Janne Oinas

THE DEGREE THEORY AND THE INDEX OF A CRITICAL POINT FOR MAPPINGS OF THE TYPE \((S_+)^n\)
JANNE OINAS

THE DEGREE THEORY AND THE INDEX OF A CRITICAL POINT FOR MAPPINGS OF THE TYPE ($\mathcal{S}_n$)

Academic dissertation to be presented, with the assent of the Faculty of Science of the University of Oulu, for public defence in Raahensali (Auditorium L10), Linnanmaa, on June 9th, 2007, at 12 noon.
Abstract

The dissertation considers a degree theory and the index of a critical point of demi-continuous, everywhere defined mappings of the monotone type.

A topological degree is derived for mappings from a Banach space to its dual space. The mappings satisfy the condition \((S)\), and it is shown that the derived degree has the classical properties of a degree function.

A formula for the calculation of the index of a critical point of a mapping \(A : X \rightarrow X^*\) satisfying the condition \((S)\) is derived without the separability of \(X\) and the boundedness of \(A\). For the calculation of the index, we need an everywhere defined linear mapping \(A' : X \rightarrow X^*\) that approximates \(A\) in a certain set. As in the earlier results, \(A'\) is quasi-monotone, but our situation differs from the earlier results because \(A'\) does not have to be the Fréchet or Gateaux derivative of \(A\) at the critical point. The theorem for the calculation of the index requires a construction of a compact operator \(T = (A' + \Gamma)^{-1}\Gamma\) with the aid of linear mappings \(\Gamma : X \rightarrow X\) and \(A'\). In earlier results, \(\Gamma\) is compact, but here it need only be quasi-monotone. Two counter-examples show that certain assumptions are essential for the calculation of the index of a critical point.

Keywords: Brouwer degree, Browder degree, degree theory, index of a critical point, index of a singular point, Leray-Schauder theory, mapping degree, mappings of monotone type, nonlinear analysis, nonlinear functional analysis, rotation of vector field, Skrypnik degree, the Brouwer-Hopf theory, the Browder theory, the Leray-Schauder degree, the Skrypnik theory, topological degree
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Oulu, May 2007

Janne Oinas
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1 Introduction

The dissertation is concerned with a degree theory and the index of a critical point of semi-continuous, everywhere defined mappings of the monotone type. The concept of monotonicity plays an important role in real analysis. For example functions that increase or decrease monotonically have many nice properties: they are measurable, they are differentiable almost everywhere, one-sided limits exist in the interior of the domain, a strictly monotone function has the inverse etc. Another important concept is compactness, which is very useful in optimization. We would like to use these concepts when we study boundary value problems. However, such applications involve infinite-dimensional spaces where the use of compactness becomes quite problematic. The situation is even worse for the concept of monotonicity because difficulties appear even in two dimensions. For the extension of monotonicity we have at least two options: we either define a suitable relation \( \preceq \) and require again that \( x \preceq y \) \( (y \preceq x) \) implies \( Ax \preceq Ay \), or we use the observation that a real function \( F \) is increasing on an interval \( J \subset \mathbb{R} \) if and only if it satisfies
\[
(Fx - Fy) \cdot (x - y) \geq 0 \quad \text{for all} \quad x, y \in J.
\]
(1.1)

Both ways have proved to be useful. The first option can be realized when we have a suitable cone in the space\(^1\), but in this thesis we do not consider that. The second option is feasible if \( F \) is a mapping with the domain \( D \) and the range in the same Hilbert space, or \( F \) is a mapping from a Banach space \( X \) to the dual space \( X' \). In the case of a Hilbert space, we replace the product in (1.1) by the inner product
\[
\langle Fx - Fy, x - y \rangle \geq 0 \quad \text{for all} \quad x, y \in D,
\]
and in the case of a Banach space, we replace the product by the so-called (natural) pairing
\[
\langle Fx - Fy, x - y \rangle \geq 0 \quad \text{for all} \quad x, y \in D.
\]

There is also the concept of mappings of a monotone type which differ from monotone mappings; their definitions suggest that such mappings are more suitable for the study of weakly convergent sequences. In their definitions, it is usually required that if a sequence \( (x_n)_{n=1}^{\infty} \subset D \) converges weakly to \( x \), then a condition similar to
\[
\lim_{n \to \infty} \langle Fx_n - Fx, x_n - x \rangle = 0
\]
\(^1\)See Deimling’s book [61, Chapter 6, pp. 217–255] for details and further information.

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is either implied by certain other conditions, or it implies something with some other conditions. Depending on the definition, \(\lim\) is replaced by \(\lim\) or \(\lim\) and \(=\) is replaced by \(\leq\) or \(\geq\).

Usually the mappings of the monotone type are associated with the study of the solvability of the nonlinear partial differential equation

\[ A(u)(x) := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha A_\alpha(x, u(x), \nabla u(x), \ldots, \nabla^m u(x)) = f(x), \quad x \in \Omega \]

with appropriate boundary conditions. Sometimes this equation is termed as a quasilinear elliptic partial differential equation in the generalized divergence form. The above equality is usually converted to the so-called weak form

\[ \int_{\Omega} \sum_{|\alpha| \leq m} \partial^\alpha A_\alpha(x, u(x), \nabla u(x), \ldots, \nabla^m u(x)) \partial^\alpha \phi(x) \, dx = \int_{\Omega} f(x) \phi(x) \, dx, \quad (1.2) \]

which is to be valid for all functions \(\phi\) in a certain set. When the functions \(u, f,\) and \(A_\alpha\), where \(|\alpha| \leq m\), and the set \(\Omega\) are sufficiently smooth, Equation (1.2) induces a mapping from one function space to another. Under certain auxiliary conditions this induced mapping will have some monotonicity properties.\(^5\) Other typical contexts where mappings of the monotone type emerge are parabolic equations, elliptic and parabolic variational inequalities, numerical analysis (usually in the connection of Galerkin’s approximation), control and network theory, etc. Since the applications of mappings of the monotone type are so manifold, the monotonicity conditions are worthy of study in the abstract setting because then the derived results will be available in the special situation of an application.

In applications, we usually want to know whether an equation has a solution, and if this question has a positive answer, then we want to know the structure of the solution set and the properties of a solution such as its smoothness. The last task is typical in the connection of partial differential equations. A topological degree is a tool intended for the examination of the structure of the solution set of an equation, and it applies to many tasks such as determining the solvability or the existence of other solutions as well as the bifurcation points of an equation. Roughly speaking, a topological degree is an integer-valued function \(\deg(f, D, p)\) whose values depend on a function \(f\), a point \(p\) of the target space of \(f\), and a set \(D\) from where we seek an element of \(f^{-1}(p)\). Its main property is that if \(\deg(f, D, p)\) is nonzero and the mapping \(f - p\) has no zeros on the boundary of \(D\), then there is a point \(x\) in \(D\) with \((f)(x) = p\). Since the mappings of the monotone type form a larger class than the compact mappings and arise naturally from elliptic partial differential equations, it is worthwhile developing degree theories for them and tools for the evaluation of the degree.

1.1 Outline of the thesis

This thesis is organized as follows.

\(^{5}\)Such results are found in the books by Pascali [145], Zeidler [195–199], and Skrypnik [167, 169, 170] for example.

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Chapter 2 is concerned with prerequisites; it is a collection of results required in the ensuing chapters. It contains some elementary facts about degree theory (Section 2.1), renorming of Banach spaces and some facts on Banach spaces (Section 2.2). Special attention is given to compact linear mappings in Section 2.3. Subsection 2.3.1 is devoted to a decomposition of a Banach space. This decomposition is achieved by using generalized eigenspaces of a compact, linear mapping, and it is needed in Chapter 4. Section 2.4 contains definitions of various monotonicity conditions and perturbation results concerning mappings of the monotone type.

In Chapter 3, the topological degree is derived for mappings satisfying the condition $(S_1)$ in a reflexive Banach space. Section 3.1 conveys some information on the degree theory of mappings of the monotone type and Section 3.2 contains the construction of the degree. The construction was already outlined by I. V. Skrypnik in 1973. Section 3.3 is devoted to proving that this degree has the so-called classical properties of a degree function. There is also a brief discussion on the other constructions of the $(S_1)$-degree in Section 3.4.

In Chapter 4, the formula for the calculation of the index of a critical point is derived when a Banach space $X$ is only reflexive and an $(S_1)$-mapping $A: X \to X^*$ might be unbounded. The result generalizes a result that was presented for the first time by I. V. Skrypnik in 1973. The value of the index is calculated by using some homotopies that are connected with an everywhere defined, linear mapping $A' : X \to X^*$ that approximates the mapping $A$ in a certain set. As in the earlier result, the linear approximation is quasi-monotone, but in our setting the mapping $A'$ does not have to be the Frechet or Gateaux derivative of $A$ at origin. The index formula requires a construction of a compact, linear operator $T = (A' + \Gamma)^{-1} \Gamma'$ with the aid of a linear mapping $\Gamma : X \to X^*$ and the linear approximation $A'$. In the earlier results, $\Gamma$ is compact, but in our setting it suffices that it is only quasi-monotone.

Section 4.1 is a brief introduction to the subject of Chapter 4. In Section 4.2, we define the concept of a critical point, and in Section 4.3, we give a method for its calculation. Section 4.4 is the longest section of Chapter 4 and the whole thesis. It solely contains the proof of the main theorem of Chapter 4 (Theorem 4.3.1). The proof is quite long and technical, though the techniques used are rather straightforward. The "core" of the proof is in Subsection 4.4.6, but for that proof we need some auxiliary results. The auxiliary results are presented in subsections 4.4.1–4.4.5 and 2.3.1.

The thesis ends with Chapter 5, where two counter-examples are presented. These counter-examples show that certain assumptions are essential in Theorem 4.3. One example was outlined by I. V. Skrypnik in 1973, and the other was given by I. V. Skrypnik & A. G. Kartsatatos in 2001; the older example shows the necessity of certain assumption and the other demonstrates that there is an assumption missing in Skrypnik’s earlier results on the index of a critical point.

In the thesis, we have tried to use a detailed way of referencing: if a result of a book or an article is mentioned, not only is the corresponding reference number given, but also the page where the result lies, or other identifications such as the number of the theorem are given when these are available. The aim is to avoid the tracing task caused by phrases like "This estimate follows from the results of Agmon, Douglas & Nirenberg [3]." and

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"For the proof of this claim see Dunford & Schwartz [68]." However, on some occasions it was not possible to give a detailed reference because a certain issue was handled in various places in a source.

In this thesis, the most essential references are the works of Berkovits [14], Berkovits & Mustonen [17], Browder [40–42, 44, 46], Cioranescu [53], Deimling [61], Diestel [63], Istrătescu [102], Lloyd [127], Schechter [161], Skrypnik [167, 169, 170], Taylor & Lay [178], and Zeidler [196, 198, 199].

Both of the references are very extensive and on some occasions such phrases cause tremendous difficulties for those who are not very familiar with partial differential equations or functional analysis. In the sarcastic article of Pétard [146], there are some phrases commonly used by mathematicians translated to “everyday language”; for example, “The details are left to the reader” = “I can’t do it”.

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2 Prerequisites

This chapter is a collection of theorems that are used in the subsequent chapters of this thesis. We begin with a brief introduction to the concept of the classical topological degree, and then we shall recall some definitions and facts about Banach spaces, stating also some general convergence results of sequences in Banach spaces. A short discussion on the geometry of Banach spaces is needed solely to demonstrate of the existence of a continuous, locally bounded, strictly monotone bijection from a reflexive Banach space to its dual space. This result is required in Chapter 3, and it ensures that there is a normalizing map for the degree function of demi-continuous \((S_+)-\)mappings. A short review of compact linear operators and the various types of continuity is given in Section 2.3. Subsection 2.3.1 contains a derivation of a decomposition of a Banach space by using invariant subspaces of a linear compact operator. The decomposition is crucial to Chapter 4. The present chapter ends with Section 2.4, where the concept of duality mapping and some monotonicity conditions are defined.

2.1 Brouwer’s degree

Here we state some facts about the degree theory of mappings between \(n\)-dimensional spaces. Other concepts that are close to this theory are the rotation of a vector field and the homotopic theory of continuous mappings (or vector fields).\(^1\) The reader will find a discussion on the relationship between degree and rotation in the wide-ranging survey by Zabreiko [189] and the 726 references therein.\(^2\)

The topological degree for continuous mappings between \(n\)-dimensional euclidean spaces was first introduced by L. E. J. Brouwer in article [28] in 1912, though already according to Zabreiko [189, p. 446], rotation and degree are not always the same. Other related concepts are the fixed point index and the winding number. Fixed point index and its connection to the mapping degree is handled in Zeidler’s book [196, pp. 519–612]. Winding number is dealt with in Henle [87, pp. 48–53], the relationship of winding number and degree is discussed in the books by Deimling [61, pp. 30–32] and [60, pp. 54–55] and the monograph of Fonseca & Gangbo [75, pp. 41–46].

The book [118] by Krasnoselskii, Perov, Povolotskiy & Zabreiko is an introduction to the concept of rotation of a vector field in the plane \(\mathbb{R}^2\). See also Smart [171, pp. 75–84] and Mortici [138, pp. 7–12] for a brief discussion on rotation and degree.

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Gauss and Kronecker used arguments of a degree theoretic type.\(^3\) The degree of \(n\)-dimensional spaces is usually called Brouwer’s degree or the finite-dimensional degree, and it is a tool intended for the examination of the solution set of an equation. This new concept proved to be very useful when the existence of solutions, an estimate for the number of solutions, or the branching points (bifurcation points) of the solution set were studied.

### 2.1.1 Constructions of Brouwer’s degree

According to P. P. Zabreiko [189, p. 339], there are three distinct ways to construct the topological degree.\(^4\) The first of them is based on a simplicial (a multidimensional analogue of the piece-wise linear) approximation of the original vector field and requires heavy machinery of combinatorics and/or algebra. This approach was used by Brouwer and Hopf in articles [26–28, 93–95], and it can be found in books on combinatorial and algebraic topology wherein the degree or the rotation is obtained as a consequence of more general results. The reader who does not fear the arduous task of first learning a vast amount of combinatorial and/or algebraic topology can study this approach from Alexandrov [5, Part 2, pp. 174–177 and 193–198; Part 3, pp. 100–127 ]; Alexandrov & Hopf [4, pp. 457–478]; Anderson, Granas & Dugundji [7, pp. 258–262]; Berger [12, pp. 56–59]; Berger & Gostiaux [11, pp. 253–262]; Blackett [20, pp. 120–124]; Boltyanskii [21, pp. 162–166]; Borisovich [22]; Brown [50, pp. 48–56]; Cronin [56, pp. 1–55]; Dold [64, pp. 62–71]; Dugundji [67, pp. 337–341]; Eilenberg & Steenrod [70, pp. 304–306]; Greub, Halperin & Vanstone [81, pp. 240–279]; Hilton [90, pp. 25–30]; Hu [98, pp. 99–105]; Hurewicz & Wallman [99]; Krasnoselskii [117, pp. 77–90 and pp. 103–122]; Lefschetz [123, pp. 253–255], [122, pp. 124–131]; Madsen & Tornehave [129, pp. 97–112]; Massey [130, pp. 40–43]; Matsushima [131, pp. 289–292]; Mauner [132, pp. 287–298]; Mayer [133, Chapter IV]; Pontrjagin [152], [151, pp. 60–66]; Rado & Reichelderfer [154, pp. 123–132]; Rinow [156, pp. 567–570]; Schubert [162, pp. 289–290]; Seifert & Thrall [164, pp. 283–285]; Spanier [174, pp. 193–199 and p. 207]; Spivak [175, Chapter 8]; Sternberg [176, pp. 120–130]; Vick [183, pp. 28–29]; and Wall [185, pp. 68–73].

The second approach uses differential geometry; it is based on smooth approximations of the original vector field and depends on a decisive result presented in Sard’s famous survey [160] which handled the measure of the critical values of a differentiable map (the so-called Sard or Brown-Sard theorem).\(^5\) This approach was for the first time presented by Nagumo [140] and can be found also in Chinn & Steenrod [52]; Fomenko [74]; Guillemin & Pollack [82, Chapter 3]; Hirsch [91, pp. 121–131]; Istrătescu [102, pp. 345–360]; Krawcewicz & Wu [120, pp. 41–56]; Milnor [135, pp. 26–31]; and Nirenberg [141, Chapter 1]. See also Dubrovin, Fomenko & Novikov [66, pp. 102–110].

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\(^{3}\)There is further interesting historical information on the development of the degree theory in Sieberg [165, 166], Zeidler [194, pp. 259–260], Browder [44], Eisenack & Fenske [71, pp. 96–98 and 136–141], and Zabreiko [189]. See also the books by Hirsch [91, pp. 140–141] and Hu & Papageorgiou [96, pp. 448–450].

\(^{4}\)In Berger & Berger’s book [13, pp. 39–43], there is a discussion on these different constructions.

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\(^{6}\)In Berger & Berger’s book [13, pp. 39–43], there is a discussion on these different constructions.
The third approach is based on the study of the so-called Kronecker integral and Sard’s theorem. This construction was first studied by Heinz in article [86] in 1959 (see also Hadamard’s treatise [83]), but it can be studied also from the works of Arăgrawal, Meehan & O’Regan [2, pp. 142–157]; Berger & Berger [13, pp. 23–45]; Cioranescu [53, pp. 114–124]; Deimling [61, pp. 12–16], [60, pp. 35–41]; Fonseca & Gangbo [75, pp. 1–34]; Fučík, Nečas, Souček & Souček [77, pp. 9–23]; Hutson & Pym [100, pp. 325–338]; Lloyd [127, pp. 2–22]; Mortici [138, pp. 18–28]; O’Regan, Cho & Chen [144, pp. 2–15]; Pasca [145, pp. 70–79]; and Schwartz [163, pp. 55–74]. See also Berger [12, pp. 52–54]; Denkowski, Migórski & Papageorgiou [62, pp. 189–199]; Eisenack & Festsch [71, pp. 75–105]; Jeggle [103, pp. 92–104]; Riedrich [155, pp. 93–113]; Smith [172, pp. 396–406]; and Zeidler [196, pp. 519–553], [194, pp. 266–270].

We note that in the modern form of the third construction of Brouwer’s degree, the degree function is first defined for continuously differentiable mappings, and then this definition is extended to continuous mappings via an approximation process with respect to the sup-norm. For an open, bounded set \( D \), a differentiable mapping \( f : D \subset \mathbb{R}^n \to \mathbb{R}^n \), the set \( \mathcal{S}(D) = \{ x \in D | J_f(x) = 0 \} \), and a point \( y \notin f(D \cup \partial \mathcal{S}(D)) \) the degree is first defined by

\[
\deg(f, D, y) = \sum_{x \in f^{-1}(y)} \sign(J_f(x)) \quad \text{(convention: } \sum_{\emptyset} = 0 \text{)},
\]

where \( J_f(x) \) is the Jacobian determinant of \( f \) at \( x \). After that, the requirement \( y \notin f(\mathcal{S}(D)) \) is removed by approximating an element of the set \( f(\mathcal{S}(D)) \) by the elements of the set \( f(D \cup \partial \mathcal{S}(D)) \); this approximation is possible due to Sard’s theorem. The details of this process are found in Deimling [61, pp. 13–16], Fonseca & Gangbo [75, pp. 6–19], Lloyd [127, pp. 8–20], Pascali [145, pp. 72–78], and Schwartz [163, pp. 55–71].

### 2.1.2 The classical properties of a degree function

Mathematicians are first interested in the existence of certain objects, and after that there is the question of the other properties such as its uniqueness. The uniqueness has significance both in theory and in practice. If one can show that a tool such as a topological degree with certain properties is unique, then it is the best possible tool with those properties and it is therefore useless to seek another tool with the same properties. Moreover, if we know that something is unique, and if we have different expressions for it, we can choose the one which is the most suitable for a certain situation. So it is reasonable to ask whether a certain topological degree is unique, and/or what properties make the degree unique.

In year 1972, Fuhrer [76] proved that certain properties of the topological degree in \( \mathbb{R}^n \) make that degree unique.\(^6\) A year after, without knowing Fuhrer’s result, Amann & Weiss [6] showed that the Leray-Schauder degree is uniquely determined by a few conditions and established the uniqueness of Brouwer’s degree in a normed space as a consequence.

\(^6\)One consequence of the uniqueness of the degree is that the topological degrees obtained by different means (analysis, differential geometry, algebraic topology) are really the same.
The method used by Amann & Weiss is found also in Lloyd [127, pp. 86–88] and the conditions used by Amann & Weiss are nowadays called the classical properties of the degree; any topological degree which has these properties is called a classical topological degree. The list of these classical properties in the $\mathbb{R}^n$-setting is given below.\footnote{Nagumo [139] was the first to assert that the uniqueness of the Brouwer degree can be proved by simplicial approximations; this assertion was made in paper [139] which appeared in 1951. In paper [143], O’Neill obtained the uniqueness of the Leray-Schauder degree by using the uniqueness of the fixed point index that was introduced via cohomology theory. O’Neill’s paper appeared in 1955. In paper [143], O’Neill obtained the uniqueness of the Leray-Schauder degree by using the uniqueness of the fixed point index that was introduced via cohomology theory. O’Neill’s paper appeared in 1955. The uniqueness of the fixed point index was obtained also by Brown [49] using a slightly different set of axioms than those of O’Neill. See also Istrățescu [102, pp. 360–370]. In article [179], which appeared in 1973, Thomas showed that classical properties make the topological degree of differentiable $k$-set contractions unique.}

For each bounded, open set $D$ of $\mathbb{R}^n$, each continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, and each $y \in \mathbb{R}^n$ with $y \notin f(\partial D)$ we associate an integer $\deg(f, D, y)$. This integer-valued function is called a classical topological degree function, provided it satisfies the following conditions:

1. If $\deg(f, D, y) \neq 0$, then the equation $f(x) = y$ has a solution in $D$.
2. If $D$ is the union of the disjoint, open sets $D_1$ and $D_2$, then
   \[ \deg(f, D, y) = \deg(f, D_1, y) + \deg(f, D_2, y). \]
3. If $y \notin K$ and $K \subset D$ is closed, then $\deg(f, D, y) = \deg(f, D, K, y)$.
4. If $H : [0, 1] \times D \to \mathbb{R}^n$ is a continuous homotopy and $y \in Y \setminus H([0, 1] \times \partial D)$, then $\deg(H(t, \cdot), D, y)$ is independent of $t \in [0, 1]$.
5. For the identity mapping $I$ of $\mathbb{R}^n$, $\deg(I, D, y) = 1$ if $y \in D$.

Conditions 2–5 make the degree of a finite-dimensional normed space unique. The first property follows from the second property (see Berkovits [14, p. 10]). The second and the third properties are called the domain decomposition and the domain excision respectively; moreover, they are usually given in one condition called the additivity:

\[ \text{additivity:} \quad \text{if } x, y \in D, \text{ and } f(x) = y, \text{ then } \deg(f, D, x) = \deg(f, D, y). \]

The fourth condition is the homotopy invariance, the fifth condition is the normalisation and the mapping which satisfies the normalisation condition is often called normalizing mapping. If we have a degree for a certain admissible class of mappings $X \to Y$, then the fifth condition is written as

\[ \text{normalization:} \quad \text{if } x \in J(D), \text{ then } \deg(f, D, x) = 1 \text{ if } y \in J(D). \]

Here we do not state the exact definition of Brouwer’s degree in $\mathbb{R}^n$ because we only need to know that it exists and has the above properties. The interested reader is encouraged to consult the references given earlier.


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Here we do not state the exact definition of Brouwer’s degree in $\mathbb{R}^n$ because we only need to know that it exists and has the above properties. The interested reader is encouraged to consult the references given earlier.

2.1.3 Brouwer’s degree in an \( n \)-dimensional normed space

In the construction of Brouwer’s degree for the mappings from and to the euclidean space \( \mathbb{R}^n \), one usually uses the natural basis \( e^1, \ldots, e^n \), where the \( i \)-th coordinate \( e^i_j \) of \( e^i \) is defined by \( e^i_j = \delta_{ij} \). Note that the natural basis is ordered. We obtain the same degree function if we consider a different ordered basis, say \( \tilde{e}^1, \ldots, \tilde{e}^n \). Let us explain this a little without offering a rigorous proof. We know from linear algebra that in this case there is the so-called transition matrix or the change-of-coordinates matrix \( A \), that is, the matrix \( A \) that corresponds to the change of bases \( e^1, \ldots, e^n \rightarrow \tilde{e}^1, \ldots, \tilde{e}^n \). The elements \( a_{ij} \) of \( A \) are defined by

\[
e^i = \sum_{j=1}^{n} a_{ij} \tilde{e}^j \quad j = 1, \ldots, n \tag{2.2}
\]

and it is known that \( \det A \neq 0 \). Hence, if \( x \) and \( \tilde{x} \) are the coordinate vectors of a vector \( x \) with respect to the bases \( e^1, \ldots, e^n \) and \( \tilde{e}^1, \ldots, \tilde{e}^n \), then \( \tilde{x} = Ax \); that is, \( Ax \) is the coordinate vector of \( x \) with respect to the basis \( \tilde{e}^1, \ldots, \tilde{e}^n \). For a detailed treatment of transition matrices, see Lancaster & Tismenetsky [121, pp. 98–100] and Hohn [92, pp. 323–331].

For an open, bounded set \( D \), a point \( x \in D \), and a function \( f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \), let \( \tilde{x} = Ax \), \( \tilde{D} = AD \), and \( g(\tilde{x}) = Af(\tilde{A}^{-1}\tilde{x}) \) for \( \tilde{x} \in \tilde{D} \) be the representations of \( x, D \), and \( f \) with respect to the new ordered basis. The diagram

\[
\begin{array}{ccc}
\text{span}\{e^1, \ldots, e^n\} & \supsetneq & D' \\
A \downarrow & & \downarrow f \\
\text{span}\{\tilde{e}^1, \ldots, \tilde{e}^n\} & \supsetneq & D
\end{array}
\]

illustrates the situation. If \( f \) is differentiable, then the chain rule yields

\[
J_f(\tilde{x}) = \det A \cdot J_f(A^{-1}\tilde{x}) \cdot \det A^{-1} = J_f(A^{-1}\tilde{x}) = J_f(x).
\]

In some books, Brouwer’s degree is defined even for a real \( n \)-dimensional topological space \( T \)-space, that is, a real \( n \)-dimensional vector space whose topology is such that each one-point set is closed, and the addition of vectors and the multiplication of vectors by scalars is continuous.\(^4\)

\(^4\)These definitions are properly presented in the books by Coroarşescu [53, pp. 127–128]; Deimling [61, pp. 28–29]; [60, pp. 56–57, 65–66]; Eisenack & Fenske [71, pp. 103–104]; Jeggle [103, pp. 117–118]; and Rothe [157, pp. 118–122, 133–134]. The definitions are also justified in Deimling’s books. An outline of the justification of the first definition is also found in Zeidler [196, p. 543]; Fučík, Nečas, Souček & Souček [77, pp. 33–34]; and Petryshyn [147, p. 103].

In some books, Brouwer’s degree is defined even for a real \( n \)-dimensional topological vector \( T \)-space, that is, a real \( n \)-dimensional vector space whose topology is such that each one-point set is closed, and the addition of vectors and the multiplication of vectors by scalars is continuous.\(^4\)

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In some books, Brouwer’s degree is defined even for a real \( n \)-dimensional topological vector \( T \)-space, that is, a real \( n \)-dimensional vector space whose topology is such that each one-point set is closed, and the addition of vectors and the multiplication of vectors by scalars is continuous.\(^4\)
Especially,
\[ \text{sign}(J_f(x)) = \text{sign}(J_f(x)). \] (2.3)
If \( y \notin f(\partial D \cup S_f(D)) \), then Equation (2.1) and Equation (2.3) imply that
\[ \text{deg}(f, \mathcal{T}, y) = \text{deg}(A_f A^{-1}, A(\mathcal{T}), A(y)) = \text{deg}(g, A(\mathcal{T}), A(y)). \] (2.4)
The requirement \( y \notin f(S_f(D)) \) can be removed again by Sard’s theorem and the case of a continuous mapping follows from this by approximating a continuous mapping by smooth mappings with respect to the sup-norm. The details of this process are found in the references given on the page 15 just before the beginning of Subsection 2.1.2. Hence, Equation (2.4) holds for continuous mappings as well.

Observe that the degree is derived in the situation
\[ f : D \subset \text{span}\{e^1, \ldots, e^n\} \to \text{span}\{e^1, \ldots, e^n\}, \]
and not in the situation
\[ f : D \subset \text{span}\{e^1, \ldots, e^n\} \to \text{span}\{e^1, \ldots, e^n\}, \]
where the ordered bases \( e^1, \ldots, e^n \) and \( e^1, \ldots, e^n \) are different.

Let \( X \) be an \( n \)-dimensional topological vector space. The details of this process are found in the references given on the page 15 just before the beginning of Subsection 2.1.2. Hence, Equation (2.4) holds for continuous mappings as well.

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Let \( X \) be an \( n \)-dimensional real normed space. It is known that the space \( X \) is linearly homeomorphic to \( \mathbb{R}^n \). This result is achieved by choosing an ordered basis \( x^1, \ldots, x_n \) for \( X \) and showing that
\[ h : \sum_{i=1}^n \alpha(x) x^i \to \sum_{i=1}^n \alpha(x) x^i = (\alpha_1(x), \ldots, \alpha_n(x))^T \]
is a homeomorphism \( X \to \mathbb{R}^n \). Here \( \alpha(x) \) are the coordinates of \( x \) with respect to the basis \( x^1, \ldots, x^n \). So what we do in practice with the homeomorphisms \( h \) that we identify \( x \) with its coordinate vector. Note that the basis has to be ordered if we want to obtain a bijection that associates a vector \( x \) with its coordinate vector \( (\alpha_1(x), \ldots, \alpha_n(x)) \). Below we use homeomorphisms instead of treating a vector and its coordinate vector as equal; the aim is to accentuate that the mapping \( F : X \to X \) and the mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) constructed from \( F \) are different mappings.

Let \( D \subset X \) be open and bounded, \( F : \overline{D} \to X \) continuous, and \( y \notin \partial D \). Then the fact “\( h \) is a homeomorphism” implies that \( \text{deg}(f, h(\overline{D}), h(y)) \) is defined for the continuous mapping \( f = h h^{-1} \); \( h(\overline{D}) \subset \mathbb{R}^n \to \mathbb{R}^n \). We clarify this with the diagram
\[
\begin{array}{ccc}
\text{span}\{e^1, \ldots, e^n\} & \xrightarrow{h(D)} & \text{span}\{e^1, \ldots, e^n\} \\
\downarrow{h^{-1}} & & \downarrow{h} \\
\text{span}\{x^1, \ldots, x^n\} & \xrightarrow{f} & \text{span}\{x^1, \ldots, x^n\}.
\end{array}
\]

For the proof, see Taylor & Lay [178, Theorem 3.1, p. 62]. Note that there is a more general result: an \( n \)-dimensional topological vector space over \( \mathbb{R} \) is homeomorphic with \( \mathbb{R}^n \). For the proof, see Edwards [69, pp. 64–65] or Taylor & Lay [178, Theorem 9.4, p. 97–98].

A homeomorphism \( h \) has nice properties such as \( h \) and \( h^{-1} \) are continuous (by the definition), \( h(D) \) is open (closed) if and only if \( D \) is open (closed), \( h(\overline{D}) = h(D) \), and \( h(D) = h(\overline{D}) \). See Willard’s book [186, Theorem 7.9, p. 46] for proofs and more information.

Especially,
\[ \text{sign}(J_f(x)) = \text{sign}(J_f(x)). \] (2.3)
If \( y \notin f(\partial D \cup S_f(D)) \), then Equation (2.1) and Equation (2.3) imply that
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Let \( X \) be an \( n \)-dimensional real normed space. It is known that the space \( X \) is linearly homeomorphic to \( \mathbb{R}^n \). This result is achieved by choosing an ordered basis \( x^1, \ldots, x_n \) for \( X \) and showing that
\[ h : \sum_{i=1}^n \alpha(x) x^i \to \sum_{i=1}^n \alpha(x) x^i = (\alpha_1(x), \ldots, \alpha_n(x))^T \]
is a homeomorphism \( X \to \mathbb{R}^n \). Here \( \alpha(x) \) are the coordinates of \( x \) with respect to the basis \( x^1, \ldots, x^n \). So what we do in practice with the homeomorphisms \( h \) that we identify \( x \) with its coordinate vector. Note that the basis has to be ordered if we want to obtain a bijection that associates a vector \( x \) with its coordinate vector \( (\alpha_1(x), \ldots, \alpha_n(x)) \). Below we use homeomorphisms instead of treating a vector and its coordinate vector as equal; the aim is to accentuate that the mapping \( F : X \to X \) and the mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) constructed from \( F \) are different mappings.

Let \( D \subset X \) be open and bounded, \( F : \overline{D} \to X \) continuous, and \( y \notin \partial D \). Then the fact “\( h \) is a homeomorphism” implies that \( \text{deg}(f, h(\overline{D}), h(y)) \) is defined for the continuous mapping \( f = h h^{-1} \); \( h(\overline{D}) \subset \mathbb{R}^n \to \mathbb{R}^n \). We clarify this with the diagram
\[
\begin{array}{ccc}
\text{span}\{e^1, \ldots, e^n\} & \xrightarrow{h(D)} & \text{span}\{e^1, \ldots, e^n\} \\
\downarrow{h^{-1}} & & \downarrow{h} \\
\text{span}\{x^1, \ldots, x^n\} & \xrightarrow{f} & \text{span}\{x^1, \ldots, x^n\}.
\end{array}
\]

For the proof, see Taylor & Lay [178, Theorem 3.1, p. 62]. Note that there is a more general result: an \( n \)-dimensional topological vector space over \( \mathbb{R} \) is homeomorphic with \( \mathbb{R}^n \). For the proof, see Edwards [69, pp. 64–65] or Taylor & Lay [178, Theorem 9.4, p. 97–98].

A homeomorphism \( h \) has nice properties such as \( h \) and \( h^{-1} \) are continuous (by the definition), \( h(D) \) is open (closed) if and only if \( D \) is open (closed), \( h(\overline{D}) = h(D) \), and \( h(D) = h(\overline{D}) \). See Willard’s book [186, Theorem 7.9, p. 46] for proofs and more information.
To be a sensible definition, the degree of \( F \) should not depend on the representative \( f \) of \( F \), or, in other words, the degree should not depend on the choice of the homeomorphism.

