\{ E_{el} \hat{\sigma}_z - E_{J0} \left[ \cos \frac{\pi \hat{\Phi}}{\Phi_0} \left( \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x - d \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_y \right) \\
- \sin \frac{\pi \hat{\Phi}}{\Phi_0} \left( \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x + d \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_y \right) \right] \right\},
\]

(3.19)

where we have separated the two flux contributions and assumed that the external flux \(\Phi_{ext} = \Phi_b\) is time-independent. We see that the coupling between the qubit and the field is strongly nonlinear. We have assumed that the circuit is capacitively symmetric \((C_1 = C_2 \text{ and } C_3 = C_4)\) and defined the oscillator capacitance \(C = (C_1 + C_3)/2\).

At this stage, we have already a kind of an analog with the elementary world. The charge-flux qubit can be interpreted as an atom and the LC-oscillator as light. There is an interaction which is nonlinear, but looks attractive enough to make us say that we are talking about some kind of matter-light interactions in a completely different context. But, we can do more. Let us linearize the Hamiltonian \(\hat{H}_{qo}\) with respect to the interaction with the oscillator flux \(\hat{\Phi}\). We obtain

\[
\hat{H}_{qo} \approx \frac{1}{2} \left\{ E_{el} \hat{\sigma}_z - E_{J0} \left[ \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x - d \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_y \right] \\
- \sin \frac{\pi \hat{\Phi}}{\Phi_0} \left( \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x + d \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_y \right) \right\} + \frac{\hat{q}^2}{2C} + \frac{\hat{\Phi}^2}{2L},
\]

(3.20)

Now, if we take \(d = 0\) and \(\Phi_b = 0.5 \Phi_0\) we obtain

\[
\hat{H}_{qo} = \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \hbar g (\hat{a}^\dagger + \hat{a}) (\hat{\sigma}_+ + \hat{\sigma}_-) + \hbar \omega_L \hat{\sigma}^\dagger \hat{\sigma},
\]

(3.21)

where we have denoted the oscillator (light) frequency with \(\omega_L = 1/\sqrt{LC}\) and the qubit energy with \(\hbar \omega_0 = E_{el}\). The creation and annihilation operators of the electric oscillator are defined as

\[
\hat{a} = \frac{1}{\sqrt{2\hbar C \omega_L}} (C \omega_L \hat{\Phi} + i \hat{q}) \\
\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar C \omega_L}} (C \omega_L \hat{\Phi} - i \hat{q}),
\]

(3.22)

The coupling constant is written as

\[
g/2\pi = \frac{E_{J0}}{2\hbar} \sqrt{\frac{\pi Z}{R_K}} \approx 0.5 \text{ GHz},
\]

(3.23)
where the characteristic impedance of the oscillator $Z = \sqrt{Z/C}$ and the resistance quantum $R_K = \frac{\hbar}{e}$. It is interesting to compare this value with those of the other similar systems. Even without optimisation of the parameters, our coupling seems to be among the strongest ever observed in natural or artificial cavity-QED systems [16]. The only thing preventing us from entering the strong coupling regime has been the coupling of our atom with undesired modes of the environment. As was discussed in the previous chapter, the Rabi coupling $g$ should be much larger than any of the relaxation rates of the system. Our qubits have relaxation times of order of nanoseconds, leading to rates of the order of the coupling which inhibits us from making strong coupling regime measurements. This problem probably has its origin in our implementation of the cavity. We use lumped LC-oscillators that may suffer from possible unwanted radiation and parasitic resonances in the wiring [62].

Equation (3.21) describes a linear interaction between an electric two-state system and an LC-oscillator. The interaction term is identical to the Hamiltonian (2.32), which appeared in the context of an atom coupled to an electromagnetic field mode. Mathematically, we cannot distinguish the electric version from the natural one, so it is safe to say that we can imitate the matter-light interactions with our circuit. There are, however, some subtleties that need to be discussed. The physics behind these systems is completely different. Whereas in the atomic case there is only one electron interacting with the electromagnetic field, there are a macroscopic number of Cooper pairs in the SCPT island. The charge states of this island differing by just one Cooper pair can be coupled with the LC-oscillator that provides the quanta in a form of excitations in the electric circuit. The most intriguing property is the possibility to control the parameters of the atoms and the strength of their coupling with the field, which are either impossible or at least very hard to do in the realm of natural atoms. The coupling with the artificial atom and light is determined by parameters $E_{J0}$, $C$ and $L$ which can be modified during the fabrication process by changing the dimensions of the elements [57]. As discussed in the previous chapter, the strength of the interaction between an atom and an electromagnetic field mode is characterized by the fine structure constant $\alpha \simeq 1/137$. The smallness of $\alpha$ makes the interaction in free space very weak and poses difficulties in reaching the strong-coupling regime also in the natural cavity-QED. The fine structure constant $\alpha_A = \sqrt{Z/R_K}$ in our circuit (and similar constants in other circuit-QED systems) can be changed in the engineering process and can have more or less arbitrary values. There is a nice comparison of the fine structure constants in cavity- and circuit-QED systems that can be found in [15].

The discussion above shows that one can mimic the matter-light interactions with electric circuits. There were theoretical proposals for coupling the
artificial atom with different kind of realisations of the photon source \cite{63,62}, but the field really started to bloom in 2004 when the Delft and Yale groups reported the first experimental observations of vacuum Rabi oscillations in solid-state systems \cite{21,22}. Since then, the vacuum Rabi oscillations have been observed also in the longitudinal dressed states of the flux qubit \cite{24}. Progress in the field has been rapid ever since and we have tried to cite the most of the important results already in the previous discussions.

### 3.3.2 Landau-Zener Picture

We have seen that it is possible to couple the artificial atom with the zero-point vibrations and also with finite photon numbers of the LC-oscillator. The short relaxation times of our qubits prevent us to make strong coupling experiments at the quantum limit of the oscillator. We have instead considered a situation where, in addition to the oscillator coupling, there is also a strong classical driving of the atom induced by the external flux. This is produced by monochromatic flux drive \( \Phi_{\text{ext}} = \Phi_b + \Phi_L \cos \omega_L t \). We emphasize that a similar situation is hard to achieve with ordinary atoms since they couple weakly to the laser fields. In the following we first neglect the oscillator and later use it as a weak probe of the energy levels of the strongly driven atom.

The energy levels of the artificial atoms can be changed by some parameter(s) (\( \Phi_b \) and \( Q_0 \), in our two-level system) that can be controlled with magnetic and electric fields. There often are values of the control parameters at which the levels intersect, as was discussed already in the previous chapter. The degeneracy can be lifted by coupling the two intersecting energy levels leading to an avoided crossing. This kind of situation can be seen also in Figure 3.5. We refer to these as adiabatic energies in contrast to the diabatic (intersecting) ones.

If the system is made to travel through the avoided crossing slowly, one expects that it remains in one of the adiabatic states. On the other hand, if the variations of the control parameter is so large that the system crosses the intersection rapidly, there is an asymptotic probability to tunnel from one adiabatic level to another that approaches one (see Appendix B.2). This Landau-Zener (LZ) tunneling was introduced in Equation (2.41). If the system goes through the avoided crossing periodically, the successive LZ-tunneling events can interfere (within the coherence time of the two-level system) leading to the St"uckelberg oscillations of the excited state population. The resulting Landau-Zener-St"uckelberg (LZS) interferometer \cite{32,64,31} gives an alternative perspective to the Rabi oscillations in terms of quantum interference. Solid-state realization of the LZS-interferometer has been achieved in References \cite{37,38}.

We test the presented LZS-model in different parameter regimes of the
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Hamiltonian of our artificial atom

$$\hat{H}_q = -\frac{E_{J0}}{2} \left[ \cos \frac{\pi \Phi_{ext}}{\Phi_0} \hat{\sigma}_x - d \sin \frac{\pi \Phi_{ext}}{\Phi_0} \hat{\sigma}_y \right],$$  \hspace{1cm} (3.24)

where we have assumed that we work at the charge degeneracy where the offset charge noise is minimized [65]. The Hamiltonian has (adiabatic) eigenenergies

$$E_{\pm} = \pm \frac{E_{J0}}{2} \sqrt{\cos^2 \frac{\pi \Phi_{ext}}{\Phi_0} + d^2 \sin^2 \frac{\pi \Phi_{ext}}{\Phi_0}},$$  \hspace{1cm} (3.25)

with minimum gap

$$\hbar \Delta = dE_{J0}$$  \hspace{1cm} (3.26)

at flux degeneracy $\Phi_{ext} = \Phi_0/2$. Now we can calculate the speed at which the avoided crossing is traversed. We define the diabatic ($d = 0$) eigenenergies $\epsilon_0 = E_{J0} \cos \frac{\pi \Phi_{ext}}{\Phi_0}$ and $\epsilon_1 = -E_{J0} \cos \frac{\pi \Phi_{ext}}{\Phi_0}$. According to Equation (2.42), we obtain

$$\nu = \frac{1}{\hbar} \frac{d(\epsilon_1 - \epsilon_0)}{dt} \bigg|_{\epsilon_1 = \epsilon_0} = \frac{E_{J0}}{\hbar} \frac{d(\cos \frac{\pi \Phi_{ext}}{\Phi_0})}{dt} \bigg|_{\Phi_{ext} = \Phi_0/2}$$  \hspace{1cm} (3.27)

$$= \frac{\omega_L E_{J0}}{2h \Phi_0} \sqrt{\Phi_0^2 - \left(\frac{\Phi_0}{2} - \Phi_b\right)^2},$$  \hspace{1cm} (3.28)

One needs to have $\Phi_L > |\Phi_0/2 - \Phi_b|$ in order that the avoided crossing is reached. We can also write the adiabatic condition (2.45) in terms of the present system parameters as

$$\beta_A \gg \pi \frac{d \Phi_0}{d \Phi_L} \sqrt{\Phi_0^2 - \left(\frac{\Phi_0}{2} - \Phi_b\right)^2},$$  \hspace{1cm} (3.29)

where we have defined the ratio $\beta_A = \Delta/\omega_L$ and see immediately, that it characterizes the level of adiabaticity of the driven atom. With increasing $\beta_A$ one needs larger and larger amplitudes $\Phi_L$ in order to violate the adiabaticity condition. Therefore, one can roughly identify $\beta_A \gg 1$ as the adiabatic limit and $\beta_A \ll 1$ as the diabatic limit. With the values of $\beta_A \sim 1$, one should observe a continuous transition from the adiabatic to diabatic limit, as the amplitude $\Phi_L$ is increased.

To visualise this, we have calculated numerically the excited state population $P_\uparrow$ as a function of $\Phi_b$ and $\Phi_L$. The computational basis is taken to be the eigenbasis defined at the oscillation center, introduced in the Appendix [B.1]. We have used the values $\beta_A = 3.5$ and 1. The results can be seen in Figure 3.7. On top of the contours we have plotted the adiabatic (magenta) and diabatic (blue) resonance conditions defined in Equations (2.50) and (2.57), respectively. Also, we have indicated with solid and dashed green
Figure 3.7: Population of the excited state $P_\uparrow$ calculated numerically using Bloch equations (2.25). We have used $\beta_A = 3.5$ (left) and $\beta_A = 1$ (right). Above the solid black line, the amplitude $\Phi_L$ is large enough for the system to reach the avoided crossing. The dashed black line indicates the border between the adiabatic and diabatic regimes. Colored lines determine the locations of resonances (see text). On the left, we see a clear transition from the adiabatic to diabatic regime. When $\beta_A = 1$, the system can be well described in terms of the diabatic conditions.

We have also plotted a solid black line above which the amplitude is large enough for the oscillation to reach the avoided crossing. The dashed black line gives a rough estimate of the border between the diabatic and adiabatic regimes and is defined by

$$\beta_A = \frac{\pi}{d\Phi_0} \sqrt{\Phi_L^2 - \left(\frac{\Phi_0}{2} - \Phi_b\right)^2}.$$  (3.33)

Evidently, in the parameter regime above the border line, we should analyze the resonances in terms of the diabatic resonance conditions. Especially, with $\beta_A = 1$ this area covers almost the whole plot and we can say we are well in the diabatic regime. As we start to lower the frequency, the area below the border becomes bigger. When $\beta_A = 3.5$ we see already clearly, that the locations of the numerical resonance lines and their extrema deviate from the analytic diabatic regime results. We have entered into the adiabatic regime where the resonances should be discussed in terms of adiabatic conditions.

We comment on a special feature seen in Figure 3.7. Contrary to previous realization of the LZS-interferometer in the diabatic limit [37], we notice...
that the resonances in our apparatus are curving. This has its origin in our nonlinear coupling of the driving field. In the Reference [37], the used flux qubit is treated in the linear approximation leads to a linear dependence on the drive amplitude. In our case, the the amplitude of the oscillating flux is strong and the coupling via the Josephson term leads to an unprecedented dependence of the atomic resonance frequency on the field amplitude. Similar phenomenon could also take place in flux qubit interferometry [37, 66] at a very high harmonic drive.

We see in Figure 3.7 that the curving occurs along the diabatic resonance line. In the case of symmetric atom \( (d = 0) \) we can calculate this analytically as

\[
\varphi_R - \varphi_L = \frac{2\pi}{\hbar \omega_L} E_{J0} J_0 \left( \frac{\pi \Phi_L}{\Phi_0} \right) \cos \frac{\pi \Phi_b}{\Phi_0} = k \times 2\pi.
\]  

(3.34)

This means that the traditional multi-photon resonance condition is now \textit{Bessel modulated}

\[
\omega_0 = k \omega_L,
\]  

(3.35)

where we have defined the undriven qubit energy

\[
\hbar \omega_0 = -E_{J0} J_0 \left( \frac{\pi \Phi_L}{\Phi_0} \right) \cos \frac{\pi \Phi_b}{\Phi_0}.
\]  

(3.36)

When the flux amplitude is increased, one has to adjust the bias flux in order to stay in resonance. We can say that due to the nonlinear drive, our atom can now be controlled also by the magnetic flux amplitude.

### 3.3.3 Rabi Picture: The Floquet Approach

The Landau-Zener approach explains well the oscillations of the excited state occupation. If one wants to study the energy structure of the strongly driven qubit one needs to rely on the probe absorption spectroscopy. For that, we have to transform the periodic time-dependent Hamiltonian into the corresponding Fourier space by using the so-called Floquet formalism [67, 68, 69] (see Appendix C). This way the time-periodicity in the system is transferred into a repeating energy spectrum, which can be seen as an example of the interconnection between energy and time (see Chapter I). Moreover, we can rely on time-independent quantum physics in solving problems related to such systems.