Assume that \( x^1, \ldots, x^n \) is another ordered basis for \( X \), \( \tilde{h} \) is the homeomorphism defined by \( \tilde{h}(\tilde{x}^i) = e^i \) for \( i = 1, \ldots, n \), and \( A \) is the transition matrix determined by the change of bases \( x^1, \ldots, x^n \rightarrow \tilde{x}^1, \ldots, \tilde{x}^n \). Observe that \( \tilde{h}(x) \) is the coordinate vector of \( x \) with respect to the ordered basis \( \tilde{x}^1, \ldots, \tilde{x}^n \), and \( h(x) \) is the coordinate vector of \( x \) with respect to the ordered basis \( x^1, \ldots, x^n \). As \( A \) is the transition matrix, \( h = \tilde{A}h \) and consequently

\[
\deg(hf^{-1}, h(D), h(y)) = \deg(\tilde{A}h \tilde{f}^{-1} A^{-1}, \tilde{A}h(D), \tilde{A}h(y)) \quad \text{by } h = \tilde{A}h
\]

\[
= \deg(hf^{-1}, h(D), h(y)) \quad \text{by (2.4)}.
\]

So we get the same integer as before. This justifies the following definition.

**Definition 2.1.2.** Let \( X \) be a real \( n \)-dimensional normed space and

\[
M = \{(F,D,y) : D \subset X \text{ open bounded}, F : \overline{D} \rightarrow X \text{ continuous, and } y \notin F(\partial D)\}.
\]

Assume that \( x^1, \ldots, x^n \) is an ordered basis for \( X \), and \( e^1, \ldots, e^n \) is the natural basis of \( \mathbb{R}^n \).

Then we define \( \deg : M \rightarrow \mathbb{Z} \) by

\[
\deg(F, \overline{D}, y) := \deg(hf^{-1}, h(D), h(y)),
\]

where \( h : X \rightarrow \mathbb{R}^n \) is the linear homeomorphism defined by \( h(x^i) = e^i \) for \( i = 1, \ldots, n \).

**Theorem 2.1.3.** The degree defined in Definition 2.1.2 has the properties of a classical degree function.

**Proof.** This proof relies heavily on the fact that the mapping \( h \) is a linear homeomorphism; see footnote 10 on page 18.

1. If \( \deg(hf^{-1}, h(D), h(y)) \neq 0 \), then there is an element \( x \in h(D) \subset \mathbb{R}^n \) satisfying \( hf^{-1}x = h(y) \), that is, \( f\tilde{x} = y \) with \( \tilde{x} = h^{-1}x \in \tilde{D} \).

2. If \( D \) is the union of the disjoint, open sets \( D_1 \) and \( D_2 \), then the fact that \( h \) is a homeomorphism implies that \( h(D) \) is the union of the disjoint, open sets \( h(D_1) \) and \( h(D_2) \). Moreover,

\[
\deg(F, \overline{D}, y) = \deg(hf^{-1}, h(D), h(y))
\]

\[
= \deg(hf^{-1}, h(D_1), h(y)) + \deg(hf^{-1}, h(D_2), h(y))
\]

\[
= \deg(F, \overline{D_1}, y) + \deg(F, \overline{D_2}, y).
\]

3. If \( y \notin K \) and \( K \) is closed, then \( h(y) \notin h(K) \) and \( h(K) \) is closed. Hence,

\[
\deg(F, \overline{D}, y) = \deg(hf^{-1}, h(D), h(y))
\]

\[
= \deg(hf^{-1}, h(D) \setminus h(K), h(y))
\]

\[
= \deg(hf^{-1}, h(D \setminus K), h(y))
\]

\[
= \deg(f, \overline{D \setminus K}, y).
\]

To be a sensible definition, the degree of \( F \) should not depend on the representative \( f \) of \( F \), or, in other words, the degree should not depend on the choice of the homeomorphism.

Assume that \( x^1, \ldots, x^n \) is another ordered basis for \( X \), \( \tilde{h} \) is the homeomorphism defined by \( \tilde{h}(\tilde{x}^i) = e^i \) for \( i = 1, \ldots, n \), and \( A \) is the transition matrix determined by the change of bases \( x^1, \ldots, x^n \rightarrow \tilde{x}^1, \ldots, \tilde{x}^n \). Observe that \( \tilde{h}(x) \) is the coordinate vector of \( x \) with respect to the ordered basis \( \tilde{x}^1, \ldots, \tilde{x}^n \), and \( h(x) \) is the coordinate vector of \( x \) with respect to the ordered basis \( x^1, \ldots, x^n \). As \( A \) is the transition matrix, \( h = \tilde{A}h \) and consequently

\[
\deg(hf^{-1}, h(D), h(y)) = \deg(\tilde{A}h \tilde{f}^{-1} A^{-1}, \tilde{A}h(D), \tilde{A}h(y)) \quad \text{by } h = \tilde{A}h
\]

\[
= \deg(hf^{-1}, h(D), h(y)) \quad \text{by (2.4)}.
\]

So we get the same integer as before. This justifies the following definition.

**Definition 2.1.2.** Let \( X \) be a real \( n \)-dimensional normed space and

\[
M = \{(F,D,y) : D \subset X \text{ open bounded}, F : \overline{D} \rightarrow X \text{ continuous, and } y \notin F(\partial D)\}.
\]

Assume that \( x^1, \ldots, x^n \) is an ordered basis for \( X \), and \( e^1, \ldots, e^n \) is the natural basis of \( \mathbb{R}^n \).

Then we define \( \deg : M \rightarrow \mathbb{Z} \) by

\[
\deg(F, \overline{D}, y) := \deg(hf^{-1}, h(D), h(y))
\]

where \( h : X \rightarrow \mathbb{R}^n \) is the linear homeomorphism defined by \( h(x^i) = e^i \) for \( i = 1, \ldots, n \).

**Theorem 2.1.3.** The degree defined in Definition 2.1.2 has the properties of a classical degree function.

**Proof.** This proof relies heavily on the fact that the mapping \( h \) is a linear homeomorphism; see footnote 10 on page 18.

1. If \( \deg(hf^{-1}, h(D), h(y)) \neq 0 \), then there is an element \( x \in h(D) \subset \mathbb{R}^n \) satisfying \( hf^{-1}x = h(y) \), that is, \( f\tilde{x} = y \) with \( \tilde{x} = h^{-1}x \in \tilde{D} \).

2. If \( D \) is the union of the disjoint, open sets \( D_1 \) and \( D_2 \), then the fact that \( h \) is a homeomorphism implies that \( h(D) \) is the union of the disjoint, open sets \( h(D_1) \) and \( h(D_2) \). Moreover,

\[
\deg(F, \overline{D}, y) = \deg(hf^{-1}, h(D), h(y))
\]

\[
= \deg(hf^{-1}, h(D_1), h(y)) + \deg(hf^{-1}, h(D_2), h(y))
\]

\[
= \deg(F, \overline{D_1}, y) + \deg(F, \overline{D_2}, y).
\]

3. If \( y \notin K \) and \( K \) is closed, then \( h(y) \notin h(K) \) and \( h(K) \) is closed. Hence,

\[
\deg(F, \overline{D}, y) = \deg(hf^{-1}, h(D), h(y))
\]

\[
= \deg(hf^{-1}, h(D) \setminus h(K), h(y))
\]

\[
= \deg(hf^{-1}, h(D \setminus K), h(y))
\]

\[
= \deg(f, \overline{D \setminus K}, y).
\]
Remark 2.1.4. As noted before, Amann & Weiss demonstrated in their paper [6] that if $X$ is a finite-dimensional normed space, and the identity mapping of $X$ is chosen as the normalizing mapping, then the classical topological degree for mappings $X \to X$ is unique. Note that that theorem does not assert anything about the uniqueness of the degree function if the normalizing mapping is chosen differently. The result from Amann & Weiss is also in the book by Lloyd [127, Theorem 5.3.2, p. 88]. A consequence of that result and the above discussion is that an $n$-dimensional normed linear space has exactly one classical topological degree with the identity mappings as normalizing mapping, and the degree can be constructed in the manner given above.

There is interesting information on the uniqueness problem of the topological degree in Istrătescu’s monograph [102, pp. 469–70, 375–388].

2.1.4 Brouwer’s degree for mappings from one $n$-dimensional normed space to another

When we consider mappings from one $n$-dimensional normed space to another, the situation is not so simple as in the previous subsection.

Assume that we have two real $n$-dimensional normed spaces $X$ and $Y$, $D$ an open, bounded subset of $X$, $F : D \to Y$ continuous, and $y \in Y \setminus F(D)$. Let $x_1, \ldots, x_n$ be an ordered basis for $X$ and $y_1, \ldots, y_n$ an ordered basis for $Y$. Then there are the corresponding homeomorphisms $h_1 : X \to \mathbb{R}^n$ and $h_1 : Y \to \mathbb{R}^n$ that associate each element $x \in X$ and each element $y \in Y$ with their coordinate vectors $h_1(x)$ and $h_1(y)$, respectively. We define $f = h_1 F h_1^{-1} : h_1(D) \subset \mathbb{R}^n \to \mathbb{R}^n$. The diagram

\[
\begin{array}{ccc}
\text{span}\{e_1, \ldots, e_n\} & \xrightarrow{\text{span}\{e_1, \ldots, e_n\}} & \text{span}\{y_1, \ldots, y_n\} \\
h_1^{-1} & \downarrow & h_1 \\
\text{span}\{x_1, \ldots, x_n\} & \xrightarrow{F} & \text{span}\{y_1, \ldots, y_n\}
\end{array}
\]

clarifies the situation. Since \( \text{deg}(f, h_1(D), h_1(y)) \) is defined for $f$, it would be tempting to define

\[
\text{deg}(F, D, y) = \text{deg}(h_1 F h_1^{-1}, h_1(D), h_1(y)).
\]
There is a slight flaw which will become apparent when we examine what happens if we change the ordered bases of $X$ and $Y$ to the ordered bases $\vec{e}^1, \ldots, \vec{e}^n$ and $\vec{y}^1, \ldots, \vec{y}^m$, respectively.

Suppose that $h_2$ is the homeomorphism $X \to \mathbb{R}^n$ defined by $h_2(x^i) = e^i$ for $i = 1, \ldots, n$ and $h_2$ the homeomorphism $Y \to \mathbb{R}^m$ defined by $h_2(y^j) = e^j$ for $i = 1, \ldots, n$. If $A$ is the transition matrix corresponding to the change of bases $\vec{e}^1, \ldots, \vec{e}^n \to x^1, \ldots, x^n$ and if $B$ is the transition matrix corresponding to the change of bases $\vec{y}^1, \ldots, \vec{y}^m \to y^1, \ldots, y^m$, then $h_1 = Ah_2, h_1 = Bh_2$, and

$$g := h_2 F h_2^{-1} = (B^{-1} B) h_2 F h_2^{-1} (A^{-1} A)$$
$$= B^{-1} (B h_2) F (h_2^{-1} A^{-1} A) B = B^{-1} h_2 F h_1^{-1} A$$
$$= B^{-1} f A$$

is the function $\mathbb{R}^n \to \mathbb{R}^n$ constructed by using $F$ and the ordered bases $\vec{e}^1, \ldots, \vec{e}^n$ and $\vec{y}^1, \ldots, \vec{y}^m$. The diagram

$$\begin{array}{ccc}
\text{span}(e^1, \ldots, e^n) & \xrightarrow{h_2^{-1}} & \text{span}(\vec{e}^1, \ldots, \vec{e}^n) \\
\xrightarrow{h_2} & & \\
\text{span}(x^1, \ldots, x^n) & \xrightarrow{\tilde{h}_2} & \text{span}(\vec{x}^1, \ldots, \vec{x}^n) \\
\end{array}$$

elucidates the situation. When $f$ and $g$ are differentiable, we can calculate as follows:

$$J_g(\tilde{x}) = \det B^{-1} \cdot J_f(\tilde{A} \tilde{x}) \det A = J_f(A \tilde{x}) \det(A B^{-1}) = J_f(\tilde{x}) \det(A B^{-1}).$$

This implies

$$\text{sign}(J_g(\tilde{x})) = \text{sign}(J_f(\tilde{x})) \text{sign}(\det A \cdot \det B) = \text{sign}(J_f(\tilde{x})) \text{sign}(\det A \cdot \det B)$$

from which we can deduce that

$$\deg(g, h_2(\mathcal{D}), \tilde{h}_2(\mathcal{D})) = \deg(B^{-1} f A, h_2(D), \tilde{h}_2(D))$$

should be equal to $\deg(f, \mathcal{D})$.

$$= \deg(B^{-1} f A, A^{-1} A h_2(D), B^{-1} B \tilde{h}_2(D))$$
$$= \text{sign}(\det A \cdot \det B) \deg(f, A h_2(D), B \tilde{h}_2(D))$$
$$= \text{sign}(\det A \cdot \det B) \deg(f, h_1(D), \tilde{h}_1(D)) \cdot$$

This means that the degree function defined in this way depends on the choice of the ordered bases.

The problem is solved by using the widely used terminology of differential geometry. This terminology is quite common in $\mathbb{R}^n$, but rarely presented in connection with general $n$-dimensional vector spaces. The following definition is used in the books by Istrătescu [102, Definition 12.1.1, p. 346] and Milnor [135, p. 26]:

There is a slight flaw which will become apparent when we examine what happens if we change the ordered bases of $X$ and $Y$ to the ordered bases $\vec{e}^1, \ldots, \vec{e}^n$ and $\vec{y}^1, \ldots, \vec{y}^m$, respectively.

Suppose that $h_2$ is the homeomorphism $X \to \mathbb{R}^n$ defined by $h_2(x^i) = e^i$ for $i = 1, \ldots, n$ and $h_2$ the homeomorphism $Y \to \mathbb{R}^m$ defined by $h_2(y^j) = e^j$ for $i = 1, \ldots, n$. If $A$ is the transition matrix corresponding to the change of bases $\vec{e}^1, \ldots, \vec{e}^n \to x^1, \ldots, x^n$ and if $B$ is the transition matrix corresponding to the change of bases $\vec{y}^1, \ldots, \vec{y}^m \to y^1, \ldots, y^m$, then $h_1 = Ah_2, h_1 = Bh_2$, and

$$g := h_2 F h_2^{-1} = (B^{-1} B) h_2 F h_2^{-1} (A^{-1} A)$$
$$= B^{-1} (B h_2) F (h_2^{-1} A^{-1} A) B = B^{-1} h_2 F h_1^{-1} A$$
$$= B^{-1} f A$$

is the function $\mathbb{R}^n \to \mathbb{R}^n$ constructed by using $F$ and the ordered bases $\vec{e}^1, \ldots, \vec{e}^n$ and $\vec{y}^1, \ldots, \vec{y}^m$. The diagram

$$\begin{array}{ccc}
\text{span}(e^1, \ldots, e^n) & \xrightarrow{h_2^{-1}} & \text{span}(\vec{e}^1, \ldots, \vec{e}^n) \\
\xrightarrow{h_2} & & \\
\text{span}(x^1, \ldots, x^n) & \xrightarrow{\tilde{h}_2} & \text{span}(\vec{x}^1, \ldots, \vec{x}^n) \\
\end{array}$$

elucidates the situation. When $f$ and $g$ are differentiable, we can calculate as follows:

$$J_g(\tilde{x}) = \det B^{-1} \cdot J_f(\tilde{A} \tilde{x}) \det A = J_f(A \tilde{x}) \det(A B^{-1}) = J_f(\tilde{x}) \det(A B^{-1}).$$

This implies

$$\text{sign}(J_g(\tilde{x})) = \text{sign}(J_f(\tilde{x})) \text{sign}(\det A \cdot \det B) = \text{sign}(J_f(\tilde{x})) \text{sign}(\det A \cdot \det B)$$

from which we can deduce that

$$\deg(g, h_2(\mathcal{D}), \tilde{h}_2(\mathcal{D})) = \deg(B^{-1} f A, h_2(D), \tilde{h}_2(D))$$

should be equal to $\deg(f, \mathcal{D})$.

$$= \deg(B^{-1} f A, A^{-1} A h_2(D), B^{-1} B \tilde{h}_2(D))$$
$$= \text{sign}(\det A \cdot \det B) \deg(f, A h_2(D), B \tilde{h}_2(D)) \cdot$$

This means that the degree function defined in this way depends on the choice of the ordered bases.

The problem is solved by using the widely used terminology of differential geometry. This terminology is quite common in $\mathbb{R}^n$, but rarely presented in connection with general $n$-dimensional vector spaces. The following definition is used in the books by Istrătescu [102, Definition 12.1.1, p. 346] and Milnor [135, p. 26]:
Definition 2.1.5. Assume that $x^1, \ldots, x^n$ and $\bar{x}^1, \ldots, \bar{x}^n$ are ordered bases for $X$. The bases $x^1, \ldots, x^n$ and $\bar{x}^1, \ldots, \bar{x}^n$ have the same orientation, if the matrix $P$, whose elements $\alpha_{ij}$ are defined by
\[ x_i = \sum_{j=1}^{n} \alpha_{ij} \bar{x}_j \quad i = 1, \ldots, n, \]
has the positive determinant. If the determinant of $P$ is negative, the bases have the opposite orientations.

Remark 2.1.6. It is clear from the definition of the transition matrix (see Equation (2.2), but replace $\bar{e}_i$ by $\delta_i$, and $e_j$ by $x_j$), that the matrix $P$ of the previous definition is the transpose of the transition matrix corresponding to the change of bases $x^1, \ldots, x^n \mapsto \bar{x}^1, \ldots, \bar{x}^n$. Since $\det B = \det B^T$ for every real $n \times n$-matrix $B$, the same orientation can be equivalently defined as follows: ordered bases $x^1, \ldots, x^n$ and $\bar{x}^1, \ldots, \bar{x}^n$ have the same orientation if the transition matrix corresponding to the change of bases $x^1, \ldots, x^n \mapsto \bar{x}^1, \ldots, \bar{x}^n$ has the positive determinant. See also Hirsch [91, pp. 103–104], Matsushima [131, pp. 253–254], and Guillemin & Pollack [82, pp. 95–96].

So, when we have fixed an ordered basis $x_1, \ldots, x_n$, the definition of the orientation gives us an equivalence relation in the set of all ordered bases of $X$, with exactly two equivalence classes. We call an equivalence class in the family of all ordered bases an orientation and the space $X$ oriented when we have chosen the admissible class of bases. If $X$ and $Y$ are oriented $n$-dimensional spaces, the topological degree of continuous maps from $X$ to $Y$ is defined since we only have $\det A > 0$ and $\det B > 0$ above. More information on the oriented spaces is found in the books by Snapper & Troyer [173], Matsushima [131], Hirsch [91], and Guillemin & Pollack [82].

Note that if a basis is not admissible, then we get an admissible basis by changing the direction of one basis vector to the opposite or by changing the order of two basis vectors. These assertions follow from the properties of the determinant.

Remark 2.1.7. Above we forbade the use of the homeomorphisms belonging to the other equivalence class in the definition of a degree function. The reason for this was that we wanted to have an integer-valued function and not a set-valued function. However, the other homeomorphisms do not cease to exist as a result of this action, and therefore we may define also other degree functions for mappings $X \mapsto Y$. Furthermore, we can show that a degree defined in this manner possesses the classical properties of a degree function. The proof of this is similar to the proof of Theorem 2.1.3, but identity mapping cannot be used as a normalization mapping. Since there are two orientations, there are at least two candidates for normalization mapping. This follows from the observation that a normalization can be given by $h^{-1} I_{2n} h$, where $h$ is an admissible homeomorphism of $Y$ and $\mathbb{R}^n$ and $h$ is an admissible homeomorphism of $X$ and $\mathbb{R}^n$.

Note that there may be also other normalization mappings. So if $X$ and $Y$ are two different $n$-dimensional normed spaces, then there are many classical degrees for mappings $X \mapsto Y$. Nevertheless, it is reasonable to ask if the choice of the normalization mapping

\[ \deg(J, D, y) = \deg(hkh^{-1}, h(D), hy) = \deg(\text{id}, h(D), hy) = 1. \]

Remark 2.1.8. It is clear from the definition of the transition matrix (see Equation (2.2), but replace $\bar{e}_i$ by $\delta_i$, and $e_j$ by $x_j$), that the matrix $P$ of the previous definition is the transpose of the transition matrix corresponding to the change of bases $x^1, \ldots, x^n \mapsto \bar{x}^1, \ldots, \bar{x}^n$. Since $\det B = \det B^T$ for every real $n \times n$-matrix $B$, the same orientation can be equivalently defined as follows: ordered bases $x^1, \ldots, x^n$ and $\bar{x}^1, \ldots, \bar{x}^n$ have the same orientation if the transition matrix corresponding to the change of bases $x^1, \ldots, x^n \mapsto \bar{x}^1, \ldots, \bar{x}^n$ has the positive determinant. See also Hirsch [91, pp. 103–104], Matsushima [131, pp. 253–254], and Guillemin & Pollack [82, pp. 95–96].

So, when we have fixed an ordered basis $x_1, \ldots, x_n$, the definition of the orientation gives us an equivalence relation in the set of all ordered bases of $X$, with exactly two equivalence classes. We call an equivalence class in the family of all ordered bases an orientation and the space $X$ oriented when we have chosen the admissible class of bases. If $X$ and $Y$ are oriented $n$-dimensional spaces, the topological degree of continuous maps from $X$ to $Y$ is defined since we only have $\det A > 0$ and $\det B > 0$ above. More information on the oriented spaces is found in the books by Snapper & Troyer [173], Matsushima [131], Hirsch [91], and Guillemin & Pollack [82].

Note that if a basis is not admissible, then we get an admissible basis by changing the direction of one basis vector to the opposite or by changing the order of two basis vectors. These assertions follow from the properties of the determinant.
determines a classical topological degree for mappings $X \to Y$ uniquely?\textsuperscript{12}

**Remark 2.1.8.** When $F$ is a mapping from a real finite-dimensional normed space $X$ to the dual space $X^*$, and $X$ has the basis $x_1, \ldots, x_n$, it is natural to choose the dual basis $x_1^*, \ldots, x_n^*$ for $X^*$. The elements of the dual basis are determined by the condition

$$\langle x_i^*, x_k \rangle = \delta_{ik} \quad \text{for all } i, k = 1, \ldots, n.$$ 

Note that this action restricts the set of the representatives $f = \tilde{h} F h^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ of $F$ because the choice of the basis of $X$ completely determines the basis of $X^*$, and vice versa (the choice of one of the homeomorphisms $h$ and $h$ determines also the other homeomorphism).

We shall show that if we first choose a basis for $X$, and then the dual basis as a basis of $X^*$, then the degree does not depend on the choice of the basis.

Let $h_1 : X \to \mathbb{R}^n$ and $h_1 : X^* \to \mathbb{R}^n$ be the homeomorphisms defined by $h_1(x_i) = e_i$ and $h_1(x_i^*) = e_i$ for $i = 1, \ldots, n$. Suppose that $\tilde{x}_1, \ldots, \tilde{x}_n$ is another basis for $X$, and $\tilde{x}_1^*, \ldots, \tilde{x}_n^*$ is the corresponding dual basis for $X^*$. Assume further that $h_2 : X \to \mathbb{R}^n$ and $h_2 : Y \to \mathbb{R}^n$ are the homeomorphisms defined by $h_2(\tilde{x}_i) = e_i$ and $h_2(\tilde{x}_i^*) = e_i$ for $i = 1, \ldots, n$. We define the mapping $h_1^* : \mathbb{R}^n \to X^*$ of $h_1$ by

$$\langle h_1^*(u), v \rangle = \langle u, h_1(v) \rangle$$

where $x \in \mathbb{R}^n$, $u \in X$, and $(\cdot, \cdot)$ is the inner product of $\mathbb{R}^n$. Since

$$\delta_{ij} = (e_i, e_j) = (e_i, h_1(x_j)) = \langle h_1^*(e_i), x_j \rangle$$

for all $i, j = 1, \ldots, n$, we have $h_1^*(e_i) = x_i^*$ for each $i = 1, \ldots, n$ and consequently

$$e_i = (h_1^*)^{-1}(x_i^*).$$

So $(h_1^*)^{-1} = h_1$. In a similar manner, we can show that $(h_2^*)^{-1} = h_2$.

On the other hand, there is the transition matrix $A$ corresponding to the base change $x_1, \ldots, x_n \to \tilde{x}_1, \ldots, \tilde{x}_n$. Then $h_2 = Ah_1$, and consequently $(h_2^*)^{-1} = (A^*)^{-1}(h_1^*)^{-1}$, that is,

$$h_2 = (A^*)^{-1}h_1.$$ 

Since

$$\tilde{h}_2 F h_2 = (A^*)^{-1} \tilde{h}_1 F h_1 A^{-1}$$

and since the determinants of the matrices $A$ and $(A^*)^{-1}$ have always the same sign, the conclusion follows.

### 2.1.5 The Leray-Schauder lemma and the Leray-Schauder degree

Since the finite-dimensional topological degree proved to be such a useful tool, and many interesting problems needed to be put in the context of infinite-dimensional (Banach)

\textsuperscript{12}There is an answer in a special setting: if we consider continuous mappings $F : F$ where $F$ is a finite-dimensional normed space and $F^*$ its topological dual, then the classical topological degree is unique when the normalization mapping is a duality mapping that is given by locally uniformly convex norms of $F$ and $F^*$. Note that different norms give different duality mapping and therefore the topological degree might be different when a different duality mapping is used as a normalization mapping. See Definition 2.4.9 and Remark 3.4.5.

### 2.1.5 The Leray-Schauder lemma and the Leray-Schauder degree

Since the finite-dimensional topological degree proved to be such a useful tool, and many interesting problems needed to be put in the context of infinite-dimensional (Banach)
spaces, there appeared a demand for a generalization of Brouwer’s degree to infinite-dimensional spaces. The first such result was achieved in 1934 when J. Schauder & J. Leray [125] presented the topological degree for compact perturbations of the identity; that degree is known as the Leray–Schauder degree, and the considered mappings have their domain and range in the same Banach space. The main technique for the construction is to approximate a compact mapping by a mapping with the range and the domain in an n-dimensional space, and then to calculate the Brouwer’s degree for these approximations. References that contain this approach include Deimling [61, pp. 56–58]; [60, pp. 66–69]; Denkowski, Migórski & Papageorgiou [62, pp. 199–208]; Fonseca & Gangbo [75, pp. 174–177]; Hutson & Pym [100, pp. 338–342]; Jeggle [103, pp. 118–123]; Joshi & Bose [104, pp. 174–177]; Lloyd [127, pp. 54–60]; Pascali [145, pp. 84–88]; and Schwartz [163, pp. 83–85]. The key result for the construction is the so-called Leray-Schauder lemma, which allows one—under certain conditions—to shift between finite-dimensional spaces without change in the value of the degree function. This Leray-Schauder lemma is also a momentous result in the construction of the topological degree for (S)\(\_\)mappings and therefore we present a variation of it below. The references given above contain proofs of the Leray-Schauder lemma.

Note that when \(m \leq n\), the space \(\mathbb{R}^n\) is identified with

\[ \mathbb{R}^n \cap \{ (x_1, \ldots, x_n) \in \mathbb{R}^m \mid x_{m+1}, \ldots, x_n = 0 \}. \]

Lemma 2.1.9. Assume that \(m \leq n\), the set \(D\) is an open, bounded subset of \(\mathbb{R}^n\), and \(\phi\) belongs to \(C(D, \mathbb{R}^m)\). Let the function \(\psi : \mathcal{D} \to \mathbb{R}^m\) be defined by

\[ \psi(x) = x + \phi(x), \quad x \in \mathcal{D}. \]

If \(p \in \mathbb{R}^m \setminus \psi(\partial D)\), then

\[ \deg(\psi, \mathcal{D}, p) = \deg(\psi|_{\mathbb{R}^m \setminus \partial D \cap \mathbb{R}^m}, p), \]

where \(\psi|_{\mathbb{R}^m \setminus \partial D}\) is the restriction of \(\psi\) to \(\mathbb{R}^m \setminus \partial D\).

At the time of the publication of Leray & Schauder’s article [125], it was not known whether the Leray-Schauder degree was unique, and whether there was a topological degree for all kinds of functions defined on an infinite-dimensional space. As noted before, the question on uniqueness was answered by Amann & Weiss in paper [6]. The second question has a negative answer, for in 1936, Leray constructed a counter-example (so-called Leray’s example) which points out that it is impossible to construct a classical topological degree for all continuous functions. The counter-example can be found in the original article [124] or in Fonseca & Gangbo [75, pp. 172–174], Istrăţescu [102, pp. 370–372], Krasnoselskii & Zabreiko [119, p. 72], and Lloyd [127, pp. 52–54]. Different counter-examples are found in Deimling [61, p. 37], [60, p. 61]; Fučík, Nečas, Souček & Souček [77, pp. 33–34]; Jeggle [103, p. 118]; Joshi & Bose [104, p. 174]; Krasnoselskii & Zabreiko [119, p. 73]; Mortici [138, pp. 52–53]; Riedrich [155, p. 80]; and Pascali [145, pp. 84–85].

Remark 2.1.10. The Leray-Schauder degree can be constructed also in other ways. See Anderson, Granas & Dugundji [7, pp. 348–352]; Berger [12, pp. 244–256]; Brown [50, 24] spaces, there appeared a demand for a generalization of Brouwer’s degree to infinite-dimensional spaces. The first such result was achieved in 1934 when J. Schauder & J. Leray [125] presented the topological degree for compact perturbations of the identity; that degree is known as the Leray–Schauder degree, and the considered mappings have their domain and range in the same Banach space. The main technique for the construction is to approximate a compact mapping by a mapping with the range and the domain in an n-dimensional space, and then to calculate the Brouwer’s degree for these approximations. References that contain this approach include Deimling [61, pp. 56–58]; [60, pp. 66–69]; Denkowski, Migórski & Papageorgiou [62, pp. 199–208]; Fonseca & Gangbo [75, pp. 174–177]; Hutson & Pym [100, pp. 338–342]; Jeggle [103, pp. 118–123]; Joshi & Bose [104, pp. 174–177]; Lloyd [127, pp. 54–60]; Pascali [145, pp. 84–88]; and Schwartz [163, pp. 83–85]. The key result for the construction is the so-called Leray-Schauder lemma, which allows one—under certain conditions—to shift between finite-dimensional spaces without change in the value of the degree function. This Leray-Schauder lemma is also a momentous result in the construction of the topological degree for (S)\(\_\)mappings and therefore we present a variation of it below. The references given above contain proofs of the Leray-Schauder lemma.

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Remark 2.1.10. The Leray-Schauder degree can be constructed also in other ways. See Anderson, Granas & Dugundji [7, pp. 348–352]; Berger [12, pp. 244–256]; Brown [50, 24]
2.2 Some definitions and results concerning normed spaces

The aim here is to remind the reader of some well-known convergence results of sequences in Banach spaces, and other facts about topological spaces and Banach spaces as well. These results are essential in the successive chapters.

Reflexivity and pairing

If \( X \) is a real normed space, the space of all continuous real-valued linear functionals is called the (topological) dual space of \( X \) and is denoted by \( X^* \). The (natural) pairing\(^{13}\) of \( X \) and \( X^* \) is the real valued mapping \( \langle \cdot, \cdot \rangle \) which is usually defined by the formula

\[
\langle x^*, x \rangle = x^*(x) \quad \text{for all } x^* \in X^* \text{ and } x \in X
\]

but also the formula

\[
\langle x, x^* \rangle = x^*(x) \quad \text{for all } x^* \in X^* \text{ and } x \in X
\]

is used. In this thesis, we use both of the formulas because it should be clear from the context which argument is linear functional and which is not. For clarity, it should be also reasonable to denote \( \langle x^*, x \rangle_{X^*,X} \) or \( \langle x, x^* \rangle_{X,X^*} \), but we usually do not use those denotations because the context should make it clear which spaces are in question. Recall that the norm of the dual space is given by

\[
\| x^* \| = \sup \{ |x^*(x)| \mid x \in X, \| x \| \leq 1 \} = \sup \{ |x^*(x)| \mid x \in X, \| x \| = 1 \}
\]

where \( \| \cdot \| \) is the norm of \( X \). We usually write \( \| \cdot \| \) for the dual norm also, for the context makes it evident which norm is in question. The pairing \( \langle \cdot, \cdot \rangle \) is bilinear and continuous with respect to both of its arguments.

The dual space of \( X^* \) is denoted by \( X^{**} \) and often called the bidual or the second dual of \( X \). When \( x \) is an element of \( X \), then we denote by \( Jx \) the continuous linear functional \( x^* \mapsto x^*(x) \). The correspondence \( x \mapsto Jx \) is called the canonical mapping of \( X \) into \( X^{**} \), and the action of the image \( Jx \) of \( x \) is given by \( (x^*, Jx) = \langle x^*, x \rangle \). Evidently \( J \) is linear and

\[
\| Jx \|_* = \sup_{\| x^* \|_* = 1} |\langle x^*, Jx \rangle| = \sup_{\| x^* \|_* = 1} |\langle x^*, x \rangle| = \| x \|. \tag{2.6}
\]

If the range of \( J \) is all of \( X^{**} \), then we say that \( X \) is reflexive. Because the dual space is always a complete vector space with respect to the norm given by (2.6)\(^{14}\), a reflexive

\[^{13}\text{The term natural pairing is adopted from Bishop & Goldberg's book [19, p. 77]. The adjective "natural" is quite a vague modifier; hence, it would be reasonable to use the term "duality pairing" instead. Note that in the literature, natural pairing is sometimes called the scalar product of vectors and dual vectors.}

\[^{14}\text{See Taylor & Lay's book [178, Theorem 4.5, p. 70].} \]
space is always complete.\textsuperscript{15} For a detailed treatment of these concepts, see the book by Taylor & Lay [178, Section III.4].

Here we do not present the definition of weak topology because we only need to know that it exists\textsuperscript{16} and has some useful properties. For the definition of weak topology and some discussion on it, see Taylor & Lay [178, pp. 156–160], Edwards [69, Chapter 8], Conway [55, Chapter V], and Dunford & Schwarz [68, pp. 413–435]. The reflexivity of a Banach space is characterized by the weak compactness of the unit ball.