Let us again study an atom that is driven with the external flux (Equation (B.34))

\[
\hat{H}_q = -\frac{E_{J0}}{2} \left[ \cos \frac{\pi \Phi_L(t)}{\Phi_0} \left( \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_z - d \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x \right) \right.
\]

\[
- \sin \frac{\pi \Phi_L(t)}{\Phi_0} \left( \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_z + d \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x \right) \left. \right] \right].
\]  

(3.37)
where we, again, have assumed that $\Phi_{\text{ext}} = \Phi_b + \Phi_L(t)$ and $\Phi_L(t) = \Phi_L \cos \omega_L t$. We have also made rotation into the $\hat{\sigma}_x$ eigenbasis. We refer to the eigenstates in this basis as bare states. For simplicity, we give here explicit formulas in the approximation that we drop terms in $\hat{H}_q$ containing the second and higher harmonics as

\[
\cos \frac{\pi \Phi_L(t)}{\Phi_0} \approx J_0 \left( \frac{\pi \Phi_L}{\Phi_0} \right) \tag{3.38}
\]

\[
\sin \frac{\pi \Phi_L(t)}{\Phi_0} \approx 2J_1 \left( \frac{\pi \Phi_L}{\Phi_0} \right) \cos \omega_L t. \tag{3.39}
\]

We end up with the dressed atom Hamiltonian

\[
\hat{H}_q = \frac{\hbar}{2} \left[ \left( \omega_0 + A \cos \omega_L t \right) \hat{\sigma}_z + \left( \Delta_0 + B \cos \omega_L t \right) \hat{\sigma}_x \right] \tag{3.40}
\]

where we have defined the effective qubit energy $\hbar \omega_0$ and the asymmetry $d$ induced coupling $\Delta_0$ as

\[
\hbar \omega_0 = -E_{J0}J_0 \left( \frac{\pi \Phi_L}{\Phi_0} \right) \cos \frac{\pi \Phi_b}{\Phi_0} \tag{3.41}
\]

\[
\hbar \Delta_0 = dE_{J0}J_0 \left( \frac{\pi \Phi_L}{\Phi_0} \right) \sin \frac{\pi \Phi_b}{\Phi_0}. \tag{3.42}
\]

Longitudinal and transverse amplitudes are defined as

\[
\hbar A = 2E_{J0}J_1 \left( \frac{\pi \Phi_L}{\Phi_0} \right) \sin \frac{\pi \Phi_b}{\Phi_0} \tag{3.43}
\]

\[
\hbar B = 2dE_{J0}J_1 \left( \frac{\pi \Phi_L}{\Phi_0} \right) \cos \frac{\pi \Phi_b}{\Phi_0}. \tag{3.44}
\]

respectively.

Let us first try to get a more qualitative picture of the situation. We first transform the atom into a frame rotating non-uniformly, by using a transformation (a similar transformation was done, for example, in Reference [37])

\[
\hat{U}^\dagger = e^{i \frac{A}{\omega_L} \sin(\omega_L t) \hat{\sigma}_z}. \tag{3.45}
\]

This compensates for the longitudinal temporal variation and can be seen as a rotation into the basis of longitudinally dressed states (LDS) $\{|\downarrow, n\rangle, |\uparrow, n\rangle\}$ [36, 39]. After doing some algebra and using the Jacobi-Anger relation, we obtain

\[
\hat{H}_{ld} = \frac{\hbar}{2} \left[ \omega_0 \hat{\sigma}_z + \sum_{k=-\infty}^{\infty} (\Omega_k \hat{\sigma}_+ + \Omega_{-k} \hat{\sigma}_-) e^{ik\omega_L t} \right], \tag{3.46}
\]

where the Rabi couplings $\Omega_k = \left( \Delta_0 + k\omega_L d \cot \left( \frac{\pi \Phi_b}{\Phi_0} \right) \right) J_k \left( \frac{A}{\omega_L} \right)$.
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Figure 3.8: Rabi couplings $\Omega_k$ with $k = 0, 1, 2, 3$ plotted as a function of the flux bias $\Phi_b$. The value of the drive amplitude $\Phi_L = 0.24\Phi_0$ is chosen to be the same as in Figure 3.(b) in Paper [LV]. The vertical lines indicate the locations where RWA-resonance condition $\omega_0 = k\omega_L$ is fulfilled with $k = 1, 2, 3$. For example, when $k = 1$ we see that $\Omega_1 \approx \Omega_2 \approx \Omega_3$. This means that the RWA is inaccurate and one needs to rely on Floquet methods.

Let us focus on a couple of details. First, the atomic energy $\hbar\omega_0$ is dependent on the drive amplitude $\Phi_L$. This was already foreseen in Equation (3.35) and has its origin in the nonlinear Josephson coupling of the flux drive. Second, this drive-amplitude-dependent atom is coupled with a transverse field consisting of multiple harmonic modes, whose amplitudes $\Omega_k$ are nonlinear functions of $\Phi_L$. This will produce rich and unprecedented effects in the observed spectra, as we will soon see.

Equation (3.46) is very similar to the Rabi Hamiltonian (2.62). When two longitudinally dressed states $\left| \uparrow, n \right>$ and $\left| \downarrow, n + k \right>$ are degenerate we say we have a $k$-photon resonance. These states are coupled by $\Omega_k$ and the neglect of the interactions with non-resonant modes corresponds to a making of a RWA. Just at resonance, $\omega_0 = k\omega_L$, and we observe $k$-photon Rabi oscillations at frequency $\Omega_k$. We see that this is exactly the same as the diabatic resonance condition (3.35) obtained in the LZ-picture. The RWA holds when the resonant coupling dominates in magnitude. It was used successfully in the description of an LZ-problem in References [36, 37, 39].

If the Rabi couplings to non-resonant modes are of the same order as the resonant one, the RWA becomes insufficient and we will need the introduction of the Floquet formalism in order to obtain an accurate resonance condition. This was the case in Paper [LV] In Figure 3.8 we have plotted Rabi couplings $\Omega_k$ with $k = 0, 1, 2, 3$ as a function of the flux bias $\Phi_b$ using the parameters of
Paper IV For example, we see an overlap of the Rabi couplings in the case of the resonance \( k = 1 \) which means that the interplay with the non-resonant modes affect the effective atom energies and lead to shifts of the resonance locations similar to the Bloch-Siegert shift. The difference is that whereas in the BS-case we had to consider only the counter-rotating term, now we have to account for all the harmonic components in Hamiltonian \((3.46)\).

**Floquet Picture**

We notice that the Hamiltonian \((3.46)\) can be written as \( \hat{H}_{ld} = \hat{H}_0 + \hat{V}(t) \). The first term

\[
\hat{H}_0 = \frac{\hbar \omega_0}{2} \hat{\sigma}_z
\]

\((3.47)\)
describes a two-level atomic system that is independent of time. The second term couples the system with harmonic modes of the classical monochromatic field

\[
\hat{V}(t) = \frac{\hbar}{2} \sum_{k=-\infty}^{\infty} (\Omega_k \hat{\sigma}_+ + \Omega_{-k} \hat{\sigma}_-) e^{i k \omega_L t}.
\]

\((3.48)\)

The Hamiltonian \( \hat{H}(t) \) is \( 2\pi/\omega_L \)-periodic which means that we can use the general Floquet formalism presented in Appendix C. In our case, the non-zero Fourier components are

\[
\hat{H}_{0}[0] = \frac{\hbar}{2} \begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix}
\]

\((3.49)\)

\[
\hat{V}[k] = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{-k} \\ \Omega_k & 0 \end{pmatrix}
\]

\((3.50)\)

We can now write down the Floquet Hamiltonian (in the Floquet (i.e. LDS) basis \( B = \{|\alpha, n\}|\alpha\rangle \in \{|\downarrow\rangle, |\uparrow\rangle\}, |n\rangle \in B_T\} \), see Appendix C)

\[
\hat{H}_F = \frac{\hbar}{2} \times \begin{pmatrix}
\omega_0 - 2\omega_L & \Omega_0 & 0 & 0 & \Omega_2 \\
\Omega_0 & -\omega_0 - 2\omega_L & \Omega_1 & 0 & 0 \\
0 & \Omega_0 & -\omega_0 & 0 & \Omega_1 \\
\Omega_1 & 0 & \Omega_0 & 0 & \Omega_1 \\
0 & \Omega_0 & 0 & \Omega_0 + 2\omega_L & 0 \\
\Omega_2 & 0 & \Omega_1 & 0 & \Omega_0 - \omega_0 + 2\omega_L \\
& & & & & \vdots
\end{pmatrix}
\]

\((3.51)\)

In time-periodic systems there exists a notion of quasienergy that is analo-
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Figure 3.9: Left: A characteristic quasienergy spectrum (red) together with the LDS (dashed black) and the bare state (dotted black) energies. The time-periodicity is reflected as \( \hbar \omega_L \)-periodic energy levels. The energy difference between two quasienergies is denoted with \( \hbar \Delta \) (due to energy-periodicity, the other relevant splitting is \( \hbar (\omega_L - \Delta) \)). The Rabi couplings lift the degeneracy of the longitudinally dressed states, e.g. \( |\uparrow, n - 1 \rangle \) and \( |\downarrow, n \rangle \), leading to avoided crossings. Here the coupling with the field affects also far away from the degeneracies due to asymmetries of Rabi couplings with respect to the bias flux \( \Phi_b \) and due to their overlapping (see text). Right: Magnification of the gray squares on the left. The black line is the RWA result obtained in the LDS-basis. Green line is the RWA in the oscillation center basis (see text). Obtained from Paper [IV].

Gous to the quasimomentum in periodic lattice potentials of condensed matter physics. Quasienergies can be solved from the eigenvalue equation of the Floquet Hamiltonian

\[
\hat{H}_F|\Phi\rangle = \varepsilon|\Phi\rangle.
\] (3.52)

We have solved this numerically in Paper [IV]. We have neglected the Floquet modes with Fourier index \( |n| \geq 5 \), which is sufficient for the convergence of the quasienergies with our parameters. We have depicted a typical (numerical) quasienergy spectrum in Figure 3.9 where we have used the parameters of Paper [IV]. Due to the \( 2\pi/\omega_L \)-periodicity of the initial Hamiltonian, the energy spectrum repeats with intervals \( \hbar \omega_L \). We are dealing with a two-state system, so there are two quasienergy levels per period. The energy difference between two quasienergies is denoted by \( \hbar \Delta \). Periodic structure allows excitation energies \( \hbar (n\omega_L \pm \Delta) \) with \( n \) being an integer. In the semi-classical limit the Floquet approach captures the essential features of the fully quan-
tized strongly driven atom [67] and the quasienergy can be considered as the characteristic energy of the combined atom+field system.

We can see from Figure 3.9 the reason why we have to rely on the Floquet solution. On the left plot, we see that the quasienergies differ from the uncoupled energies. In Chapter 2 this was referred to as the dynamic Stark effect. The deviation occurs also in between the degeneracy points which in linearly driven atoms would be a clear indication of overlapping of the Rabi couplings and the resulting insufficiency of RWA. Here one has to be more careful, since the couplings are asymmetric with respect to the degeneracy points (see Figure 3.8) which can lead to similar kind of effects. We have concentrated on two of the avoided crossings (right plot), in order to study this more carefully.

One can also obtain analytic results by using the so-called generalized van Vleck perturbation theory for nearly degenerate Floquet basis states which corresponds to finding a unitary transformation that block-diagonalises the Floquet Hamiltonian leading to an eigenvalue equation for an effective $2 \times 2$-matrix [70, 71, 69]. We will not go through the theory here but simply present a couple of relevant results. We assume that two states, e.g. $|\uparrow, 0\rangle$ and $|\downarrow, k\rangle$ are (nearly) degenerate. The first order perturbative correction corresponds to calculating the eigenvalues of the effective Floquet matrix

$$\hat{H}_F \approx \hbar \left( \frac{\omega_0}{\Omega_1} \frac{\Omega_1}{-(\omega_0 - 2k\omega_L)} \right),$$

which in the case of $k = 1$ corresponds to the bold-face elements in (3.51).

We see that the nearly degenerate first order perturbation theory reproduces the Rabi Hamiltonian (2.63) and corresponds to making an RWA. The second order correction can be written as

$$\omega_k \hat{\sigma}_z = \frac{1}{4} \sum_{n \neq k \Omega_n^2 \omega_0 - n\omega_L} \hat{\sigma}_z.$$

We see that the first correction to the RWA affects the unperturbed energies $\hbar\omega_0$ leading to a change in the resonance frequency. The situation is similar to the Bloch-Siegert shift (Section 2.2.2) but in this case the couplings to non-resonant modes other than the counter-rotating one have also been included. Accordingly, we call the second order correction the generalized Bloch-Siegert shift.

One can extract three different components of the strong driving that affect on the location of the minimum energy splitting and the resonance frequency. First, we see that the degeneracy point of the LDS energies is shifted from that of the bare states. This is due to the nonlinear coupling of the drive flux leading to the Bessel-modulated atom splitting (3.41) and can be seen as rectification of the ac-drive leading to an effective dc-bias of

\[\text{CHAPTER 3. PHYSICS OF THE ARTIFICIAL} \]
the energy levels. It has a large effect on the location of the resonance and is different from the usual dynamic Stark effect since it does not involve any exchange of energy between the atom and the field. The additional deviations due to the drive are measured with respect to the Bessel-modulated degeneracy point. Second, we see in Figure 3.9(b) that the avoided crossing due to the resonant Rabi coupling $\Omega_2$ is located at different flux bias point than the degeneracy. This has its origin in the asymmetric flux bias dependence of the Rabi coupling which can be seen in Figure 3.8. Nevertheless, the highest population in the excited state $P_{\uparrow} = \frac{1}{2}$ still occurs at the degeneracy which is where the Rabi oscillations have the largest amplitude. Lastly, the $\Omega_k$'s overlap (see again Figure 3.8) which means that there is also coupling to the non-resonant modes. This is seen as a deviation of the numerical quasienergy from the RWA result. This shift contains the generalized Bloch-Siegert shift and also possible higher order corrections.