**Theorem 2.2.1.** The normed linear space $X$ is reflexive if and only if its unit ball is compact with respect to the weak topology of $X$ (that is, the unit ball is weakly compact).

**Proof.** See Taylor & Lay [178, Theorem 10.8, p. 178].

For reflexive spaces, we have the following profound and useful result called James’ characterization of reflexivity. It is needed in the proof of Lemma 4.4.9.

**Theorem 2.2.2 (James’ characterization of reflexivity).** The Banach space $X$ is reflexive if and only if each linear functional $x^*$ of $X$ attains its norm on the unit sphere of $X$ (that is, there is an element $x \in \partial B(0,1)$ such that $\langle x^*, x \rangle = \|x^*\|$).

**Proof.** For a proof of the necessity, see Deimling [61, Proposition 12.1, p. 113]. A complete proof is found in Diestel [63, Theorem 5, p. 12] and Cioranescu [53, Theorem 3.1, p. 56].

**On convergent and weakly convergent sequences**

(Strong) convergence or convergence with respect to norm is denoted by $\rightarrow$. A sequence $(x_n)_{n=1}^\infty$ converges weakly to $x \in X$ if

$$\langle x^*, x_n - x \rangle \xrightarrow{n \to \infty} 0$$

for every $x^* \in X^*$. We denote $x_n \rightharpoonup x$ when $x_n$ converges weakly to $x$.

**Remark 2.2.3.** A space is said to be weakly sequentially complete if the existence of $\lim_{n \to \infty} \langle x^*, x_n \rangle$ for each $x^* \in X^*$ implies that there is $x \in X$ with $x_n \rightharpoonup x$ in $X$. There are spaces that are not weakly sequentially complete; according to Yosida [188, Example, p. 120], Conway [55, Example 4.5, p. 136], and Taylor & Lay [178, Problem 9, p. 180] the Banach space $C[a,b]$ is such a space.\textsuperscript{17} Although a reflexive space is always sequentially weakly complete (see Yosida [188, Theorem 7, p. 124]), the reflexivity is not necessary: the nonreflexive spaces $L^1(a,b)$ and $l^1$ are weakly sequentially complete (see Taylor & Lay [178, Problem 9, p. 180]).

\textsuperscript{15}Usually, reflexivity is defined only for Banach spaces, but the remark just given shows that it is unnecessary to assume the completeness because it is a consequence of the existence of the isometric isomorphism. See also Yosida’s book [188, p. 113].

\textsuperscript{16}In this thesis, we speak of weak closures of sets and weak compactness, but we do not need to use the definition of weak topology.

\textsuperscript{17}According to Taylor & Lay [178, Problem 1 b. p. 70], the space $C[a,b]$ is complete.
The following result is useful in the study of the mappings of the monotone type and it is tacitly used in many books and articles.

**Lemma 2.2.4.** Assume that \( (x_n)_{n=1}^\infty \subset X \) is a sequence whose any subsequence has in turn a subsequence that converges to the limit \( x_0 \in X \), and the limit \( x_0 \) is the same for all these convergent subsequences, then the whole sequence \( (x_n)_{n=1}^\infty \) converges to \( x_0 \).

**Proof.** If \( (x_n)_{n=1}^\infty \) does not converge to \( x_0 \), then there is a positive number \( \varepsilon_0 \) and a subsequence \( (x_{n_k})_{k=1}^\infty \) such that
\[
\|x_{n_k} - x_0\| \geq \varepsilon_0
\]
for all \( k \in \mathbb{Z}_+ \). According to our assumption, \( (x_{n_k})_{k=1}^\infty \) has in turn a subsequence which converges to \( x_0 \), and hence the previous inequality is violated. Thus, the claim is true.

For weakly convergent sequences, we have the estimate given by the following lemma.

**Lemma 2.2.5.** If \( X \) is a Banach space and \( u_n \xrightarrow{n \to \infty} u \) in \( X \), then \( \lim_{n \to \infty} \|u_n\| \geq \|u\| \) and the sequence \( (u_n)_{n=1}^\infty \) is bounded.

**Proof.** See Zeidler [196, Proposition 21.23 (c), p. 258] and Yosida [188, Theorem 1 ii), p. 120].

The reflexivity of a Banach space can be characterized also by weakly convergent sequences.

**Theorem 2.2.6.**
1. If a Banach space \( X \) is reflexive, then its every bounded sequence has a subsequence that converges weakly to an element of \( X \).
2. If \( X \) is a Banach space whose every bounded sequence has a weakly convergent subsequence, then the space \( X \) is reflexive.

**Proof.** The claims are proved in Taylor & Lay’s book [178]. For claim 1. see Theorem 10.6 on page 177 and for claim 2. see Theorem 10.9 on page 178.

The following proposition shows that the weak closure of a bounded subset of a reflexive space can be completely characterized by weakly convergent sequences. On the other hand, even in an infinite-dimensional Hilbert space, it is impossible to characterize the weak closure of each unbounded set by weakly convergent sequences; von Neumann offered an example of such set in the space \( \ell^2 \). Von Neumann’s example is found in de Figueiredo’s treatise [58, Example, p. 5]; see also Conway [55, Exercise 2, p. 167], Royden [158, Problem 38, pp. 202–203], and Dunford & Schwartz [68, Exercise 38, p. 438].

**Proposition 2.2.7** (Browder). Let \( X \) be a reflexive Banach space, and \( x_0 \) a point in the weak closure of \( D \), where \( D \) is a bounded subset of \( X \). Then there exists an infinite sequence \( (x_n)_{n=1}^\infty \) in \( D \) converging weakly to \( x_0 \) in \( X \).

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**Proof.** If \( (x_n)_{n=1}^\infty \) does not converge to \( x_0 \), then there is a positive number \( \varepsilon_0 \) and a subsequence \( (x_{n_k})_{k=1}^\infty \) such that
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For weakly convergent sequences, we have the estimate given by the following lemma.

**Lemma 2.2.5.** If \( X \) is a Banach space and \( u_n \xrightarrow{n \to \infty} u \) in \( X \), then \( \lim_{n \to \infty} \|u_n\| \geq \|u\| \) and the sequence \( (u_n)_{n=1}^\infty \) is bounded.

**Proof.** See Zeidler [196, Proposition 21.23 (c), p. 258] and Yosida [188, Theorem 1 ii), p. 120].

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**Proof.** A proof of this proposition can be found in the books by Browder [40, Proposition 7.2, p. 81], de Figueiredo [58, Theorem 1.5, pp. 5–6], and Zeidler [199, Problem 32.1 (with solution), p. 931–912].

**Remark 2.2.8.** In Edwards’ book [69, Theorem 8.12.4 (b), pp. 549–550], there is a more general theorem than the previous proposition:

Let $E$ be a metrizable locally convex topological vector space, $H$ a weakly relatively compact subset of $E$, and $x_0$ an element in the weak closure of $H$. Then there is a sequence $(x_m)_{m=1}^\infty \subset H$ with $x_m \xrightarrow{w} x_0$ in $E$.

Note that the first edition of Edwards’ book was published in 1965, and Browder’s result appeared later. For a similar result, see Rudin [159, Exercise 28, p. 86].

**Remark 2.2.9.** In infinite-dimensional normed linear spaces, weak topology is always weaker than norm topology; see Taylor & Lay [178, p. 173]. However, there are infinite-dimensional normed linear spaces where the strong and the weak convergence of a sequence are equivalent; the space $\ell^1$ is an example of such space, and for details see Schechter [161, Example 7, p. 202] and Taylor & Lay [178, Problem 6, p. 180].

**On the geometry of Banach spaces**

The results presented here are needed for proving that there is a normalizing mapping for the degree of $(S_\lambda)$-mappings.

First we recall that if $X$ is a normed space which has norms $\| \cdot \|_1$ and $\| \cdot \|_2$, then these norms are equivalent if there is such a positive number $a$ that

$$a^{-1}\|x\|_1 \leq \|x\|_2 \leq a\|x\|_1 \quad \text{for all } x \in X.$$ 

Note that when $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent norms of $X$, they define the same norm topology, and consequently the same dual space, which in turn implies that also the corresponding weak topologies of $X$ are the same.

In analysis, we are often interested in norms that are somehow round or convex because roundedness is related to the differentiability of a norm, and to the number of points where a functional attains its norm. These issues are dealt with in Cioranescu [53], Diestel [63], and Day [57]. A very concise treatment is found in Deimling’s book [61, Section 12.2, pp. 111–114].

**Definition 2.2.10.** A norm of the space $X$ is

1. **locally uniformly convex** if for every number $\varepsilon \in [0,2]$ and every element $x$ lying on the unit sphere of the space $X$ there is such a number $\delta(x,\varepsilon) > 0$ that the two conditions $\|y\| = 1$ and $\|y-x\| \geq \varepsilon$ imply the inequality $\| (y+x)/2 \| \leq 1 - \delta(x,\varepsilon)$.
2. **strictly convex** if $x$ and $y$ are two distinct elements on the unit sphere, then the inequality $\|Ax+(1-\lambda)y\| < 1$ holds for all $\lambda \in [0,1]$.

**Remark 2.2.11.** The notion of local uniform convexity was introduced by A. R. Lovaglia in [128] in 1955. There is a stronger convexity condition of a norm than
local uniform convexity; it is called uniform convexity and it was introduced by James A. Clarkson in [54] in 1936. In that notion, the number $\delta$ depends only on $r$.

2. The definition of uniform convexity is easier to remember in its geometric form. Suppose that the endpoints of an arbitrary chord of the unit sphere of $X$ are $x$ and $y$; then the norm of $X$ is uniformly convex if the midpoint of the chord cannot approach the sphere if $y$ does not approach $x$.

3. Strict convexity is also easier to remember in its geometrical form: the norm of $X$ is strictly convex if the boundary of the unit sphere of the space does not contain line segments.

The uniform convexity of a norm has an influence on the structure of the dual space; according to the Milman-Pettis theorem, a Banach space with a uniformly convex norm is reflexive. For the proof, see Milman [134], Pettis [149], Yosida [188, Theorem 2, p. 127], Cioranescu [53, Theorem 2.9, p. 49], or Diestel [63, Theorem 2, p. 37]. As the convexity properties of a norm are connected with reflexivity, it is relevant to ask whether reflexivity has some relation to the convexity of a norm. The following theorem is usually called Trojanski’s theorem, and it is a partial answer to the previously posed question and is essential in some points of this thesis.

**Theorem 2.2.12.** Any reflexive Banach space $X$ has such an equivalent norm that both it and the corresponding norm of $X^*$ are locally uniformly convex.

This theorem is a consequence of a more general theorem due to Trojanski (which is also called Trojanski’s theorem) and Asplund’s averaging process. For these results, and more general results of this kind, see Trojanski [180], Asplund [8], Istrătescu [102, pp. 46–71], Diestel [63, pp. 113 and Theorem 1, p. 164] and Cioranescu [53, Theorem 1.8, p. 98, and Theorem 2.10, p. 108]. In Diestel’s book, Trojanski’s theorem is in the more general form.

The geometry of the norm also has an effect on the relationship of the weak convergence and the strong convergence. Recall that the normed space $X$ has the Kadec property if $u_j \to u$ and $\|u_j\| \to \|u\|$ imply $u_j \to u$. Note that in the book [102, p. 57] by Istrătescu, the Kadec property is called the property $(H)$. It is known that Hilbert spaces have the Kadec property, and according to Istrătescu [102, p. 56] the following theorem was established by J. Radon & F. Riesz for $L^p$ spaces, but it holds for locally uniformly convex spaces as well. Spaces having Kadec property were studied by Ky Fan & I. Glicksberg in article [72].

**Theorem 2.2.13.** A locally uniformly convex Banach space $X$ has the Kadec property.

**Proof.** See Cioranescu [53, 2.8 Proposition, p. 49], Istrătescu [102, Theorem 2.5.17, p. 56], and Zeidler [196, Proposition 21.23 (d), p. 258].

18The more general form of Trojanski’s theorem concerns weakly compactly generated Banach spaces.

18It holds even for the so-called $k$-locally uniformly convex spaces; see Istrătescu’s book [102, Theorem 2.5.17, p. 56].
Hahn-Banach theorem and projections

The results presented here are basic material in the functional analysis. The reason for including them here is that there was a certain assumption missing in the original results on the index of a critical point of mappings of class ς and that assumption is connected with projections. For further comments, see Remark 4.4.2 on page 74.

One of the most important theorems is the Hahn-Banach theorem. However, the term Hahn-Banach theorem refers to a collection of theorems, where one theorem is the Hahn-Banach theorem and others are its consequences. In Taylor & Lay’s book [178], the following theorem is called the Hahn-Banach theorem. Recall that a subspace \( M \) of \( X \) is a proper subspace if \( M \subsetneq X \).

**Theorem 2.2.14.** Assume that \( M \) is a proper subspace of the normed linear space \( X \). If \( m^* \in M^* \), then \( X^* \) contains an element \( x^* \) such that \( \|x^*\| = \|m^*\| \) and \( x^*(x) = m^*(x) \) for all \( x \in M \).

**Proof.** See the book [178, Theorem 3.1, p. 134] by Taylor & Lay.

There is more discussion on the Hahn-Banach theorem in Taylor & Lay’s book [178, Section III.2 and Section III.3].

If \( X \) is a linear space, then a linear mapping \( P \) is called a projection of \( X \) if \( P^2 = P \). Each projection determines a direct sum decomposition of the space: \( X = \mathcal{R}(P) \oplus \mathcal{N}(P) \), where \( \mathcal{R}(P) \) denotes the range of \( P \) and \( \mathcal{N}(P) \) denotes the kernel of \( P \). Conversely, each direct sum decomposition of \( X \) determines a projection: if \( X = M_1 \oplus M_2 \), then mapping \( P \), defined by

\[
P x = x_1 \quad \text{for} \quad x = x_1 + x_2 \quad \text{where} \quad x_1 \in M_1, \quad x_2 \in M_2,
\]

is the desired projection. In this case, we say that \( P \) is the projection of \( X \) onto \( M_1 \) along \( M_2 \). It is easy to see that \( I - P \) is the projection of \( X \) onto \( M_2 \) along \( M_1 \). However, if we want the projection to be continuous, then we must require more properties for the spaces \( M_1 \) and \( M_2 \).

**Lemma 2.2.16.** 1. If \( X \) is a Hausdorff topological linear space, and if \( P \) a continuous projection of \( X \), then \( \mathcal{R}(P) \) and \( \mathcal{N}(P) \) are closed.

2. If \( X \) is a Banach space, and if \( M_1 \) and \( M_2 \) are closed subspaces such that \( X = M_1 \oplus M_2 \), then the projection \( P \) of \( X \) onto \( M_1 \) along \( M_2 \) is continuous.

**Proof.** The proofs are in Taylor & Lay’s book [178]. For claim 1. see Theorem 12.1 on page 247, and for claim 2. see Theorem 12.2 on page 247.

Hahn-Banach theorem and projections

The results presented here are basic material in the functional analysis. The reason for including them here is that there was a certain assumption missing in the original results on the index of a critical point of mappings of class ς and that assumption is connected with projections. For further comments, see Remark 4.4.2 on page 74.

One of the most important theorems is the Hahn-Banach theorem. However, the term Hahn-Banach theorem refers to a collection of theorems, where one theorem is the Hahn-Banach theorem and others are its consequences. In Taylor & Lay’s book [178], the following theorem is called the Hahn-Banach theorem. Recall that a subspace \( M \) of \( X \) is a proper subspace if \( M \subsetneq X \).

**Theorem 2.2.14.** Assume that \( M \) is a proper subspace of the normed linear space \( X \). If \( m^* \in M^* \), then \( X^* \) contains an element \( x^* \) such that \( \|x^*\| = \|m^*\| \) and \( x^*(x) = m^*(x) \) for all \( x \in M \).

**Proof.** See the book [178, Theorem 3.1, p. 134] by Taylor & Lay.

There is more discussion on the Hahn-Banach theorem in Taylor & Lay’s book [178, Section III.2 and Section III.3].

If \( X \) is a linear space, then a linear mapping \( P \) is called a projection of \( X \) if \( P^2 = P \). Each projection determines a direct sum decomposition of the space: \( X = \mathcal{R}(P) \oplus \mathcal{N}(P) \), where \( \mathcal{R}(P) \) denotes the range of \( P \) and \( \mathcal{N}(P) \) denotes the kernel of \( P \). Conversely, each direct sum decomposition of \( X \) determines a projection: if \( X = M_1 \oplus M_2 \), then mapping \( P \), defined by

\[
P x = x_1 \quad \text{for} \quad x = x_1 + x_2 \quad \text{where} \quad x_1 \in M_1, \quad x_2 \in M_2,
\]

is the desired projection. In this case, we say that \( P \) is the projection of \( X \) onto \( M_1 \) along \( M_2 \). It is easy to see that \( I - P \) is the projection of \( X \) onto \( M_2 \) along \( M_1 \). However, if we want the projection to be continuous, then we must require more properties for the spaces \( M_1 \) and \( M_2 \).

**Lemma 2.2.16.** 1. If \( X \) is a Hausdorff topological linear space, and if \( P \) a continuous projection of \( X \), then \( \mathcal{R}(P) \) and \( \mathcal{N}(P) \) are closed.

2. If \( X \) is a Banach space, and if \( M_1 \) and \( M_2 \) are closed subspaces such that \( X = M_1 \oplus M_2 \), then the projection \( P \) of \( X \) onto \( M_1 \) along \( M_2 \) is continuous.

**Proof.** The proofs are in Taylor & Lay’s book [178]. For claim 1. see Theorem 12.1 on page 247, and for claim 2. see Theorem 12.2 on page 247.
In this thesis, we are mainly interested in the continuous projections that are onto a certain space and along a certain space. We also need the following basic fact about the existence of projections.

**Lemma 2.2.17.** Let $M$ be a finite-dimensional subspace of a Hausdorff locally convex space $X$. Then there exists a continuous projection of $X$ onto $M$.

**Proof.** See Taylor & Lay [178, Theorem 12.3, p. 247].

Sometimes we need a concrete formula for the action of the projection $P$ of $X$ onto $M_1$ along $M_2$. This is always possible if the subspace $M_2$ of a normed linear space $X$ is finite-dimensional. If $x_1, \ldots, x_n$ is a basis of $M_1$, then the Hahn-Banach theorem implies the existence of the linear functionals $x_1^*, \ldots, x_n^*$ with

$$(x_i^*, x_j) = \delta_{ij} \quad \text{and} \quad (x_i^*, x) = 0 \quad \text{for} \quad x \in M_2.$$

Now the projection $P$ has the expression

$$Px = \sum_{i=1}^n (x_i^*, x)x_i.$$

If we consider the projection $P^*$ of $X^*$ onto $N_1^*$ along $N_2^*$ where $N_1^*$ is finite-dimensional, then

$$P^*x^* = \sum_{i=1}^n (x_i^*, x^*)x_i^*$$

where $x_1^*, \ldots, x_n^*$ is a basis of $N_1^*$ and the elements $x_1^*, \ldots, x_n^* \in X^{**}$ satisfy $(x_i^*, x_i^*) = \delta_{ij}$ and $(x_i^*, x^*) = 0$ for $x^* \in N_2^*$. Note that if the space $X$ is reflexive, then there are elements $x_i \in X$ with $Jx_i = x_i^*$ where $J$ is the canonical mapping of $X$ into $X^{**}$; so we may write

$$P^*x^* = \sum_{i=1}^n (x_i, x^*)x_i$$

where the elements $x_i \in X$ satisfy the conditions $(x_i, x_i^*) = \delta_{ij}$ and $(x_i, x^*) = 0$ for $x^* \in N_2^*$.

Detailed proofs and more information on projections can be found in the book by Taylor & Lay [178, Section IV.12].

**The adjoint of a linear mapping and annihilators**

We need the concept of the adjoint (or the conjugate) mapping of a continuous, everywhere defined, linear mapping.

**Definition 2.2.18.** Let $X$ and $Y$ be normed, linear spaces, and let $T : X \rightarrow Y$ be a continuous, everywhere defined, linear mapping. Then the adjoint mapping of $T$ is the linear mapping $T^* : Y^* \rightarrow X^*$ whose action is defined by the formula

$$\langle T^*v, u \rangle = \langle v, Tu \rangle, \quad u \in X \quad \text{and} \quad v \in Y^*.$$
In the general setting (\( T \) is only densely defined and/or unbounded), the domain of \( T^* \) consists of those \( v \in Y^* \) for which the mapping \( v \circ T \) is continuous on \( D \), where \( D \) is the domain of \( T \). Note that \( T^* \) is not necessarily unique if \( \mathcal{D} \neq X \). Clearly, if \( T \) is continuous and everywhere defined, the domain of \( T^* \) is the whole space \( Y^* \). These mappings and their properties are investigated in Taylor & Lay’s book [178, Chapter IV, pp. 188–263], even in the setting where \( T \) is densely defined or unbounded, and \( X, Y \) are normed spaces.

**Remark 2.2.19.** Usually the terms “adjoint” and “conjugate” are considered as synonyms, and usually the term adjoint is used, but in the book by Taylor & Lay [178, Section IV 11 pp. 242–246] these two concepts are distinguished, with a good reason. Conjugate mapping is defined in that book as we define the adjoint herein, but the term adjoint mapping is reserved for the situation where \( T \) is a mapping from a Hilbert space \( X \) into a Hilbert space \( Y \). In the case of Hilbert spaces, the conjugate mapping \( T^* \) of \( T \) maps the topological dual \( Y^* \) of \( Y \) to the topological dual \( X^* \) of \( X \). The spaces \( X \) and \( X^* \) are different, as are the spaces \( Y \) and \( Y^* \), but the Frechet-Riesz theorem offers a possibility to eliminate the spaces \( X^* \) and \( Y^* \) by imposing the existence of linear homeomorphisms \( E_X: X \to X^* \) and \( E_Y: Y \to Y^* \). Here \( E_X \) associates to each \( z \in X \) the linear functional \( x \mapsto (x, z) \), and \( E_Y \) associates to each \( v \in Y \) the linear functional \( y \mapsto (y, v) \). However, if we make the required identifications with those homeomorphisms, then we study the mapping \( E_X^{-1} T^* E_Y: Y \to X \) instead of \( T^*: Y^* \to X^* \). For this reason, the term adjoint is reserved for \( E_X^{-1} T^* E_Y: Y \to X \) in Taylor & Lay’s book. See also Remark 2.4.13.

The following concept is salient in the study of a linear mapping and its adjoint.

**Definition 2.2.20.** If \( A \subseteq X \), then the annihilator \( A^\perp \) of \( A \) is the set
\[
A^\perp = \{ x^* \in X^* \mid \langle x^*, x \rangle = 0, \text{ for all } x \in A \}. 
\]

If \( C \subseteq X^* \), then the annihilator \( C^\perp \) of \( C \) is the set
\[
C^\perp = \{ x \in X \mid \langle x^*, x \rangle = 0, \text{ for all } x^* \in C \}. 
\]

Recall that a mapping \( T: D \subseteq X \to Y \), where \( X \) and \( Y \) are topological spaces, is called closed if its graph \( \{(x, Tx) \mid x \in D \} \) is a closed set in the product topology of the space \( X \times Y \).

**Theorem 2.2.21.** If \( T: D \subseteq X \to Y \) is a densely defined, linear operator, then
1. \( \mathcal{A}(T^*) = \mathcal{A}(T)^\perp = \mathcal{M}(T^*) \),
2. \( \mathcal{A}(T) = \mathcal{M}(T^*)^\perp \),
3. if \( T \) is closed, \( \mathcal{A}(T^*)^\perp \cap D = \mathcal{M}(T) \).

**Proof.** See Taylor & Lay’s book [178, Theorem 8.4 and Theorem 8.5, p. 232].

In the general setting (\( T \) is only densely defined and/or unbounded), the domain of \( T^* \) consists of those \( v \in Y^* \) for which the mapping \( v \circ T \) is continuous on \( D \), where \( D \) is the domain of \( T \). Note that \( T^* \) is not necessarily unique if \( \mathcal{D} \neq X \). Clearly, if \( T \) is continuous and everywhere defined, the domain of \( T^* \) is the whole space \( Y^* \). These mappings and their properties are investigated in Taylor & Lay’s book [178, Chapter IV, pp. 188–263], even in the setting where \( T \) is densely defined or unbounded, and \( X, Y \) are normed spaces.

**Remark 2.2.19.** Usually the terms “adjoint” and “conjugate” are considered as synonyms, and usually the term adjoint is used, but in the book by Taylor & Lay [178, Section IV 11 pp. 242–246] these two concepts are distinguished, with a good reason. Conjugate mapping is defined in that book as we define the adjoint herein, but the term adjoint mapping is reserved for the situation where \( T \) is a mapping from a Hilbert space \( X \) into a Hilbert space \( Y \). In the case of Hilbert spaces, the conjugate mapping \( T^* \) of \( T \) maps the topological dual \( Y^* \) of \( Y \) to the topological dual \( X^* \) of \( X \). The spaces \( X \) and \( X^* \) are different, as are the spaces \( Y \) and \( Y^* \), but the Frechet-Riesz theorem offers a possibility to eliminate the spaces \( X^* \) and \( Y^* \) by imposing the existence of linear homeomorphisms \( E_X: X \to X^* \) and \( E_Y: Y \to Y^* \). Here \( E_X \) associates to each \( z \in X \) the linear functional \( x \mapsto (x, z) \), and \( E_Y \) associates to each \( v \in Y \) the linear functional \( y \mapsto (y, v) \). However, if we make the required identifications with those homeomorphisms, then we study the mapping \( E_X^{-1} T^* E_Y: Y \to X \) instead of \( T^*: Y^* \to X^* \). For this reason, the term adjoint is reserved for \( E_X^{-1} T^* E_Y: Y \to X \) in Taylor & Lay’s book. See also Remark 2.4.13.

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\]

If \( C \subseteq X^* \), then the annihilator \( C^\perp \) of \( C \) is the set
\[
C^\perp = \{ x \in X \mid \langle x^*, x \rangle = 0, \text{ for all } x^* \in C \}. 
\]

Recall that a mapping \( T: D \subseteq X \to Y \), where \( X \) and \( Y \) are topological spaces, is called closed if its graph \( \{(x, Tx) \mid x \in D \} \) is a closed set in the product topology of the space \( X \times Y \).

**Theorem 2.2.21.** If \( T: D \subseteq X \to Y \) is a densely defined, linear operator, then
1. \( \mathcal{A}(T^*) = \mathcal{A}(T)^\perp = \mathcal{M}(T^*) \),
2. \( \mathcal{A}(T) = \mathcal{M}(T^*)^\perp \),
3. if \( T \) is closed, \( \mathcal{A}(T^*)^\perp \cap D = \mathcal{M}(T) \).

**Proof.** See Taylor & Lay’s book [178, Theorem 8.4 and Theorem 8.5, p. 232].

2.3 On compact mappings and various types of continuity

In this section, we recollect facts on compact, linear operators, certain continuities and related concepts. Special attention is paid to invariant subspaces of a compact, linear operator and a decomposition of a Banach space by using invariant subspaces. This decomposition is needed in Chapter 4.

**Definition 2.3.1.** Assume that $X$ and $Y$ are normed spaces and $T : D \subset X \to Y$. Then $T$ is

1. **compact** if for each bounded subset $B$ of $X$ the set $T(B \cap D)$ is relatively compact in $Y$, that is, $T(B \cap D)$ is compact in $Y$.
2. **bounded** if the image of any bounded subset of $D$ is bounded.
3. **locally bounded** if for every $x \in D$ there is a neighborhood $V$ of $x$ such that the set $T(V \cap D)$ is bounded in $Y$.

In infinite-dimensional spaces, there are two very important types of convergence (the strong and the weak convergence) for sequences. The following continuity conditions have proved to be useful in nonlinear functional analysis.

**Definition 2.3.2.** Let $X$ and $Y$ be normed spaces, $T : D \subset X \to Y$ and $(x_n)_{n=1}^{\infty} \cup \{x\} \subset D$. We say that $T$ is

1. **completely continuous** if $T(x_n) \to T(x)$ in $Y$ whenever $x_n \to x$ in $X$.
2. **continuous** if $T(x_n) \to T(x)$ in $Y$ whenever $x_n \to x$ in $X$.
3. **demi-continuous** if $T(x_n) \to T(x)$ in $Y$ whenever $x_n \to x$ in $X$.
4. **weakly continuous** if $T(x_n) \to T(x)$ in $Y$ whenever $x_n \to x$ in $X$.

For linear operators, $D$ is a linear subspace of $X$. These different continuities are presented and their relationships are tackled in detail in the books by Cioranescu [53], Deimling [61], Joshi & Bose [104], and Zeidler [199].

**Theorem 2.3.3.** If $X$ and $Y$ are Banach spaces and $T : X \to Y$, then the following assertions hold:

1. If $T$ is a continuous, linear operator, then it is weakly continuous.
2. If $T$ is linear, then it is continuous if and only if it is bounded.
3. If $T$ is a compact, linear operator, then it is bounded.
4. If $X$ is reflexive and $T$ is linear, then $T$ is completely continuous if and only if it is compact.
5. If $T$ is demi-continuous, then it is locally bounded.

**Proof.** For the proofs of the weak continuity and assertion 4., see Joshi & Bose [104, Theorem 1.1.1, p. 2, and Theorem 1.1.5, p. 5]. The proofs of 2. and 3. are found in Taylor & Lay’s book [178, Theorem 1.1, p. 54, and Theorem 7.1, p. 294]. For the proof of 5., see the report by Berkovits & Mustonen [17, Remark 4°, p. 5].

**Remark 2.3.4.** For an example showing that continuous nonlinear mapping need not be weakly continuous, see Joshi & Bose [104, Example 1.1.3, p. 2] for example.
2.3.1 A decomposition of a Banach space

The purpose of this subsection is to present a certain decomposition of a (real or complex) Banach space that uses properties of compact, linear mapping. The results presented here apply also to a normed space containing nonzero elements. Hereafter, in this subsection, we use the following convention: if \( X \) is a complex (real) normed space, then a scalar \( \lambda \) denotes a complex (real) number.

The next definition plays a central role in the analysis of linear operators; the definition is used by Taylor & Lay [178, p. 264]. In practice we do not use it, but for the completeness of this treatise we include it here.

**Definition 2.3.5.** Assume that \( X \) is a normed linear space containing nonzero elements. The **resolvent set** of a linear operator \( T : \mathcal{D}(T) \subset X \to X \) is the set \( \rho(T) \) of all \( \lambda \) such that the range of \( \lambda I - T \) is dense in \( X \) and \( \lambda I - T \) has a continuous inverse. For \( \lambda \in \rho(T) \), the operator \((\lambda I - T)^{-1}\) is called the **resolvent operator** and is often denoted by \( R_\lambda \). The spectrum of \( T \) is the set \( \sigma(T) \) of all scalar values not in \( \rho(T) \).

**Remark 2.3.6.**

1. If \( X \) is a complex Banach space and \( T : X \to X \) a continuous, linear mapping, then the spectrum of \( T \) is not empty; see Taylor & Lay [178, Theorem 3.2, p. 278]. However, if \( X \) is real Banach space, then the spectrum might be empty.
2. If \( X \) is a Banach space and \( \lambda \in \rho(T) \), then
   \[
   (\lambda I - T)^{-1} : (\lambda I - T)(X) \subset X \to X
   \]
   has the unique, everywhere defined, continuous extension \( X \to X \). Moreover, if \( T \) is closed, then \((\lambda I - T)^{-1}\) is defined on the whole space \( X \); for the details, see Taylor & Lay [178, Theorem IV.5.2, p. 211].

In this thesis, one of the most used properties of compact, linear mappings (acting on a real reflexive Banach space) is their characterization by their continuity properties. The other properties that are used in this thesis are purely and simply connected with a certain decomposition of a real reflexive space; that decomposition is needed for the formulation of the main theorem of Chapter 4. The decomposition must be such that the subspaces contain their own images. Such spaces are also of interest in various theoretical examinations and applications; therefore, these spaces have their own designation.

**Definition 2.3.7.** Assume that \( X \) is a linear space and \( T : \mathcal{D}(T) \subset X \to X \). We call a subspace \( M \) of \( X \) an **invariant subspace** of \( T \) if \( T(M \cap \mathcal{D}(T)) \subset M \).

It is clear that if a subspace is an invariant subspace of \( T \), it is also an invariant subspace of \( T^n \) for every \( n \in \mathbb{Z}_+ \), and an invariant subspace of \( \mu I - T \) for any scalar \( \mu \). For linear mappings one can try to form invariant subspaces by seeking nonzero vectors \( x \) with the property \( Tx = \lambda x \), and then forming a linear span of a selection of such vectors. The scalar \( \lambda \) is called an **eigenvalue** of \( T \) and \( x \) an **eigenvector** of \( T \). Note that in the

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More information on invariant subspaces is found in Radjavi & Rosenthal’s book [153] and the references therein. See also Taylor & Lay [178, p. 303] and Conway [55, pp. 182-187].
Russian literature, we often encounter the term *characteristic value* which is defined as the reciprocal of a nonzero eigenvalue; if \( \lambda \) is a characteristic value of \( T \), then \( \lambda^{-1} \) is an eigenvalue of \( T \). The *point spectrum* of a linear operator consists of all eigenvalues of that operator. The space spanned by the eigenvectors corresponding to an eigenvalue is called the *eigenspace* corresponding to that eigenvalue; that is, the eigenspace corresponding to an eigenvalue \( \lambda \) of \( T \) is defined as the set
\[
\{ u \in X \mid (\lambda I - T)u = 0 \}.
\]
The dimension of this space is called the geometric multiplicity of \( \lambda \).