Finally, we stress that the RWA is basis dependent. If we write the Hamiltonian (3.46) in the basis defined at the bias point $\Phi_b$ (this is the so-called oscillation center basis, see Appendix B.1.2) we observe different results in the RWA. The resulting quasienergies are depicted in Figures 3.9(b) and (c) by green lines. In (b) this basis seems to work well but in (c) the LDS one is better. With our parameters, it seems hard to find a single basis that would give correct RWA-quasienergies in the whole $\Phi_b - \Phi_L$ -plane. Therefore, it is more practical to rely on the numerical results that are independent of the chosen basis and take into account energy corrections in all orders.

3.4 Artificial Molecules

The analogy of an atom interacting with cavity photons has been fruitful in understanding numerous solid-state experiments. It often is the case with analogies, that there exists also other possibilities that may show their strength compared with the old workhorse. Alternative viewpoints on the familiar subjects can give new insights and also lead to simpler and more intuitive interpretations. Here we present an option that identifies the electric qubit-oscillator system with a diatomic molecule.

The analogy goes as follows. The qubit states correspond to two electronic ones in a diatomic molecule (see also Section 2.3). In addition to electronic degrees of freedom, there are vibrations of the nuclei with respect to some equilibrium separation. In our circuit, these are produced by the flux $\Phi$ of the $LC$-oscillator. We have studied in Papers I and III our atom-oscillator system in the dispersive limit where the oscillator frequency is much smaller than the atomic splitting. This means that the electronic motion in our artificial molecule is much faster than the vibrations of the nuclei. This is the case also with natural molecules. We are now in place to make an approximation which
is similar to the Born-Oppenheimer one in Equation (2.65). We diagonalise the electronic part separately leading to adiabatic energies
\[ E_{\pm} = \pm \frac{1}{2} \sqrt{E_{\text{el}}^2 + E_J(\Phi)^2}. \]  
(3.55)

The potential energy of the total system now has two possibilities depending on the state of the qubit
\[ U_{\pm}(\Phi) = \pm \frac{1}{2} \sqrt{E_{\text{el}}^2 + E_J(\Phi)^2 + (\Phi - \Phi_b)^2}. \]  
(3.56)

Here the $\Phi$ denotes the flux through the qubit, not the oscillator as in the previous discussions (see Appendix B.2). The two potentials are plotted in Figure 3.10 with parameters taken from Paper I. Without the qubit there is a potential minimum at the flux bias point $\Phi_b$. The effect of the qubit is to shift this minimum in opposite directions which depend on the state of the qubit [36, 39]. This is because of the different curvatures of the qubit energies. Accordingly, the oscillator resonance frequencies in the two qubit states are different [22, 72, 73] (see also Section 4.3.1). We see from Figure 3.10 that the situation is very similar to the natural molecules presented in Section 2.3. We will show in Chapter 4 the experimental data of the absorption spectroscopy illustrating the vibronic transitions discussed in the context of natural molecules. Finally, we emphasise that many of the previous experiments discussed in terms of cavity QED [74, 21, 22, 75, 72, 73, 24, 76, 77, 78] and including strongly driven qubits [36, 79, 37, 38, 39] could also be interpreted using the molecular analogy.
Chapter 4

Absorption Spectroscopy

Details about the internal structure of the natural elements can be examined by studying their interaction with electromagnetic fields. An everyday example of this is the sense of sight. If the light reflected from the surface of an object reaches the eye it turns into an image in the brain. The reflection occurs at a boundary of two substances with different refractive indices $n_0$ and $n$ (we assume here that the light enters from the medium with refractive index $n_0$). The refractive index $n$ tells how the speed $v$ of light "slows down" in the medium compared with its speed $c$ in vacuum. It is defined as

$$n = \frac{c}{v}. \quad (4.1)$$

If the angle of incidence is normal (the light enters perpendicularly to the boundary) the ratio $\Gamma$ between the reflected and incident electromagnetic field amplitudes, $E_R$ and $E_I$ respectively, can be written solely in terms of the refractive indices as

$$\Gamma = \frac{E_R}{E_I} = \frac{n - n_0}{n + n_0}. \quad (4.2)$$

We see that if the medium has $n = n_0$ there is no reflection.

Similar to natural systems, we can study electric circuits by measuring the reflection spectrum. The important thing is that we have much more control over the "refractive index" of a circuit during a measurement than in the classic optical case. The electromagnetic field is modeled as an infinite set of harmonic oscillators, as was discussed in Chapter 2. We want to imitate the field using circuit elements, which requires electrical waves propagating in the circuit. This means that we to have go beyond the normal lumped element model which assumes that the characteristic size of the network is much smaller than the electrical wavelength. Fortunately, there exists such a distributed component which is called the transmission line.

Everything that has been presented in the preceding chapters serves as an introduction to the original Papers I-IV. The purpose of this last chapter
is to act as a transition in between them. We will not go through the same discussion that can be found in the Papers. Nevertheless, we will highlight the main findings, relate them to the presented concepts and concentrate on a couple of aspects that caught only little attention in the Papers.

4.1 Transmission-Line Spectrometer

We have schematised in Figure 4.1 a transmission line connected to a system under study built of lumped components. We assume that the line consists of an infinite set of electric harmonic oscillators described by the Lagrangian

$$L = \sum_{i=1}^{\infty} \Delta x \left[ \frac{C_T \phi_i^2}{2} - \frac{(\phi_{i+1} - \phi_i)^2}{2(\Delta x)^2 l_T} \right], \quad (4.3)$$

where $C_T$ and $l_T$ are the capacitance and the inductance per unit length, respectively. The node fluxes $\phi_i$ between inductors are defined as in Figure 4.1.

Let us first assume that the line is infinite at both ends (i.e. not terminated to anything). We can go to continuum limit by assuming $\Delta x \to 0$ resulting in

$$L = \int_{0}^{\infty} dx \left[ \frac{C_T \phi^2}{2} - \frac{1}{2l_T} \left( \frac{\partial \phi}{\partial x} \right)^2 \right]. \quad (4.4)$$

We will follow Hamilton’s principle and assume that the evolution of any system is such that the time integral of the Lagrangian has an extremum. This leads to Lagrange’s equations, which in this case reduce to

$$\frac{\partial^2 \phi}{\partial t^2} - v^2 \frac{\partial^2 \phi}{\partial x^2} = 0. \quad (4.5)$$
This describes a 1-D wave of propagating flux in a medium with velocity \( v = 1/\sqrt{c_T T} \). The wave equation has a solution of the form

\[
\phi(x, t) = \phi_{in}\left(\frac{x}{v} + t\right) + \phi_{out}\left(-\frac{x}{v} + t\right),
\]

which has components going to the negative and positive \( x \)-directions, "in" and "out", respectively.

Next we assume that the transmission line is terminated at \( x = 0 \) to the circuit element whose properties we wish to study. We assume that the system is small compared with wavelengths used in the measurement and to the length of the transmission line. This means that we can call it a lumped element. At the end of the transmission line \( (x = 0) \), the voltage can be written as

\[
\dot{\phi}(\omega) \equiv V(\omega) = V_{in}(\omega) + V_{out}(\omega),
\]

where we have made a Fourier transform into the frequency space. The system now imposes boundary conditions on the propagating wave in the transmission line. This means that the out-going wave is determined by the so-called complex reflection coefficient \( \Gamma \)

\[
V_{out} = \Gamma V_{in}.
\]

Let us first assume that the voltage-current relationship of the system is linear \( V = ZI \). According to the Kirchhoff loop rule for currents, we have that (the signs in the transmission line currents are determined by the directions of the arrows in Figure 4.1)

\[
\frac{V_{in} + V_{out}}{Z} = \frac{V_{in} - V_{out}}{Z_0} \Rightarrow \Gamma = \frac{Z - Z_0}{Z + Z_0}.
\]

We see now that the impedance behaves much like the refractive index in the optical case. Unfortunately, the impedance is defined in the linear regime and holds only in the limit where the amplitude of the incoming voltage is small. In general we have to calculate \( \Gamma \) numerically.

Our measurement scheme goes qualitatively as follows. We send in a monochromatic signal \( V_{in} \). Depending on the measurement frequency \( \omega_{LF} \), we get a strong absorption if the impedance of the system matches with that of the transmission line. We measure the reflected signal \( V_{out} \) with the same frequency. By changing the system bias parameters during the experiment, we obtain an absorption spectrum which contains information on the inner structure of the system.

### 4.1.1 Semi-Classical Reflection Measurement

We use the above presented transmission-line spectrometer in measuring the properties of our qubit-oscillator system. We use a capacitive coupling (see
Figure 4.2: Circuit diagram of the measurement in Paper I.

Figure 4.2) and with the help of the network theory we obtain the Hamiltonian operator for the whole reflection measurement setup. We will skip the lengthy derivation and just present the results. The Hamiltonian consists of the qubit-oscillator part $\hat{H}_{qo} = \hat{H}_q + \hat{H}_o$, transmission line part and a part describing the capacitor producing the coupling between the two

$$\hat{H} = \hat{H}_q + \hat{H}_o + \hat{H}_{tl} + \hat{H}_c$$  \hspace{1cm} (4.10)

$$\hat{H}_q = \frac{1}{2} \left\{ E_{ct} \sigma_z \right\} \hspace{1cm} (4.11)$$

$$\hat{H}_o = \frac{\hat{q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} - \hat{q} \hat{\Phi}_L(t)$$ \hspace{1cm} (4.13)

$$\hat{H}_{tl} = \frac{1}{2 \Delta x} \sum_{i=1}^{\infty} \left[ \frac{\hat{q}_{i+1}^2}{C_T} + \frac{(\hat{\phi}_{i+1} - \hat{\phi}_i)^2}{2l_T} \right]$$ \hspace{1cm} (4.14)

$$\hat{H}_c = \frac{\hat{q}_1^2}{2C_c} \left( 1 + \frac{C_c}{2C} \right) + \frac{\hat{q}_1 \hat{\dot{q}}}{2C_c}$$ \hspace{1cm} (4.15)

where we have assumed that $C_g \ll C, C_c$. Charge $\hat{q}_1$ and flux $\hat{\phi}_1$ are defined at the end of the transmission line. The coupling Hamiltonian $\hat{H}_c$ determines the boundary condition for the line. To see this, let us write down the Heisenberg equations of motion for the degrees of freedom at the end of the line

$$\dot{\phi}_1 = \frac{\partial \hat{H}}{\partial \hat{q}_1} = \frac{\hat{q}_1}{C_c} \left( 1 + \frac{C_c}{2C} \right) + \frac{\hat{\dot{q}}}{2C}$$ \hspace{1cm} (4.16)

$$\dot{\phi}_1 = -\frac{\partial \hat{H}}{\partial \phi}_1 \bigg|_{x=0} = \frac{\hat{\phi}_2 - \hat{\phi}_1}{l_T \Delta x} \rightarrow \frac{1}{l_T} \frac{\partial \hat{\phi}_1}{\partial x} \bigg|_{x=0}.$$ \hspace{1cm} (4.17)
Based on Equation (4.6), we can write
\[
\frac{\partial \hat{\phi}_1}{\partial x} \bigg|_{x=0} = \frac{1}{v} \left( \frac{\partial \hat{\phi}_{in}}{\partial t} - \frac{\partial \hat{\phi}_{out}}{\partial t} \right),
\]
(4.18)
where \(\hat{\phi}_{in}\) and \(\hat{\phi}_{out}\) are defined at the end of the line. Our measurement procedure was such, that we know the time-evolution of the incoming signal \(\hat{\phi}_{in}\) beforehand. Thus, we can solve for the outgoing wave \(\hat{V}_{out}(t) = \partial \hat{\phi}_{out}/\partial t\) from equations of motion (4.16) and (4.17) combined with (4.18)
\[
\hat{V}_{out}(t) = -\hat{V}_{in}(t) + \frac{\hat{q}_1}{C_c \left(1 + \frac{C_c}{2C}\right)} + \frac{\hat{q}}{2C},
\]
(4.19)
\[
\dot{\hat{q}}_1 = \left(\hat{V}_{in}(t) - \hat{V}_{out}(t)\right)/Z_0,
\]
(4.20)
where we have defined the characteristic impedance of the transmission line as \(Z_0 = \sqrt{l_T/c_T}\) (which typically has a value \(\sim 50 \Omega\)). We stress that since we are measuring \(\hat{V}_{out}\), the outcome should be related to the expectation value \(\langle \hat{V}_{out} \rangle \equiv V_{out}\). We have, therefore, made the semi-classical approximation and assumed that there is no quantum entanglement between the transmission line and system variables. We see that the second equation describes the net current flow at the end of the line. The first equation is the boundary condition and is related to the system operator \(\hat{q}\) describing the charge on the harmonic oscillator.

**System Evolution**

We need the expectation value of the oscillator charge operator \(\hat{q}\) for the calculation of the reflection coefficient. We use again the Heisenberg equations of motion, take the averages of all observables, and obtain
\[
\dot{\Phi} = \frac{q}{C} + \frac{q_1}{2C} - \dot{\Phi}_{ext}
\]
(4.21)
\[
\dot{\hat{q}} = -\left\langle \frac{\partial H_q}{\partial \Phi} \right\rangle - \frac{\Phi}{L} - \frac{\dot{\Phi}}{R}
\]
(4.22)
We have introduced a phenomenological damping term \(R\) into the equation of motion of the charge. This describes the unavoidable losses in the oscillator due to, e.g. dielectric losses in the capacitors. We stress the similarity with equations of motion (2.2) and (2.3) of the driven and damped harmonic oscillator. The inclusion of dissipation can be done also in the Hamiltonian formalism (see Paper III) by introducing another transmission line describing a resistor with characteristic resistance \(R = \sqrt{l_R/c_R}\).

Equations of motion (4.21) and (4.22) are a restatement of the Kirchhoff loop rules for voltage and current, respectively. The operator \(\hat{I}_q = \partial H_q/\partial \Phi\)
describes the current flowing in the charge-flux qubit (the quantum generalisation of Equation (3.10)). We make another semi-classical approximation and assume that the oscillator and the qubit are not entangled. This means that the state vector of the qubit-oscillator system is separable and the expectation values can be written as $\langle \hat{O}\hat{Q} \rangle = \langle \hat{O} \rangle \langle \hat{Q} \rangle$, where $\hat{O}$ and $\hat{Q}$ act in the Hilbert spaces of the oscillator and the qubit, respectively. Accordingly, the time-evolution of $\langle \hat{I}_q \rangle = \text{Tr}(\hat{\rho}_q \hat{I}_q)$ ($\hat{\rho}_q$ is the density operator of the qubit) can be solved with the Bloch equations (2.25).

Let us finally sum up the results. The time-evolution of the reflected signal can be solved from the set of first-order differential equations consisting of the semi-classical equations for the line and the oscillator, and of the Bloch equations for the qubit

$$
\begin{align*}
\dot{V}_{\text{out}}(t) &= -V_{\text{in}}(t) + \frac{q_1}{C_c} \left( 1 + \frac{C_c}{2C} \right) \frac{q}{2C} \\
\dot{q}_1 &= \left( V_{\text{in}}(t) - V_{\text{out}}(t) \right)/Z_0 \\
\dot{\Phi} &= \frac{q}{C} + \frac{q_1}{2C} - \dot{\Phi}_{\text{ext}} \\
\dot{\Phi} &= -\left( \frac{\partial \hat{H}_q}{\partial \Phi} \right) - \frac{\Phi}{L} - \frac{\dot{\Phi}}{R} \\
\dot{S}_x &= (\Omega \times S)_x - \frac{S_x}{T_2} \\
\dot{S}_y &= (\Omega \times S)_y - \frac{S_y}{T_2} \\
\dot{S}_z &= (\Omega \times S)_z - \frac{S_z - S_{z0}}{T_1}.
\end{align*}
$$

We can solve this set with standard numerical methods (e.g. 4th order Runge-Kutta [40]). In the end of the calculation we calculate the Fourier component of $V_{\text{out}}(t)$ at the measurement frequency $\omega_{LF}$ and calculate the reflection coefficient

$$
\Gamma = \frac{V_{\text{out}}(\omega_{LF})}{V_{\text{in}}(\omega_{LF})}.
$$

This method was presented also in Paper II and used in the interpretation of the experiments in Paper II.

4.2 Probe Absorption Spectroscopy of an Artificial Atom

In Section 3.3.3 we discussed a charge-flux atom driven strongly with external flux. We showed that the effect of the drive on the atomic energies was so
pronounced, that the usual RWA became insufficient. Accordingly, one has to rely on Floquet methods in order to calculate the characteristic quasienergy bands that describe the total energy of the combined atom and the strong field. In Paper IV, we have shown that by coupling such a driven two-level system with a weakly driven oscillator, we can probe the quasienergies of the dressed atom.

To see that our circuit in Figure 3.6 really describes such a system, let us study the qubit-part of Hamiltonian (B.50) at charge degeneracy

\[
\hat{H} = \frac{E J_0}{2} \left[ \cos \frac{\pi \Phi(t)}{\Phi_0} \left( \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_z - d \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x \right) \right. \\
\left. - \sin \frac{\pi \Phi(t)}{\Phi_0} \left( \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_z + d \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x \right) \right], \tag{4.25}
\]

where \( \Phi(t) = \Phi_P(t) + \Phi_L(t) \) and we have made the semi-classical approximation and assumed that oscillator coupling is equivalent to an oscillator induced flux drive \( \Phi_P(t) = \Phi_P \cos \omega_P t \). As before, the external flux through the inductance loop is \( \Phi_{ext}(t) = \Phi_b + \Phi_L(t) \) and \( \Phi_L(t) = \Phi_L \cos \omega_L t \). We have also performed rotation into the \( \hat{\sigma}_x \) eigenbasis, similarly as in Section 3.3.3.

We assume that the flux amplitude \( \Phi_P \) is small which allows us to use the oscillator as a weak probe of the quasienergies. Thus, we can write

\[
\cos \frac{\pi \Phi(t)}{\Phi_0} \approx \cos \frac{\pi \Phi_L(t)}{\Phi_0} - \frac{\pi \Phi_P(t)}{\Phi_0} \sin \frac{\pi \Phi_L(t)}{\Phi_0}, \\
\sin \frac{\pi \Phi(t)}{\Phi_0} \approx \sin \frac{\pi \Phi_L(t)}{\Phi_0} + \frac{\pi \Phi_P(t)}{\Phi_0} \cos \frac{\pi \Phi_L(t)}{\Phi_0}. \tag{4.26}
\]

We can now rewrite the Hamiltonian into two parts as \( \hat{H} = \hat{H}_q + \hat{H}_P \) where the Hamiltonian \( \hat{H}_q \) of the strongly driven atom is defined as in Equation (3.37) and that of the weak probe as

\[
\hat{H}_P = \frac{E J_0 \pi \Phi_P(t)}{2 \Phi_0} \left[ \sin \frac{\pi \Phi_L(t)}{\Phi_0} \left( \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_z - d \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x \right) \right. \\
\left. + \cos \frac{\pi \Phi_L(t)}{\Phi_0} \left( \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_z + d \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x \right) \right] \tag{4.27}
\]

We use the approximations (4.26) and by moving into the rotating frame (3.45) we obtain

\[
\hat{H}_P = \frac{\hbar \pi \Phi_P(t)}{2 \Phi_0} \cos(\omega_P t) \left[ \left( \frac{\Delta\phi}{d} + \frac{B}{2d} (e^{i \omega_L t} + e^{-i \omega_L t}) \right) \hat{\sigma}_z \right. \\
\left. - \sum_{n=-\infty}^{\infty} \left( \delta_n \hat{\sigma}_+ + \delta_{-n} \hat{\sigma}_- \right) e^{i n \omega_L t} \right], \tag{4.28}
\]

where we have used notations (3.41) and (3.43) and denoted \( \delta_n = d(\omega_0 + n \omega_L)J_n(A/\omega_L) \).
Figure 4.3: Left: The landscape of the quasienergy splitting $\Delta$ of Paper IV. Solid dots are the experimental resonances. Blue and green contours indicate the locations of the resonances in Equation (4.29). For more information, see Paper IV. Right: The argument of the reflection coefficient obtained from the semi-classical simulations using the parameters of Paper IV. Dark corresponds to lower resonance frequency of the system.

In the Floquet formalism the probe Hamiltonian acts as a harmonic perturbation. Thus, we can rely on Fermi’s golden rule. We are studying probe induced transitions from one quasienergy state $\varepsilon_0$ into another $\varepsilon_1$. In Paper IV we used $\omega_P < \omega_L$. In such a case, due to the $\omega_L$ periodicity of the quasienergy ladder, the absorption can occur when

$$\omega_P = \begin{cases} \Delta & \text{(blue)} \\ \omega_L - \Delta & \text{(green)} \end{cases},$$

(4.29)

where the color coding is related to Figures 3.9 and 4.3. The intensity of the transition is determined by the matrix element $\langle \varepsilon_1 | \hat{H}_P | \varepsilon_0 \rangle$.

In Figure 4.3 we have shown the landscape of the quasienergy splitting $\Delta$ (defined as in Figure 3.9) as a function of flux bias $\Phi_b$ and flux amplitude $\Phi_L$. We have indicated the probe resonance conditions (4.29) with blue and green contours. We have measured the probe absorption using the reflection spectroscopy presented in the previous section. The experimental resonances (marked with dots) are in good agreement with the theoretical curves. We have also checked that their spectral weights are given by the corresponding matrix elements. We have also plotted the argument of the reflection coefficient $\arg(\Gamma)$ that is obtained from the semi-classical simulations (see Equations 4.23). This relates the probe absorption spectrum with a measurable quantity. The dots in the quasienergy landscape have been picked from the experimental $\arg(\Gamma)$ data.

Finally, we comment on the curving of the resonances which has its origin in the rectification effect that was based on the nonlinear coupling of
4.3 Vibronic Spectroscopy of an Artificial Molecule

In Paper I we have studied the vibronic transitions of the artificial molecule. We will shortly review the results here and put an emphasis on a couple of details that were mentioned only briefly in the original Paper. In the following, we follow the nomenclature of Section 3.3.1 and relate the oscillations of the oscillator flux with the vibrations of the nuclear separation. Moreover, we identify the qubit states with two electronic states in a molecule. Accordingly, we can call our circuit as an artificial molecule. The experimental vibrational frequency \( \nu_P = 874 \text{ MHz} \) is much smaller than the minimum gap of the qubit \( \Delta/h = dE_J/h \approx 2.7 \text{ GHz} \). This means that the molecular terminology is particularly apt (see Section 3.3.1) and, especially, we can characterize the vibronic transitions in terms of the Franck-Condon principle (Section 2.3).
Our experimental apparatus is depicted in Figure 4.2. We use the reflection spectroscopy method presented in Section 4.1. The frequency $\nu_{LF} = 872$ MHz of the measurement signal is only slightly detuned from the vibrational frequency of the molecule. It can therefore act as a source for vibrational quanta and in circuit language “feel” the changes in the reactance of the oscillator induced by the qubit (see following sections). We use also another microwave source at higher frequency $\nu_{HF} = 22$ GHz. This is produced by a voltage drive on the capacitor island (see Figure 4.2) and effectively leads into a high frequency flux drive. The high-frequency signal can be used to induce vibronic transitions, in which both the vibrational and electronic states change simultaneously.

Because of the large detuning between the vibrational and electronic energy scales, the vibrational degrees of freedom do not have time to react when the electronic state is changed. This means that the Franck-Condon principle holds and the most intense transitions occur between such vibrational states whose wave functions overlap. This is illustrated in Figure 4.4. The transitions can also be seen as interference in the phase space [81, 82] as discussed in Paper II.

The measured reflection coefficient can be seen in Figure 4.5. The observed spectrum has two major contributions. First of all, the vibronic transitions can be seen as concentric fringes in the spectrum. The locations of the resonances are described nicely by the Franck-Condon principle, as discussed in Paper I. Second, the spectrum is asymmetric with respect to the pure electronic transition $\nu_{HF} = \nu_0$ (dashed black arc in Figure 4.5), where $\nu_0$ is the qubit splitting frequency. This can be explained as amplification/damping of the vibrations. Due to the conical shape of the electronic energy bands, the transitions outside the pure electronic transition require more than a quantum of energy from the high-frequency drive. This energy is taken from the nuclear vibrations leading to damping of the oscillations and seen as an increase in the steady state absorption. Analogously, the transitions inside
4.3. VIBRONIC SPECTROSCOPY OF AN ARTIFICIAL MOLECULE

Figure 4.6: Ground state resonance as a function of the gate charge $q_g$ and flux bias $\Phi$. In the measurement, $\omega_{LF}$ and $\omega_P$ are fixed, and we can control the ground state shift $\chi$ with $q_g$ and $\Phi$. We see a resonance (dip in the reflection coefficient $|\Gamma|$) when $\omega_{LF} = \omega_P - \chi$.

the pure electronic one deposit energy to the vibrations which is shown as a decrease in absorption. The amplification/damping effect can be seen in the energy flow between the qubit and the oscillator. Numerically, the reflection coefficient and the flow can be calculated using the steady state of equations of motion (4.23). We presented such simulations in Paper I and the results are shown also in the center and right panels of Figure 4.5.

4.3.1 Motional Averaging

A Josephson junction behaves as a non-linear inductance, as was shown in Equation (3.13). We can generalize this result by stating that in each energy band $E_n$ the SCPT has a curvature-dependent inductance

$$L_n^{-1} \equiv \frac{d^2 E_n}{d\Phi^2}. \quad (4.30)$$

In the two-state approximation the inductances in the two states are equal but opposite, and by denoting $L_J^{-1} = d^2 E_+/d\Phi^2$ we have

$$L_{\pm}^{-1} = \pm L_J^{-1}. \quad (4.31)$$

In the above, $+$ denotes the excited state and $-$ the ground state. When the qubit is coupled with an oscillator, the inductances in different states change the oscillator resonance frequency $\omega_P$ by an equal amount $\chi$ into opposite directions. If the inductance of the bare oscillator $L \ll L_J$, we have that the
Figure 4.7: Motional averaging problem. Qubit induced frequency shift $\chi \gg \kappa$, and one expects minimal absorption of the measurement signal $\omega_{LF} \approx \omega_P$.

The shift is

$$\chi \approx \frac{1}{2L_J} \sqrt{\frac{L}{C}},$$

when the two inductances are in parallel. If we do not drive the electronic transition, the qubit is in the ground state and the oscillator frequency becomes $\omega_P - \chi$. If the measurement frequency is in resonance, $\omega_{LF} \approx \omega_P - \chi$, and we see a strong absorption. In Figure 4.6 we have depicted this situation using the parameters of Paper I. We fix the measurement and the bare oscillator frequencies $\omega_{LF}$ and $\omega_P$, respectively, and control $\chi$ with the gate charge $q_g$ and the flux bias $\Phi$. We see that the characteristic V-shaped resonance in Figure 4.5 is due to the ground-state induced shift [83, 84] in the oscillator frequency. We have used the ground state spectroscopy in the fitting of the circuit parameters.

In Paper I we rely on the concept of motional averaging in explaining the observed absorption spectrum. The problem goes as follows. Let us consider the resonance located near $n_g = 1.32$ and $\Phi_b = \Phi_0/2$ (see Figure 4.5). The equal but opposite band curvatures in the ground and excited states of the qubit induce opposite shifts $\chi$ in the resonance frequency $\omega_P$ of the oscillator (see Figure 4.7) and we say that the oscillator is dressed by the qubit. When the system is measured with a frequency $\omega_{LF}$ near the bare resonator frequency, one should observe no absorption because the oscillator linewidth $\kappa$ is much smaller than the frequency shift due to the qubit in the ground state. The explanation of the measured resonance is based on a simultaneous resonant excitation of the qubit by a separate drive.
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The effective resonance frequency of the harmonic oscillator becomes, at sufficiently high Rabi frequencies \( \gtrsim \chi \) \cite{[85]}, an average of the two dressed ones weighted by the populations of the qubit states. This can be interpreted as a kind of motional averaging phenomenon which is well known from two-level systems \cite{[86]}. In the harmonic oscillator, the effective resonance frequency becomes an average of the two dressed frequencies. We emphasize that in the previous discussions we concentrated on the effect of the strong driving on the qubit energies (dynamic Stark effect). The situation can be also seen in the point of view of the oscillator in which the qubit state alters its resonance frequency.