For a linear mapping \( T : X \to X \), a scalar \( \lambda \), and \( n \in \mathbb{N} \), we introduce the following notations
\[
\mathcal{N}(\lambda I - T)^n = \{ u \in X \mid (\lambda I - T)^n u = 0 \}
\]
and
\[
\mathcal{R}(\lambda I - T)^n = \{ v \in X \mid \exists u \in X : v = (\lambda I - T)^n u \}.
\]
If \( \lambda \) is an eigenvalue of \( T \), then the vectors in \( \mathcal{N}(\lambda I - T)^n \), where \( n \geq 1 \), are called the (generalized) eigenvectors of \( T \). It follows from the formula
\[
(\lambda I - T)^n(\lambda I - T) = (\lambda I - T)^{n+1},
\]
that
\[
\mathcal{R}(\lambda I - T)^n \subset \mathcal{R}(\lambda I - T)^{n+1} \quad \text{and} \quad \mathcal{N}(\lambda I - T)^n \subset \mathcal{N}(\lambda I - T)^{n+1}
\]
for every \( n \in \mathbb{N} \). See the book by Taylor & Lay [178, pp. 289–293] for more information on these spaces. Recall that if \( T : X \to X \) is a linear mapping and \( \lambda \) its eigenvalue, then the algebraic multiplicity \( \mu \) of \( \lambda \) is the dimension of the space spanned by all (generalized) eigenvectors associated with \( \lambda \), that is,
\[
\mu = \dim \bigcup_{n=1}^{\infty} \mathcal{N}(\lambda I - T)^n.
\]
When \( T \) is compact, the subspaces \( \mathcal{N}(\lambda I - T)^n \) and \( \mathcal{R}(\lambda I - T)^n \) have intriguing properties.

**Theorem 2.3.8.** Assume that \( X \) is a normed linear space. If \( T : X \to X \) is a compact, linear mapping, then
1. the spectrum of \( T \) contains at most a countable set of points, and the only possible accumulation point of the spectrum is zero. Moreover, each nonzero point of the spectrum is an eigenvalue of \( T \).

Assume further that \( \lambda \) is a nonzero scalar, then
2. \( \mathcal{N}(\lambda I - T)^n \) is finite-dimensional for every \( n \geq 1 \).
3. there is a positive integer \( p \) such that for all \( m, n \geq p \)
\[
\mathcal{R}(\lambda I - T)^m = \mathcal{R}(\lambda I - T)^n, \quad \mathcal{N}(\lambda I - T)^m = \mathcal{N}(\lambda I - T)^n,
\]
and
\[
X = \mathcal{R}(\lambda I - T)^n \oplus \mathcal{N}(\lambda I - T)^n,
\]
where both subspaces are closed.

Russian literature, we often encounter the term *characteristic value* which is defined as the reciprocal of a nonzero eigenvalue; if \( \lambda \) is a characteristic value of \( T \), then \( \lambda^{-1} \) is an eigenvalue of \( T \). The *point spectrum* of a linear operator consists of all eigenvalues of that operator. The space spanned by the eigenvectors corresponding to an eigenvalue is called the *eigenspace* corresponding to that eigenvalue; that is, the eigenspace corresponding to an eigenvalue \( \lambda \) of \( T \) is defined as the set
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\[
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\]
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\[
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for every \( n \in \mathbb{N} \). See the book by Taylor & Lay [178, pp. 289–293] for more information on these spaces. Recall that if \( T : X \to X \) is a linear mapping and \( \lambda \) its eigenvalue, then the algebraic multiplicity \( \mu \) of \( \lambda \) is the dimension of the space spanned by all (generalized) eigenvectors associated with \( \lambda \), that is,
\[
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2. \( \mathcal{N}(\lambda I - T)^n \) is finite-dimensional for every \( n \geq 1 \).
3. there is a positive integer \( p \) such that for all \( m, n \geq p \)
\[
\mathcal{R}(\lambda I - T)^m = \mathcal{R}(\lambda I - T)^n, \quad \mathcal{N}(\lambda I - T)^m = \mathcal{N}(\lambda I - T)^n,
\]
and
\[
X = \mathcal{R}(\lambda I - T)^n \oplus \mathcal{N}(\lambda I - T)^n,
\]
where both subspaces are closed.
Remark 2.3.9. 1. It follows from Theorem 2.3.8 and the calculation
\[ \lambda x = Tx \quad \Rightarrow \quad |\lambda| \leq \|T\| \]
that any set whose closure does not contain zero contains only a finite number of eigenvalues of a compact, linear operator.

2. The smallest integer \( q \) for which the equation
\[ \mathcal{R}(\lambda I - T)^q = \mathcal{R}(\lambda I - T) \]
holds for all \( n > q \) is called the descent of \( T : X \to X \) associated with \( \lambda \) and the smallest number \( p \) for which
\[ \mathcal{N}(\lambda I - T)^p = \mathcal{N}(\lambda I - T)^p \]
holds for all \( n > p \) is called the ascent of \( T \) associated with \( \lambda \). If \( T \) is only linear, the ascent (descent) might not exist; in that case one says that the ascent (descent) of \( T \) is infinite. It is known that if \( p, q < \infty \), then \( q \geq p \); moreover, if \( p, q < \infty \) and \( T \) is everywhere defined, the equality \( q = p \) holds and
\[ X = \mathcal{R}(\lambda I - T)^p \oplus \mathcal{N}(\lambda I - T)^q \]
where \( n \geq p \). For the proof, see Taylor & Lay [178, Theorem 6.2, p. 290].

3. The algebraic multiplicity of an eigenvalue \( \lambda \) of a compact, linear operator \( T \) is always equal or larger than the ascent associated with \( \lambda \): if \( n \in \mathbb{N} \) is smaller than the ascent \( p \), then the space
\[ \mathcal{N}(\lambda I - T)^{p+1} \setminus \mathcal{N}(\lambda I - T)^p \]
may be spanned by more than one vector (there may be more than one eigenvector corresponding to an eigenvalue). Note that
\[ \mathcal{N}(\lambda I - T)^p = \mathcal{N}(\lambda I - T)^p \]
if \( \mu \) is the algebraic multiplicity of \( \lambda \) and \( p \) the ascent of \( T \) associated with \( \lambda \).

4. In matrix theory, the algebraic multiplicity can be defined in another way too: if \( A \) is a square matrix, then the algebraic multiplicity of the eigenvalue \( \lambda_j \) is the exponent \( \mu_j \) of the factor \( \lambda - \lambda_j \) of the characteristic polynomial
\[ \det(\lambda I - A) = \prod_{i=1}^{n}(\lambda - \lambda_i)^{\mu_i}, \]
where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \) and \( \lambda_i \neq \lambda_j \) when \( i \neq j \). Since the characteristic polynomial of an \( n \times n \)-matrix \( A \) is of the order \( n \), it follows that the sum of the algebraic multiplicities of the eigenvalues of \( A \) is equal to \( n \).

Proof. A reference that contains proofs of these assertions is Taylor & Lay [178]. For claims 1., 2., 3., and 4., see respectively Theorem 7.10 on page 301, Theorem 7.6 on page 299, Theorem 7.9 on page 300, and Theorem 7.7 on page 299.

Remark 2.3.9. 1. It follows from Theorem 2.3.8 and the calculation
\[ \lambda x = Tx \quad \Rightarrow \quad |\lambda| \leq \|T\| \]
that any set whose closure does not contain zero contains only a finite number of eigenvalues of a compact, linear operator.

2. The smallest integer \( q \) for which the equation
\[ \mathcal{R}(\lambda I - T)^q = \mathcal{R}(\lambda I - T)^p \]
holds for all \( n > q \) is called the descent of \( T : X \to X \) associated with \( \lambda \) and the smallest number \( p \) for which
\[ \mathcal{N}(\lambda I - T)^p = \mathcal{N}(\lambda I - T)^p \]
holds for all \( n > p \) is called the ascent of \( T \) associated with \( \lambda \). If \( T \) is only linear, the ascent (descent) might not exist; in that case one says that the ascent (descent) of \( T \) is infinite. It is known that if \( p, q < \infty \), then \( q \geq p \); moreover, if \( p, q < \infty \) and \( T \) is everywhere defined, the equality \( q = p \) holds and
\[ X = \mathcal{R}(\lambda I - T)^p \oplus \mathcal{N}(\lambda I - T)^q \]
where \( n \geq p \). For the proof, see Taylor & Lay [178, Theorem 6.2, p. 290].

3. The algebraic multiplicity of an eigenvalue \( \lambda \) of a compact, linear operator \( T \) is always equal or larger than the ascent associated with \( \lambda \): if \( n \in \mathbb{N} \) is smaller than the ascent \( p \), then the space
\[ \mathcal{N}(\lambda I - T)^{p+1} \setminus \mathcal{N}(\lambda I - T)^p \]
may be spanned by more than one vector (there may be more than one eigenvector corresponding to an eigenvalue). Note that
\[ \mathcal{N}(\lambda I - T)^p = \mathcal{N}(\lambda I - T)^p \]
if \( \mu \) is the algebraic multiplicity of \( \lambda \) and \( p \) the ascent of \( T \) associated with \( \lambda \).

4. In matrix theory, the algebraic multiplicity can be defined in another way too: if \( A \) is a square matrix, then the algebraic multiplicity of the eigenvalue \( \lambda_j \) is the exponent \( \mu_j \) of the factor \( \lambda - \lambda_j \) of the characteristic polynomial
\[ \det(\lambda I - A) = \prod_{i=1}^{n}(\lambda - \lambda_i)^{\mu_i}, \]
where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \) and \( \lambda_i \neq \lambda_j \) when \( i \neq j \). Since the characteristic polynomial of an \( n \times n \)-matrix \( A \) is of the order \( n \), it follows that the sum of the algebraic multiplicities of the eigenvalues of \( A \) is equal to \( n \).
5. In matrix theory, \( \mathcal{N}(\lambda I - T)^n \) is called the root space or the generalized eigenspace of \( T \) associated with the eigenvalue \( \lambda \) when \( n \) is the ascent of \( T \) associated with \( \lambda \). If \( m \) is smaller than the ascent, it is said that \( \mathcal{N}(\lambda I - T)^m \) is the generalized eigenspace of \( T \) of the order \( m \). The interested reader may study the books by Lancaster & Tismenetsky [121] and Golberg, Lancaster & Rodman [79].

It is known that \( \mu_1 \) is equal to the dimension of \( \mathcal{N}(\lambda I - T)^{\mu_1} \); that claim is proved in Lancaster & Tismenetsky [121, Proposition 1, p. 239] and Golberg, Lancaster & Rodman [79, Proposition 2.2.5, p. 56].

6. Observe that if \( T : X \to X \) is compact and \( Y \subset X \) is an invariant subspace of \( T \), then the restriction \( T|_Y \) maps \( Y \) to \( Y \) and is compact; moreover, we may apply Theorem 2.3.8 to \( T|_Y \).

7. Since \( u \in \mathcal{N}(\lambda I - T)^n \) implies that \( (\lambda I - T)^n u = 0 \), the space \( \mathcal{N}(\lambda I - T)^n \) is an invariant subspace of \( T \).

8. If \( u \in \mathcal{R}(\lambda I - T)^n \), and \( n \) is the corresponding descent of the linear operator \( T \), then

\[
Tu = -\mathcal{R}(\lambda I - T)^{n+1} + \lambda u = \mathcal{R}(\lambda I - T)^n.
\]

This means that \( \mathcal{R}(\lambda I - T)^n \) is an invariant subspace of \( T \).

9. Let \( T \) be compact and linear, \( \lambda_1 \) and \( \lambda_2 \) be distinct eigenvalues of \( T \), and \( \mu_1 \) and \( \mu_2 \) be the corresponding algebraic multiplicities. Since \( \mathcal{R}(\lambda_1 I - T)^{\mu_1} \) and \( \mathcal{R}(\lambda_2 I - T)^{\mu_2} \) are invariant subspaces of \( T \), it follows that \( \mathcal{R}(\lambda_1 I - T)^{\mu_1} \cap \mathcal{R}(\lambda_2 I - T)^{\mu_2} = \{0\} \).

Recall that algebraic multiplicity is always equal to or larger than the ascent. If \( u \in \mathcal{N}(\lambda_2 I - T)^{\mu_2} \setminus \{0\} \), then \( (\lambda_2 I - T)^{k+1} u = 0 \) and \( (\lambda_2 I - T)^{k} u \neq 0 \) for some \( 0 \leq k \leq \mu_2 - 1 \). Note that here \( \mu_2 \) can be replaced by the ascent of \( T \) associated with \( \lambda_2 \).

From the calculation

\[
(\lambda_2 I - T)^{k}(\lambda_1 I - T)^{\mu_1} u = (\lambda_2 I - T)^{k}((\lambda_1 - \lambda_2) I + (\lambda_2 I - T))^{\mu_1} u
\]

\[
= (\lambda_2 I - T)^{k} \sum_{j=0}^{\mu_1} \binom{\mu_1}{j} (\lambda_1 - \lambda_2)^{j} (\lambda_2 I - T)^{\mu_1-j} u
\]

\[
= \sum_{j=0}^{\mu_1} \binom{\mu_1}{j} (\lambda_1 - \lambda_2)^{j} (\lambda_2 I - T)^{\mu_1-j} u
\]

\[
= (\lambda_1 - \lambda_2)^{\mu_2} (\lambda_2 I - T)^{\mu_2} u
\]

\[
\neq 0,
\]

we deduce that if \( u \in \mathcal{N}(\lambda_2 I - T)^{\mu_2} \setminus \{0\} \), then \( (\lambda_2 I - T)^{\mu_2} u \neq 0 \), which means that \( u \notin \mathcal{N}(\lambda_1 I - T)^{\mu_1} \).

11. Suppose that \( \lambda_1 \) and \( \lambda_2 \) are distinct eigenvalues of a compact, linear operator \( T \), and \( \mu_1 \) and \( \mu_2 \) are the corresponding algebraic multiplicities. Then we have the inclusion

\[
\mathcal{N}(\lambda_2 I - T)^{\mu_2} \subset \mathcal{R}(\lambda_1 I - T)^{\mu_1}.
\]
This is seen in the following manner. Because \( \mathcal{N}(\lambda_2 I - T)^{m_2} \) is a finite-dimensional invariant subspace of the mapping \((\lambda_2 I - T)^{m_2}\) and the restriction

\[
(\lambda_2 I - T)^{m_2}|_{\Phi(\lambda_2 I - T)^{m_2}} : \mathcal{N}(\lambda_2 I - T)^{m_2} \to \mathcal{N}(\lambda_2 I - T)^{m_2}
\]

is an injective mapping, this restriction is also surjective; hence,

\[
\mathcal{N}(\lambda_2 I - T)^{m_2} = (\lambda_2 I - T)^{m_2}(\mathcal{N}(\lambda_2 I - T)^{m_2}) \subset \mathcal{R}(\lambda_2 I - T)^{m_2}.
\]

This proves the claim. Note that the surjectivity is really needed for the conclusion.

12. It follows from the previous remark that if \( \lambda_1 \) and \( \lambda_2 \) are distinct eigenvalues of a compact, linear operator \( T \), and if \( \mu_1 \) and \( \mu_2 \) are the corresponding algebraic multiplicities, then \( \lambda_2 \) is an eigenvalue of the restriction

\[
T|_{\Phi(\lambda_1 I - T)^{\mu_1}} : \mathcal{R}(\lambda_1 I - T)^{\mu_1} \to \mathcal{R}(\lambda_1 I - T)^{\mu_1}.
\]

Moreover, the algebraic multiplicity of \( \lambda_2 \) as an eigenvalue of the compact mapping

\[
T|_{\Phi(\lambda_1 I - T)^{\mu_1}} : \mathcal{R}(\lambda_1 I - T)^{\mu_1} \to \mathcal{R}(\lambda_1 I - T)^{\mu_1},
\]

is an injective mapping, this restriction is also surjective; hence,

\[
\mathcal{N}(\lambda_2 I - T)^{m_2} = (\lambda_2 I - T)^{m_2}(\mathcal{N}(\lambda_2 I - T)^{m_2}) \subset \mathcal{R}(\lambda_2 I - T)^{m_2}.
\]

This proves the claim. Note that the surjectivity is really needed for the conclusion.

Now we have gathered all the results which are needed for the decomposition of a (real or complex) normed space \( X \). Since there are more general “decomposition results” for compact mappings \( X \to X \) when \( X \) is a complex Banach space and complex eigenvalues are considered (see Taylor & Lay’s book [178, chapters V and VI]), the decomposition results of a real Banach space have received only little attention and are therefore hard to find. For example, the author did not find the next theorem anywhere although it seems to be well-known. Note that the theorem holds for a (real or complex) normed space that is not necessarily complete.

**Theorem 2.3.10.** Let \( X \) be a normed space and \( T : X \to X \) a compact, linear mapping. Assume that \( \lambda_1, \ldots, \lambda_m \) is a finite set of distinct, nonzero eigenvalues of \( T \), and \( \mu_1, \ldots, \mu_m \) are the corresponding algebraic multiplicities, then \( X \) has the decomposition

\[
X = \bigoplus_{n=1}^{m} \mathcal{N}(\lambda_n I - T)^{\mu_n} \oplus \bigcap_{n=1}^{m} \mathcal{R}(\lambda_n I - T)^{\mu_n}
\]

where \( \bigoplus_{n=1}^{m} \mathcal{N}(\lambda_n I - T)^{\mu_n} \) and \( \bigcap_{n=1}^{m} \mathcal{R}(\lambda_n I - T)^{\mu_n} \) are closed, invariant subspaces of \( T \). Moreover, the dimension of \( \bigoplus_{n=1}^{m} \mathcal{N}(\lambda_n I - T)^{\mu_n} \) is \( \sum_{n=1}^{m} \mu_n \).

The idea of the proof is simple: we decompose \( X \) using Theorem 2.3.8 and \( T \), and then we decompose the space \( \mathcal{N}(\lambda_n I - T)^{\mu_n} \) using the compact mapping

\[
T|_{\mathcal{R}(\lambda_n I - T)^{\mu_n}} : \mathcal{R}(\lambda_n I - T)^{\mu_n} \to \mathcal{R}(\lambda_n I - T)^{\mu_n}.
\]

After some calculations, we get

\[
\mathcal{N}(\lambda_2 I - T)^{m_2} = \mathcal{N}(\lambda_2 I - T)^{m_2} \oplus \mathcal{R}(\lambda_2 I - T)^{m_2} \cap \mathcal{R}(\lambda_2 I - T)^{m_2},
\]

then we decompose the space \( \mathcal{R}(\lambda_2 I - T)^{m_2} \cap \mathcal{R}(\lambda_2 I - T)^{m_2} \) using the compact mapping

\[
T|_{\mathcal{R}(\lambda_2 I - T)^{m_2} \cap \mathcal{R}(\lambda_2 I - T)^{m_2}} : \mathcal{R}(\lambda_2 I - T)^{m_2} \cap \mathcal{R}(\lambda_2 I - T)^{m_2} \to \mathcal{R}(\lambda_2 I - T)^{m_2},
\]

and so on until we reach the index \( m \). Note that the justifications of many of the calculations are presented in Remark 2.3.9.
Proof. Step 1. We prove that
\[ X = \bigoplus_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n} \oplus \bigcap_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n}. \]

According to Theorem 2.3.8, we have the decomposition
\[ X = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1}. \]

If we study the restriction of \( T \) to the space \( \mathcal{A}(\lambda_1 I - T)^{\mu_1} \), we see that \( \lambda_2 \) is an eigenvalue of that restriction and the algebraic multiplicity is \( \mu_2 \). Therefore, the decomposition
\[ \mathcal{A}(\lambda_1 I - T)^{\mu_1} = \mathcal{A}(\lambda_2 I - T)^{\mu_2} \oplus \mathcal{A}(\lambda_2 I - T)^{\mu_2} \] (2.8)
holds. The terms of this expression are not suitable for our needs, and therefore we alter them a little. By using the inclusion \( \mathcal{A}(\lambda_2 I - T)^{\mu_2} \subset \mathcal{A}(\lambda_1 I - T)^{\mu_1} \), we may write
\[ \mathcal{A}(\lambda_2 I - T)^{\mu_2} = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1}. \]

Secondly, since \( \mathcal{A}(\lambda_1 I - T)^{\mu_1} \) and \( \mathcal{A}(\lambda_1 I - T)^{\mu_1} \) are invariant subspaces of the mapping \( \lambda_2 (T)^{\mu_2} \) and \( X = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1} \),
\[ \mathcal{A}(\lambda_2 I - T)^{\mu_2} = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1}. \]

Now we may state that (2.8) is the same as
\[ \mathcal{A}(\lambda_1 I - T)^{\mu_1} = \mathcal{A}(\lambda_2 I - T)^{\mu_2} \oplus \mathcal{A}(\lambda_2 I - T)^{\mu_2}. \]

Thus,
\[ X = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_2 I - T)^{\mu_2} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1} \cap \mathcal{A}(\lambda_2 I - T)^{\mu_2}. \]

By continuing this till we reach subindex \( m \), we get
\[ X = \bigoplus_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n} \oplus \bigcap_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n}. \]

The claim \( \dim \mathbb{P}_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n} = \sum_{n=1}^{m} \mu_n \) follows from the definitions of the direct sum, \( \mathcal{A}(\lambda_n I - T)^{\mu_n} \) and the algebraic multiplicity of an eigenvalue.

Proof. Step 1. We prove that
\[ X = \bigoplus_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n} \oplus \bigcap_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n}. \]

According to Theorem 2.3.8, we have the decomposition
\[ X = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1}. \]

If we study the restriction of \( T \) to the space \( \mathcal{A}(\lambda_1 I - T)^{\mu_1} \), we see that \( \lambda_2 \) is an eigenvalue of that restriction and the algebraic multiplicity is \( \mu_2 \). Therefore, the decomposition
\[ \mathcal{A}(\lambda_1 I - T)^{\mu_1} = \mathcal{A}(\lambda_2 I - T)^{\mu_2} \oplus \mathcal{A}(\lambda_2 I - T)^{\mu_2} \] (2.8)
holds. The terms of this expression are not suitable for our needs, and therefore we alter them a little. By using the inclusion \( \mathcal{A}(\lambda_2 I - T)^{\mu_2} \subset \mathcal{A}(\lambda_1 I - T)^{\mu_1} \), we may write
\[ \mathcal{A}(\lambda_2 I - T)^{\mu_2} = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1}. \]

Secondly, since \( \mathcal{A}(\lambda_1 I - T)^{\mu_1} \) and \( \mathcal{A}(\lambda_1 I - T)^{\mu_1} \) are invariant subspaces of the mapping \( \lambda_2 (T)^{\mu_2} \) and \( X = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1} \),
\[ \mathcal{A}(\lambda_2 I - T)^{\mu_2} = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1}. \]

Now we may state that (2.8) is the same as
\[ \mathcal{A}(\lambda_1 I - T)^{\mu_1} = \mathcal{A}(\lambda_2 I - T)^{\mu_2} \oplus \mathcal{A}(\lambda_2 I - T)^{\mu_2}. \]

Thus,
\[ X = \mathcal{A}(\lambda_1 I - T)^{\mu_1} \oplus \mathcal{A}(\lambda_2 I - T)^{\mu_2} \oplus \mathcal{A}(\lambda_1 I - T)^{\mu_1} \cap \mathcal{A}(\lambda_2 I - T)^{\mu_2}. \]

By continuing this till we reach subindex \( m \), we get
\[ X = \bigoplus_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n} \oplus \bigcap_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n}. \]

The claim \( \dim \mathbb{P}_{n=1}^{m} \mathcal{A}(\lambda_n I - T)^{\mu_n} = \sum_{n=1}^{m} \mu_n \) follows from the definitions of the direct sum, \( \mathcal{A}(\lambda_n I - T)^{\mu_n} \) and the algebraic multiplicity of an eigenvalue.
Step 2. We prove that \( \mathcal{A}_{n=1}^{m} \langle \lambda, I - T \rangle^{\mu} \) and \( \bigcap_{n=1}^{m} \mathcal{A}(\lambda, I - T)^{\mu} \) are closed, invariant subspaces of \( T \).

The first claim is evident according to the remarks before this theorem and the fact that each of the spaces \( \mathcal{A}(\lambda, I - T)^{\mu} \) are finite-dimensional. Remember that the space \( \mathcal{A}(\lambda, I - T)^{\mu} \) is a closed, invariant subspace of \( T \). Hence, if \( u \in \mathcal{A}(\lambda, I - T)^{\mu} \) for each \( n = 1, \ldots, m \), then \( Tu \in \mathcal{A}(\lambda, I - T)^{\mu} \) for each \( n = 1, \ldots, m \). Moreover, the intersection of closed sets is also closed. The second claim is proved also. \( \square \)

Remark 2.3.11. If the space \( X \) is a Banach space, the proof of Theorem 2.3.10 is much simpler. Inasmuch as

\[
X = \mathcal{A}^{\mu}(\lambda, I - T)^{\mu} \oplus \mathcal{A}(\lambda, I - T)^{\mu},
\]

and \( \mathcal{A}^{\mu}(\lambda, I - T)^{\mu} \) and \( \mathcal{A}(\lambda, I - T)^{\mu} \) are closed subspaces of \( X \), the projection \( P_{n} \) of \( X \) onto \( \mathcal{A}^{\mu}(\lambda, I - T)^{\mu} \) along \( \mathcal{A}(\lambda, I - T)^{\mu} \) is continuous; see Lemma 2.2.16. Since the inclusion \( \mathcal{A}(\lambda, I - T)^{\mu} \subset \mathcal{A}(\lambda, I - T)^{\mu} \) holds when \( n \neq k \), the mapping defined by

\[
P_{n} = \sum_{i=1}^{m} P_{n, i}
\]

is the continuous projection of \( X \) onto \( \bigoplus_{n=1}^{m} \mathcal{A}^{\mu}(\lambda, I - T)^{\mu} \) along \( \bigcap_{n=1}^{m} \mathcal{A}(\lambda, I - T)^{\mu} \). The continuity of \( P \) implies that these spaces are closed; see again Lemma 2.2.16.

Observe that \( X \) has to be a Banach space if we want to use this technique (the projections \( P_{i} \) have to be continuous).

Step 2. We prove that \( \mathcal{A}_{n=1}^{m} \langle \lambda, I - T \rangle^{\mu} \) and \( \bigcap_{n=1}^{m} \mathcal{A}(\lambda, I - T)^{\mu} \) are closed, invariant subspaces of \( T \).

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Observe that \( X \) has to be a Banach space if we want to use this technique (the projections \( P_{i} \) have to be continuous).
2.4 On mappings of the monotone type

There exist a number of different monotonicity notions, but here we restrict ourselves to the notions of monotonicity in the sense of Kachurovskii, and the generalized monotonicity concepts of Browder, Brezis, Minty, Webb, Calvert, Hess, and Skrypnik.

The concept of monotonicity is relatively old, for as early as 1935 M. Golomb [80, pp. 66–72] used monotonicity conditions for operators of a Hilbert space when he examined a nonlinear Hammerstein integral equation. In 1960, R. I. Kachurovskii [105] introduced the concept of monotonicity for mappings which map a Banach space into its dual space. According to Kachurovskii, a mapping $F:X \rightarrow X^*$ is monotone if and only if it satisfies the condition

$$\langle Fx - Fy, x - y \rangle \geq 0 \quad \forall x, y \in X.$$

The systematic study of monotone mappings (and mappings of the monotone type) initiated in the mid 1960s when Browder and Minty derived theorems on characteristic features of these mappings and applied these results to nonlinear functional equations; see [29, 35, 36, 39, 136, 137]. In the late 1960s and the beginning of 1970s, the monotonicity condition was modified in different ways, and various classes of the mappings of the monotone type such as pseudo-monotone, quasi-monotone, and the mappings of class $\alpha$ and $(S_\alpha)$ were introduced. Since mappings satisfying monotonicity conditions apply well to certain partial differential equations, but some partial differential equations induce single-valued operators whose domain is not the whole space, there is nowadays a tendency to generalize these monotonicity conditions to mappings which are not everywhere defined and perhaps multivalued. Since the space of infinitely differentiable functions with compact support is dense in Sobolev spaces, the mappings that are densely defined are of special interest.

Since the handling of the definitions and the basic properties of monotone mappings and mappings of the monotone type is intended to be concise, below is a brief list of books and surveys for further reference and for the reader’s convenience.

The interested reader will find Zeidler [195–199], Deimling [61], Skrypnik [167, 169, 170], Joshi & Bose [104], Pascali [145] valuable. In those books, there is an abundance of theoretical results, exercises and applications concerning these mappings and also other monotonicity conditions than those presented in this thesis. Also Barbu [10]; Borwein & Lewis [23]; Brezis [25]; Cioranescu [53]; Doležal [65]; Gasiński & Papageorgiou [78]; Hu & Papageorgiou [96, 97]; Hyers, Isac & Rassias [101]; Jeggle [103]; Kachurovskii [106]; Phelps [150]; Vainberg [181, 182]; and Volpert, Volpert & Volpert [184] contain useful information. Purely theoretical results on monotone mappings and mappings of the monotone type are derived in Browder and Hess’ articles [36, 39, 47]. In Petryshyn’s book [147] the mappings of the monotone type are investigated using, or are used in the connection of various “approximation proper” (A-proper) conditions.

There is also an exceptional article [187] by Wille; it is written in German and considers monotone operators, but the whole article is written as a poem. It is quite a refreshing experience to read such an article.

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There is also an exceptional article [187] by Wille; it is written in German and considers monotone operators, but the whole article is written as a poem. It is quite a refreshing experience to read such an article.
2.4.1 Monotonicity conditions and duality mapping

This subsection contains the definitions of the monotonicity conditions and the results that are needed later in this thesis.

Mappings of monotone type

Below we list some of the most familiar monotonicity conditions and some other useful definitions which are used in the connection of mappings of the monotone type. Many of the definitions are not needed in this thesis, but are included in here just for completeness. In this thesis, pseudo-monotonicity and the conditions \((S)_1\), \((S)_0\), \((S)\), and \((M)\) are not used.

Definition 2.4.1. Let \(X\) be a reflexive Banach space, \(D\) some subset of \(X\), \((u_n)_{n=1}^{\infty} \subset D\), and \(A : D \subset X \rightarrow X^*\). Then the mapping \(A\) is

1. monotone if \(\langle Ax - Ay, x - y \rangle \geq 0\) for all \(x, y \in D\), and strictly monotone if the equality holds only when \(x = y\).
2. of class \((S)_+\) if \(u_n \rightharpoonup u\) in \(X\) and \(\lim_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0\) yield \(u_n \rightharpoonup u\).
3. of class \((S)_0\) if \(u_n \rightharpoonup u\) in \(X\) and \(\lim_{n \to \infty} \langle Au_n, u_n - u \rangle = 0\) imply \(u_n \rightharpoonup u\).
4. of class \((S)\) if \(u_n \rightharpoonup u\) in \(X\), \(Au_n \rightharpoonup b\) in \(X^*\), and \(\lim_{n \to \infty} \langle Au_n, u_n \rangle = \langle b, u \rangle\) give \(u_n \rightharpoonup u\).
5. of class \((S)_1\) if \(u_n \rightharpoonup u\) in \(X\) and \(Au_n \rightharpoonup b\) in \(X^*\) yield \(u_n \rightharpoonup u\).
6. pseudo-monotone if from the conditions \(u_n \rightharpoonup u\) and \(\lim_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0\), it follows that \(\langle Au_n, u_n - u \rangle \rightarrow 0\). Moreover, if \(u \in D\), then \(Au_n \rightharpoonup Au\).
7. quasi-monotone if \(a_n \rightharpoonup a\) implies \(\lim_{n \to \infty} \langle Au_n, u_n - u \rangle \geq 0\).
8. of class \((M)\) if \(u_n \rightharpoonup u\) in \(X\), \(Au_n \rightharpoonup y\) in \(X^*\), and \(\lim_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle y, u \rangle\) yield \(Au \rightharpoonup y\).

Instead of saying “a mapping is of class \((S)_+\)” we usually say that “it is an \((S)_+\)-mapping” or “a mapping satisfies the condition \((S)_+\)” or “a mapping has the property \((S)_+\)” throughout; furthermore, we sometimes say that a mapping satisfies a condition in (or on), or has a property in (or on) a certain set because want to stress that the mapping satisfies that condition in that set—not necessarily in the whole domain.

Remark 2.4.2. 1. In Zeidler’s book [199, Definition 27.5, p. 586] and Hess’ articles [88, 89], pseudo-monotone mappings are called mappings satisfying the condition (\(P\)) but in this thesis we prefer the term quasi-monotone. The condition (\(P\)) was for the first time used in Hess [88] and in Calvert & Webb [51].
2. Quasi-monotonicity has another equivalent definition: A is quasi-monotone if and only if \(a_n \rightharpoonup a\) implies \(\lim_{n \to \infty} \langle Au_n, u_n - u \rangle \geq 0\). For the proof, see Berkovits’ Ph.D. thesis [14, Lemma 1.1, pp. 13–14].
3. For the condition \((S)_1\) there is different terminology in the Russian literature. Note that these mappings are more general than mappings that satisfy the condition \((S)_1\) everywhere.

Let \(X\) be a reflexive Banach space, \(F \subset D \subset X\), and \(A : D \subset X \rightarrow X^*\). The mapping \(A\) is...
if a mapping possesses the property \( D \), we have the following implications and also some perturbation results; these facts are needed in Chapter 4. The follow-

4. The notion of the pseudo-monotonicity and the condition \((M)\) were first used by Brezis in article [24, pp. 123–124] in 1968, though he used filters instead of sequences in the definition of pseudo-monotonicity; see [24, p. 132]. Browder defined the condition \((S)\) in paper [38] that appeared in 1968; also the conditions \((S), (S)_{0}\), and \((S)_{1}\) were used in that article, but their earlier history is unknown to the author. The concepts of semi-continuity and hemi-continuity were first used and studied by Browder, Minty, and Kato in articles [29–35, 112, 136, 137] in years 1963–1964.

5. Usually the monotonicity conditions are applied to and connected with nonlinear mappings. Linear mappings with monotonicity conditions have certain desirable properties. For example, de Figueiredo & Gupta [59, Proposition 1, p. 39] proved that \( X \) is a real Banach space, then a linear mapping \( T \) from \( X \) to \( X^* \) is continuous if and only if it satisfies the condition \((M)\). See also Joshi & Bose [104, Proposition 3.6.2, pp. 78–79].

6. The condition \((S)_{1}\) can be modified so that it applies to mappings \( X \to Y \). Suppose that \( X \) and \( Y \) are Banach spaces, \((u_{n})_{n=1}^{\infty} \subset D \subset X\), and \( A: D \subset X \to Y \). Then \( A \) is of class \((S)_{1}\), if \( u_{n} \to u \) in \( X \) and \( Au_{n} \to b \) in \( Y \) yield \( u \to u \).