The interaction between a qubit and an oscillator at the flux degeneracy was described in Equation (3.21) by

\[
\hat{H}_{qo} = \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \hbar g (\hat{a}^\dagger + \hat{a})(\hat{\sigma}_+ + \hat{\sigma}_-) + \hbar \omega_P \hat{a}^\dagger \hat{a},
\]

where the coupling constant

\[
g/2\pi = \frac{E_J}{2\hbar} \sqrt{\frac{\pi Z}{R_K}}.
\]

Additionally, we have a two-tone driving of the oscillator with the low-frequency measurement signal \( \omega_{LF} \approx \omega_P \) and with a high-frequency gate-drive \( \omega_{HF} = \omega_0 \) that is in resonance with the qubit, i.e.

\[
\hat{H}_d = \hbar \Omega_{HF}(\hat{a}^\dagger + \hat{a}) \cos \omega_0 t + \hbar \Omega_{LF}(\hat{a}^\dagger + \hat{a}) \cos \omega_{LF} t.
\]

Here attention should be paid to the fact that in the experiment of Paper II the high-frequency drive was fed via the ground plane of the oscillator island leading to an effective driving of the oscillator flux \( \Phi \).

In the dispersive limit \( \omega_0 - \omega_P \gg g \) we can make the Schrieffer-Wolf transformation \( U = e^{\hat{S}} \) into the dressed eigenbasis of Hamiltonian (4.33) with

\[
\hat{S} = \frac{g}{\omega_0 - \omega_P} (\hat{a} \hat{\sigma}_+ - \hat{a}^\dagger \hat{\sigma}_-) + \frac{g}{\omega_0 + \omega_P} (\hat{a}^\dagger \hat{\sigma}_+ - a \hat{\sigma}_-).
\]

This leads to the driven and dressed JC-Hamiltonian

\[
\hat{H} = \hbar \left( \omega_P + \frac{2g^2}{\omega_0} \hat{\sigma}_z \right) \hat{a}^\dagger \hat{a} + \frac{\hbar}{2} \left[ \omega_0 + \frac{2g^2}{\omega_0} (1 + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) \right] \hat{\sigma}_z \\
+ \hbar \Omega_{HF}(\hat{a}^\dagger + \hat{a}) \cos \omega_0 t + \hbar \Omega_{LF}(\hat{a}^\dagger + \hat{a}) \cos \omega_{LF} t \\
- \hbar \Omega_{HF} \frac{2g \omega_P}{\omega_0^2} (\hat{\sigma}_+ + \hat{\sigma}_-) \cos \omega_0 t \\
- \hbar \Omega_{LF} \frac{2g \omega_P}{\omega_0^2} (\hat{\sigma}_+ + \hat{\sigma}_-) \cos \omega_{LF} t.
\]
The effect of the non-RWA terms in Hamiltonian (4.33) can be seen as a factor 2 in the term causing the cavity pull. This is related to the Bloch-Siegert shift that describes the inaccuracy of the RWA.

Let us then remove the time dependence by making a unitary transformation into a rotating frame. We use rotation $U = \exp(i\hat{a}^\dagger \hat{a} \omega_L t + i\hat{\sigma}_z \omega_0 t/2)$, which leads to (in RWA)

$$
\hat{H} = \hbar (\omega_P - \omega_L) \hat{a}^\dagger \hat{a} + \hbar \frac{\chi}{2} \hat{\sigma}_z
- \hbar \Omega_1 (\hat{\sigma}_+ + \hat{\sigma}_-) + \frac{\hbar \Omega_{1F}}{2} (\hat{a}^\dagger + \hat{a}).
$$

(4.38)

Here, $\Omega_1 = g \Omega_{HF} \omega_P / \omega_0^2$ is the Rabi frequency. The Lamb shift of the qubit, $\chi/2 = g^2/\omega_0$, is due to the vacuum fluctuations in the resonator. The dispersive shift $\chi \hat{\sigma}_z$ in the resonator frequency caused by the qubit will play a crucial role in our motional averaging scheme. The dynamics of the system is governed by the master equation

$$
\frac{\partial \hat{\rho}}{\partial t} = -i [\hat{H}, \hat{\rho}] + \frac{\kappa}{2} (2\hat{a}\hat{\rho}^\dagger - \hat{a}^\dagger \hat{\rho} - \hat{\rho} \hat{a}^\dagger \hat{a})
+ \frac{\gamma}{2} (2\hat{\sigma}_- \hat{\rho} \hat{\sigma}_+ - \hat{\sigma}_+ \hat{\rho} \hat{\sigma}_- - \hat{\rho} \hat{\sigma}_+ \hat{\sigma}_-)
- \frac{\gamma\phi}{4} [\hat{\sigma}_z, [\hat{\sigma}_z, \hat{\rho}]].
$$

(4.39)

Here we assume that the relaxation rates for the dressed oscillator and the qubit are the ones that are obtained from the measurements. We again make the semi-classical approximation and assume that there is no entanglement between the oscillator and the qubit. We can now write the equations of motion

$$
\langle \dot{\hat{a}} \rangle = -i (\omega_P - \omega_L) \langle \hat{a} \rangle - i \chi \langle \hat{\sigma}_z \rangle \langle \hat{a} \rangle - \frac{\kappa}{2} \langle \hat{a} \rangle - i \frac{\Omega_{1F}}{2}
$$

(4.40)

$$
\dot{N} = -\kappa N + i \frac{\Omega_{1F}}{2} (\langle \hat{a} \rangle - \langle \hat{a}^\dagger \rangle)
$$

(4.41)

$$
\dot{S}_x = -2\chi (N + \frac{1}{2}) S_y - \frac{S_x}{T_2}.
$$

(4.42)

$$
\dot{S}_y = 2\chi (N + \frac{1}{2}) S_x + 2\Omega_1 S_z - \frac{S_y}{T_2}.
$$

(4.43)

$$
\dot{S}_z = -2\Omega_1 S_y - \frac{S_z - S_{20}}{T_1}.
$$

(4.44)

where $N = \langle a^\dagger a \rangle$, $1/T_1 = \gamma$ and $1/T_2 = 1/2T_1 + \gamma\phi$ are the inverse relaxation and decoherence times of the qubit (at zero temperature), respectively. These are the so-called Cavity-Bloch equations in the semi-classical limit of large photon number.
4.3. VIBRONIC SPECTROSCOPY OF AN ARTIFICIAL MOLECULE

We are interested in the oscillator absorption with nearly resonant measurement signal. For that, we need to know the corresponding component of the Fourier transform of the correlation function \( \langle \hat{a}(t)\hat{a}^\dagger(0) \rangle \). This can be solved using the quantum regression theorem \([9]\), which gives (the incoherent part)

\[
\langle \hat{a}(t)\hat{a}^\dagger(0) \rangle = Ne^{-i(\omega_P + \chi S_z - \omega_{LF})t - \frac{\kappa}{2}|t|}.
\]

(4.45)

Here \( N \) is the steady state occupation of the oscillator. We observe emission to the environment peaked at frequency \( \omega_P + \chi S_z \) with the width of \( \kappa/2 \). Emitted energy has to be provided by the measurement signal which can be seen as increased absorption in the reflection measurement. This result means that the absorption spectrum of the dressed oscillator depends on the steady state population \( S_z \) and we should see strong absorption only when

\[
|\omega_{LF} - \omega_P - \chi S_z| \lesssim \kappa/2.
\]

In the measurement of Paper [1] the bare resonance frequency of the resonator \( \omega_r = 1/\sqrt{LC} \approx 2\pi \times 874 \text{ MHz} \). The oscillator linewidth is (full width at half maximum) \( \kappa/2\pi = 1/2\pi R_{\text{eff}}C \approx 3.5 \text{ MHz} \). Here \( R_{\text{eff}}^{-1} = R^{-1} + (\omega_P^2C_e^2Z_0)/4(1 + (\omega_cC_eZ_0)^2) \) where we have included the effect of the capacitively coupled transmission line (the derivation is a lengthy but straightforward circuit theoretical calculation, and omitted here). The dispersive shift in the oscillator frequency \( \chi = 2g^2/\omega_0 = 2\pi \times 11 \text{ MHz} \approx \frac{1}{2L_J} \sqrt{\frac{L}{C}} \) where \( L_J \) is the Josephson inductance obtained from the curvature of the energy band. It is worthwhile to notice that the coupling capacitor also affects the oscillator resonance frequency as...
\[ \omega_P = \frac{1}{\sqrt{(C + C_c/4(1 + (\omega_r C_c Z_0)^2))L}} \]

\[ \approx \omega_r (1 - \frac{C_c}{8C(1 + (\omega_r C_c Z_0)^2)}) \approx 2\pi \times 873.2 \text{ MHz}. \] (4.46)

The features of the above discussion are shown in the Figure 4.8 where we have plotted the simulated magnitude of the reflection coefficient as a function of qubit occupation \( S_z \). We calculate \(|\Gamma|\) as described in Section 4.1.1 and simultaneously we can extract \( S_z \). We have a full control over the occupation via the Rabi frequency \( \Omega_1 \). Experimentally this can be done (and is done in simulations also) by changing the amplitude \( \Omega_{HF} \) of the qubit drive. We measure the system at frequency \( \omega_{LF} = 2\pi \times 872 \text{ MHz} \) and observe the strongest absorption when \( S_z \approx (\omega_{LF} - \omega_P)/\chi = -0.11 \), which is in correspondence with the motional averaging picture above. Also, we see that the linewidth is given approximately by \( \kappa \).
This thesis has presented a theoretical modeling of novel experiments with superconducting circuits in terms of analogies with elementary objects of natural physics. We have reviewed the similarities between atoms and circuits built out of Josephson junctions. We have shown that, even without the optimisation of our circuit parameters, the charge-flux qubit we used can be coupled strongly with an LC-oscillator meaning that we can mimic the matter-light interactions on a circuit board. The only thing preventing us from entering the strong coupling regime of cavity-QED is the poor decoherence time of our qubit.

The absorption spectrum can reveal the internal energy structure of a physical object. Transmission lines can be used as the circuit equivalents of the electromagnetic field. When a transmission line is terminated to an electric apparatus, the reflection at the boundary is similar to that of light from a surface between media with different refractive indices. The reflection measurement was analysed in Papers I and II in terms of the semi-classical equations of motion.

Driving via the magnetic flux couples nonlinearly to the energy levels of the superconducting charge-flux qubit. This leads to nonlinear effects unforeseen in natural systems. The first contribution is the resonance frequency shift exceeding the original resonance frequency. This is mostly due to the rectification of the ac drive producing an effective dc bias on the energies. The second effect is the strong coupling to non-resonant harmonic modes of the driving field. This leads to the inaccuracy of the rotating wave approximation which results in additional, Bloch-Siegert type, correction of the resonance frequency. The inclusion of the non-resonant coupling terms forces us to use the Floquet method in solving the energy states of the strongly driven qubit. The structure of the resulting quasienergy ladders was studied in Paper IV using probe absorption spectroscopy.

We have shown that when the oscillator frequency is much smaller than the minimum energy splitting in the qubit, the circuit-QED system can also be discussed in terms of an analogue to a diatomic molecule. The separation between the two nuclei can be identified with the oscillator flux and the two electronic states correspond to the qubit states. We have shown in Papers II and III that, when the oscillator absorption is probed with a reflection mea-
surement, the resulting vibronic spectrum shows good agreement with the Franck-Condon principle of natural molecules.

This thesis is based on theoretical research done with electric circuits made of superconducting components. The present laws of physics were tested by making measurements in novel parameter regimes. The boundaries of the applicability of quantum mechanics were pushed a little bit further, as the experiments showed an agreement with the theory. The studied systems show a great promise in being used as elements in the foreseen quantum computer. The construction of such a device would serve as an ultimate test of principles of quantum physics. But, even if the quantum computer will never be built, the field will most likely produce other implementations to be used, for example, in metrology and electric appliances.

"...After many years of exploration of this peculiar place, I have learned to not trust my instincts. Or, better said, I have managed to create some kind of intuition of what will not happen in certain circumstances. I am afraid that my entrance was too late in life that I could ever fully understand the structure behind its whole beauty. I would imagine that a mind of a child would not be so chained to the Old World, but more open to see this one as it is. However, now that I have learned to find my way around here, I think I will continue gathering the pieces of information and gluing them together. I hope that one day it will be of use for the others."

- The End -
Appendix A

Landau-Zener Problem

In this Appendix we derive the rotation matrix $U_{LZ}$ appearing in Equation (2.43). We follow closely the discussion in [26] but with slightly modified notation. We go through the lengthy derivation here because the original paper is hard to obtain and we also extract the phase. An alternative version can be found in Reference [31]. The level crossing problem is discussed in terms of the Hamiltonian

$$\hat{H}_{LZ} = \frac{\hbar}{2} \begin{pmatrix} \varepsilon(t) & \Delta \\ \Delta^* & -\varepsilon(t) \end{pmatrix},$$

where the two (diabatic) basis states, $\{|0\rangle, |1\rangle\}$, are coupled with static $\Delta$. The time dependence is given by diagonal $\varepsilon(t) = \nu t$. This means that without the coupling, the energy levels cross at $t = 0$ and change their polarity. The coupling is complex in general, i.e. $\Delta = |\Delta| \exp(i\zeta)$. Nevertheless, the phase $\zeta$ can be generated by unitary rotation as

$$\hat{H}_{LZ} = \frac{\hbar}{2} \begin{pmatrix} \varepsilon(t) & \Delta \\ \Delta^* & -\varepsilon(t) \end{pmatrix} \begin{pmatrix} e^{i\zeta/2} & 0 \\ 0 & e^{-i\zeta/2} \end{pmatrix} \begin{pmatrix} \varepsilon(t) & |\Delta| \\ |\Delta| & -\varepsilon(t) \end{pmatrix} \begin{pmatrix} e^{-i\zeta/2} & 0 \\ 0 & e^{i\zeta/2} \end{pmatrix},$$

and can, thus, be added on top of a calculation ignoring it. We assume, therefore, in the following, that $\Delta$ is real and non-negative.