7. If \( X \) is reflexive, then a demi-continuous mapping \( A : X \to Y \) satisfying the condition \((S)_{1}\) is proper on closed, bounded subsets of \( X \), that is, for any closed, bounded set \( C \subset X \) and any compact set \( K \subset Y \), the set \( A^{-1}(K) \cap C \) is compact. This assertion is seen as follows:

\[ \text{Suppose that } (u_{n})_{n=1}^{\infty} \subset A^{-1}(K) \cap C. \text{ Then the sequence } (y_{n})_{n=1}^{\infty}, \text{ where } y_{n} = Ax_{n}, \text{ lies in the compact set } K \text{ and thus we may assume that } \exists x_{n} \to x \in K. \text{ Furthermore, by using the boundedness of } C \text{ and the reflexivity, we may suppose that } x_{n} \to x. \text{ Now } x_{n} \to x \in K \text{ because } A \text{ has the property } (S)_{1} \text{ and } C \text{ is closed. Using the demi-continuity of } A, \text{ we conclude that } Ax_{n} \to Ax. \text{ Since the weak limit is unique and } Ax_{n} \to y \in K, \text{ we have } Ax = y \in K. \text{ So } x \in A^{-1}(K) \cap C \text{ as required.} \]

We need some facts about the relationships between the different monotonicity conditions and also some perturbation results; these facts are needed in Chapter 4. The following theorem illustrates certain relations of the monotonicity conditions.

**Theorem 2.4.3.** We have the following implications \((S)_{1} \Rightarrow (S) \Rightarrow (S)_{0} \Rightarrow (S)_{1}\), that is, if a mapping possesses the property \((S)_{1}\) in a set, then it possesses also the property \((S)\) in that set etc. If a mapping is compact in a set, then it is quasi-monotone in that set. If the mapping is semi-continuous and satisfies the condition \((S)_{1}\) in a set, then it is quasi-monotone in that set.

**Proof.** All other assertions bar the last two claims are evident in the light of the definitions.

The second last claim is proved as follows. Suppose that \( A \) is a compact mapping in a set \( D \), \((u_{n})_{n=1}^{\infty} \subset D \), and \( u_{n} \to u_{0} \). We have to prove that \( \lim_{n \to \infty} (Au_{n}, u_{n} - u_{0}) \geq 0 \).

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shall use the subsequence argument to the sequence \( (\langle A_n, u_n - u_0 \rangle)_{n=1}^{\infty} \) and therefore we may drop to a subsequence whenever needed. Since \( A \) is compact, we may assume that \( A_n \to w \) in which case
\[
\lim_{n \to \infty} \langle A_{n}, u_n - u_0 \rangle = 0.
\]
So any subsequence of \( (\langle A_n, u_n - u_0 \rangle)_{n=1}^{\infty} \) has a subsequence having 0 as limit. Now the claim follows from the subsequence argument.

Theorem 2.4.4. Suppose that all the mappings below are demi-continuous and map \( X \) into \( X^* \). Then

1. if a mapping satisfies the condition \( S_+ \) on a set, and if another mapping is quasi-monotone in that set, then their sum satisfies the condition \( S_+ \) in that set.
2. if two mappings satisfy the condition \( S_+ \) on the same set, then their sum has the property \( S_+ \) in that set.
3. the sum of a compact mapping and an \( (S_+) \)-mapping is an \( (S_+) \)-mapping.

Proof. Assertion 1. is evident, and 2. follows from this and Theorem 2.4.3. The proof of the third claim follows from assertion 1. of this theorem, and the observation that a mapping is quasi-monotone in a set where it is compact.

Later in this thesis, we want to apply a degree theory for families of mappings. The next condition is for families of mappings; it is used for the so-called admissible homotopies for our degree function.

Definition 2.4.5. The family \( \{ A_t : t \in [0,1] \} \) of mappings \( A_t : D \subset X \to X^* \) satisfies the condition \( S_+ \) on the set \( D \) if the conditions \( (u_n)_{n=1}^{\infty} \subset D, (t_n)_{n=1}^{\infty} \subset [0,1], \)
\[
u_n \to u_0, \quad t_n \to t_0, \quad \text{and} \quad \lim_{n \to \infty} \langle A_t u_n, u_n - u_0 \rangle \leq 0, \quad (2.9)
\]
imply \( u_n \to u_0 \).

Remark 2.4.6. Skrypnik used the above condition but named it \( \alpha' \) (or \( \alpha'(D) \)) in his monograph [167, p. 129] that was published in 1973. In books [169, Definition 1.3.1, p. 22] and [170, Definition 4.1, p. 42], Skrypnik considered a different kind of condition for homotopies; that condition is named \( \alpha'_D(F) \) and it reads as follows.

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Let $X$ be a real, reflexive Banach space, $F \subset D \subset X$ and $A_t: D \subset X \to X^*$. The family $\{A_t : t \in [0,1]\}$ is of class $\alpha^{(l)}_0(F)$ if from $(t_u)_{u \to t} \subset [0,1]$, $(u)_{u \to t} \subset F$, $u_u \to u$, $A_u u_u \to 0$, and $\lim_{u \to t}(A_u u_u - u) = 0$ follows that $u_u \to u$.

That condition seems to be suitable for the homotopies between bounded mappings, and guarantees that the class of admissible homotopies is very large; however, the condition $\alpha^{(l)}_0(F)$ is too general for our purposes.

**Definition 2.4.7.** Let $A^0: \overline{D} \subset X \to X^*$ and $A^1: \overline{D} \subset X \to X^*$ be semi-continuous mappings satisfying the condition $(S_o)$ on $\partial D$ where $D \subset X$ is an open, bounded set. Then the mappings $A^0$ and $A^1$ are called homotopic with respect to the set $D$ if there is a parameter family $\{A_t : t \in [0,1]\}$ of mappings $A_t: \overline{D} \subset X \to X^*$ meeting the requirements:

1. $A^0 = A_0$ and $A^1 = A_1$;
2. the relation $A_t \neq 0$ holds for all $u \in \partial D$ and $t \in [0,1]$;
3. the family $\{A_t : t \in [0,1]\}$ satisfies the condition $(S_o)$ on $\partial D$;
4. the conditions $t_n \to t_0$ and $u_n \to u_0$ imply $A_t u_n \to A_t u_0$.

The family $\{A_t : t \in [0,1]\}$ having these properties is called an admissible homotopy of $A_0$ and $A_1$.

For theoretical purposes and applications, one of the most applicable families of mappings are the so-called affine homotopies.

**Lemma 2.4.8.** Assume that $A^0: \overline{D} \subset X \to X^*$ and $A^1: \overline{D} \subset X \to X^*$ are semi-continuous mappings satisfying the condition $(S_o)$ on the closed set $\overline{D}$. Then the affine homotopy

$$A_{\lambda u} := \lambda A^0 u + (1-\lambda)A^1 u$$

satisfies the condition $(S_o)$ on $\overline{D}$.

*Proof.* Suppose that the sequences $(t_u)_{u \to t} \subset [0,1]$ and $(u)_{u \to t} \subset D$ are such that $t_u \to t_0$, $u_u \to u_0$, and

$$0 \geq \lim_{n \to \infty} (A_{t_n} u_{t_n} - u) = \lim_{n \to \infty} (t_u A^0 u_u + (1-t_u)A^1 u_u - u_0).$$

Here we can use the subsequence argument, and therefore we can assume that one of the conditions

$$\lim_{n \to \infty} (A^0 u_n - u_0) \leq 0 \quad \text{and} \quad \lim_{n \to \infty} (A^1 u_n - u_0) \leq 0$$

holds. If the first inequality is true, then the $(S_o)$-property of $A^0$ yields that $u_n \to u_0$.

If the second limit holds, the same conclusion follows. Now the assertion is a direct consequence of the subsequence argument.

**Duality mapping**

We now turn our attention to duality mapping. The notion of duality mapping was first introduced by Breuer & Livingston in article [18] in 1962, and it was extensively studied by Browder [36, 37], Asplund [9], and Kachurovskii [106] in the last half of 1960s.

Duality mapping is usually multivalued.
Definition 2.4.9. The (normalized) duality mapping \( J : X \to 2^{X^*} \) of \( X \) is defined by
\[ J u = \{ w \in X^* \mid \langle w, u \rangle = \| u \|_2, \| w \|_* = \| u \| \} \] (2.10)
for all \( u \in X \).

Remark 2.4.10. It follows from the Hahn-Banach theorem that \( J u \neq \emptyset \) for all \( u \in X \). A duality mapping is completely determined by Equation (2.10) but observe that if we consider a different norm in \( X \) or in \( X^* \), then we get a different duality mapping.

We shall list some properties of duality mapping which are needed for the normalization property of the degree function and the derivation of the index theorem. As we shall see in the next theorem, many properties of duality mapping are connected with the geometric properties of the norms of \( X \) and \( X^* \).

Theorem 2.4.11. Let \( X \) be a Banach space and \( J : X \to 2^{X^*} \) its duality mapping. Then
1. \( J \) is a single-valued mapping if and only if the norm of the space \( X^* \) is strictly convex.
2. \( X \) is reflexive if and only if
\[ X^* = \bigcup_{u \in X} J u. \]
3. if the norm of \( X \) is locally uniformly convex, then \( J \) is continuous; furthermore, if also the norm of \( X^* \) is locally uniformly convex, then \( J^{-1} \) is continuous.
4. if the norm of \( X \) is strictly convex, then \( J \) is monotone and demi-continuous. Moreover, \( J \) is strictly monotone if the norm of \( X \) is strictly convex.
5. if the norms of \( X \) and \( X^* \) are locally uniformly convex, then \( J \) satisfies the condition \((S_1)\).


Duality mapping is also studied in the books by Cioranescu [53] and Zeidler [199, Section 32.3d, pp. 860–866].

Remark 2.4.12. It is known that there is a normalizing mapping for the ordinary \((S_1)\) degree if the norms of a Banach space and its dual space are locally uniformly convex; therefore, to ensure the existence of such a mapping, one assumes the norms to be locally uniformly convex. Since the main application of the degree theory is connected with solvability results of boundary value problems, the spaces under consideration are usually Sobolev spaces with norms that are locally uniformly convex, and hence that assumption is a minor restriction. Another justification for such an assumption is that a reflexive Banach space can be given an equivalent norm which is locally uniformly convex and causes the norm of the dual space to be locally uniformly convex; see Theorem 2.2.12.

On the other hand, we do not need to make such assumptions. According to Theorem 2.2.12, any reflexive Banach space \( X \) with a norm \( \| \cdot \|_1 \) has an equivalent norm \( \| \cdot \|_2 \) such

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that both it and the corresponding norm of $X^*$ are locally uniformly convex. Theorem 2.4.11 implies that the duality mapping $J_2$ of the space $(X, \| \cdot \|_2)$ is a continuous mapping satisfying the condition $(S_+)$. Since $\| \cdot \|_1$ and $\| \cdot \|_2$ induce the same norm topology and define the same dual space, a sequence converges weakly or converges strongly at the same time in both spaces. So we conclude that $J_2$ is continuous, and it satisfies the condition $(S_+)$ when it is considered as a mapping from $(X, \| \cdot \|_1)$ to $(X, \| \cdot \|_1)^*$.

Remark 2.4.13. Note that in some contexts it is claimed that if $X$ is a real Hilbert space, the duality mapping given by the inner product of $X$ is in fact the identity mapping of $X$, but strictly speaking that is not the case.

Although the Frechet-Riesz theorem states that for each $x^* \in X^*$ there is the unique $x \in X$ such that $\| x^* \| = \| x \|$ and $x^* (x) = \langle x, x \rangle$ for every $x \in X$, it does not mean that $X^* = X$, it only says that $X^*$ and $X$ are isometrically isomorphic (in the case of a complex Hilbert space the isometry is ”conjugate-linear”; see Taylor & Lay’s book [178, pp. 142 and 242–246]).

However, duality mapping has nice properties in Hilbert spaces and this connected with the Frechet-Riesz theorem. For example, one can show that duality mapping $J$ is linear if and only if $X$ is a real Hilbert space, and that the inverse mapping $J^{-1}$ of $J$ is the isomorphism given by the Frechet-Riesz theorem; see Berkovits’ Ph.D. thesis [14, Remark 1.7, p. 16]. See also Remark 2.2.19 and Cioranescu [53, Proposition 4.8, p. 29].

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3 Topological degree for mapping of type $(S_+)$

In this chapter, we present a topological degree for a class of mappings of the monotone type. The mappings considered satisfy the condition $(S_+)$, but here the mappings need not to be bounded and the space need not be separable. This chapter is organized as follows. Section 3.1 conveys some information on the history of the degree theory of mappings of the monotone type, and in Section 3.2 we derive a topological degree for $(S_+)$-mappings. The classical properties of this degree are shown in Section 3.3, and there is a short discussion on the other constructions of the degree function in Section 3.4.

3.1 Background

Ever since its construction, Brouwer’s degree has been successfully applied to numerous problems and has been generalized to various infinite-dimensional spaces. The first such generalization is the Leray-Schauder degree that has also been successfully applied to functional equations, but as the class of mappings for which the degree is defined is frankly narrow, there is a need for degree theories for more extensive classes of mappings. Several generalizations of the Leray–Schauder degree were introduced in the 1960s and 1970s: the degree of condensing mappings, the degree of $k$-set contractions, the degree of $A$-proper mappings, and some classes of multivalued mappings. See the books by Deimling [60, 61]; Hu & Papageorgiou [96]; Joshi & Bose [104]; Kamenskii, Obukhovskii & Zecca [107]; Krasnoselskii & Zabreiko [119]; Krawcewicz & Wu [120]; Lloyd [127]; Mortici [138]; O’Regan, Cho & Chen [144]; Petryshyn [147, 148]; or Zeidler [196, pp. 600–612], [199, pp. 997–1008] for details. Survey [189] by Zabreiko is recommended for the reader who wants to attain a clear image of the various degree theories quickly.

Mappings of the monotone type arise naturally from elliptic partial differential equations, and therefore mathematicians have applied various degree theories for these operators. Perhaps the first successful application was obtained by Browder in the late 1960s. In publication [40, Chapter 17, pp. 273–284], Browder showed that bounded, continuous $(S)$-mappings $X \to X^*$ are $A$-proper mappings provided that $X$ and $X^*$ are separable, reflexive Banach spaces with the so-called injective approximation scheme, and he used the multivalued topological degree of $A$-proper mappings to derive certain theorems for

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In this chapter, we present a topological degree for a class of mappings of the monotone type. The mappings considered satisfy the condition $(S_+)$, but here the mappings need not to be bounded and the space need not be separable. This chapter is organized as follows. Section 3.1 conveys some information on the history of the degree theory of mappings of the monotone type, and in Section 3.2 we derive a topological degree for $(S_+)$-mappings. The classical properties of this degree are shown in Section 3.3, and there is a short discussion on the other constructions of the degree function in Section 3.4.

3.1 Background

Ever since its construction, Brouwer’s degree has been successfully applied to numerous problems and has been generalized to various infinite-dimensional spaces. The first such generalization is the Leray–Schauder degree that has also been successfully applied to functional equations, but as the class of mappings for which the degree is defined is frankly narrow, there is a need for degree theories for more extensive classes of mappings. Several generalizations of the Leray–Schauder degree were introduced in the 1960s and 1970s: the degree of condensing mappings, the degree of $k$-set contractions, the degree of $A$-proper mappings, and some classes of multivalued mappings. See the books by Deimling [60, 61]; Hu & Papageorgiou [96]; Joshi & Bose [104]; Kamenskii, Obukhovskii & Zecca [107]; Krasnoselskii & Zabreiko [119]; Krawcewicz & Wu [120]; Lloyd [127]; Mortici [138]; O’Regan, Cho & Chen [144]; Petryshyn [147, 148]; or Zeidler [196, pp. 600–612], [199, pp. 997–1008] for details. Survey [189] by Zabreiko is recommended for the reader who wants to attain a clear image of the various degree theories quickly.

Mappings of the monotone type arise naturally from elliptic partial differential equations, and therefore mathematicians have applied various degree theories for these operators. Perhaps the first successful application was obtained by Browder in the late 1960s. In publication [40, Chapter 17, pp. 273–284], Browder showed that bounded, continuous $(S)$-mappings $X \to X^*$ are $A$-proper mappings provided that $X$ and $X^*$ are separable, reflexive Banach spaces with the so-called injective approximation scheme, and he used the multivalued topological degree of $A$-proper mappings to derive certain theorems for
(S,"\text{\textprime}"")-mappings and pseudo-monotone mappings (see [40, Chapter 17, pp. 273–284]).

But (S,"\text{\textprime}"")-mappings have their own topological degree that is integer-valued. A method used in the derivation of the degree of (S,"\text{\textprime}"")-mappings (or α-mappings) has been presented in Skrypnik’s book [167, Section 3.2] as early as 1973. This is a fact that is seldom stated. A reason for this might be that in his book Skrypnik speaks about rotation of the vector field (вращение векторного поля) instead of a topological degree, and then the book is written in Russian. In 1982, Skrypnik [168, pp. 196–200] presented again the degree theory for mappings of class α, and in 1983, Browder [42] (see also [41, 44, 46]) derived a degree theory for (bounded) mappings satisfying the condition (S,"\text{\textprime}""). Both men justified the definitions of these degrees using Brouwer’s degree, but their constructions differ from each other at certain points. Yet another construction based on Brouwer’s degree is due to Zeidler [199, Problems 36.3–36.3b, pp. 1003–1004]. Unfortunately, each of these constructions contains some vagueness. In the mid 1980s, J. Berkovits & V. Mustonen presented an alternative construction that uses the Leray-Schauder degree and Browder-Tonelli’s embedding theorem; the details are found in Mustonen & Berkovits’ article [16], Berkovits’ Ph.D. thesis [14], and Mortici’s monograph [138, pp. 75–77].

The (S,"\text{\textprime}"")-degree has proved to be useful because it can be used in the construction of degree theories for other mappings of the monotone type. In 1983, Browder defined a degree theory for maximal monotone operators by using the (S,"\text{\textprime}"")-degree and the so-called Yosida approximation (see [43]), and in 1986, Berkovits also extended the (S,"\text{\textprime}"")-degree to quasi-monotone mappings (see [14]). The degree of quasi-monotone mappings is no longer classical, but is a so-called weak topological degree, and the method for this extension is based on the perturbation result: the sum of a bounded, demi-continuous (S,"\text{\textprime}"")-mapping and a demi-continuous, quasi-monotone mapping is again a semi-continuous (S,"\text{\textprime}"")-mapping.1 These weak degrees have the weakness that if the degree of a mapping A at y with respect to the set D is nonzero, then y ∈ A(D), that is, D might not contain solutions of Au = y after all. Moreover, the degree is defined only for an open, bounded set D, a point y, and a mapping A with y ∈ A(D). The last condition, which is usually hard to prove, has to be imposed because the image A(∂D) of ∂D might not be closed.

Typically the above mentioned degrees can be applied to everywhere defined and locally bounded mappings. Since certain partial differential equations generate mappings of the monotone type that are not everywhere defined or locally bounded, more general classes of mappings of the monotone type and degree theories for such mappings have been introduced.

1This result was proved by Calvert & Webb in 1971 (see [51]) when the semi-continuous (S,"\text{\textprime}"")-mapping is a duality mapping, and this result was used by Browder in paper [40, pp. 273–284] where he derived results for continuous, pseudo-monotone mappings using the degree of A-proper mappings. The idea of perturbation method was also known to Skrypnik in 1973 when he remarked that the (S,"\text{\textprime}"")-degree can be extended to the sum of a completely continuous mapping and a semi-continuous mapping with semi-bounded variation; see [167, p. 126]. In [170, p. 41], Skrypnik noted that this method applies to demi-continuous, pseudo-monotone mappings as well. However, these results are weaker than Berkovits & Mustonen’s result because a demi-continuous, pseudo-monotone mapping is quasi-monotone; see Berkovits [14, p. 13] and Berkovits & Mustonen [17, Proposition 3, p. 13].
3.2 Construction of the degree function

In this section, we construct the degree theory for semi-continuous \((S_\alpha)\)-mappings although there are many different constructions for the degree of ordinary semi-continuous \((S_\alpha)\)-mappings.

Proposition 3.2.4 is a slight variation of a result that concerns \(\alpha\)-mappings and that has already been presented in Skrypnik’s book [167, Lemma 7, p. 125]. It is the paramount result for the construction of the degree function. The original result with its proof appears also in Skrypnik’s books [169, Theorem 1.2.1, p. 17] and [170, Theorem 3.1, p. 39]. Similar argumentation is useful also for other mappings of the monotone type; see Browder [44, p. 28], Browder & Hess [47, pp. 272, 278, and 290] (pseudo-monotone mappings), Hess [89, pp. 141, 142–143], Kartsatos & Skrypnik’s article [108, Lemma 3.1, pp. 423–427, and Lemma 4.3, pp. 433–436], Kenmochi [113, pp. 239–240] (mappings of type \(M\)), Kobayashi & Ôtani’s article [114, proof of Lemma 3.12, p. 161] (maximal monotone perturbations of an \((S_\alpha)\)-mapping), and O’Regan, Cho & Chen [144, Lemma 6.2.3, pp. 144–145] (multivalued \((S_\alpha)\)-mappings). Also we use an analogous argumentation twice in the proof of Theorem 4.3.1. Before we state or prove Proposition 3.2.4, we need some theoretical framework.

The so-called finite intersection property is connected with the compactness of sets, but there are different definitions for it. The next definition is used by Willard [186, Definition 17.3, p. 117], Royden [158, p. 158], and Bishop & Goldberg [19, p. 16].

Definition 3.2.1. A family \(\mathcal{E}\) of sets has the finite intersection property if the intersection of any finite subcollection of sets from \(\mathcal{E}\) is nonempty.

A characterization of the compactness can be given in terms of closed subsets and their finite intersections.

Theorem 3.2.2. A topological space \(X\) is compact if and only if every family \(\mathcal{E}\) of closed subsets of \(X\) with the finite intersection property has nonempty intersection.

Proof. See Willard [186, Theorem 17.4, p. 119], Royden [158, Proposition 1], or Bishop & Goldberg [19, Proposition 0.2.8.1, p. 16].

Notation 3.2.3. The set of finite-dimensional subspaces of \(X\) is denoted by \(F(X)\).

Proposition 3.2.4. Assume that \(K \subseteq \mathcal{T} \subset X\) where \(D\) is an open subset and \(K\) is a closed, bounded subset of a reflexive space \(X\). Let the family \(\{A_t \mid t \in [0, 1]\}\) consist of mappings \(A_t : \mathcal{T} \subset X \to \mathbb{X}\) with the properties

1. \(\{A_t \mid t \in [0, 1]\}\) satisfies the condition \((S_\alpha)\) on the set \([0, 1] \times K\),
2. if the sequences \((t_n)_n\) and \((u_n)_n\) in \(D\) are such that \(t_n \to t_0\) and \(u_n \to u_0\), then \(A_{t_n}u_n \to A_{t_0}u_0\),
3. \(A_{t_0}u \neq 0\) for all \(t \in [0, 1]\) and \(u \in K\).

Then there exists \(F_0 \in F(X)\) such that

\[
Z(F_0, \mathcal{T}) := \{(u, t) \in K \cap F \times [0, 1] \mid \langle A_t u, v \rangle \leq 0 \text{ and } \langle A_t u, v \rangle = 0 \text{ for all } v \in F_0\} = \emptyset
\]
for any $F \in \mathcal{F}(X)$ with $F_0 \subset F$.

The plan of the proof is the following: We suppose that the claim is not true and construct a family of nonempty bounded sets whose weak closures have the finite intersection property. Then we conclude that the intersection of those weak closures has at least one element in common. The proof culminates in the contradiction that is achieved by showing that any of the common elements is a zero of the family $\{A_t \mid t \in [0,1]\}$ in the set $K \times [0,1]$.  

**Proof.** Step 1. We set the stage for the rest of the proof.

To prove the claim, we suppose the contrary: for any $F_0 \in \mathcal{F}(X)$ there is some subspace $F_1 \in \mathcal{F}(X)$ with $F_1 \supset F_0$ and $Z(F_0,F_1) \neq \emptyset$. For each $F_0$ we define

$$G(F_0) := \bigcup_{F \in \mathcal{F}(X)} Z(F_0,F)$$

and let $w-cl(G(F_0))$ be the weak closure of $G(F_0)$ in $X \times \mathbb{R}$. Observe that $G(F_0) \neq \emptyset$ for every $F_0 \in \mathcal{F}(X)$.

Step 2. We prove that the family

$$\{w-cl(G(F)) \mid F \in \mathcal{F}(X)\}$$

has the finite intersection property.  

We first take a finite selection $F_1,\ldots,F_I$ of finite-dimensional subspaces of $X$ and let $F'$ be the sum of these subspaces. Note that $Z(F_i,F_i) \supset Z(F',F_i)$ for all $i = 1, \ldots, I$ and all $F \in \mathcal{F}(X)$ with $F \supset F'$. According to the definition of $G(F)$, we have the inclusion $G(F') \subset G(F)$ for every $i = 1, \ldots, I$ and

$$\bigcap_{1 \leq i \leq I} w-cl(G(F_i)) \supset \bigcap_{1 \leq i \leq I} G(F_i) \supset G(F') \neq \emptyset$$

which shows that the assertion of this step is true.

Step 3. We show that the sets $w-cl(G(F))$, where $F \in \mathcal{F}(X)$, have at least one element in common.

It follows from the boundedness of $K$ that the sets $w-cl(G(F))$ are weakly closed subsets of $\overline{B(0,K)} \subset X \times \mathbb{R}$ with some $K > 0$. As $X$ is reflexive, the space $X \times \mathbb{R}$ is reflexive (use item 2 of Theorem 2.2.6) and thus $\overline{B(0,K)}$ is weakly compact (see Theorem 2.2.1). Because the family $\{w-cl(G(F)) \mid F \in \mathcal{F}(X)\}$ has the finite intersection property and the sets $w-cl(G(F))$ are weakly closed subsets of the weakly compact set $\overline{B(0,K)}$, Theorem 3.2.2 implies the existence of an element $(x_0,t_0) \in X \times \mathbb{R}$ with

$$(x_0,t_0) \in \bigcap_{F \in \mathcal{F}(X)} w-cl(G(F)).$$

Step 4. Our claim is that the element $x_0$ belongs to $K$ and satisfies $A_{x_0}t_0 = 0$. These results violate the conditions of the theorem and therefore they constitute a desired contradiction.

for any $F \in \mathcal{F}(X)$ with $F_0 \subset F$.

The plan of the proof is the following: We suppose that the claim is not true and construct a family of nonempty bounded sets whose weak closures have the finite intersection property. Then we conclude that the intersection of those weak closures has at least one element in common. The proof culminates in the contradiction that is achieved by showing that any of the common elements is a zero of the family $\{A_t \mid t \in [0,1]\}$ in the set $K \times [0,1]$.  

**Proof.** Step 1. We set the stage for the rest of the proof.

To prove the claim, we suppose the contrary: for any $F_0 \in \mathcal{F}(X)$ there is some subspace $F_1 \in \mathcal{F}(X)$ with $F_1 \supset F_0$ and $Z(F_0,F_1) \neq \emptyset$. For each $F_0$ we define

$$G(F_0) := \bigcup_{F \in \mathcal{F}(X)} Z(F_0,F)$$

and let $w-cl(G(F_0))$ be the weak closure of $G(F_0)$ in $X \times \mathbb{R}$. Observe that $G(F_0) \neq \emptyset$ for every $F_0 \in \mathcal{F}(X)$.

Step 2. We prove that the family

$$\{w-cl(G(F)) \mid F \in \mathcal{F}(X)\}$$

has the finite intersection property.  

We first take a finite selection $F_1,\ldots,F_I$ of finite-dimensional subspaces of $X$ and let $F'$ be the sum of these subspaces. Note that $Z(F_i,F_i) \supset Z(F',F_i)$ for all $i = 1, \ldots, I$ and all $F \in \mathcal{F}(X)$ with $F \supset F'$. According to the definition of $G(F)$, we have the inclusion $G(F') \subset G(F)$ for every $i = 1, \ldots, I$ and

$$\bigcap_{1 \leq i \leq I} w-cl(G(F_i)) \supset \bigcap_{1 \leq i \leq I} G(F_i) \supset G(F') \neq \emptyset$$

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It follows from the boundedness of $K$ that the sets $w-cl(G(F))$ are weakly closed subsets of $\overline{B(0,K)} \subset X \times \mathbb{R}$ with some $K > 0$. As $X$ is reflexive, the space $X \times \mathbb{R}$ is reflexive (use item 2 of Theorem 2.2.6) and thus $\overline{B(0,K)}$ is weakly compact (see Theorem 2.2.1). Because the family $\{w-cl(G(F)) \mid F \in \mathcal{F}(X)\}$ has the finite intersection property and the sets $w-cl(G(F))$ are weakly closed subsets of the weakly compact set $\overline{B(0,K)}$, Theorem 3.2.2 implies the existence of an element $(x_0,t_0) \in X \times \mathbb{R}$ with

$$(x_0,t_0) \in \bigcap_{F \in \mathcal{F}(X)} w-cl(G(F)).$$

Step 4. Our claim is that the element $x_0$ belongs to $K$ and satisfies $A_{x_0}t_0 = 0$. These results violate the conditions of the theorem and therefore they constitute a desired contradiction.
We pick an arbitrary element \( w \) of \( X \) and then let \( F_n^w \in F(X) \) be such that \( u_n, w \in F_n^w \). Here the superindices \( w \) indicates that \( F_n^w \) is connected with \( w \). By Equation (3.2), we have \( (u_n, t_n^w) \in w \cdot \mathrm{cl}(G(F_n^w)) \). Since \( G(F_n^w) \) is a bounded set, we may use Proposition 2.2.7 (or the result stated in Remark 2.2.8) to obtain such a sequence \((u_n^w, t_n^w)_{n=1}^\infty \in G(F_n^w)\) that
\[
(u_n^w, t_n^w) \xrightarrow{\text{a.e.}} (u_0, t_0)
\]
in \( X \times \mathbb{R} \). Especially,
\[
u_n^w \xrightarrow{\text{a.e.}} u_0 \quad \text{in } X \quad \text{and} \quad t_n^w \xrightarrow{\text{a.e.}} t_0.
\]
If we take into account the observation
\[
Z(F_n^w, F) \subset Z(F_n^w, F^n) \quad \text{whenever} \quad F \subset F^n,
\]
and the definition of \( G(F_n^w) \), we conclude that there is a sequence \((F_n^w)_{n=1}^\infty \subset F(X) \) with
\[
(A_{w} u_n^w, v) \leq 0 \quad \text{and} \quad (A_{w} u_n^w, v) = 0 \quad \text{for all } v \in F_n^w.
\]
(3.3)
Because \( u_0 \in F_n^w \), it follows from (3.3) that
\[
(A_{w} u_n^w, u_n^w - u_0) \leq 0
\]
for all \( n \). This with the \((S_1)\)-property of \( A_t, t \in [0,1] \) forces \( u_n^w \) to converge strongly to \( u_0 \) in \( X \). Then \( u_0 \) is in \( K \) because \( K \) is closed. By using the continuity property of \( A \) in \( u \) and \( t \), we get \( A_{w} u_n^w \to A_{w} u_0 \). Especially,
\[
0 = (A_{w} u_n^w, w) \xrightarrow{\text{a.e.}} (A_{w} u_0, w)
\]
because \((u_n^w, t_n^w) \in Z(F_n^w, F^n)\) and \( w \in F_n^w \).

As \( w \in X \) was arbitrary in the previous inference and in any case the achieved result is 0 = \( (A_{w} u_0, w) \), the conclusion is that \( A_{w} u_0, w = 0 \) for every \( w \in X \). Thus, \( A_{w} u_0 = 0 \). Since \( u_0 \in K \), we have an outcome that violates the assumptions of this proposition, and the proof is now complete. \( \square \)

**Remark 3.2.5.** Although Skrypnik does not emphasize (see [170, pp. 39–40] and [169, pp. 17–19]) that \( A_t \) need not be bounded in the previous proposition, this fact was known to him in 1973. See Skrypnik’s book [167, Замечание 1, p. 125].

Given \( A : D \subset X \to X', F \in F(X) \) and a basis \( v_1, \ldots, v_m \) of \( F \), we define the finite-dimensional mapping \( A_F \) from \( D \cap F \) to \( F \) by
\[
A_F u = \sum_{i=1}^m (Au, v_i) v_i \quad \text{for all } u \in D \cap F.
\]
(3.4)
Remark 3.2.6. 1. Note that

\[ A_{F}u = \sum_{i=1}^{\lambda_0} (Au,v_i)v_i. \]

is not an approximation of \( A: A \) is a mapping \( X \to X' \) and \( A_{F} \) is a mapping \( F \to F \).

2. The demi-continuity of \( A \) guarantees the continuity of \( A_{F} \), and therefore we can calculate Brouwer’s degree of \( A_{F} \) at zero with respect to the set \( D \cap F \) provided that

\( D \cap F \) is bounded and \( A_{F} \neq 0 \) for all \( u \in \partial F \), where \( \partial D \) denotes the boundary of \( D \cap F \) in the relative topology of \( F \).

3. Assume that \( X \) is a topological space, \( F \) is a subspace of \( X \) with the relative topology, and \( D \) is a subset of \( X \). If \( \partial D \) denotes the boundary of \( D \cap F \) in \( F \) and \( \partial F \) the boundary of \( D \) in \( X \), then \( \partial D \subset F \cap \partial F \). This result is stated in Willard’s book [186, Example 6.4 b), p. 42], and we need it to show that \( A_{F} \) of \( A \) does not attain zero in \( \partial D \) if \( A \) has no zeros in \( \partial D \). Usually the subindex \( X \) is omitted and \( \partial D \) denotes the boundary of \( D \) in \( X \).

Since \( (A_{F})u = 0 \) for some \((u,t) \in K \cap F \times [0,1] \) implies that \( \langle A_{u},u \rangle = \langle A_{u},v \rangle = 0 \) for every \( v \in F \), we immediately infer the following corollary from Proposition 3.2.4 and Remark 3.2.6.

Corollary 3.2.7. Let \( A: \mathcal{T} \subset X \to X' \) satisfy the conditions of Proposition 3.2.4 with \( D \) bounded and \( K = \partial D \). Then there exists a finite-dimensional subspace \( F_0 \) of \( X \) such that \( (A_{F})u \neq 0 \) when \((u,t) \in \partial D \times [0,1] \) for any finite-dimensional subspace \( F \) of \( X \) containing \( F_0 \).