Let us then write the state vector in terms of the diabatic states ($|0\rangle = (1 \ 0)^T$, $|1\rangle = (0 \ 1)^T$) as

$$|\Psi(t)\rangle = C_0(t)e^{-i\int\varepsilon(t)/2dt}|0\rangle + C_1(t)e^{i\int\varepsilon(t)/2dt}|1\rangle.$$ (A.3)

We are searching for an asymptotic result, which means that starting from one of the diabatic states in the distant past ($t = -\infty$), we derive the state vector far in the future ($t = \infty$). Let us thus assume first that

$$C_0(-\infty) = 0, \quad |C_1(-\infty)| = 1.$$ (A.4)
APPENDIX A. LANDAU-ZENER PROBLEM

The time-evolution of the state vector is calculated from the Schrödinger equation

\[ H_{LZ} \Psi(t), t) = i\hbar \frac{\partial}{\partial t} \Psi(t). \]  

We note that Zener used a different sign convention (minus on the right-hand side) which leads to different signs in the phase definitions. It can interpreted as a reversal of the time axis. Even though it gives a numerous possibilities for making mistakes, we switch to notation commonly used nowadays.

If we insert the state vector we obtain two coupled differential equations

\[ i \frac{\partial C_0}{\partial t} = \frac{\Delta}{2} e^{i \int \epsilon(t) dt} C_1(t), \]  

\[ i \frac{\partial C_1}{\partial t} = \frac{\Delta}{2} e^{-i \int \epsilon(t) dt} C_0(t). \]

We can eliminate the first equation and we end up with a single second order equation

\[ \frac{\partial^2 C_1}{\partial t^2} = -i \epsilon \frac{\partial C_1}{\partial t} - \frac{\Delta^2}{4} C_1. \]  

Let us make the substitution

\[ C_1(t) = e^{-i \int \epsilon/2 dt} U_1, \]

which reduces the differential equation to

\[ \frac{\partial^2 U_1}{\partial t^2} + \left( \frac{\Delta^2}{4} - \frac{\nu^2}{2} + \frac{\nu^2}{4} t^2 \right) U_1 = 0, \]

where \( \nu = \dot{\epsilon} \). If we then define

\[ z = \sqrt{\nu} e^{-i\pi/4} t, \]

\[ n = i \frac{\Delta^2}{4 \nu}, \]

we end up with the Weber equation in the standard form

\[ \frac{\partial^2 U_1}{\partial z^2} + \left( n + \frac{1}{2} - \frac{z^2}{4} \right) U_1 = 0. \]

The Weber equation has four possible solutions \( W_n(\pm z) \) and \( W_{-n-1}(\pm iz) \), where \( W_n(z) \) is the Weber function. We are looking for a solution that satisfies the boundary condition \( \Psi(-\infty) = 1 \), i.e. does not vanish when \( t \to -\infty \). It turns out that the solution can be written as (when \( t < 0 \))

\[ U_1(z) = B_\pm W_n(-z) = B_\pm W_n(|z| e^{\pi i/4}) \to B_\pm e^{\gamma - i \pi} e^{\pi i |z|^2/4} |z|^n, \quad |z| \to \infty, \]  

(A.14)
where we have defined \( R = |z| \). We denote with upper and lower signs the regions where \(|0\rangle\) approaches \(|1\rangle\) from below \((\nu > 0)\) and above \((\nu < 0)\), respectively. By inserting \( n \), we obtain

\[
C_1(-\infty) = B_{\pm} e^{\mp \gamma} e^{\pm \frac{\nu|z|^2}{4}} |z|^{\mp i\gamma}, \quad (A.15)
\]

where we define \( \gamma = \Delta^2/4|\nu| \). Finally, we find

\[
|B_{\pm}| = e^{-\frac{\nu}{2}}. \quad (A.16)
\]

In order to solve the total state vector, we need to study the other differential equation

\[
C_0(t) = \frac{2i}{\Delta} e^{i \int \varepsilon(t) dt} \frac{\partial C_1}{\partial t} = \frac{2i}{\Delta} e^{\frac{i}{2} \int \varepsilon(t) dt} \left( -i \frac{\varepsilon}{2} U_1 + \frac{\partial U_1}{\partial t} \right) \quad (A.17)
\]

\[
= \pm B_{\pm} \gamma e^{\mp i\frac{\nu}{2}} e^{\pm \frac{i\varepsilon|z|^2}{4}} |z|^{\mp i\gamma}. \quad (A.18)
\]

We have used in the above the recurrence relation \( W'_n(-z) - \frac{\nu}{2} W_n(-z) = nW_{n-1}(-z) \) obtained from [10]. Now, in the distant past this has a limit

\[
C_0(-\infty) = \pm B_{\pm} \gamma e^{\mp i\frac{\nu}{2}} e^{\pm \frac{i\varepsilon|z|^2}{4}} |z|^{\mp i\gamma} \quad (A.19)
\]

We notice that \( C_0(-\infty) \to 0 \), so it obeys the initial condition. Altogether,

\[
C_1(t) = e^{-\frac{\nu}{2}} e^{-\frac{i\varepsilon}{2} \int \varepsilon(t) dt} W_n(-z) \quad (A.20)
\]

\[
C_0(t) = \pm e^{-\frac{\nu}{2}} e^{\mp i\frac{\nu}{2}} e^{\pm \frac{i\varepsilon|z|^2}{4}} \int \varepsilon(t) dt W_{n-1}(-z). \quad (A.21)
\]

Now, in the other limit, where \( t \to \infty \), we have that

\[
W_n(-z) = W_n(|z| e^{\pm i\frac{\nu}{2}}) \to e^{-\frac{\gamma}{4}} e^{\pm i\frac{|z|^2}{4}} |z|^{\mp i\gamma} \quad (A.22)
\]

\[
W_{n-1}(-z) = W_{n-1}(|z| e^{\pm i\frac{\nu}{2}}) \to \frac{\sqrt{2\pi}}{\Gamma(1 - i\gamma)} e^{-\frac{\gamma}{4}} e^{\mp i\frac{|z|^2}{4}} |z|^{\mp i\gamma} \quad (A.23)
\]

We can calculate the populations in the two states. Those reduce to the famous result obtained first independently by Landau, Zener, Stückelberg and Majorana

\[
|C_1(\infty)|^2 = e^{-2\pi\gamma} = P_{LZ} \quad (A.24)
\]

\[
|C_0(\infty)|^2 = 1 - P_{LZ} \quad (A.25)
\]

where \( P_{LZ} \) is defined in Equation (2.41). Additionally, the phase difference between the two states is

\[
\phi = \mp \left( \frac{\pi}{4} + \arg(\Gamma(1 - i\gamma)) + \frac{|z|^2}{2} + 2\gamma \ln(|z|) \right). \quad (A.26)
\]
This can be simplified by calculating (with Mathematica) the difference
\[ \int_0^T \left( \sqrt{\varepsilon^2 + \Delta^2} - |\varepsilon| \right) dt \approx \frac{(1 - 2 \ln \Delta + 2 \ln(2|\nu|T)) \Delta^2}{4|\nu|}, \quad T \to \infty \] (A.27)
and by noticing that \(|z| = \sqrt{|\nu|} t| and \int_0^t |\varepsilon| = |z|^2/2, we obtain
\[ \frac{|z|^2}{2} + 2 \gamma \ln(|z|) = \int_0^T \sqrt{\varepsilon^2 + \Delta^2} dt + \gamma(\ln \gamma - 1), \] (A.28)
where \( \sqrt{\varepsilon^2 + \Delta^2} = E_+(t) - E_-(t) \) can be written in terms of adiabatic
eigenenergies \( E_{\pm}(t) = \pm \frac{1}{2} \sqrt{\varepsilon^2 + \Delta^2} \). We can sum up the results obtained
above as a unitary rotation matrix that determines the asymptotic time evo-
lution from \( t = -\infty \) to \( t = \infty \)
\[ U_{LZ}^\pm = \begin{pmatrix} \sqrt{PLZ} & \pm \sqrt{1 - PLZ} \exp(\pm i\phi_S) \\ \mp \sqrt{1 - PLZ} \exp(\mp i\phi_S) & \sqrt{PLZ} \end{pmatrix}, \] (A.29)
where \( \phi_S = \frac{\pi}{4} + \text{arg}(\Gamma(1 - i\gamma)) + \gamma(\ln \gamma - 1) \) is the so called Stokes phase.
The upper sign is related to a region where the state \( |0\rangle \) approaches the state \( |1\rangle \) from below \((\nu > 0)\) and the lower to that from above \((\nu < 0)\).

For clarity, let us write this in adiabatic basis. It turns out that we do
not need to worry anymore about the sign of the velocity \( \nu \), since
\[ U_{LZ} = \begin{pmatrix} \sqrt{1 - PLZ} \exp(-i\phi_S) & \sqrt{PLZ} \\ -\sqrt{1 - PLZ} \exp(i\phi_S) & \sqrt{1 - PLZ} \end{pmatrix}, \] (A.30)
where \( (10)^T \) corresponds to eigenvalue \( E_+ = \frac{1}{2} \sqrt{\varepsilon^2 + \Delta^2} \) and \((01)^T \) to
\( E_- = -\frac{1}{2} \sqrt{\varepsilon^2 + \Delta^2} \).

In addition to tunneling, we have free phase evolution defined by
\[ U_\varphi = \begin{pmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{pmatrix}, \] (A.31)
where \( \varphi = \int (E_+ - E_-) dt \). This way the whole LZ process, starting from the
distant past \((t_1 \ll -\Delta/|\nu|)\) and ending far in the future \((t_2 \gg \Delta/|\nu|)\), can
be characterized by the unitary time evolution operator
\[ U = U_\varphi_2 U_{LZ} U_\varphi_1 = \begin{pmatrix} e^{-i\frac{\varphi_1}{2}} & 0 \\ 0 & e^{i\frac{\varphi_1}{2}} \end{pmatrix} \times \]
\[ \begin{pmatrix} \sqrt{1 - PLZ} \exp(-i\phi_S) & \sqrt{PLZ} \\ -\sqrt{1 - PLZ} \exp(i\phi_S) & \sqrt{1 - PLZ} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\varphi_2}{2}} & 0 \\ 0 & e^{i\frac{\varphi_2}{2}} \end{pmatrix}, \] (A.32)
where \( \varphi_1 = \int_0^{t_1} (E_+ - E_-) dt \) and \( \varphi_2 = \int_0^{t_2} (E_+ - E_-) dt \). This an asymptotic
result and holds when \(|t_1|, t_2 \to \infty \).

A Similar transfer matrix approach to the curve crossing problem has been
reported by Child [89]. The Weber equation and closely related probability
amplitudes and the Stokes phase turn up naturally when studying scattering
from an inverted parabolic potential barrier [90].
Appendix B

Quantum Network Theory

Electrical circuits can be described in the Hamiltonian formalism of classical dynamics. The foundation of the procedure was laid by Yurke and Denker [60] and later on generalized by Devoret [61]. In this Appendix, we will follow Devoret’s notes and derive the Hamiltonian operators that are relevant for this thesis.

Hamiltonian formalism requires a choice of coordinate system. We use the description where any electrical network can be described as branches consisting of two-terminal electrical elements. We define the branch flux and charge passed through such an element by

\[ \Phi_b(t) = \int_{-\infty}^{t} V_b(t') dt' \quad (B.1) \]
\[ Q_b(t) = \int_{-\infty}^{t} I_b(t') dt'. \quad (B.2) \]

In the above, \( V_b \) is the voltage across and \( I_b \) the current through the branch. For example, for a capacitor with capacitance \( C \) this means that using these variables we get the familiar relation for the stored energy as

\[ E_C = \int_{-\infty}^{t} V_b(t') I_b(t') dt' = \int_{-\infty}^{t} \frac{Q dQ}{C} dt' = \int_{0}^{Q_C} \frac{Q}{C} dQ = \frac{Q^2}{2C}. \quad (B.3) \]

Similarly, that of an inductance \( L \) is

\[ E_L = \int_{-\infty}^{t} \frac{\Phi d\Phi}{L} dt' = \int_{0}^{\Phi_L} \frac{d\Phi}{L} = \frac{\Phi^2}{2L}. \quad (B.4) \]

In addition to these standard linear circuit elements, we use Josephson junctions, which have a nonlinear current-branch-flux relation

\[ I_J = I_c \sin \frac{2\pi \Phi_J}{\Phi_0}. \quad (B.5) \]
This means that the inductive energy stored in JJ is

\[ U_J = \int_{-\infty}^{t} I_c \sin \frac{2\pi \Phi}{\Phi_0} dt' dt' = \int_{0}^{\Phi_J} I_c \sin \frac{2\pi \Phi}{\Phi_0} d\Phi = -E_J \cos \frac{2\pi \Phi_J}{\Phi_0}, \quad (B.6) \]

where we have defined \( E_J = I_c \Phi_0 / 2\pi \). In addition to this, JJ also stores capacitive energy described by the ordinary capacitive relation

\[ E_C = \frac{Q_J^2}{2C_J}. \quad (B.7) \]

These are the circuit elements used in this thesis. We follow the convention, used at least by the Chalmers group [91, 92], and choose the branch fluxes as our canonical coordinates. Since the inductive terms depend on our coordinates, they can be identified as potential energy terms. The charge on a capacitor is written in terms of flux as \( Q_C = C \dot{\Phi}_C \), and can, thus, be identified as the coordinate velocity. Capacitive energies correspond, therefore, to the kinetic energy of the system. For any circuit, we can write down the kinetic energy \( T \) as the sum capacitive energies in each of the circuit elements. Similarly, the potential energy \( V \) is the sum of the inductive energies. According to the classical Hamiltonian formalism, we can define the Lagrangian \( L \) as the difference between the kinetic and potential energies as

\[ L = T - V. \quad (B.8) \]

The chosen set of branch fluxes is not necessarily independent. There are constraints, imposed by the topology of the circuit and by the Kirchhoff laws, saying that the sum of fluxes around any loop in the circuit should be equal to the flux \( \Phi_l \) through the loop, i.e.

\[ \sum_{\text{all } b \text{ around a loop}} \Phi_b = \Phi_l. \quad (B.9) \]

Taking into account all of these constraints, we end up with a set of \( N \) independent coordinates \( \{ \Phi_i | i = 1, \ldots, N \} \). For each coordinate, we can define corresponding canonical momentum as

\[ q_i = \frac{\partial L}{\partial \dot{\Phi}_i}. \quad (B.10) \]