Theorem 3.2.8. Let \( D \subset X \) be a bounded, open set, \( A: \mathcal{T} \subset X \to X' \) be a demi-continuous mapping that satisfies the condition \( (S_{\alpha}) \) on \( \partial D \), and \( Au \neq 0 \) when \( u \in \partial D \). Assume that \( F_0 \in F(X) \) is as in Proposition 3.2.4, that is,

\[ Z(F_0,F) = \{ u \in \partial D \cap F \mid \langle Au,u \rangle \leq 0 \text{ and } \langle Au,v \rangle = 0 \text{ for all } v \in F_0 \} = \emptyset \]

for all \( F \in F(X) \) with \( F_0 \subset F \). Then

\[ \deg(A_{F},\mathcal{T} \cap F,0) = \deg(A_{F_0},\mathcal{T} \cap F_0,0) \]

for all \( F \in F(X) \) with \( F_0 \subset F \).

Proof. We let \( F \supset F_0 \) be a finite-dimensional subspace of \( X \) and then choose a basis for \( F \) of the form \( v_1, \ldots, v_{\lambda_0}, w_1, \ldots, w_{\mu} \), where \( v_1, \ldots, v_{\lambda_0} \) is a basis of \( F_0 \). We consider on \( \mathcal{T} \cap F \) two mappings

\[ A_{F}u = \sum_{i=1}^{\lambda_0} (Au,v_i)v_i + \sum_{i=1}^{\mu} (Au,w_i)w_i, \]

\[ A_{F}u = \sum_{i=1}^{\lambda_0} (Au,v_i)v_i + \sum_{i=1}^{\mu} f_{i,F}(u)w_i, \]

where \( f_{i,F} \) is an element of \( X' \) satisfying the conditions \( \langle f_{i,F},w_i \rangle = \delta_{ij} \) for \( j = 1, \ldots, \mu \) and \( \langle f_{i,F},v_k \rangle = 0 \) for \( k = 1, \ldots, \lambda_0 \).

Remark 3.2.6. 1. Note that

\[ A_{F}u = \sum_{i=1}^{\lambda_0} (Au,v_i)v_i. \]

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\( D \cap F \) is bounded and \( A_{F} \neq 0 \) for all \( u \in \partial F \), where \( \partial D \) denotes the boundary of \( D \cap F \) in the relative topology of \( F \).

3. Assume that \( X \) is a topological space, \( F \) is a subspace of \( X \) with the relative topology, and \( D \) is a subset of \( X \). If \( \partial D \) denotes the boundary of \( D \cap F \) in \( F \) and \( \partial F \) the boundary of \( D \) in \( X \), then \( \partial D \subset F \cap \partial F \). This result is stated in Willard’s book [186, Example 6.4 b), p. 42], and we need it to show that \( A_{F} \) of \( A \) does not attain zero in \( \partial D \) if \( A \) has no zeros in \( \partial D \). Usually the subindex \( X \) is omitted and \( \partial D \) denotes the boundary of \( D \) in \( X \).

Since \( (A_{F})u = 0 \) for some \((u,t) \in K \cap F \times [0,1] \) implies that \( \langle A_{u},u \rangle = \langle A_{u},v \rangle = 0 \) for every \( v \in F \), we immediately infer the following corollary from Proposition 3.2.4 and Remark 3.2.6.

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Theorem 3.2.8. Let \( D \subset X \) be a bounded, open set, \( A: \mathcal{T} \subset X \to X' \) be a demi-continuous mapping that satisfies the condition \( (S_{\alpha}) \) on \( \partial D \), and \( Au \neq 0 \) when \( u \in \partial D \). Assume that \( F_0 \in F(X) \) is as in Proposition 3.2.4, that is,

\[ Z(F_0,F) = \{ u \in \partial D \cap F \mid \langle Au,u \rangle \leq 0 \text{ and } \langle Au,v \rangle = 0 \text{ for all } v \in F_0 \} = \emptyset \]

for all \( F \in F(X) \) with \( F_0 \subset F \). Then

\[ \deg(A_{F},\mathcal{T} \cap F,0) = \deg(A_{F_0},\mathcal{T} \cap F_0,0) \]

for all \( F \in F(X) \) with \( F_0 \subset F \).

Proof. We let \( F \supset F_0 \) be a finite-dimensional subspace of \( X \) and then choose a basis for \( F \) of the form \( v_1, \ldots, v_{\lambda_0}, w_1, \ldots, w_{\mu} \), where \( v_1, \ldots, v_{\lambda_0} \) is a basis of \( F_0 \). We consider on \( \mathcal{T} \cap F \) two mappings

\[ A_{F}u = \sum_{i=1}^{\lambda_0} (Au,v_i)v_i + \sum_{i=1}^{\mu} (Au,w_i)w_i, \]

\[ A_{F}u = \sum_{i=1}^{\lambda_0} (Au,v_i)v_i + \sum_{i=1}^{\mu} f_{i,F}(u)w_i, \]

where \( f_{i,F} \) is an element of \( X' \) satisfying the conditions \( \langle f_{i,F},w_i \rangle = \delta_{ij} \) for \( j = 1, \ldots, \mu \) and \( \langle f_{i,F},v_k \rangle = 0 \) for \( k = 1, \ldots, \lambda_0 \).
We prove the claim $\deg(A_\mathcal{F}, \mathcal{D} \cap F, 0) = \deg(\tilde{A}_\mathcal{F}, \mathcal{D} \cap F, 0) = \deg(A_\mathcal{D}, \mathcal{D} \cap F_0, 0)$. The first equality holds if
\[
t_0 f_\mathcal{F} u + (1 - t) \tilde{f}_\mathcal{F} u \neq 0 \quad \text{for} \quad u \in \partial D \cap F, \quad t \in [0, 1].
\] (3.5)
If $P_0$ is the continuous projection of $F$ onto $F_0$ along span$\{w_1, \ldots, w_\mu\}$, then $P_0 f_\mathcal{F} = A_\mathcal{D}$ (when we think of $A_\mathcal{F}$ as a mapping $D \cap F \subset F \rightarrow F$) and $\tilde{A}_\mathcal{F} = I_\mathcal{F} + P_0 (A_\mathcal{F} - I_\mathcal{F})$
where $I_\mathcal{F}$ is the identity mappings of $F$. Hence, the equality
\[
\deg(\tilde{A}_\mathcal{F}, \mathcal{D} \cap F, 0) = \deg(A_\mathcal{D}, \mathcal{D} \cap F_0, 0)
\]
is implied by the Leray-Schauder lemma (Lemma 2.1.9) if Equation (3.5) holds for $t = 0$. So it suffices to prove (3.5). If Equation (3.5) is not valid, for some $\mu_0 \in \partial D \cap F$ and $t_0 \in [0, 1]$ we have
\[
\langle A_\mu_0, v \rangle = 0, \quad i = 1, \ldots, \lambda_0,
\]
to $(A_\mu_0, \nu_0) + (1 - t_0)(f_\mathcal{F}, u_\mu_0) = 0, \quad i = 1, \ldots, \mu.$
(3.6)
Especially, $\langle A_\mu_0, v \rangle = 0$ for all $v \in F_0$. If $t_0 = 1$, then $\langle A_\mu_0, v \rangle = 0$ for every $v \in F$. Since $u_\mu \in F$ and $u_\mu \in \partial D$, this is impossible (otherwise $Z(F, F) \neq \emptyset$ with $F \supset F_0$). Thus, $t_0 \neq 1$. Let
\[
u_0 = \sum_{j=1}^{\lambda_0} a_j \nu_j + \sum_{j=1}^{\mu} b_j w_j.
\]
Then $b_j = \langle f_\mathcal{F}, u_\mu_0 \rangle$ for all $j = 1, \ldots, \mu$
and
\[
t_0 \left( A_\mu_0, \nu_0 \right) = \frac{\mu_0}{1 - t_0} \left( A_\mu_0, \nu_0 \right) + \frac{\mu}{1 - t_0} \sum_{j=1}^{\mu} b_j \langle A_\mu_0, w_j \rangle
\]
\[
= \mu_0 \left( f_\mathcal{F}, u_\mu_0 \right) + \mu \sum_{j=1}^{\mu} b_j \langle A_\mu_0, w_j \rangle
\]
\[
= \mu \sum_{j=1}^{\mu} b_j \langle A_\mu_0, w_j \rangle \leq 0.
\]
The last equality follows from Equation (3.6). The existence of the element $u_\mu_0$ satisfying $\langle A_\mu_0, u_\mu_0 \rangle \leq 0$ and $\langle A_\mu_0, v \rangle = 0$ for all $v \in F_0$ contradicts with the result $Z(F_0, F) = \emptyset$ of Proposition 3.2.4. Thus, (3.5) holds, which proves the theorem.

Remark 3.2.9. The proof of Theorem 3.2.8 shows that if we have chosen a basis for the space $F_0$, it does not matter how we complete the basis of $F_0$ to the basis of $F$ if we leave the basis of $F_0$ intact. However, the proof of Theorem 3.2.8 does not tell us anything about whether the degree remains the same if the basis of $F_0$ is changed. This question is difficult because if we consider a different basis $w_1, \ldots, w_\mu$ for $F_0$, then we have a different mapping $A_\mathcal{D} = \sum_{j=1}^{\mu} \langle A_\mu_0, w_j \rangle w_j$ and $u \in D \cap F_0$.
In this section, we prove that the previously constructed degree function possesses the classical properties of a degree function. We start this section by proving that the degree whose degree at 0 relative to \( D \cap F_0 \) is well-defined by Proposition 3.2.4 and Theorem 3.2.8, provided that \( A \) has no zeros in \( \partial D \). We will return to this question in Section 3.4.

Remark 3.2.10. Proposition 3.2.4 does not say anything about the uniqueness of \( F_0 \)—even if it is required that there are no spaces \( F' \subseteq F_0 \) with \( Z(F', F) = \emptyset \) for all finite-dimensional spaces \( F \supset F' \). A careful study of Theorem 3.2.8 reveals that if \( F_0 \) is not unique, then some problems appear. If there are two candidates \( F_1 \) and \( F_2 \) for the space \( F_0 \) and the sum \( F_1 + F_2 \) is direct, then there are no problems because we may complete the basis of \( F_1 \) to the basis of \( F_1 \oplus F_2 \) by adding the basis vectors of \( F_2 \) to the basis of \( F_1 \). With that action, we get the equalities

\[
\deg(A_{F_1}, \overline{D} \cap F_1, 0) = \deg(A_{F_1 \oplus F_2}, \overline{D} \cap (F_1 \oplus F_2), 0) = \deg(A_{F_2}, \overline{D} \cap F_2, 0).
\]

The problems arise if \( F_1 \cap F_2 \neq \{0\} \) and if some basis vectors of \( F_1 \) that span \( F_1 \cap F_2 \) are different from some basis vectors of \( F_2 \) that span \( F_1 \cap F_2 \). The problem is that the “approximations” formed with respect to these bases are different.

However, it makes sense to speak about degree when the space \( F_0 \) is fixed and the basis of \( F_0 \) is fixed. So, in a sense, the following definition is justified. In Section 3.3, we prove that the degree given by Definition 3.2.11 possesses the classical properties of a degree function, and in Section 3.4, we will use the uniqueness results of a degree function to give a partial answer to the question whether the degree function depends on the choice of \( F_0 \) and the basis of \( F_0 \).

Definition 3.2.11. Let \( D \subseteq X \) be an open, bounded set, and let \( A : \overline{D} \subseteq X \to X^* \) be a semi-continuous mapping that satisfies the condition \((S_+)\) on \( \partial D \) and has no zeros in \( \partial D \). Assume that \( A_F \) is defined in (3.4) and \( F_0 \) is a finite-dimensional subspace of \( X \) given by Corollary 3.2.7. Then the number

\[
\deg(A, \overline{D}, 0) = \deg(A_F, \overline{D} \cap F, 0), \quad F_0 \subset F
\]

is called the degree of \( A \) at 0 in \( X^* \) with respect to \( \overline{D} \). For any \( y \in X^* \setminus A(\partial D) \), we set

\[
\deg(A, \overline{D}, y) := \deg(A - y, \overline{D}, 0).
\]

Remark 3.2.12. Because the mapping \( x \mapsto y \) is compact and semi-continuous, \( A - y \) is a semi-continuous \((S_+)\)-mapping if \( A \) is such a mapping. So the definition

\[
\deg(A, \overline{D}, y) := \deg(A - y, \overline{D}, 0)
\]

makes sense. Using this, many properties of the degree function can be proved in the setting where the degree is calculated at the origin; in the case of the normalization property, we have to use the additivity, the homotopy invariance, and the bijectivity of the duality mapping \( J \) that is chosen as normalization mapping.

3.3 Classical properties of the degree function

In this section, we prove that the previously constructed degree function possesses the classical properties of a degree function. We start this section by proving that the degree whose degree at 0 relative to \( D \cap F_0 \) is well-defined by Proposition 3.2.4 and Theorem 3.2.8, provided that \( A \) has no zeros in \( \partial D \). We will return to this question in Section 3.4.

Remark 3.2.10. Proposition 3.2.4 does not say anything about the uniqueness of \( F_0 \)—even if it is required that there are no spaces \( F' \subseteq F_0 \) with \( Z(F', F) = \emptyset \) for all finite-dimensional spaces \( F \supset F' \). A careful study of Theorem 3.2.8 reveals that if \( F_0 \) is not unique, then some problems appear. If there are two candidates \( F_1 \) and \( F_2 \) for the space \( F_0 \) and the sum \( F_1 + F_2 \) is direct, then there are no problems because we may complete the basis of \( F_1 \) to the basis of \( F_1 \oplus F_2 \) by adding the basis vectors of \( F_2 \) to the basis of \( F_1 \). With that action, we get the equalities

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\deg(A_{F_1}, \overline{D} \cap F_1, 0) = \deg(A_{F_1 \oplus F_2}, \overline{D} \cap (F_1 \oplus F_2), 0) = \deg(A_{F_2}, \overline{D} \cap F_2, 0).
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The problems arise if \( F_1 \cap F_2 \neq \{0\} \) and if some basis vectors of \( F_1 \) that span \( F_1 \cap F_2 \) are different from some basis vectors of \( F_2 \) that span \( F_1 \cap F_2 \). The problem is that the “approximations” formed with respect to these bases are different.

However, it makes sense to speak about degree when the space \( F_0 \) is fixed and the basis of \( F_0 \) is fixed. So, in a sense, the following definition is justified. In Section 3.3, we prove that the degree given by Definition 3.2.11 possesses the classical properties of a degree function, and in Section 3.4, we will use the uniqueness results of a degree function to give a partial answer to the question whether the degree function depends on the choice of \( F_0 \) and the basis of \( F_0 \).

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\[
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\]

is called the degree of \( A \) at 0 in \( X^* \) with respect to \( \overline{D} \). For any \( y \in X^* \setminus A(\partial D) \), we set

\[
\deg(A, \overline{D}, y) := \deg(A - y, \overline{D}, 0).
\]

Remark 3.2.12. Because the mapping \( x \mapsto y \) is compact and semi-continuous, \( A - y \) is a semi-continuous \((S_+)\)-mapping if \( A \) is such a mapping. So the definition

\[
\deg(A, \overline{D}, y) := \deg(A - y, \overline{D}, 0)
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makes sense. Using this, many properties of the degree function can be proved in the setting where the degree is calculated at the origin; in the case of the normalization property, we have to use the additivity, the homotopy invariance, and the bijectivity of the duality mapping \( J \) that is chosen as normalization mapping.

3.3 Classical properties of the degree function

In this section, we prove that the previously constructed degree function possesses the classical properties of a degree function. We start this section by proving that the degree
defined in Definition 3.2.11 is homotopy invariant with respect to admissible homotopies. The admissible homotopy is defined in Definition 2.4.7 on page 45.

**Theorem 3.3.1.** Let $D$ be an open, bounded set, and assume that $A^0 : \overline{D} \subset X \rightarrow X^*$ and $A^1 : \overline{D} \subset X \rightarrow X^*$ are demi-continuous mappings that satisfy the condition $(S_1)$ on $\partial D$ and have no zeros in $\partial D$. If there is an admissible homotopy between $A^0$ and $A^1$ with respect to the set $D$, then

$$\text{Deg}(A^0, \overline{D}, 0) = \text{Deg}(A^1, \overline{D}, 0).$$

(3.7)

**Proof.** Let $\{A_t | t \in [0,1]\}$ be a family of mappings that is a homotopy between $A^0$ and $A^1$ in the sense of Definition 2.4.7. Proposition 3.2.4 guarantees that there is a subspace $F_0$ such that for any finite-dimensional subspace $F$ containing $F_0$ and any $t \in [0,1]$, the mapping $(A_t)_F$ has no zeros in the boundary of $D$. Consequently,

$$\text{deg}(A^0_t, \overline{F} \cap F, 0) = \text{deg}(A^1_t, \overline{F} \cap F, 0)$$

for any $F \subseteq F(X)$ containing $F_0$. This means that

$$\text{Deg}(A^0, \overline{D}, 0) = \text{Deg}(A^1, \overline{D}, 0),$$

which was to be proved. \qed

**Theorem 3.3.2.** Let $A : \overline{D} \subset X \rightarrow X^*$ be a demi-continuous mapping satisfying the condition $(S_1)$ on the boundary $\partial D$ of an open, bounded set $D$. Then

$$Au \neq 0 \quad \text{for all} \quad u \in \overline{D}$$

imply $\text{Deg}(A,D,0) = 0$. 

**Proof.** According to Proposition 3.2.4, there is a finite-dimensional subspace $F_0$ with the property that the degree $\text{deg}(A_F, \overline{D} \cap F, 0)$ is well-defined when $F \supseteq F_0$ and $F \in F(X)$. Moreover, $A_F u \neq 0$ for all $u \in \overline{D} \cap F$ and, consequently, $\text{deg}(A_F, \overline{D} \cap F, 0) = 0$ for every $F \subseteq F(X)$ with $F \supseteq F_0$. This implies that $\text{Deg}(A,D,0) = 0$. \qed

We immediately obtain the following corollary from the previous theorem.

**Corollary 3.3.3.** Let $A : \overline{D} \subset X \rightarrow X^*$ be a demi-continuous mapping that satisfies the condition $(S_1)$ and has no zeros in the boundary $\partial D$ of an open, bounded set $D$. If

$$\text{Deg}(A, \overline{D}, 0) \neq 0,$$

then the equation $Au = 0$ has at least one solution in $D$.

**Theorem 3.3.4.** Assume that $D \subset X$ is an open, bounded set, and that $A : \overline{D} \subset X \rightarrow X^*$ is a demi-continuous mapping satisfying the condition $(S_1)$ on $\overline{D}$. If $D_1$ and $D_2$ are nonempty, open, disjoint subsets of $D$ with $Au \neq 0$ for all $u \in \overline{D} \setminus (D_1 \cup D_2)$, then

$$\text{Deg}(A, \overline{D}, 0) = \text{Deg}(A, \overline{D_1}, 0) + \text{Deg}(A, \overline{D_2}, 0).$$

defined in Definition 3.2.11 is homotopy invariant with respect to admissible homotopies. The admissible homotopy is defined in Definition 2.4.7 on page 45.

**Theorem 3.3.1.** Let $D$ be an open, bounded set, and assume that $A^0 : \overline{D} \subset X \rightarrow X^*$ and $A^1 : \overline{D} \subset X \rightarrow X^*$ are demi-continuous mappings that satisfy the condition $(S_1)$ on $\partial D$ and have no zeros in $\partial D$. If there is an admissible homotopy between $A^0$ and $A^1$ with respect to the set $D$, then

$$\text{Deg}(A^0, \overline{D}, 0) = \text{Deg}(A^1, \overline{D}, 0).$$

(3.7)

**Proof.** Let $\{A_t | t \in [0,1]\}$ be a family of mappings that is a homotopy between $A^0$ and $A^1$ in the sense of Definition 2.4.7. Proposition 3.2.4 guarantees that there is a subspace $F_0$ such that for any finite-dimensional subspace $F$ containing $F_0$ and any $t \in [0,1]$, the mapping $(A_t)_F$ has no zeros in the boundary of $D$. Consequently,

$$\text{deg}(A^0_t, \overline{F} \cap F, 0) = \text{deg}(A^1_t, \overline{F} \cap F, 0)$$

for any $F \subseteq F(X)$ containing $F_0$. This means that

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which was to be proved. \qed

**Theorem 3.3.2.** Let $A : \overline{D} \subset X \rightarrow X^*$ be a demi-continuous mapping satisfying the condition $(S_1)$ on the boundary $\partial D$ of an open, bounded set $D$. Then

$$Au \neq 0 \quad \text{for all} \quad u \in \overline{D}$$

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**Proof.** According to Proposition 3.2.4, there is a finite-dimensional subspace $F_0$ with the property that the degree $\text{deg}(A_F, \overline{D} \cap F, 0)$ is well-defined when $F \supseteq F_0$ and $F \in F(X)$. Moreover, $A_F u \neq 0$ for all $u \in \overline{D} \cap F$ and, consequently, $\text{deg}(A_F, \overline{D} \cap F, 0) = 0$ for every $F \subseteq F(X)$ with $F \supseteq F_0$. This implies that $\text{Deg}(A,D,0) = 0$. \qed

We immediately obtain the following corollary from the previous theorem.

**Corollary 3.3.3.** Let $A : \overline{D} \subset X \rightarrow X^*$ be a demi-continuous mapping that satisfies the condition $(S_1)$ and has no zeros in the boundary $\partial D$ of an open, bounded set $D$. If

$$\text{Deg}(A, \overline{D}, 0) \neq 0,$$

then the equation $Au = 0$ has at least one solution in $D$.

**Theorem 3.3.4.** Assume that $D \subset X$ is an open, bounded set, and that $A : \overline{D} \subset X \rightarrow X^*$ is a demi-continuous mapping satisfying the condition $(S_1)$ on $\overline{D}$. If $D_1$ and $D_2$ are nonempty, open, disjoint subsets of $D$ with $Au \neq 0$ for all $u \in \overline{D} \setminus (D_1 \cup D_2)$, then

$$\text{Deg}(A, \overline{D}, 0) = \text{Deg}(A, \overline{D_1}, 0) + \text{Deg}(A, \overline{D_2}, 0).$$
\textbf{Remark 3.3.5.} In the previous theorem, it suffices to assume that A satisfies the condition \((S_+)\) on \(D_1 \cup D_2\).

\textbf{Theorem 3.3.6.} Assume that \(J\) is the duality mapping of \(X\) generated by a norm which is locally uniformly convex and makes the corresponding norm of \(X^*\) locally uniformly convex. If \(D\) is an open, bounded subset of \(X\) and \(y \in J(D)\), then \(\text{Deg}(J, \overline{D}, y) = 1\).

\textbf{Proof.} Step 1. We demonstrate that it is sufficient to prove that \(\text{Deg}(J, \overline{D}, 0) = 1\) if \(0 \in D\).

Since \(J\) is bijective mapping \(X \rightarrow X^*\) and \(y \in J(D)\), there are no points of \(J^{-1}(y)\) outside of \(D\). Thus, the additivity of the degree implies that
\[
\text{Deg}(J, \overline{D}, y) = \text{Deg}(J, \overline{D}, y) 
\]
for any open, bounded set \(D' \supset D\). According to the definition of duality mapping, we have \(||Ju|| = ||u||\). We choose so large \(R\) that \(R > ||y||\) and \(D \subset B(0, R)\). Then
\[
\tau Ju + (1 - \tau)(Ju - y) = Ju + (1 - \tau)y \neq 0 \quad \text{for all} \quad u \in \partial B(0, R).
\]
Note that \(J^{-1}(0) = 0\). The homotopy invariance and Equation (3.9) yields
\[
\text{Deg}(J, \overline{D}, y) = \text{Deg}(J, \overline{B}(0, R), y) = \text{Deg}(J, \overline{B}(0, R), 0)
\]
as required.

\textbf{Remark 3.3.5.} In the previous theorem, it suffices to assume that \(A\) satisfies the condition \((S_+)\) on \(\overline{D_1 \cup D_2}\).

\textbf{Theorem 3.3.6.} Assume that \(J\) is the duality mapping of \(X\) generated by a norm which is locally uniformly convex and makes the corresponding norm of \(X^*\) locally uniformly convex. If \(D\) is an open, bounded subset of \(X\) and \(y \in J(D)\), then \(\text{Deg}(J, \overline{D}, y) = 1\).

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\]
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\[
\text{Deg}(J, \overline{D}, y) = \text{Deg}(J, \overline{B}(0, R), y) = \text{Deg}(J, \overline{B}(0, R), 0)
\]
as required.
Step 2. We prove that $\operatorname{Deg}(J, \overline{T}, 0) = 1$ if $0 \in D$.
Since $J$ is strictly monotone, any non-zero, finite-dimensional subspace $F_0$ of $X$ serves as the space $F_0$ given by Proposition 3.2.4. Let $J_F$ be the “approximation” of $J$ with respect to some finite-dimensional subspace $F$ of $X$, and let the vectors $v_1, \ldots, v_n$ form a basis of $F$. We consider the homotopy

$$H_{J_F}(u) = t I_F u + (1 - t) J_F u, \quad t \in [0, 1], \quad u \in D \cap F,$$

where $I_F$ is the identity mapping of $F$. To prove the theorem, we need to demonstrate that $H_{J_F}(u) \neq 0$ for all $u \in \partial D \cap F$ and $t \in [0, 1]$.

If this is not the case, there are $t \in [0, 1]$ and $u \in \partial D \cap F$ for which

$$0 = H_{J_F}(u) = t \sum_{i=1}^{n} (h_i, u) v_i + (1 - t) \sum_{i=1}^{n} (J_F u, v_i)$$

hold. Here the elements $h_i$ belong to $X^*$ and satisfy the condition

$$\langle h_j, v_i \rangle = \begin{cases} 1, & \text{when } j = i, \\ 0, & \text{when } j \neq i. \end{cases}$$

Note that $u$ has the representation $u = \sum_{i=1}^{n} (h_i, u) v_i$. Since $v_1, \ldots, v_n$ are linearly independent,

$$t \langle h_i, u \rangle = -(1 - t) \langle J_F u, v_i \rangle, \quad \text{for } i = 1, \ldots, n.$$

This implies that

$$0 \leq t \sum_{i=1}^{n} (h_i, u)^2 = -(1 - t) \sum_{i=1}^{n} \langle J_F u, v_i \rangle \langle h_i, u \rangle = -(1 - t) \left( \langle J_F u, \sum_{i=1}^{n} (h_i, u) v_i \rangle \right)$$

$$= -(1 - t) \langle J_F u, u \rangle = -(1 - t) \|u\|^2 \leq 0.$$

Consequently, $\|u\| = 0$ or $(h_i, u) = 0$ for $i = 1, \ldots, n$. In either case, the conclusion is $u = 0$ and the assumption $\|u\| > 0$ is violated.

Hence, $H_{J_F}(u) \neq 0$ when $t \in [0, 1]$ and $u \in \partial D \cap F$. Since Brouwer’s degree is invariant under homotopies, we conclude that

$$1 = \operatorname{deg}(J_F, D \cap F, 0) = \operatorname{deg}(J_F, \overline{T}, 0).$$

As this is valid for every $F \in F(X)$ with $F \supset F_0$, we have $1 = \operatorname{Deg}(J, \overline{T}, 0)$. □

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Consequently, $\|u\| = 0$ or $(h_i, u) = 0$ for $i = 1, \ldots, n$. In either case, the conclusion is $u = 0$ and the assumption $\|u\| > 0$ is violated.

Hence, $H_{J_F}(u) \neq 0$ when $t \in [0, 1]$ and $u \in \partial D \cap F$. Since Brouwer’s degree is invariant under homotopies, we conclude that

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As this is valid for every $F \in F(X)$ with $F \supset F_0$, we have $1 = \operatorname{Deg}(J, \overline{T}, 0)$. □
### 3.4 The other constructions of the $(S_+)$-degree

In this section, we give information on the other constructions of the $(S_+)$-degree. Only the main ideas of the constructions are sketched, but appropriate references are given for the reader who wishes to see rigorous proofs, and exact theorems and definitions. We also give a partial answer to the question on whether the degree is independent of the basis. This question was posed on page 54.

#### 3.4.1 Skrypnik’s other construction

Skrypnik gave another construction for the degree function of bounded, demi-continuous $(S_+)$-mappings in separable, reflexive Banach spaces. This construction appeared for the first time in 1973, and the details of this method are found in Skrypnik’s works [167, pp. 121–123], [168, pp. 196–200], [169, pp. 12–17], and [170, pp. 35–38].

In this definition, $X$ is a reflexive, separable Banach space and $v_1, v_2, \ldots$ is a linearly independent, complete system in $X$. Skrypnik defines the finite-dimensional space $F_n$ by

$$ F_n = \text{span}\{v_1, \ldots, v_n\} $$

and the finite-dimensional approximation $A_{F_n}$ of $F_n \cap F_n$ $\to F_n$ by

$$ A_{F_n}u = \sum_{i=1}^n (Au)v_i, \quad u \in F_n \cap F_n. $$

(3.10)

The definition of the degree is based on the observation that if $Au \neq 0$ on $\partial D$, then there is $N_1 \in \mathbb{Z}_+$ so that $\deg(A_{F_n}, F_n \cap \partial D, 0)$ is well-defined and independent of $n$ when $n \geq N_1$.

The degree of $A$ at $0$ with respect to $D$ is defined as

$$ D\{v_i\} := \lim_{n \to \infty} \deg(A_{F_n}, F_n \cap D, 0). $$

#### Remark 3.4.1

Skrypnik claims that the degree is independent of the chosen basis; the proof of that claim is found in article [168, Lemma 4, pp. 198–200] and in books [167, Lemma 6, pp. 122–123], [169, Theorem 1.1.2, pp. 15–17], and [170, Theorem 2.2, pp. 37–38].

The claim is proved using the observation that if we have two systems $v_1, v_2, \ldots$ and $v_1', v_2', \ldots$, then there is an auxiliary system $v_1'', v_2'', \ldots$ with

$$ \{v_1, \ldots, v_n, v_1'', v_2'', \ldots\} \quad \text{and} \quad \{v_1', \ldots, v_n', v_1', v_2', \ldots\} $$

linearly independent for every $n \in \mathbb{Z}_+$. By using the systems $v_1, v_2, \ldots$ and $v_1', v_2', \ldots$, Skrypnik proves that

$$ D\{v_i\} = D\{v_i''\}. $$

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2 Also some extensions of the $(S_+)$-degree have their own construction; see the books by O’Regan, Cho & Chen [144, Chapter 6, pp. 127–168] and Hu & Papageorgiou [96].

3 Skrypnik does not say what he means with a complete system, but if we examine the method he uses to justify the definition of the degree, we can conclude that it means that the condition $(x', v_i) = 0$ for all $i \in \mathbb{Z}$ implies $x' = 0$. In the connection of the so-called minimal systems, such a system is called fundamental; see Lindenstrauss & Tzafriri’s book [126, Definition 1.f.2, p. 45] and Subsection 3.4.2.

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However, it is unclear whether there is an auxiliary system \( v'_1, v''_1, \ldots \) which has the required linear independence property and which is also complete. A careful study of the construction of the degree function reveals that the auxiliary system has to be complete if we want \( D\{v'_1\} \) to be well-defined. Note that a similar argument appears in Kartatsos & Skrypnik’s article [108, proof of Theorem 2.2, p. 419].

**Remark 3.4.2.** It can be shown that the degree formed with respect to a fixed linearly independent, complete system \( v_1, v_2, \ldots \) has the classical properties of a degree function when a duality mapping that is given by locally uniformly convex norms is chosen as a normalization mapping; see Skrypnik’s books [169, pp. 22–28] and [170, pp. 42–47]. The additivity and normalization are not proved, but they can be proved as in the previous section. Note that Skrypnik considers a larger class of admissible homotopies than those that we, Browder, and Berkovits & Mustonen use; see Remark 2.4.6.

In 1983, Browder and in 1986, Berkovits & Mustonen showed (see Subsections 3.4.4 and 3.4.3) that the degree of bounded, demi-continuous \( (S_\epsilon) \)-mappings is uniquely characterized by the classical properties of a degree function. These uniqueness results can be applied to prove that the degree defined by Skrypnik for bounded, demi-continuous \( (S_\epsilon) \)-mappings in reflexive, separable Banach spaces does not depend on the choice of the complete system \( v'_1, v''_2, \ldots \).

### 3.4.2 Zeidler’s construction

A construction of the \( (S_\epsilon) \)-degree is given also in Zeidler’s book [199, problems 36.3–36.3b, pp. 1003–1004].

In this construction, Zeidler selects a complete system \( v_1, v_2, v_3, \ldots \) for a reflexive, separable Banach space \( X \) and let

\[
F_n = \text{span}\{v_1, \ldots, v_n\}.
\]

Then he chooses the functionals \( v'_1, v''_2, \ldots \in X^* \) with

\[
\langle v'_i, v_j \rangle = \delta_{ij}
\]

for all \( i, j \). It is said that the sequences \((v_i)_{i=1}^\infty\) and \((v'_i)_{i=1}^\infty\) with the property (3.11) form a biorthogonal system. By using the biorthogonal system, Zeidler forms the approximation

\[
A_n u = \sum_{i=1}^n \langle Au, v'_i \rangle \langle v'_i, v_i \rangle^{\ast} u \in D \cap F_n
\]

for a bounded, demi-continuous \( (S_\epsilon) \)-mapping \( A : D \subset X \to X^* \). Zeidler states that the degree of this approximation does not depend on the choice of \((v_i)_{i=1}^\infty\).

Zeidler left the construction as an exercise but gave hints to the reader on that how the construction can be carried out. The hints imply that Zeidler’s construction conforms to the method used by Skrypnik for bounded, demi-continuous \( (S_\epsilon) \)-mappings in separable, reflexive Banach spaces. Thus, the biorthogonal system \((v_i)_{i=1}^\infty, (v'_i)_{i=1}^\infty\) has to be fundamental (see footnote 3 on page 59).