If denote define the velocity vector \( \dot{\Phi} = (\dot{\Phi}_1, \dot{\Phi}_2, \ldots, \dot{\Phi}_N)^T \) and the momentum vector \( P = (q_1, q_2, \ldots, q_N)^T \) we can write the canonical momenta as a linear transformation (since the kinetic energies are at most quadratic in \( \dot{\Phi}_i \))

\[ P = C \dot{\Phi} - a. \quad (B.11) \]
In the above, the (symmetric) \( N \times N \)-matrix \( C \) is called as the capacitance matrix. The coordinate independent \( a \) is named as the vector potential, since it is analogous to that in the Hamiltonian of an electron in a magnetic field. Using this notation, we write the Lagrangian function as

\[
L = \frac{1}{2} \dot{\Phi}^T C \dot{\Phi} - \dot{\Phi}^T a - V. \tag{B.12}
\]

Because

\[
\dot{\Phi} = C^{-1}(P + a), \tag{B.13}
\]

we have that the Legendre transformation leading to Hamiltonian function can be written as (possible constant energy terms are omitted, because they just fix the energy origin)

\[
H = \dot{\Phi}^T P - L = \dot{\Phi}^T P - \frac{1}{2} \dot{\Phi}^T C \dot{\Phi} + \dot{\Phi}^T a + V. \tag{B.14}
\]

We want to express the Hamiltonian in terms of the momenta so we will replace the phase derivatives with \( \{B.13\} \). Then,

\[
H = \dot{\Phi}^T P - L = \dot{\Phi}^T P - \frac{1}{2} \dot{\Phi}^T C \dot{\Phi} + \dot{\Phi}^T a + V
\]

\[
= \frac{1}{2}(C^{-1}(P + a))^T C C^{-1}(P + a) + (C^{-1}(P + a))^T a + V
\]

\[
= \frac{1}{2}(P + a)^T C^{-1}(P + a) + V, \tag{B.15}
\]

where we have used the fact that the capacitance matrix is symmetric, i.e. \( (C^{-1})^T = C^{-1} \). This is the Hamiltonian function of the classical mechanics. Finally, we move into the quantum description by assuming that the classical variables are replaced by corresponding operators

\[
\Phi \rightarrow \hat{\Phi}
\]

\[
q \rightarrow \hat{q}
\]

\[
H \rightarrow \hat{H} \tag{B.16}
\]

and that the operators corresponding to canonically conjugated variables \( \Phi_i \) and \( q_i \) obey the commutation relation

\[
[\hat{\Phi}_i, \hat{q}_i] = i\hbar. \tag{B.17}
\]

In the following sections, we use this method and derive Hamiltonian operators for circuits used in this thesis.
Appendix B. Quantum Network Theory

Figure B.1: Single Cooper pair transistor. Arrows denote the branch fluxes.

B.1 Single Cooper Pair Transistor

According to the procedure, the kinetic energy consists of the sum of the charging energies at the capacitances (remembering that the charges on the capacitances can be written in terms of the branch fluxes as $Q = C\dot{\Phi}$)

$$T_{\text{scpt}} = \frac{1}{2}C_1\dot{\Phi}_1^2 + \frac{1}{2}C_2\dot{\Phi}_2^2 + \frac{1}{2}C_g\dot{\Phi}_g^2 \quad (B.18)$$

We have chosen the branch fluxes our coordinates, denoted with arrows in Figure B.1. The potential energy is solely due to the Josephson energies

$$V_{\text{scpt}} = -E_{J1}\cos\frac{2\pi\Phi_1}{\Phi_0} - E_{J2}\cos\frac{2\pi\Phi_2}{\Phi_0}. \quad (B.19)$$

Let us make a convenient change of variables. We define

$$\begin{align*}
\Phi &= \Phi_1 + \Phi_2 \\
\Theta &= \frac{\Phi_2 - \Phi_1}{2}
\end{align*} \implies \begin{align*}
\Phi_1 &= \frac{\Phi - \Theta}{2} \\
\Phi_2 &= \frac{\Phi + \Theta}{2}.
\end{align*} \quad (B.20)$$

We will assume at this point that the flux $\Phi$ through the SCPT is fixed as constant by some surrounding circuitry. Thus, we have only one canonical coordinate $\Theta$. In the next section, we will define the rest of the circuit more precisely which will upgrade $\Phi$ to a variable. If we choose the electric ground at the down going wire in Fig. B.1, we get the constraint

$$\dot{\Phi}_g = \dot{\Phi}_2 + V_g = \dot{\Theta} + V_g. \quad (B.21)$$

Thus, after a little algebra and trigonometry, we obtain the Lagrangian

$$L_{\text{scpt}} = T_{\text{scpt}} - V_{\text{scpt}} = \frac{1}{2}C_1\dot{\Theta}^2 + \frac{1}{2}C_2\dot{\Theta}^2 + \frac{1}{2}C_g\left(\dot{\Theta} + V_g\right)^2 \quad (B.22)$$

$$+ E_{J0}\cos\frac{\pi\Phi}{\Phi_0}\cos\frac{2\pi\Theta}{\Phi_0} + dE_{J0}\sin\frac{\pi\Phi}{\Phi_0}\sin\frac{2\pi\Theta}{\Phi_0},$$

where we have defined the $E_{J0} = E_{J1} + E_{J2}$ and the asymmetry $d = (E_{J1} - E_{J2})/E_{J0}$. In the symmetric case, $d = 0$, we notice that the SCPT behaves
as a single JJ with a flux controllable coupling energy. For the Hamiltonian, we need the canonical momentum

$$Q = \frac{\partial L_{\text{scpt}}}{\partial \dot{\Theta}} = (C_1 + C_2 + C_g)\dot{\Theta} + Q_0,$$

where $Q_0 = C_g V_g$.

Since we have only one variable, it is straightforward to write down the Hamiltonian of the SCPT

$$H_{\text{scpt}} = \left( Q - Q_0 \right)^2 - E_{J_0} \cos \frac{\pi \Phi}{\Phi_0} \cos \frac{2\pi \Theta}{\Phi_0} - dE_{J_0} \sin \frac{\pi \Phi}{\Phi_0} \sin \frac{2\pi \Theta}{\Phi_0},$$

where we have denoted $C_{\Sigma} = C_1 + C_2 + C_g$. This is the ”classical” Hamiltonian of the SCPT.

Let us then move into the quantum world. We quantize the system similarly as in the single junction case and end up with Hamiltonian operator

$$\hat{H}_{\text{scpt}} = \left( \hat{Q} - Q_0 \right)^2 - E_{J_0} \cos \frac{\pi \Phi}{\Phi_0} \cos \frac{2\pi \hat{\Theta}}{\Phi_0} - dE_{J_0} \sin \frac{\pi \Phi}{\Phi_0} \sin \frac{2\pi \hat{\Theta}}{\Phi_0},$$

where $[\hat{\Theta}, \hat{Q}] = i\hbar$.

The eigenstates $|N\rangle$ of the dimensionless charge operator $\hat{Q}/2e$ describe the number of excess Cooper pairs on the island. We use them as a basis and obtain

$$\hat{H}_{\text{scpt}} = \sum_n \left[ 4E_C(N - N_0)^2|N\rangle - \frac{E_i(\Phi)}{2} \left( e^{i\phi}|N + 1\rangle + e^{-i\phi}|N - 1\rangle \right) \right]\langle N|,$$

where $E_C = e^2/2C_{\Sigma}$ and $\tan \phi = -d\tan \frac{\pi \Phi}{\Phi_0}$. In the charge limit $E_{J_0} \ll E_C$ (see Chapter 3) and only two charge states are relevant. The SCPT Hamiltonian reduces to an effective $2 \times 2$-matrix

$$\hat{H}_q \approx \frac{1}{2} \left[ E_{\text{el}} \hat{\sigma}_z - E_i(\Phi) \left( \cos \phi \hat{\sigma}_x + \sin \phi \hat{\sigma}_y \right) \right],$$

where $E_{\text{el}} = 4E_C(1 - 2N_0)$.

### B.1.1 Adiabatic Basis

If the drive frequency and amplitude are small, the qubit follows the adiabatic energy eigenstates. In this limit, one benefits from using those as basis states when solving the time-evolution from the master equation (2.25) (the relaxation times $T_1$ and $T_2$ that are measured in experiments describe the decoherence of the adiabatic states in this limit). This choice of basis was used for the two-level system in Paper I.
Let us rewrite the Hamiltonian (B.27) as

\[
\hat{H}_q = \frac{1}{2} \sqrt{E_{cl}^2 + E_J(\Phi)^2} \left[ \cos \theta \hat{\sigma}_z + \sin \theta \left( \cos \phi \hat{\sigma}_x + \sin \phi \hat{\sigma}_y \right) \right],
\]

(B.28)

where \( \tan \theta = -\frac{E_J(\Phi)}{E_{cl}} \). Now we can diagonalise this into the adiabatic eigenbasis by a unitary transformation \( \hat{U}^\dagger \hat{H}_q \hat{U} + i\hbar \frac{\partial \hat{U}^\dagger}{\partial t} \hat{U} \) where

\[
\hat{U} = \begin{pmatrix}
  e^{-i\phi/2} \cos \frac{\theta}{2} & -e^{-i\phi/2} \sin \frac{\theta}{2}
  \\
  e^{i\phi/2} \sin \frac{\theta}{2} & e^{i\phi/2} \cos \frac{\theta}{2}
\end{pmatrix}.
\]

(B.29)

With these notations, Pauli spin matrices are transformed as (this is equivalent to a transformation from Cartesian coordinates into spherical coordinate system)

\[
\hat{U}^\dagger \hat{\sigma}_x \hat{U} = \cos \theta \cos \phi \hat{\sigma}_x' - \sin \phi \hat{\sigma}_y' + \sin \theta \cos \phi \hat{\sigma}_z'
\]

(B.30)

\[
\hat{U}^\dagger \hat{\sigma}_y \hat{U} = \cos \theta \sin \phi \hat{\sigma}_x' + \cos \phi \hat{\sigma}_y' + \sin \theta \sin \phi \hat{\sigma}_z'
\]

(B.31)

\[
\hat{U}^\dagger \hat{\sigma}_z \hat{U} = -\sin \theta \hat{\sigma}_x' + \cos \theta \hat{\sigma}_z'.
\]

(B.32)

We end up with the rotated Hamiltonian as

\[
\hat{H}_q^{\text{rot}} = \frac{1}{2} \left[ \hbar \Omega_0 \hat{\sigma}_z - \hbar \dot{\theta} \hat{\sigma}_y + \hbar \dot{\phi} \left( \sin \theta \hat{\sigma}_x - \cos \theta \hat{\sigma}_z \right) \right],
\]

(B.33)

where \( \hbar \Omega_0 = \sqrt{E_{cl}^2 + E_J(\Phi)^2} \) is the qubit energy splitting.

### B.1.2 Oscillation Center Basis

We assume here that the flux can be driven externally as \( \Phi = \Phi_b + \Phi_L(t) \) (see next section). When the drive amplitude becomes large enough the system follows the diabatic states. Then the use of adiabatic states becomes questionable in terms of relaxation processes. We solve this problem by introducing the so-called oscillation center basis which can be defined when at least one of the control parameters is oscillating periodically. In the case of rapid oscillations we can assume that the decoherence processes occur at the oscillation center. This basis was used in the experiment of Paper IV where the drive amplitude was large.

Let us first separate the static flux from the SCPT-Hamiltonian and obtain

\[
\hat{H}_q = -\frac{E_{J0}}{2} \left[ \cos \frac{\pi \Phi_L(t)}{\Phi_0} \left( \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x - \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_y \right) 
  \\
  - \frac{\pi \Phi_L(t)}{\Phi_0} \left( \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x + \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_y \right) \right],
\]

(B.34)
B.2. SCPT COUPLED TO LC-RESONATOR

where we have biased the system at charge degeneracy in order to get rid of the first order decoherence induced by charge fluctuators [65]. This includes also the interaction between the oscillator and the qubit. At first, we consider the zeroth order term in the qubit Hamiltonian

$$\hat{H}_0 = -\frac{E_{J_0}}{2} \left[ \cos \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_x - d \sin \frac{\pi \Phi_b}{\Phi_0} \hat{\sigma}_y \right].$$  (B.35)

This can be written in form

$$\hat{H}_0 = \frac{E_J(\Phi_b)}{2} \left( \begin{array}{cc} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{array} \right),$$  (B.36)

where $\tan \phi = -d \tan(\pi \Phi_b/\Phi_0)$ and

$$E_J(\Phi_b) = E_{J_0} \sqrt{\cos^2(\pi \Phi_b/\Phi_0) + d^2 \sin^2(\pi \Phi_b/\Phi_0)}$$  (B.37)

is the flux $\Phi_b$ controlled Josephson energy. Let us now diagonalize $\hat{H}_0$ by making a unitary rotation $\hat{H}_q \to \hat{U}^\dagger \hat{H}_q \hat{U}$ where

$$\hat{U} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} e^{-i\phi/2} & -e^{-i\phi/2} \\ e^{i\phi/2} & e^{i\phi/2} \end{array} \right).$$  (B.38)

The Pauli matrices transform as

$$\hat{U}^\dagger \hat{\sigma}_x \hat{U} = \cos \phi \hat{\sigma}_z - \sin \phi \hat{\sigma}_y,$$
$$\hat{U}^\dagger \hat{\sigma}_y \hat{U} = \sin \phi \hat{\sigma}_z + \cos \phi \hat{\sigma}_y$$  (B.39)
$$\hat{U}^\dagger \hat{\sigma}_z \hat{U} = -\hat{\sigma}_x,$$

where $\cos \phi = -E_{J_0} \cos(\pi \Phi_b/\Phi_0)/E_J(\Phi_b)$ and $\sin \phi = dE_{J_0} \sin(\pi \Phi_b/\Phi_0)/E_J(\Phi_b)$.

We plug these into Hamiltonian $\hat{H}_q$ and after some straightforward algebra we obtain

$$\hat{H}_q = \frac{1}{2} \left[ \cos \frac{\pi \Phi_L(t)}{\Phi_0} E_J(\Phi_b) \hat{\sigma}_z + \sin \frac{\pi \Phi_L(t)}{\Phi_0} \frac{E_{J_0}^2}{E_J(\Phi_b)} \left( \frac{d^2 - 1}{2} \sin \frac{2\pi \Phi_b}{\Phi_0} \hat{\sigma}_z - d \hat{\sigma}_y \right) \right].$$  (B.40)

B.2 SCPT Coupled to LC-resonator

Next we couple the SCPT to an LC-oscillator. The circuit diagram is depicted in Figure B.2. The green arrows indicate the flux coordinates and we obtain the kinetic energy

$$T = \frac{1}{2} C_1 \dot{\Phi}_1^2 + \frac{1}{2} C_2 \dot{\Phi}_2^2 + \frac{1}{2} C_3 \dot{\Phi}_3^2 + \frac{1}{2} C_4 \dot{\Phi}_4^2 + \frac{1}{2} C_g \dot{\Phi}_g^2$$  (B.41)
Figure B.2: SCPT coupled to an electric resonator. Arrows denote the branch fluxes.

The potential energy includes Josephson and linear inductive terms

\[ V = -E_{J1} \cos \frac{2\pi \Phi_1}{\Phi_0} - E_{J2} \cos \frac{2\pi \Phi_2}{\Phi_0} + \frac{1}{2}LI^2 + \frac{1}{2}L_{ext}I_{ext}^2 + MII_{ext}. \]  

(B.42)

The auxiliary coil is introduced to produce flux control to the qubit. We assume that the current source \( I_{ext} \) determines fully the flux through \( L_{ext} \) and therefore induces the external flux \( \Phi_{ext} = M_{ext} \) to the system via the mutual inductance \( M \). In other words, there is an induced electromotive force \( \dot{\Phi}_{ext} = M \dot{I}_{ext} \) due to the time dependent current \( I_{ext} \) in the external loop. Accordingly, the flux through the coil \( L \) is \( \Phi_5 = LI + \Phi_{ext} \). Notice, that \( \Phi_5 + \Phi_{ext} = \Phi_1 + \Phi_2 \), so we have to choose the flux coordinate in terms of which we will express our Lagrangian. If we use \( \Phi = \Phi_1 + \Phi_2 \), the effect of the external flux can be seen only in the inductor part (this was done in Papers I and II where it did not make any difference since the flux was time-independent). On the other hand, if we use \( \Phi = \Phi_5 \) the flux dependence is partly transferred to the Josephson part. The final Hamiltonians seem at the first sight totally different since in the first case the auxiliary field drives the resonator but in the latter case it also modulates the qubit energy. Regardless of this apparent difference the Hamiltonians should describe the same physics. In this thesis we choose \( \Phi \equiv \Phi_5 \) as our flux coordinate, similarly as in Paper IV.

At this point we make another convenient change of variables (in addition to Eq. (B.20) by

\[ \Theta_1 = \frac{\Phi_4 - \Phi_3}{2}. \]  

(B.43)

Remembering that the flux through the Josephson junctions is now \( \Phi + \Phi_{ext} \), we get that

\[ \Phi_1 = \frac{\Phi + \Phi_{ext}}{2} - \Theta, \quad \Phi_2 = \frac{\Phi + \Phi_{ext}}{2} + \Theta, \quad \Phi_3 = \frac{\Phi + \Phi_{ext}}{2} - \Theta_1, \quad \Phi_4 = \frac{\Phi + \Phi_{ext}}{2} + \Theta_1. \]  

(B.44)

We now have the constraints

\[ \Phi_1 + \Phi_2 = \Phi_3 + \Phi_4 = \Phi + \Phi_{ext}. \]
\[ \dot{\Phi}_g = \dot{\Phi}_2 - \dot{\Phi}_4 - V_g = \dot{\Theta} - \dot{\Theta}_1 + V_g. \] (B.45)

Again, by making the variable changes (B.20) and (B.43) and using the constraints (B.45) we obtain the canonical momenta (we skip here the lengthy but straightforward algebra)

\[
\begin{align*}
q &= \frac{\partial L_{qo}}{\partial \dot{\Phi}} = C(\dot{\Phi} + \dot{\Phi}_{\text{ext}}) \\
Q &= \frac{\partial L_{qo}}{\partial \dot{\Theta}} = C_{\Sigma} \dot{\Theta} - C_g \dot{\Theta}_1 + Q_0 \\
p &= \frac{\partial L_{qo}}{\partial \dot{\Theta}_1} = -C_g \dot{\Theta} + C_{\sigma} \dot{\Theta}_1 - Q_0,
\end{align*}
\]

where Lagrangian \( L_{qo} = T - V \). We have assumed the capacitive symmetry \( C_1 = C_2 \) and \( C_3 = C_4 \). We have also denoted \( C = (C_1 + C_3)/2, C_{\Sigma} = C_1 + C_2 + C_g \) and \( C_{\sigma} = C_3 + C_4 + C_g \). Finally, we obtain the Hamiltonian for the circuit described in Figure 3.6 (assuming that \( C_g \ll C_1, C_3 \), which holds in the systems discussed in this thesis)

\[
H_{qo} = \frac{(Q - Q_0)^2}{2C_{\Sigma}} - E_{J_0} \cos \frac{\pi(\Phi + \Phi_{\text{ext}})}{\Phi_0} \cos \frac{2\pi \Theta}{\Phi_0} - dE_{J_0} \sin \frac{\pi(\Phi + \Phi_{\text{ext}})}{\Phi_0} \sin \frac{2\pi \Theta}{\Phi_0} + \frac{q^2}{2C} + \frac{\Phi^2}{2L} - q\dot{\Phi}_{\text{ext}}, \]
\]

where we have noticed that the system does not explicitly depend on \( \Theta_1 \), which implies, according to Lagrange’s equation, that the corresponding momentum is a constant of motion and we can set \( p = \text{constant} = Q_0 \). We notice that Hamiltonian \( H \) consists of two parts. First, there is the SCPT-part

\[
H_q = \frac{(Q - Q_0)^2}{2C_{\Sigma}} \]

\[
- E_{J_0} \cos \frac{\pi(\Phi + \Phi_{\text{ext}})}{\Phi_0} \cos \frac{2\pi \Theta}{\Phi_0} - dE_{J_0} \sin \frac{\pi(\Phi + \Phi_{\text{ext}})}{\Phi_0} \sin \frac{2\pi \Theta}{\Phi_0}, \]
\]

which is very similar to the Hamiltonian (B.24). The difference is that the flux through the SCPT is now fixed explicitly by the coil \( \Phi \) and the external flux \( \Phi_{\text{ext}} \). The other part consists of a driven harmonic oscillator

\[
H_o = \frac{q^2}{2C} + \frac{\Phi^2}{2L} - q\dot{\Phi}_{\text{ext}}. \]
\]

Let us then quantize the Hamiltonian. We assume that \( \Phi_{\text{ext}} = \Phi_0 + \Phi_L(t) \) and end up with
\[
\hat{H}_{q_0} = \frac{q^2}{2C} + \frac{\dot{q}^2}{2L} - \dot{q}\Phi_L(t) + \frac{1}{2}\left\{ E_{el}\hat{\sigma}_z - E_{J0}\left[ \cos \frac{\pi(\dot{\Phi} + \Phi_L(t))}{\Phi_0}\left( \cos \frac{\pi\Phi_b}{\Phi_0}\hat{\sigma}_x - d\sin \frac{\pi\Phi_b}{\Phi_0}\hat{\sigma}_y \right) - \sin \frac{\pi(\dot{\Phi} + \Phi_L(t))}{\Phi_0}\left( \sin \frac{\pi\Phi_b}{\Phi_0}\hat{\sigma}_x + d\cos \frac{\pi\Phi_b}{\Phi_0}\hat{\sigma}_y \right) \right] \right\}, \tag{B.50}
\]

where we have made the two-state approximation for the SCPT-part and separated the static flux \(\Phi_b\).
Appendix C

Floquet Formalism

In this Appendix we analyze the time-periodic Hamiltonians (e.g. Equations (2.62) and (3.46)) in terms of the Floquet theory [68] (we assume here, for simplicity, that the system is 2-dimensional which can be straightforwardly generalized). We assume that the Hamiltonian \( \hat{H}(t) \) is \( \tau \)-periodic, i.e. \( \hat{H}(t + \tau) = \hat{H}(t) \). We denote the corresponding angular frequency by \( \omega_L = \frac{2\pi}{\tau} \). The periodicity allows us to write the Hamiltonian as the Fourier series

\[
\hat{H}(t) = \sum_{n=-\infty}^{\infty} \hat{H}^{[n]} e^{in\omega_L t}, \tag{C.1}
\]

where the Fourier components are given by

\[
\hat{H}^{[n]} = \frac{1}{\tau} \int_{0}^{\tau} dt \hat{H}(t)e^{-in\omega_L t}. \tag{C.2}
\]

According to the Floquet theorem, we can write the solution of the Schrödinger equation

\[
\hat{H}\Psi(t) = i\hbar \frac{\partial}{\partial t} \Psi(t) \tag{C.3}
\]

in the form

\[
\Psi(t) = e^{-i\hat{H}t}\Phi(t), \tag{C.4}
\]

where \( \Phi(t) \) is \( \tau \)-periodic, i.e.

\[
\Phi(t + \tau) = \Phi(t), \tag{C.5}
\]

and \( \varepsilon \) is a real parameter called the quasienergy. Now, if we substitute (C.4) into the Schrödinger equation (C.3), we obtain an eigenvalue equation for the quasienergy

\[
\hat{\mathcal{H}}\Phi_{\gamma}(t) \equiv \left( \hat{H} - i\hbar\frac{\partial}{\partial t} \right) \Phi_{\gamma}(t) = \varepsilon_\gamma \Phi_{\gamma}(t). \tag{C.6}
\]
Also, we notice that the transformation \((m)\) is an arbitrary integer

\[
\varepsilon' = \varepsilon + m\hbar\omega, \quad \Phi'(t) = \exp(i\omega t)\Phi(t) \tag{C.7}
\]
leads to

\[
\Psi'(t) = e^{-i\varepsilon' t}\Phi'(t) = e^{-i(\varepsilon + m\hbar\omega) t}e^{i\omega t}\Phi(t) = \Psi(t). \tag{C.9}
\]

Thus, we see that the Floquet states with quasienergies differing by \(m\hbar\omega\) are physically equivalent.

We can now introduce the Hilbert spaces \(\mathcal{H}_S\) and \(\mathcal{H}_T\) for the qubit part and for the temporal part, respectively. \(\mathcal{H}_S\) is spanned by \(B_S = \{|\uparrow\rangle, |\downarrow\rangle\}\) and \(\mathcal{H}_T\) by \(B_T = \{ |n\rangle|\langle t|n\rangle = \exp(i\omega t)\}\). In \(B_T\), the inner product is defined as

\[
\langle m|n \rangle = \frac{1}{\tau} \int_0^\tau e^{i(n-m)\omega t} dt = \delta_{nm}. \tag{C.10}
\]

Now, because \(\langle t|n \rangle = \exp(i\omega t)\) we have that

\[
\frac{1}{\tau} \int_0^\tau e^{i(m-n)\omega t} dt = \frac{1}{\tau} \int_0^\tau \langle m|t \rangle\langle t|n \rangle dt \tag{C.11}
\]

\[= \langle m| \left( \frac{1}{\tau} \int_0^\tau dt\langle t|\langle t|n \rangle \right) |n \rangle = \langle m|n \rangle \tag{C.12}\]

\[\Rightarrow \frac{1}{\tau} \int_0^\tau dt\langle t|\langle t| = \hat{I} \tag{C.13}\]

Let us then define a composite Hilbert space \(\mathcal{H}_F = \mathcal{H}_S \otimes \mathcal{H}_T\), which we will refer to as the Floquet space. It is spanned by \(B = \{ |\alpha, n\rangle = |\alpha\rangle \otimes |n\rangle \}|\alpha \rangle \in B_S, |n\rangle \in B_T\}\) with the inner product

\[
\langle \beta, m|\alpha, n \rangle = \delta_{\alpha\beta}\delta_{nm}. \tag{C.14}\]

In this basis, we can rewrite

\[
|\Psi\rangle = e^{-i\hat{\Phi} t} \sum_{\beta} \sum_{m} \langle \beta, m|\Phi\rangle |\beta, m \rangle \tag{C.15}\]

\[\Rightarrow \Psi(t) = \langle t|\Psi\rangle = e^{-i\hat{\Phi} t} \sum_{m} \Phi^{(m)} e^{i\omega t}, \tag{C.16}\]

with

\[
\Phi^{(m)} = \sum_{\beta} \langle \beta, m|\Phi\rangle |\beta \rangle = \sum_{\beta} \Phi^{(m)}_{\beta} |\beta \rangle. \tag{C.17}\]
Now, by inserting this into the Schrödinger equation and by taking the inner product with \( |\alpha, n\rangle \) we obtain

\[
\langle \alpha, n | \hat{H} | \Psi \rangle = \langle \alpha, n | (\hat{H} - i\hbar \frac{\partial}{\partial t}) | \Psi \rangle \tag{C.18}
\]

\[
e^{-i\epsilon \hbar t} \sum_m \sum_\beta \left\langle \alpha, n \left| \left[ \hat{H} - \varepsilon - i\hbar \frac{\partial}{\partial t} \right] \beta, m \right\rangle \Phi^{(m)}_\beta \right. \tag{C.19}
\]

\[
e^{-i\epsilon \hbar t} \sum_m \sum_\beta \left[ \hat{H}^{[m-n]}_{\alpha \beta} - (\varepsilon - m\hbar \omega_L) \delta_{nm} \delta_{\alpha \beta} \right] \Phi^{(m)}_\beta \tag{C.20}
\]

\[
= 0, \tag{C.21}
\]

where

\[
\hat{H}^{[m-n]}_{\alpha \beta} = \langle \alpha, n | \hat{H} | \beta, m \rangle = \frac{1}{\tau} \int_0^\tau dt \langle \alpha | \hat{H} | \beta \rangle e^{i(m-n)\omega_L t}. \tag{C.22}
\]

Thus, we obtain

\[
\sum_m \sum_\beta \left[ \hat{H}^{[m-n]}_{\alpha \beta} + m\hbar \omega_L \delta_{nm} \delta_{\alpha \beta} \right] \Phi^{(m)}_\beta = \varepsilon \Phi^{(n)}_\alpha, \tag{C.23}
\]

which is the eigenvalue equation for the time-independent Floquet Hamiltonian

\[
\langle \alpha, n | \hat{H}^F | \beta, m \rangle = \hat{H}^{[m-n]}_{\alpha \beta} + m\hbar \omega_L \delta_{nm} \delta_{\alpha \beta}. \tag{C.24}
\]
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