**Remark 3.4.2.** It can be shown that the degree formed with respect to a fixed linearly independent, complete system \( v_1, v_2, \ldots \) has the classical properties of a degree function when a duality mapping that is given by locally uniformly convex norms is chosen as a normalization mapping; see Skrypnik’s books [169, pp. 22–28] and [170, pp. 42–47]. The additivity and normalization are not proved, but they can be proved as in the previous section. Note that Skrypnik considers a larger class of admissible homotopies than those that we, Browder, and Berkovits & Mustonen use; see Remark 2.4.6.

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Remark 3.4.3. There are gaps in Zeidler’s construction.

1. Because the set \( v_1, v_2, v_3, \ldots \) is infinite, the existence of the functionals \( v_1^*, v_2^*, v_3^*, \ldots \) with the property (3.11) is not evident. In Lindenstrauss & Tzafriri’s book [126, Section 1.g, pp. 42–47], there are results in this direction—for example, for a separable space there is at least one biorthogonal system which is fundamental—but there seems to be no result which states that for each complete system of \( X \) there is a corresponding sequence of functionals so that a biorthogonal system is formed. However, we know that the degree defined in this way exists when a Banach space \( X \) is separable and reflexive. The interested reader is advised to study the given references.

2. It is not evident how one proves that the degree is independent of the chosen basis. The hints given by Zeidler (see [199, Problem 36.3b, p. 1004] and [196, p. 543]) imply that the problem can be solved by using the biorthogonal system \(((v_i)_{i=1}^\infty, (v_i')_{i=1}^\infty)\) and the definition of Brouwer’s degree of mappings between different \( n\)-dimensional spaces. For more discussion, see Remark 2.1.8.

3. In Zeidler’s construction, the existence of normalizing mapping is left open; problem 36.3h on page 1006 of [199] offers a result in that direction but the solution given to that problem is incorrect.\(^4\)

\[3.4.3 \text{ Berkovits and Mustonen’s construction} \]

This construction is found in Berkovits & Mustonen’s publications [14, 16, 17] and Mortici’s monograph [138, pp. 71–75]. It is based on the Leray-Schauder degree and the so-called Browder-Ton embedding theorem according to which to every reflexive, separable Banach space \( X \) there is a separable Hilbert space \( H \) and a compact, injective, linear mapping \( \psi : H \rightarrow X \) with \( \psi(H) \) dense in \( X \). For a bounded, semi-continuous \((S_\epsilon)\)-mapping \( f : \overline{D} \subset X \rightarrow X^* \), Mustonen & Berkovits defined the “approximation”\(^6\)

\[ f_\epsilon = I + \frac{1}{\epsilon} \psi \psi f \quad \text{for any } \epsilon > 0, \tag{3.13} \]

where the mapping \( \psi : X^* \rightarrow H \) is defined by

\[ (\psi(w), v)_{H, H} = (w, \psi(v))_{X, X} \quad \text{for all } v \in H, w \in X^*. \]

The mapping \( f_\epsilon : \overline{D} \subset X \rightarrow X \) is the sum of a compact mapping and the identity mapping, and therefore the Leray-Schauder degree is available. It can be proved that if \( f \) does not have zeros on \( \partial D \), then the Leray-Schauder degree

\[ \deg_{LS}(f_\epsilon, \overline{D}, 0) = \deg_{LS}(I + \frac{1}{\epsilon} \psi \psi f, \overline{D}, 0) \]

\[6\text{Zeider uses approximations } A_{\varepsilon \alpha}, F_\varepsilon \rightarrow F_\alpha \text{ and the solution involves the study of the equation } A_{\varepsilon \alpha} w = \alpha w \text{ with a scalar } \alpha \text{ and } w \in D \cap F_\alpha. \]

\[\text{The Browder-Ton embedding theorem was stated by Browder & Ton in article [48, Theorem 1, p. 178, and proof of Theorem 1, p. 183], but the proof that they gave is incorrect; a correct proof can be found in Berkovits’ article [15, Theorem 2.1, pp. 2964–2965].} \]

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is defined and does not depend on $\varepsilon$ when $\varepsilon$ is sufficiently small. The degree of $f$ at zero with respect to an open, bounded set $D \subset X$ is defined as

$$\text{Deg}(f, \mathcal{T}, 0) = \lim_{\varepsilon \to 0^+} \deg_{\mathcal{BR}}(f, \mathcal{T}, 0).$$

They also proved that the degree defined in this way is a classical degree, when a duality mapping is taken as a normalization mapping, and that this degree is uniquely characterized by the classical properties.

In Berkovits’ Ph.D. thesis [14, Section 2.3, pp. 24–28], this degree was extended to all demi-continuous $(S_1)$-mappings and it was proved that this extension is unique. The extension is based on the finding that if $A : D \subset X \to X^*$ is a demi-continuous $(S_1)$-mapping that has no zeros on the boundary of the open, bounded set $D$, then there is an open set $D'$ such that

$$\mathcal{T} \cap A^{-1}(0) \subset D' \subset D$$

and $A(D')$ is bounded.

For the proof of this result, see Berkovits [14, Section 2.3, pp. 24–28]. It is worth noting that this result does not depend on the separability of the reflexive Banach space $X$ and that by using this result one can always calculate the topological degree of a demi-continuous $(S_1)$-mapping in a set where that mapping is bounded. The uniqueness of the extension can be proved also by using the previously mentioned result; see Remark 3.4.4.

### 3.4.4 Browder’s construction

In 1983, Browder gave yet another construction for the degree of demi-continuous $(S_1)$-mappings. Browder’s construction can be found in paper [46, Section 5, pp. 21–31], and it is outlined in article [42]. The proof of the uniqueness of this degree is found in articles [45] and [46, Section 5].

The approximations used by Browder differ from the “approximation” used by Skrypnik. In [42, p. 1772], [46, p. 209], and [44, Definition 4, p. 21], the Galerkin approximation is

$$A_0(u) = \phi^*(A(u)) \quad \text{for} \quad u \in F \cap \mathcal{T}.$$  

Here $\phi^* : X^* \to F^*$ is the adjoint of the injection mapping3 $\phi$ of $F$ into $X$, that is,

$$\langle \phi^* w, u \rangle_{F^*, F} = \langle w, \phi u \rangle_{X^*, X} \quad \text{for all} \quad w \in X^*, \quad u \in F.$$  

But in [41, Theorem 4, p. 21] Browder considers the approximation

$$\phi ^* A (\phi ^* |_{\mathcal{T} \cap F}) : \mathcal{T} \cap F \to F^*$$

where $A$, $\phi$, and $\phi^*$ are as above. These approximations are continuous mappings from an $n$-dimensional space $F$ to the dual space $F^*$.

3The injection mapping is sometimes called the inclusion map or the imbedding operator, and it is defined as follows: if $A \subset X$ is a subset, then the inclusion mapping $i : A \to X$ is defined by $(i a) = a$ for all $a \in A$.

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3The injection mapping is sometimes called the inclusion map or the imbedding operator, and it is defined as follows: if $A \subset X$ is a subset, then the inclusion mapping $i : A \to X$ is defined by $(i a) = a$ for all $a \in A$. 
Thus, one can first construct the topological degree for bounded, demi-continuous $(S_*)$-mappings, and then extend this to unbounded mappings in such a way that the extension has the additivity property (that causes the extension to be unique when the degree for bounded mappings is unique). For details, see Berkovits’ Ph.D. thesis [14, Section 2.3, pp. 24–28] and Mustonen & Berkovits’ report [17, pp. 31–34].
Remark 3.4.5. Browder asserts that the degree for the approximation $A_0$ of a demi-continuous $(S_\ast)$-mapping $A$ is unique with the duality mapping of $F$ as a normalization mapping, and makes a reference to the results Section 1 of his article [46, pp. 4–10]. Those results concern the uniqueness of the topological degree in $\mathbb{R}^n$, and it is unclear how those results apply the uniqueness of the topological degree of mappings $F \rightarrow F^\ast$.

Moreover, Browder justifies the uniqueness by stating that a finite-dimensional space $F$ is equivalent to a Hilbert space (he identifies $F^\ast$ with $F$); see Browder [46, p. 23]. It is true that any finite-dimensional normed space is in fact an inner product space: as $\mathbb{R}^n$ is an inner product space and there is a continuous, linear homeomorphism $h : F \rightarrow \mathbb{R}^n$, an inner product $(\cdot, \cdot)_F$ for $F$ is defined by

$$(y, x)_F = (h(y), h(x))_{\mathbb{R}^n},$$

where $(\cdot, \cdot)_{\mathbb{R}^n}$ is the (usual) inner product of $\mathbb{R}^n$. Note that there are many linear homeomorphisms and therefore many inner products for $F$. Since $F$ is a finite-dimensional normed space, all its norms are equivalent and thus induce the same dual space. So $F^\ast$ can be identified with $F$ by using the linear homeomorphism induced by the Frechet-Riesz theorem. Note that also other linear homeomorphisms may be used. However, if we make such identifications, then we do not study a mapping $A$ from $F$ into the topological dual $F^\ast$ anymore, but a mapping $hA : F \rightarrow F$ where $h$ is a homeomorphism $F^\ast \rightarrow F$. The diagram demonstrates the situation. Recall what we have expressed in Subsection 2.1.4: in general it is not known if a classical topological degree for continuous mappings between different $n$-dimensional normed spaces is unique, even if the normalization mapping is given. In that subsection, we also saw that the choice of the homeomorphisms has an influence on the value of the degree. See also Remark 2.4.13 and Remark 2.2.19 for more discussion on homeomorphisms.

On the other hand, Berkovits & Mustonen have proved that the degree for bounded, demi-continuous $(S_\ast)$-mappings from a separable, reflexive Banach space to its dual is a unique classical topological degree with a duality mapping as a normalizing mapping. Since a finite-dimensional space is a separable, reflexive Banach space and the Galerkin approximations of bounded, demi-continuous $(S_\ast)$-mappings with respect to a finite-dimensional space are bounded, continuous $(S_\ast)$-mappings, Mustonen & Berkovits’ result is available. In the light of this, it seems that Browder’s claim on the uniqueness of the degree is true.

### 3.4.5 Does the degree depend on the choice of $F_0$ and the basis of $F_0$?

In some sense this question can be considered irrelevant: it can be sufficient that the results hold for the chosen space $F_0$ and the chosen basis. We can always comfort ourselves with this thought, and be satisfied with the situation. This is the case for the degree of

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In some sense this question can be considered irrelevant: it can be sufficient that the results hold for the chosen space $F_0$ and the chosen basis. We can always comfort ourselves with this thought, and be satisfied with the situation. This is the case for the degree of
the A-proper mappings; see Deimling [61, pp. 267–277], Zeidler [199, pp. 997–1007], and Petryshyn [147, 148]. However, in the applications it almost impossible to know beforehand the basis which suits one’s purposes, and therefore it is useful to know whether the degree depends on the basis chosen. Since the degree defined in Definition 3.2.11 has the classical properties of a degree function and since it is known that the degree of demi-
continuous \((S_+)-\)mappings is unique at least when \(X\) is a separable and reflexive Banach space (see Subsection 3.4.3), the degree does not depend on the choice of \(F_0\) or the choice of the basis if the Banach space \(X\) is separable and reflexive.
In this chapter, we derive a method for the calculation of the index of a critical point of a mapping of the type \((S_+)^{-}\) acting from a reflexive Banach space to its dual space. The index is defined via the degree theory of such mappings, and for the calculation of the index we use a linear approximation of the mapping at the critical point.

This chapter is organized as follows. Section 4.1 is included to impart some background information of the present subject to the reader. In Section 4.2, we define the concept of the index of a critical point, and in Section 4.3, we formulate the main theorem of this chapter. Section 4.4 is the longest section of this chapter and the whole thesis. It solely contains the proof of the main theorem, which is quite long and technical, though the techniques used are rather straightforward. In the proof, we need some auxiliary results that are presented in subsections 4.4.1–4.4.5 and 2.3.1. The “core” of the proof is in Subsection 4.4.6, where we apply the auxiliary results to prove the index theorem.

Auxiliary results are presented in the following order. Subsection 4.4.1 gives some information on a certain decomposition of the dual of a Banach space. This decomposition relates to the decomposition presented in 2.3.1, and most of the results of this chapter are founded on these two decompositions. A useful estimate concerning the decomposition of the Banach space \(X\) is presented in Section 4.4.2. Subsection 4.4.3 introduces some projections and a particular functional that are needed in the subsequent subsections. In Subsection 4.4.4 and Subsection 4.4.5, we seek certain subspaces that are used in the construction of the homotopies of Section 4.4.6.

After this chapter, Chapter 5 presents two counter-examples. The counter-examples show the necessity of certain assumptions. Both of the examples stem from I. V. Skrypnik, and one example demonstrates that there is an assumption missing in Skrypnik’s earlier results on the index of a critical point.

4.1 Background

Sometimes we are interested in the value of the degree in a neighborhood of an isolated solution. Although the topological degree is a powerful tool for the examination of the solution set of an equation, it is sometimes hard to calculate the degree using the classical
properties of the degree. In such a case, if a mapping is sufficiently smooth, it is reasonable to ask whether the degree can be computed by using a linear approximation of the mapping. When the mapping is from $\mathbb{R}^n$ to $\mathbb{R}^n$, this question has been answered in the case of isolated solutions. The solution is given by the ensuing theorem, whose proof can be found in Lloyd’s book [127, Theorem 2.2.3, p. 28].

Before we present the solution, we recall the concept of the index of a critical point in $\mathbb{R}^n$: the index $\text{Ind}(\phi, a, p)$ of $\phi$ at $a$ is the common value of $\text{deg}(\phi; U, p)$ for those $U \subset \mathbb{R}^n$ whose closures contain no other points $x$ with $\phi(x) = p$ than $a$.

**Theorem 4.1.1.** Assume that $\phi : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}^n$ is continuously differentiable and that $a \in \phi^{-1}(p)$ is a regular value of $\phi$, that is, the determinant of the Jacobian matrix $J_{\phi(a)}$ of $\phi$ at $a$ is nonzero. Then $\text{Ind}(\phi, a, p) = (-1)^v$, where $v$ is the sum of the algebraic multiplicities of the critical points of $\phi$ in a neighborhood of $a$.

Since this result has proved to be useful, it has been generalized to other classes of mappings. In 1946, Leray & Schauder derived a similar result for mappings which operate on infinite-dimensional Banach space and are of the form ‘identity + compact’ and $\nu$-mappings. In 1946, Leray & Schauder derived a similar result for mappings which operate on infinite-dimensional Banach space and are of the form ‘identity + compact’ and $\nu$-mappings. In 1946, Leray & Schauder derived a similar result for mappings which operate on infinite-dimensional Banach space and are of the form ‘identity + compact’ and $\nu$-mappings.

**Theorem 4.1.2.** Assume that $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable and that $a \in \phi^{-1}(p)$ is a regular value of $\phi$, that is, the determinant of the Jacobian matrix $J_{\phi(a)}$ of $\phi$ at $a$ is nonzero. Then $\text{Ind}(\phi, a, p) = (-1)^v$, where $v$ is the sum of the algebraic multiplicities of the critical points of $\phi$ in a neighborhood of $a$.

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**Theorem 4.1.3.** Assume that $X$ is a reflexive, separable Banach space, $U$ is a neighborhood of zero in $X$, and $A : U \rightarrow X^*$ is a bounded, semi-continuous mapping that satisfies the condition $\text{at}(U)$ (see item 3 of Definition 2.4.2 on page 42) and has the Frechet derivative $A' = A'(0) : X \rightarrow X^*$ at zero. Assume further that the following conditions are satisfied:

1. the equation $A'u = 0$ has only the zero solution;
2. there is a linear, everywhere defined, completely continuous mapping $\Gamma : X \rightarrow X^*$ such that $A' + \Gamma$ is strictly monotone and $T = (A' + \Gamma)^{-1} : X \rightarrow X$ is completely continuous;
3. for a sufficiently small $\varepsilon > 0$ the condition $0 \notin \text{w-cl}(\sigma_\varepsilon)$ holds where

$$\sigma_\varepsilon = \{ v \mid v = u/\|u\|, \|A'u\|Au = -\|AuA'u\|, 0 < \|u\| \leq \varepsilon \}.$$ 

Then zero is an isolated critical point of $A$ and its index is equal to $(-1)^\nu$, where $\nu$ is the sum of the algebraic multiplicities of the characteristic values of $T$, lying in the interval $[0, 1]$.

A proof of this theorem is found in Skrypnik’s books [170, Theorem 5.2, p. 49], [169, Theorem 1.4.2, p. 31], and [167, Theorem 4, p. 134]. The last book is written in the Russian language and was published in 1973.

Remark 4.1.4. 1. Note that the construction of $T : X \to X$ is essential because we need to use the theory of compact mappings $X \to X$, and both $A'$ and $A$ have the domain in $X$ and the range in $X'$.

2. It can be proved that if $X$ is a real Banach space, the Frechet derivative of a compact mapping is also compact; see Krasnoselkii [117, Lemma 4.1, p. 135]. Similar results hold for strict set contractions and $\alpha$-set contractions; see Nussbaum [142] and Istrățescu [102, Theorem 12.5.24, p. 383]. Therefore, the Leray-Schauder degree, the degree of strict set contractions, and the degree of $\alpha$-set contractions are directly applicable to the Frechet derivative. For this reason, Theorem 4.1.1 is much easier to generalize for mappings of the form ‘identity + compact’, ‘identity + strict set contraction’, or ‘identity + $\alpha$-set contraction’. In the case of $A$-proper mappings, some additional assumptions have to be made because the Frechet derivative of an $A$-proper mapping is not necessarily an $A$-proper mapping (there is a similar situation with $(S_\ast)$-mappings).

4.2 Definition of the index of $(S_\ast)$-mapping

Next we introduce the concept of a critical point, and later derive a method for the calculation of the index of a critical point. Note that the definition is specious, for the term critical point usually refers to a zero of a Gateaux or a Frechet derivative of a mapping. For lucidity of the text, it would be expedient to use the term zero of a mapping (or special point or singular point), but in this monograph we adhere to the tradition and speak about critical points. It seems that the motivation for the use of the term critical point is that it is quite natural in the case of a potential operator (that is, an operator that is the gradient of a functional); see the books [169, p. 29] and [170, p. 48] by Skrypnik.

Definition 4.2.1. A point $u_0 \in D$ is a critical point of $A : D \subset X \to X'$ if $Au_0 = 0$. A critical point $u_0 \in D$ is isolated if there is a ball $B(u_0, r) \subset D$ which contains no other critical points of $A$.

Theorem 3.3.4 implies that the degree $\text{Deg}(A, D)$ is the same for all open, bounded sets $D' \subset D$ that contain the same critical point $u_0$ of $A$ but no other critical points. Thus, it is natural to introduce the following definition.

3. for a sufficiently small $\varepsilon > 0$ the condition $0 \notin \text{w-cl}(\sigma_\varepsilon)$ holds where

$$\sigma_\varepsilon = \{ v \mid v = u/\|u\|, \|A'u\|Au = -\|AuA'u\|, 0 < \|u\| \leq \varepsilon \}.$$ 

Then zero is an isolated critical point of $A$ and its index is equal to $(-1)^\nu$, where $\nu$ is the sum of the algebraic multiplicities of the characteristic values of $T$, lying in the interval $[0, 1]$.

A proof of this theorem is found in Skrypnik’s books [170, Theorem 5.2, p. 49], [169, Theorem 1.4.2, p. 31], and [167, Theorem 4, p. 134]. The last book is written in the Russian language and was published in 1973.

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Definition 4.2.1. A point $u_0 \in D$ is a critical point of $A : D \subset X \to X'$ if $Au_0 = 0$. A critical point $u_0 \in D$ is isolated if there is a ball $B(u_0, r) \subset D$ which contains no other critical points of $A$.

Theorem 3.3.4 implies that the degree $\text{Deg}(A, D)$ is the same for all open, bounded sets $D' \subset D$ that contain the same critical point $u_0$ of $A$ but no other critical points. Thus, it is natural to introduce the following definition.
Definition 4.2.2. Assume that $A : \mathcal{D} \subset X \to X^\ast$ is a demi-continuous $(S_\ast)$-mapping and $u_0$ is an isolated critical point of $A$. If $D' \subset D$ is an open, bounded set that contains $u_0$ but no other critical points of $A$, then the number
\[
\text{Ind}(A, u_0) := \text{Deg}(A, D', 0),
\]
is the index of the isolated critical point $u_0$ of $A$.
\[
\text{Ind}(A, u_0) := \text{Deg}(A, D', 0),
\]
is the index of the isolated critical point $u_0$ of $A$.

Remark 4.2.3. In his works [169, Definition 1.4.2, p. 30], [170, Definition 5.2, p. 48], [109, Definition 2.2, p. 113], [110, Definition 1.4], and [111, Definition 1.6, p. 192], Skrypnik defines the index of the isolated critical point in a different way: the number
\[
\text{Ind}(A, u_0) := \lim_{p \to 0} \text{Deg}(A, B(u_0, p), 0)
\]
is the index of the isolated critical point $u_0$ of $A$. Evidently these definitions coincide.

In the case where $A$ has only isolated critical points, the following theorem is valid.

Theorem 4.2.4. Assume that a demi-continuous $(S_\ast)$-mapping $A : \mathcal{D} \subset X \to X^\ast$ has only isolated critical points in the open, bounded set $D$ and $Au \neq 0$ whenever $u \in \partial D$. Then $A$ has only a finite set $\{u_i\}_{i=1}^\infty$ of critical points in $D$ and the equality
\[
\text{Deg}(A, D, 0) = \sum_{i=1}^\infty \text{Ind}(A, u_i)
\]
holds.

Proof. All critical points lie in $D$ because $A$ has no zeros in the boundary of $D$.

Next, we prove that the set of all critical points is finite. To do this, we suppose that the contrary is true, that is, the set of critical points is infinite. We can then select a sequence $(u_n)_{n=1}^\infty$ from $D$ in such a way that $Au_n = 0$ for every $n = 1, 2, \ldots$. We may suppose that $u_i$ converges weakly to $u_0$ because $X$ is reflexive and $(u_n)_{n=1}^\infty$ is included in the bounded set $D \subset X$. Thus,
\[
u_n \to u_0, \quad \lim_{n \to \infty} (Au_n, u_n - u_0) = 0.
\]
Since $A$ possesses the property $(S_\ast)$ on $\mathcal{D}$, we have $u_n \to u_0 \in \mathcal{D}$. Then $Au_0 = 0$ by the demi-continuity of $A$. This means that $u_0$ is a non-isolated critical point of $A$ in $\mathcal{D}$, which is absurd because $A$ has only isolated critical points in that set.

Let $u_1, \ldots, u_I$ be the critical points of $A$ in $D$ and choose $\rho > 0$ so small that the balls $B(u_i, \rho), i = 1, \ldots, I$, do not contain other critical points of $A$ and are pairwise disjoint and included in $D$. Then $A$ has no critical points in the set
\[
D_{\rho} = D \setminus \bigcup_{i=1}^I B(u_i, \rho).
\]
Then
\[
\text{Deg}(A, D, 0) = \sum_{i=1}^I \text{Deg}(A, B(u_i, \rho), 0) = \sum_{i=1}^I \text{Ind}(A, u_i).
\]
\[
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\]
4.3 Formulation of the index theorem

In this section, we formulate the main theorem of this chapter; it is similar to Theorem 4.1.3 with the exception that it involves \((S_\infty)\)-mappings that are not necessarily bounded, \(X\) is not necessarily separable, \(\Gamma\) does not have to be compact, and \(A'\) is not necessarily the Frechet derivative of \(A\) at zero. The theorem provides a way of calculating the index of a mapping of class \((S_\infty)\) under certain hypotheses.

In the sequel, we speak about a linearization \(A'\) of a mapping \(A\). By linearization \(A'\) we mean a linear mapping that approximates a mapping \(A\) in a certain sense at a certain point. For example, the Gateaux or the Frechet derivative of a mapping at a point are linearizations of that mapping at that point. For the formulation of the main result of this chapter, we introduce certain subspaces of \(X\) connected with the mapping \(T\).

Suppose that linear mappings \(A'\) and \(\Gamma\) are such that \(A' + \Gamma\) is strictly monotone and

\[
T = (A' + \Gamma)^{-1}\Gamma
\]

is everywhere defined and compact. Hereafter, \(\lambda_1, \lambda_2, \ldots, \lambda_m\) denote the eigenvalues of \(T\) that lie in the interval \([1, \infty]\), \(\mu_1, \ldots, \mu_n \in \mathbb{Z_+}\) denote the corresponding algebraic multiplicities, and \(f_1, \ldots, f_M \in X\), where \(M = \sum_{i=1}^{m} \mu_i\), denote the corresponding generalized eigenvectors. With these assumptions and notations, Theorem 2.3.10 gives us the decomposition

\[
X = F_0 \oplus R_0
\]

The previous theorem is usually applied in the following situation. If a point is an isolated critical point of a mapping, then we may calculate the index of that critical point in a ball. In some applications, this critical point happens to be “trivial”, that is, it is zero, but it can be a nontrivial critical point also. In either case, there is also the question of the existence of other solutions. Usually, we try to answer this question by calculating the degree of that mapping in a ball with a large radius, and then comparing that degree with the index of the critical point. However, this procedure requires a great deal of luck if we do not know that the set of critical points of the mapping is bounded.

For the Brouwer and the Leray–Schauder degree, there are methods for the calculation of the degree when the radius tends to infinity. These methods almost always involve assumptions that make the set of the critical points bounded and that a mapping is defined and nonzero for all elements of \(X\) with sufficiently large norm. In that case, we say that infinity is an isolated critical (singular) point of \(A\) and we know that the degree \(\text{Deg}(A, B(0, r), 0)\) is the same for all sufficiently large \(r\). This common value is called the index of the point at infinity or the asymptotical index and is denoted by \(\text{Ind}(A, \infty)\). This concept is dealt with in book [119] by Krasnoselskii & Zabreiko; see also Zeidler’s book [196, Problem 14.9, pp. 650–651]. For recent applications of the index at infinity, see Kovalenok [115, 116] and Zabreiko & Kovalenok [190–193].
Note that in this chapter, $F_0$ has a different meaning from in the previous chapter because here $F_0$ has no connection to the set $Z(F_0, F)$ of Proposition 3.2.4. According to Theorem 2.3.10, $F_0$ and $R_0$ are closed, invariant subspaces of $T$. We denote by $\Pi$ the projection of $X$ onto $F_0$ along $R_0$, that is,

$$
\Pi(f + r) = f \quad \text{for} \quad f \in F_0, \quad r \in R_0.
$$

$\Pi$ is completely continuous because $F_0$ is a finite-dimensional subspace and $R_0$ is a closed subspace of the Banach space $X$; see Lemma 2.2.16.

**Theorem 4.3.1.** Let $X$ be a reflexive Banach space, $D \subset X$ an open, bounded set containing the origin, $A' : X \rightarrow X'$, and $A : D \subset X \rightarrow X'$. For $\varepsilon > 0$ we define the sets

$$
Z_\varepsilon = \{ u \in D | A_iu = \lambda_iAu + (1 - r)A'u = 0, \quad 0 < \|u\| \leq \varepsilon, \quad \text{for some} \quad t \in [0, 1] \},
$$

$$
\sigma_\varepsilon = \{ v | v = \|u\|u, \quad u \in Z_\varepsilon \}.
$$

Assume that the following conditions are satisfied:

1. $A$ is demi-continuous in $D$ and $A$ satisfies the condition $(S_+)$ in a ball $B(0, r_2) \subset D$;
2. $A'$ is everywhere defined, injective, linear, continuous, and quasi-monotone;
3. there is such an everywhere defined, linear, continuous, quasi-monotone mapping $\Gamma : X \rightarrow X'$ such that $\Gamma$ is strictly monotone and $T = (A' + \Gamma)^{-1} \Gamma : X \rightarrow X$

is everywhere defined and completely continuous;
4. $\Pi$ is the continuous projection of $X$ onto $F_0$ along $R_0$;
5. the mapping

$$
\Pi(A' + \Gamma)^{-1} : (A' + \Gamma)(X) \subset X' \rightarrow X
$$

is continuous;
6. there is a constant $\varepsilon_0 > 0$ such that $0 \notin w\text{-}cl(\sigma_{\varepsilon_0})$ when $\varepsilon_0 \geq \varepsilon > 0$;
7. $\|Au - A'u\|/\|u\| \rightarrow 0$ as $\|u\| \rightarrow 0$ in $Z_\varepsilon$.

Then zero is an isolated critical point of $A$ and its index is equal to $(-1)^M$, where $M$ is the sum of the algebraic multiplicities of the eigenvalues $\lambda \in ]1, \infty[ \cup ]1, \infty[$ of $T$.

**Remark 4.3.2.**

1. Note that we do not assume that $A(0) = 0$, which is a consequence of the result.
2. In Skrypnik’s results, \( A' \) is not assumed to be quasi-monotone; however, this is a consequence of the other assumptions: if \( u_n \to u_0 \), then it follows from the strict monotonicity of \( A' + \Gamma \) and the compactness of \( \Gamma \) that

\[
\lim_{n \to \infty} \langle (A' + \Gamma)u_n, u_0 - u_0 \rangle = \lim_{n \to \infty} \langle (A' + \Gamma)(u_n - u_0), u_0 - u_0 \rangle \geq 0.
\]

3. In the earlier results, \( \Gamma \) is assumed to be compact and \( A \) bounded, but in our setting \( \Gamma \) is only quasi-monotone and \( A \) does not need to be bounded.

4. The theorem does not give any method for finding \( A' \) and \( \Gamma \); moreover, there are many assumptions concerning the properties of these mappings, and the assumptions are somehow intertwined with each other.

First of all, the injective, continuous, quasi-monotone, linear mapping \( A' \) is not necessarily uniquely determined, and \( A' \) itself has an effect on the set \( \mathbb{Z}_e \) where it should approximate \( A \). Moreover, \( A' \) has also an effect on the set \( \sigma_e \), whose weak closure may not contain zero; see Equation (4.3).

Secondly, for given \( A' \) we have to find a linear, continuous, quasi-monotone mapping \( \Gamma \), with \( A' + \Gamma \) strictly monotone and \( T = (A' + \Gamma)^{-1} \Gamma \) everywhere defined and compact. So the choice of \( A' \) has an effect to the possible candidates of \( \Gamma \) and vice versa.

Thirdly, after the choice of \( A' \) and \( \Gamma \), we need to find \( R_0 \) and \( F_0 \) and the continuous projection \( \Pi \) of \( X \) onto \( F_0 \) along \( R_0 \). Then we have to show that \( \Pi(A' + \Gamma)^{-1} \) is continuous, and calculate the sum of the algebraic multiplicities of the eigenvalues of \( T \) that lie in the interval \([1, \infty]\).

The proof of Theorem 4.3.1 is complicated and requires many auxiliary results with long proofs. The proof is therefore divided into parts.

We first decompose the dual space \( X^* \) using a decomposition of \( X \) and strictly monotone mapping \( A' + \Gamma \) (the result is given in a more general setting). We also derive a certain inequality connected with the decomposition of \( X \). After this, we use the decompositions of \( X \) and \( X^* \) and the achieved inequality to find the subspaces of \( X \) that are connected with some desirable properties of certain mappings (that are later used in the construction of some homotopies). By using these subspaces, we construct two subspaces \( F_1, F_2 \) which are used in the construction of needed homotopies.

Auxiliary results and the “core” of the proof are presented in following manner. Subsection 4.4.1 contains the decomposition of the dual space. The estimate concerning the decomposition of \( X \) is presented in Section 4.4.2. In Subsection 4.4.3, some projections and a functional \( \delta \) are defined. These facts and results are essential in the construction of \( F_1 \) and in the choice of a basis for \( X \). We also study the main properties of \( \delta \) in that subsection. In subsections 4.4.4 and 4.4.5, we seek certain subspaces that are used in the construction of the subspace \( F_2 \). The “core” of the proof is in Subsection 4.4.6.

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4.4 Proof of the index theorem

4.4.1 Auxiliary results 1: a decomposition of the dual space

In this section, we decompose the dual space $X^*$ of a (reflexive) Banach space $X$ with the aid of a linear mapping $K : X \to X^*$ and the decomposition $X = F_0 \oplus R_0$, where $F_0$ is a finite-dimensional subspace of $X$. To perform this, we may not use any linear mapping, but we stipulate that the adjoint $K^* : X^* \to X^*$ is an injective mapping and the inverse $K^{-1} : K(X) \subset X^* \to X$ exists and has certain continuity property. Observe that in the later sections, $K$ is replaced by an everywhere defined and strictly monotone mapping $A' + F$.

Proof. Step 1. To prove that the sum $F_0 + R_0$ is direct and closed, we suppose that the sum does not span $X$, and after that we use the Hahn-Banach theorem to obtain a contradiction with the injectivity of $K^*$. Let $f \in F_0$. The linearity of $K$ implies that also $f$ is nonzero. Since $Kf$ belongs to the closure of $K(R_0)$, there is a sequence $(r_n)_{n=1}^{\infty}$ in $R_0$ with $Kf_n \to Kf$. Thus,

$$
\lim_{n \to \infty} K(f - r_n) = 0.
$$

By the continuity of $K^{-1} : K(X) \subset X^* \to X$,

$$
0 = \lim_{n \to \infty} K^{-1}(f - r_n) = \lim_{n \to \infty} \Pi(f - r_n) = f.
$$

This contradicts the fact $f \neq 0$, and therefore the sum is direct.

Step 2. Next we demonstrate that the space $F_0 \oplus R_0$ is closed.

Lemma 4.4.1. Let $F_0$ and $R_0$ be from (4.4) and $\Pi$ be the projection of $X$ onto $F_0$ along $R_0$. Assume that $K : X \to X^*$ is such a linear mapping that $K^* : X^* \to X^*$ is injective and $\Pi K^{-1} : K(X) \subset X^* \to X$ is continuous. Then we have

$$
X^* = F_0 \oplus R_0.
$$

The plan of the proof is the following: we first show that the sum $F_0 + R_0$ is direct and closed, then we suppose that the sum does not span $X$, and after that we use the Hahn-Banach theorem to obtain a contradiction with the injectivity of $K^*$.

Proof. Step 1. To prove that the sum $F_0 + R_0$ is direct and closed, we suppose the contrary. So the intersection $F_0 \cap R_0 = K(F_0) \cap K(R_0)$ contains a nonzero element $Kf$, where $f \in F_0$. The linearity of $K$ implies that also $f$ is nonzero. Since $Kf$ belongs to the closure of $K(R_0)$, there is a sequence $(r_n)_{n=1}^{\infty}$ in $R_0$ with $Kf_n \to Kf$. Thus,

$$
\lim_{n \to \infty} K(f - r_n) = 0.
$$

By the continuity of $K^{-1} : K(X) \subset X^* \to X$,

$$
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$$

This contradicts the fact $f \neq 0$, and therefore the sum is direct.

Step 2. Next we demonstrate that the space $F_0 \oplus R_0$ is closed.

4.4.1 Auxiliary results 1: a decomposition of the dual space

In this section, we decompose the dual space $X^*$ of a (reflexive) Banach space $X$ with the aid of a linear mapping $K : X \to X^*$ and the decomposition $X = F_0 \oplus R_0$, where $F_0$ is a finite-dimensional subspace of $X$. To perform this, we may not use any linear mapping, but we stipulate that the adjoint $K^* : X^* \to X^*$ is an injective mapping and the inverse $K^{-1} : K(X) \subset X^* \to X$ exists and has certain continuity property. Observe that in the later sections, $K$ is replaced by an everywhere defined and strictly monotone mapping $A' + F$.

Proof. Step 1. To prove that the sum $F_0 + R_0$ is direct and closed, we suppose the contrary. So the intersection $F_0 \cap R_0 = K(F_0) \cap K(R_0)$ contains a nonzero element $Kf$, where $f \in F_0$. The linearity of $K$ implies that also $f$ is nonzero. Since $Kf$ belongs to the closure of $K(R_0)$, there is a sequence $(r_n)_{n=1}^{\infty}$ in $R_0$ with $Kf_n \to Kf$. Thus,

$$
\lim_{n \to \infty} K(f - r_n) = 0.
$$

By the continuity of $K^{-1} : K(X) \subset X^* \to X$,

$$
0 = \lim_{n \to \infty} K^{-1}(f - r_n) = \lim_{n \to \infty} \Pi(f - r_n) = f.
$$

This contradicts the fact $f \neq 0$, and therefore the sum is direct.

Step 2. Next we demonstrate that the space $F_0 \oplus R_0$ is closed.
If \( w^* \in \hat{F}_0 \oplus \hat{R}_0 \), then there is a sequence \((w^*_n)_n\) \( \subset \hat{F}_0 \oplus \hat{R}_0 \) such that \( w^*_n \to w^* \) in \( X^* \). For each \( n \in \mathbb{Z}_+ \), we have the equation
\[
w^*_n = f^*_n + r^*_n
\]
where \( f^*_n \in \hat{F}_0 \) and \( r^*_n \in \hat{R}_0 \). Since \( \hat{R}_0 = K(\hat{R}_0) \), we may assume that \( r^*_n \in K(\hat{R}_0) \) for each \( n \in \mathbb{Z}_+ \). The injectivity of \( K \) and the definitions of \( \hat{F}_0 \) and \( \hat{R}_0 \) imply that there are unique sequences \((f^*_n)_n\) \( \subset \hat{F}_0 \) and \((r^*_n)_n\) \( \subset \hat{R}_0 \) with \( K(f^n) = f^*_n \) and \( K(r^n) = r^*_n \). Since \((w^*_n)_n\) \( \subset K(\hat{X}) \) converges in \( X^* \), it is a Cauchy sequence in \( X^* \). Observe that
\[
\Pi(r^n + f^n) = \Pi f^n = f^n
\]
and recall that \( \Pi K^{-1}: K(\hat{X}) \subset X^* \to X \) is continuous. The calculation
\[
f^n - f = \Pi(f^n + r^n - f^n - r^n) = \Pi K^{-1} K(f^n + r^n - f^n - r^n) = \Pi K^{-1}(w^n - w^n)
\]
implies that \((f^*_n)_n\) is a Cauchy sequence in \( \hat{F}_0 \). Because \( \hat{F}_0 \) is finite-dimensional, there is an element \( f \in \hat{F}_0 \) with \( f^n \to f \), and the restriction of \( K \) to \( \hat{F}_0 \) is a continuous mapping \( \hat{F}_0 \to \hat{F}_0 \subset X^* \). Then
\[
f^*_n = K f^n \xrightarrow{\text{def.}} K f =: f^* \in \hat{F}_0 = K(\hat{F}_0).
\]
Since
\[
\hat{R}_0 \ni r^*_n = w^* - f^* - f^* - f^* \quad \text{and since } \hat{R}_0 \text{ is closed, we have } w^* - f^* \in \hat{R}_0.
\]
Step 3. Now we show that \( X^* \) \( \neq \hat{F}_0 \oplus \hat{R}_0 \).

The inclusion \( \hat{F}_0 \oplus \hat{R}_0 \subset X^* \) is patently true, so we have to prove that \( X^* \subset \hat{F}_0 \oplus \hat{R}_0 \). We suppose the contrary, that is, \( \hat{F}_0 \oplus \hat{R}_0 \) is a proper subspace of \( X^* \) and then show that this hypothesis results in a contradiction. Since the sum is a closed, proper subspace of \( X^* \), we may use the Hahn-Banach theorem to find a nonzero element \( w \) of \( X^* \) that satisfies the condition
\[
(v^*, w) = 0 \quad \text{for all } v^* \in \hat{F}_0 \oplus \hat{R}_0.
\]
Especially,
\[
(Kw, w) = 0 \quad \text{for all } w \in X.
\]
This means that \( \langle K^* w, u \rangle = 0 \) for every \( u \in X \), that is, \( K^* w = 0 \) and the injectivity of \( K^* \) forces \( w \) to be zero—a contradiction. Thus, \( X^* = \hat{F}_0 \oplus \hat{R}_0 \).

Remark 4.4.2. At the end of this thesis, there is a counter-example which was originally given by Skrypnik & Kartasatos in paper [110, Propostion 3.5, pp. 1613–1614]. It demonstrates that if \( \Pi (A' + \Gamma)^{-1} \) is not continuous, then the decomposition \( X^* = \hat{F}_0 \oplus \hat{R}_0 \) might not hold.

This assumption does not appear in the earlier works of Skrypnik although it seems that he is forced to use this decomposition: in his books [167, p. 138], [169, p. 35], and [170, p. 51], he deduces the result (4.14) on page 84 in a similar manner to the proof of Lemma 4.4.11, but in order to do that, he needs the continuous projection of \( X^* \) along \( \hat{R}_0 \) along \( \hat{R}_0 \).
Remark 4.4.3. By using the quotient spaces of a normed linear space, one can show the following result:

If $X$ is a normed linear space, $F$ a finite-dimensional subspace of $X$, and $M$ a closed subspace of $X$, then $M + F$ is a closed in $X$.

For the proof, see Taylor & Lay [178, Theorem 5.3, p. 73]. So the result "Skrypnik, Kartsatos & Shramenko [111, Equation (2.45)] can be proved also in another way.

4.4.2 Auxiliary results 2: an inequality related to $X = F_0 \oplus R_0$

In this section, we present a result which pertains to the decomposition $X = F_0 \oplus R_0$. This result is essential when we seek finite-dimensional subspaces $F_1, F_2 \subset X$ that are suitable for the proof of Theorem 4.3.1; see subsections 4.4.4 and 4.4.5.

Recall that $T: X \rightarrow X$ is a compact operator, and $F_0$ is the subspace of $X$ spanned by the generalized eigenvectors corresponding to the eigenvalues $\lambda \in \mathbb{C}$ of $T$. Assume that $K: X \rightarrow X^*$ is injective, everywhere defined, continuous, linear mapping. The boundedness of $T$ and $K$ imply that for each $f \in F_0$ the mapping

$$t \rightarrow \|K((2t-1)f - tTf)\|$$

is continuous on the compact set $[0,1]$, and thus attains its minimum in $[0,1]$. For this minimum, we have the following result which was known by Skrypnik, Kartatsos & Shramenko2 but its proof was not given. The following result can also be generalized to the situation where $K$ is densely defined and $F_0$ is a finite-dimensional subspace of the domain of $K$. Observe that in the proof of Theorem 4.3.1 there is a strictly monotone, bounded, linear, and everywhere defined mapping $A^* + T$ in the place of $K$.

Lemma 4.4.4. Let $K: X \rightarrow X^*$ and $T: X \rightarrow X$ be such linear mappings that $T$ is compact and $K$ is injective and continuous. Assume that $F_0$ is the space spanned by the generalized eigenvectors corresponding to the eigenvalues $\lambda \in \mathbb{C}$ of $T$. Then there is a positive constant $c_1$ for which the inequality

$$\|f\| \leq c_1 \min_{\lambda \in \mathbb{C}, \|\lambda\| = 1} \|K((2t-1)f - tTf)\|$$

holds for all $f \in F_0$.

Proof. To prove the claim, we suppose the contrary and then derive a contradiction with the definition of $F_0$. Note that if $F_0 = \{0\}$, then the assertion is certainly true.

If (4.5) is not true, there is a sequence $(f_n)_{n=1}^{\infty}$ in $F_0$ with

$$0 \leq n \min_{\lambda \in \mathbb{C}, \|\lambda\| = 1} \|K((2t-1)f_n - tTf_n)\| < \|f_n\|.$$
Then, for each $n$ we have $f_n \neq 0$ and
\[ 0 \leq \min_{0 \leq t \leq 1} \left\| K \left( (2t - 1) \frac{f_n}{\|f_n\|} - tT \frac{f_n}{\|f_n\|} \right) \right\| < \frac{1}{n} \xrightarrow{n \to \infty} 0. \tag{4.6} \]
Recall that $F_0$ is finite-dimensional. Since $\left( f_n/\|f_n\| \right)_{n=1}^\infty$ belongs to the unit sphere of $F_0$, we may assume that $(f_n/\|f_n\|)_{n=1}^\infty$ converges to an element $f_0 \in F_0$ with $\|f_0\| = 1$. Note that for each $n$ the minimum is in fact assumed at a point $t_n \in [0, 1]$. We may assume that $(t_n)_{n=1}^\infty$ converges to $t_0 \in [0, 1]$. It follows from the previous reasoning, from the continuity of $K$ and $T$, and from Equation (4.6), that $f_0$ and $t_0$ also satisfy the equality
\[ K((2t_0 - 1)f_0 - t_0Tf_0) = 0. \]
Then the injectivity of $K$ implies that $(2t_0 - 1)f_0 - t_0Tf_0 = 0$. This equation has only the zero solution for any $t \in [0, 1]$. This fact is a consequence of the definition of $F_0$ and the fact that the solution set of $0 < t/(2t - 1) < 1$ is the union of $]-\infty, 0[ \cup ]1, \infty[$. \hfill \Box

### 4.4.3 Auxiliary results 3: projections and a functional

In this subsection, all the assumptions of Theorem 4.3.1 are assumed to hold. The results presented here are mostly used in the construction of the finite-dimensional subspaces $F_1, F_2, F_3, F_4 \subset X$ that are used in the construction of certain homotopies. We find $F_1$ in Subsection 4.4.4 and $F_2$ in Subsection 4.4.5. In Subsection 4.4.6, these subspaces are used in the definition of $F_3$ and $F_4$. We also use results of this subsection to find a suitable basis for $F_5$; see Lemma 4.4.20.

**Projection $\Pi$ and decomposition $X = F_0 \oplus R_0$**

Recall that $\Pi$ is the continuous projection of $X$ onto $F_0$ along $R_0$, that is,
\[ \Pi(f + r) = f \quad \text{for} \quad f \in F_0, \quad r \in R_0, \]
$\Pi$ is compact because $F_0$ is a finite-dimensional. The action of $\Pi$ is given by
\[ \Pi u = \sum_{i=1}^M (h_i, u) f_i, \tag{4.7} \]
where the functionals $h_1, \ldots, h_M \subset X^*$ satisfy the conditions:
\[ \langle h_j, f_i \rangle = \delta_{ij}, \quad \langle h_i, r \rangle = 0 \quad \text{for} \quad i, j = 1, \ldots, M, \quad r \in R_0. \tag{4.8} \]

**Decomposition $X^* = F_0^* \oplus \hat{R}_0$ and projection $P^*$**

Since $A'$ and $\Gamma$ are linear mappings with $A' + \Gamma$ strictly monotone and $\Pi \Gamma (A' + \Gamma)^{-1}$ continuous, the requirements of Lemma 4.4.1 are satisfied by $K = A' + \Gamma$. Therefore, we have
\[ X^* = F_0^* \oplus \hat{R}_0, \]
Then, for each $n$ we have $f_n \neq 0$ and
\[ 0 \leq \min_{0 \leq t \leq 1} \left\| K \left( (2t - 1) \frac{f_n}{\|f_n\|} - tT \frac{f_n}{\|f_n\|} \right) \right\| < \frac{1}{n} \xrightarrow{n \to \infty} 0. \tag{4.6} \]
Recall that $F_0$ is finite-dimensional. Since $\left( f_n/\|f_n\| \right)_{n=1}^\infty$ belongs to the unit sphere of $F_0$, we may assume that $(f_n/\|f_n\|)_{n=1}^\infty$ converges to an element $f_0 \in F_0$ with $\|f_0\| = 1$. Note that for each $n$ the minimum is in fact assumed at a point $t_n \in [0, 1]$. We may assume that $(t_n)_{n=1}^\infty$ converges to $t_0 \in [0, 1]$. It follows from the previous reasoning, from the continuity of $K$ and $T$, and from Equation (4.6), that $f_0$ and $t_0$ also satisfy the equality
\[ K((2t_0 - 1)f_0 - t_0Tf_0) = 0. \]
Then the injectivity of $K$ implies that $(2t_0 - 1)f_0 - t_0Tf_0 = 0$. This equation has only the zero solution for any $t \in [0, 1]$. This fact is a consequence of the definition of $F_0$ and the fact that the solution set of $0 < t/(2t - 1) < 1$ is the union of $]-\infty, 0[ \cup ]1, \infty[$. \hfill \Box

### 4.4.3 Auxiliary results 3: projections and a functional

In this subsection, all the assumptions of Theorem 4.3.1 are assumed to hold. The results presented here are mostly used in the construction of the finite-dimensional subspaces $F_1, F_2, F_3, F_4 \subset X$ that are used in the construction of certain homotopies. We find $F_1$ in Subsection 4.4.4 and $F_2$ in Subsection 4.4.5. In Subsection 4.4.6, these subspaces are used in the definition of $F_3$ and $F_4$. We also use results of this subsection to find a suitable basis for $F_5$; see Lemma 4.4.20.

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Recall that $\Pi$ is the continuous projection of $X$ onto $F_0$ along $R_0$, that is,
\[ \Pi(f + r) = f \quad \text{for} \quad f \in F_0, \quad r \in R_0, \]
$\Pi$ is compact because $F_0$ is a finite-dimensional. The action of $\Pi$ is given by
\[ \Pi u = \sum_{i=1}^M (h_i, u) f_i, \tag{4.7} \]
where the functionals $h_1, \ldots, h_M \subset X^*$ satisfy the conditions:
\[ \langle h_j, f_i \rangle = \delta_{ij}, \quad \langle h_i, r \rangle = 0 \quad \text{for} \quad i, j = 1, \ldots, M, \quad r \in R_0. \tag{4.8} \]

**Decomposition $X^* = F_0^* \oplus \hat{R}_0$ and projection $P^*$**

Since $A'$ and $\Gamma$ are linear mappings with $A' + \Gamma$ strictly monotone and $\Pi \Gamma (A' + \Gamma)^{-1}$ continuous, the requirements of Lemma 4.4.1 are satisfied by $K = A' + \Gamma$. Therefore, we have
\[ X^* = F_0^* \oplus \hat{R}_0, \]
where 
\[ F_0 = (A' + \Gamma)F_0 \quad \text{and} \quad R_0 = (A' + \Gamma)R_0. \]

Since \(X\) is complete, \(F_0\) finite-dimensional, and \(R_0\) closed, the projection \(P'\) of \(X'\) onto \(F_0\) along \(R_0\) is continuous and compact. The reflexivity of \(X\) implies that the action of \(P'\) is given by the formula
\[ P'h' = \sum_{i=1}^{M} (h', w_i)(A' + \Gamma)f_i, \quad (4.9) \]
where the vectors \(w_1, \ldots, w_M \subset X\) satisfy
\[ \langle (A' + \Gamma)f_j, w_i \rangle = \delta_{ij}, \quad \langle r^*, w_i \rangle = 0, \quad (4.10) \]
for \(i, j = 1, \ldots, M\) and \(r^* \in R_0\).

Another decomposition of \(X\) and the projection \(P\)

We need a decomposition of \(X\) connected with \(P'\). If
\[ Pu = \sum_{i=1}^{M} \langle (A' + \Gamma)f_i, u \rangle w_i, \quad u \in X, \]
then it is easy to see that \(P\) is a projection of \(X\) onto \(\text{span}\{w_1, \ldots, w_M\}\). The calculation
\[ \langle h', Pf \rangle = \left(h', \sum_{i=1}^{M} \langle (A' + \Gamma)f_i, f \rangle w_i \right) = \sum_{i=1}^{M} \langle (A' + \Gamma)f_i, h' \rangle \langle w_i, f \rangle = \langle P'h', f \rangle \]
shows that \(P'^*\) is the adjoint of \(P\). Moreover, \(P\) is also compact because it is everywhere defined, continuous, and has a finite-dimensional range. According to Theorem 2.2.21, we have \(\text{span}\{w_1, \ldots, w_M\} = \mathcal{H}(P) = \mathcal{A}'(P'^*) = (\mathcal{R}_0)^\perp\) and \(\mathcal{A}'(P) = \mathcal{A}(P'^*) = (\mathcal{F}_0)^\perp\). Therefore,
\[ X = \mathcal{H}(P) \oplus \mathcal{A}'(P) = (\mathcal{R}_0)^\perp \oplus (\mathcal{F}_0)^\perp. \]

Functional \(\delta\)

Now we direct our attention to a functional \(\delta\) defined by the mappings \(P' : X' \to X^*, A : \mathcal{D} \subset X \to X^*, \) and \(T : (A' + \Gamma)^{-1} : X \to X\). For fixed \(u \in \mathcal{D}\), the function
\[ \tau \mapsto \langle (I - P')Au, u - \tau Tu \rangle \]
is continuous in the compact set \([0, 1]\) and thus attains its minimum in that interval. So we may define the functional \(\delta : \mathcal{D} \subset X \to \mathbb{R}_0 = \mathbb{R}_+ \cup \{0\}\) by
\[ \delta(u) = \max \left\{ 0, c \min_{0 \leq \tau \leq 1} \langle (I - P')Au, (I - \tau T)u \rangle \right\}, \quad (4.11) \]
where \(c > 0\) is a constant. The admissible values for the constant \(c\) are restricted to a certain interval, we return to this question in Remark 4.4.7.
Lemma 4.4.5. The functional ϐ is continuous in $\mathcal{D}$.

Proof. Suppose that $(u_n)_{n=1}^\infty \subset \mathcal{D}$ converges to $u_0 \in \mathcal{D}$. To prove the theorem, it is sufficient to examine the behavior of the sequence

$$\left( \min_{\theta \in C_1} (|I - P^\tau|Au_n, |I - \tau T|u_n) \right)_{n=1}^\infty.$$ 

Let $\tau_0 \in [0, 1]$ be a number such that

$$\langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle = \min_{\theta \in C_1} (|I - P^\tau|Au_n, |I - \tau_0 T|u_n).$$

Note that this $\tau_0$ might not be unique.

We shall use the subsequence argument. We thus pick an arbitrary subsequence of $\left( (|I - P^\tau|Au_n, |I - \tau T|u_n) \right)_{n=1}^\infty$ and denote it again by $\left( (|I - P^\tau|Au_n, |I - \tau_0 T|u_n) \right)_{n=1}^\infty$. Since $(\tau_0)_{n=1}^\infty$ is bounded in $\mathbb{R}$, we may assume that it converges to some number $\tau'$. This is possible because we are using the subsequence argument. It follows from the definition of $\tau_0$ that

$$\langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle \leq \langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle$$

holds for every $n \in \mathbb{Z}^+$. This, the continuity of $T$ and $P^\tau$, and the demi-continuity of $A$ imply that

$$\langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle \leq \langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle.$$

Since $\tau_0$ is a point where $\tau \mapsto \langle (I - P^\tau)Au_n, u_0 - \tau Tu_0 \rangle$ attains its minimum, $\tau'$ is also a point where this function attains its minimum. Note that $\tau_0$ does not have to be equal to $\tau'$. However, the reasoning shows that

$$\langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle \xrightarrow{n \to \infty} \langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle,$$

which is sufficient for our purposes.

We have proved that any subsequence of

$$\left( \min_{\theta \in C_1} (|I - P^\tau|Au_n, |I - \tau T|u_n) \right)_{n=1}^\infty$$

has a subsequence that converges to $\langle (I - P^\tau)Au_0, u_0 - \tau_0 Tu_0 \rangle$. Consequently, the subsequence argument yields that the original sequence converges to that limit. Therefore, $\delta(u_n) \to \delta(u_0)$ as $n \to \infty$. □

Lemma 4.4.6. If $A : \mathcal{D} \subset X \to X^*$ and $\mathcal{D}$ are bounded, then $\delta : \mathcal{D} \subset X \to \mathbb{R}_0$ is bounded.

Proof. The claim follows from the calculation

$$|\delta(u)| \leq c \min_{\theta \in C_1} (|I - P^\tau|Au_n, |I - \tau T|u_n) \leq c \|I - P^\tau\| \|Au\| \|u\|.$$

Remark 4.4.7. 1. The continuity of $\delta$ is also used (without proving the continuity) in Skrypnik’s books [167, pp. 136–139], [169, pp. 32–37], and [170, pp. 50–53]. In the Kartsatos, Skrypnik & Shramenko’s articles [110, 111], a more general result is used.

Lemma 4.4.5. The functional $\delta$ is continuous in $\mathcal{D}$.

Proof. Suppose that $(u_n)_{n=1}^\infty \subset \mathcal{D}$ converges to $u_0 \in \mathcal{D}$. To prove the theorem, it is sufficient to examine the behavior of the sequence

$$\left( \min_{\theta \in C_1} (|I - P^\tau|Au_n, |I - \tau T|u_n) \right)_{n=1}^\infty.$$ 

Let $\tau_0 \in [0, 1]$ be a number such that

$$\langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle = \min_{\theta \in C_1} (|I - P^\tau|Au_n, |I - \tau_0 T|u_n).$$

Note that this $\tau_0$ might not be unique.

We shall use the subsequence argument. We thus pick an arbitrary subsequence of $\left( (|I - P^\tau|Au_n, |I - \tau_0 T|u_n) \right)_{n=1}^\infty$ and denote it again by $\left( (|I - P^\tau|Au_n, |I - \tau_0 T|u_n) \right)_{n=1}^\infty$. Since $(\tau_0)_{n=1}^\infty$ is bounded in $\mathbb{R}$, we may assume that it converges to some number $\tau'$. This is possible because we are using the subsequence argument. It follows from the definition of $\tau_0$ that

$$\langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle \leq \langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle$$

holds for every $n \in \mathbb{Z}^+$. This, the continuity of $T$ and $P^\tau$, and the demi-continuity of $A$ imply that

$$\langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle \leq \langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle.$$

Since $\tau_0$ is a point where $\tau \mapsto \langle (I - P^\tau)Au_n, u_0 - \tau Tu_0 \rangle$ attains its minimum, $\tau'$ is also a point where this function attains its minimum. Note that $\tau_0$ does not have to be equal to $\tau'$. However, the reasoning shows that

$$\langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle \xrightarrow{n \to \infty} \langle (I - P^\tau)Au_n, u_0 - \tau_0 Tu_0 \rangle,$$

which is sufficient for our purposes.

We have proved that any subsequence of

$$\left( \min_{\theta \in C_1} (|I - P^\tau|Au_n, |I - \tau T|u_n) \right)_{n=1}^\infty$$

has a subsequence that converges to $\langle (I - P^\tau)Au_0, u_0 - \tau_0 Tu_0 \rangle$. Consequently, the subsequence argument yields that the original sequence converges to that limit. Therefore, $\delta(u_n) \to \delta(u_0)$ as $n \to \infty$. □

Lemma 4.4.6. If $A : \mathcal{D} \subset X \to X^*$ and $\mathcal{D}$ are bounded, then $\delta : \mathcal{D} \subset X \to \mathbb{R}_0$ is bounded.

Proof. The claim follows from the calculation

$$|\delta(u)| \leq c \min_{\theta \in C_1} (|I - P^\tau|Au_n, |I - \tau T|u_n) \leq c \|I - P^\tau\| \|Au\| \|u\|.$$

Remark 4.4.7. 1. The continuity of $\delta$ is also used (without proving the continuity) in Skrypnik’s books [167, pp. 136–139], [169, pp. 32–37], and [170, pp. 50–53]. In the Kartsatos, Skrypnik & Shramenko’s articles [110, 111], a more general result is used.
2. As $A$ is an everywhere defined, demi-continuous mapping, it is locally bounded. So there is a constant $r_1 > 0$ with $A(B(0,r_1))$ bounded. In such a case, Lemma 4.4.6 implies that $\delta$ is bounded in $B(0,r_1).

3. In the proof of Theorem 4.3.1, the constant $c$ in the definition of $\delta$ is required to satisfy the condition

$$c \leq \epsilon_1 := \frac{1}{2} (c_1 \epsilon_2^2 ||I-P^*||(1+||T||))^{-1}$$

where $c_1$ is the constant from Lemma 4.4.4, $c_2 = \sup\{||Au|| : u \in B(0,r_1)\}$, and $r_1 > 0$ is as in the previous item of this remark.

### 4.4.4 Auxiliary results 4: the subspace $F_1$

In this subsection, we assume that all the assumptions of Theorem 4.3.1 hold. Recall that $r_2 > 0$ is such that $A(B(0,r_1))$ is bounded; see Remark 4.4.7. We choose the constant $c$ in the definition of $\delta$ (see Equation (4.11) on page 77) so that

$$c \leq \epsilon_1 := \frac{1}{2} (c_1 \epsilon_2^2 ||I-P^*||(1+||T||))^{-1}$$

where $c_1$ is the constant from Lemma 4.4.4 and $c_2 = \sup\{||Au|| : u \in B(0,r_1)\}$. Keep in mind that the constant $c$ has the same value also in Subsection 4.4.5 and Subsection 4.4.6.

In this subsection, we employ the auxiliary results presented in subsections 4.4.1, 4.4.2, and 4.4.3. The main result of this subsection is that there is a finite-dimensional subspace $F_1 \subset X$ with the property: if

$$E_1^1u = (\delta(u) + r)Au + (1-r)A'u,$$

then

$$Z'(F_1,F) := \{(u,t) \in \partial B(0,r_2) \cap F \times [0,1] \mid \langle E_1^1u,u \rangle \leq 0 \text{ and } \langle E_1^1u,v \rangle = 0 \text{ for all } v \in F_1\}$$

is empty for any finite-dimensional subspace $F \supset F_1$. This result is crucial in the proof of Lemma 4.4.17.

**Lemma 4.4.8.** Assume that all the assumptions of Theorem 4.3.1 hold. Denote

$$E_1^1u := (\delta(u) + r)Au + (1-r)A'u.$$  

There is a constant $r_2 \in [0,r_1]$ such that $E_1^1u \neq 0$ for all $t \in [0,1]$ and $0 < ||u|| \leq r_2$. Especially, if $0 < r \leq r_2$, then $E_1^1u \neq 0$ for all $(u,t) \in \partial B(0,r) \times [0,1]$.

**Proof.** We suppose that the assertion of this lemma is false. This implies that there are sequences $(u_n)_{n=1}^{\infty}$ in $[0,1]$ and $(u_n)_{n=1}^{\infty}$ in $D$ fulfilling the conditions:

$$(\delta(u_n) + r_n)Au_n + (1-r_n)A'u_n = 0, \quad u_n \neq 0, \quad t_n \xrightarrow{\text{w}} t_0, \quad u_n \xrightarrow{\text{w}} 0.$$
Observe that
\[ \frac{\delta(u_0) + t_n}{1 + \delta(u_0)} + \frac{1 - t_n}{1 + \delta(u_0)} = 1. \]
Recall that there is such a positive constant \( \varepsilon_0 \) that \( 0 \notin \text{w-cl}(\sigma_g) \) when \( 0 < \varepsilon \leq \varepsilon_0 \). Let \( 0 < \varepsilon \leq \varepsilon_0 \). Evidently, there is an integer \( N \) such that \( 0 < \|u_0\| \leq \varepsilon \) when \( n \geq N \). This implies that \( u_n \) is an element of \( ZE \) when \( n \geq N \). Inasmuch as
\[ \{ v'_n \mid v'_n = u_n/\|u_n\|, n \geq N \} \]
is a subset of \( \sigma_g \), its weak closure cannot contain zero. On the basis of this, the reflexivity of \( X \), and the boundedness of \( (v_n)_{n=1}^{\infty} \), we may assume that \( v'_n \rightarrow v'_0 \) as \( n \to \infty \).
Since \( (u_n)_{n=1}^{\infty} \) is contained in \( ZE, u_n \to 0 \), and \( (Au - A' u)/\|u\| \to 0 \) as \( \|u\| \to 0 \) in \( ZE \), we deduce that
\[ A' u_n - A u_n \xrightarrow{\|u_n\|} 0. \]
Furthermore, according to Lemma 4.4.5, \( \delta(u_n) \to \delta(0) = 0 \). A consequence of these results and the continuity of \( A' \) is that the computation
\[
\langle A' v'_n, v \rangle = \lim_{n \to \infty} \left( (1 + \delta(u_n)) A' u_n - (1 - \delta(u_n)) A' u_n, v \right) = 0
\]
holds for all \( v \in X \). Hence, \( A' v'_0 = 0 \) with \( v'_0 \neq 0 \), which is against the injectivity of \( A' \).
Consequently, the assertion of the lemma is proved.

**Lemma 4.4.9.** Assume that all the assumptions of Theorem 4.3.1 hold and that \( r_2 \) is given by Lemma 4.4.8. If
\[ E^1_2 u = (\delta(u) + t) A u + (1 - t) A' u, \]
then there exists \( F_1 \in F(X) \) such that
\[ Z'(F_1, F) := \{ (u, t) \in \partial R(0, r_2, r_3) \cap F \times [0, 1] \mid \langle E^1_2 u, u \rangle \leq 0 \text{ and } \langle E^1_2 u, v \rangle = 0 \text{ for all } v \in F_1 \} \]
is empty for any \( F \in F(X) \) with \( F \supseteq F_1 \).

**Remark 4.4.10.** The conclusion of Lemma 4.4.9 does not follow from Proposition 3.2.4 because \( E^1_2 \) is not necessarily an \( (S_+) \)-mapping when \( t = 0 \), and hence \( E^1_2 \) might not satisfy the condition \( (S_+) \).

The proof of Lemma 4.4.9 goes as follows. We suppose that the claim fails to hold, and construct a family of nonempty bounded sets whose weak closures have the finite

Observe that
\[ \frac{\delta(u_0) + t_n}{1 + \delta(u_0)} + \frac{1 - t_n}{1 + \delta(u_0)} = 1. \]
Recall that there is such a positive constant \( \varepsilon_0 \) that \( 0 \notin \text{w-cl}(\sigma_g) \) when \( 0 < \varepsilon \leq \varepsilon_0 \). Let \( 0 < \varepsilon \leq \varepsilon_0 \). Evidently, there is an integer \( N \) such that \( 0 < \|u_0\| \leq \varepsilon \) when \( n \geq N \). This implies that \( u_n \) is an element of \( ZE \) when \( n \geq N \). Inasmuch as
\[ \{ v'_n \mid v'_n = u_n/\|u_n\|, n \geq N \} \]
is a subset of \( \sigma_g \), its weak closure cannot contain zero. On the basis of this, the reflexivity of \( X \), and the boundedness of \( (v_n)_{n=1}^{\infty} \), we may assume that \( v'_n \rightarrow v'_0 \) as \( n \to \infty \).
Since \( (u_n)_{n=1}^{\infty} \) is contained in \( ZE, u_n \to 0 \), and \( (Au - A' u)/\|u\| \to 0 \) as \( \|u\| \to 0 \) in \( ZE \), we deduce that
\[ A' u_n - A u_n \xrightarrow{\|u_n\|} 0. \]
Furthermore, according to Lemma 4.4.5, \( \delta(u_n) \to \delta(0) = 0 \). A consequence of these results and the continuity of \( A' \) is that the computation
\[
\langle A' v'_n, v \rangle = \lim_{n \to \infty} \left( (1 + \delta(u_n)) A' u_n - (1 - \delta(u_n)) A' u_n, v \right) = 0
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holds for all \( v \in X \). Hence, \( A' v'_0 = 0 \) with \( v'_0 \neq 0 \), which is against the injectivity of \( A' \).
Consequently, the assertion of the lemma is proved.

**Lemma 4.4.9.** Assume that all the assumptions of Theorem 4.3.1 hold and that \( r_2 \) is given by Lemma 4.4.8. If
\[ E^1_2 u = (\delta(u) + t) A u + (1 - t) A' u, \]
then there exists \( F_1 \in F(X) \) such that
\[ Z'(F_1, F) := \{ (u, t) \in \partial R(0, r_2, r_3) \cap F \times [0, 1] \mid \langle E^1_2 u, u \rangle \leq 0 \text{ and } \langle E^1_2 u, v \rangle = 0 \text{ for all } v \in F_1 \} \]
is empty for any \( F \in F(X) \) with \( F \supseteq F_1 \).

**Remark 4.4.10.** The conclusion of Lemma 4.4.9 does not follow from Proposition 3.2.4 because \( E^1_2 \) is not necessarily an \( (S_+) \)-mapping when \( t = 0 \), and hence \( E^1_2 \) might not satisfy the condition \( (S_+) \).

The proof of Lemma 4.4.9 goes as follows. We suppose that the claim fails to hold, and construct a family of nonempty bounded sets whose weak closures have the finite
intersection property. As the weak closures are included in a weakly compact set, they have at least one element \((u_0, t_0)\) in common. We demonstrate that for every choice of a finite-dimensional subspace \(F^w\) of \(X\) there exists a sequence \((u_n^w, t_n^w)_{n=1}^\infty\) with some sought-after properties, as well as

\[
\frac{u_n^w}{t_n^w} \rightarrow u_0 \quad \text{and} \quad \frac{t_n^w}{t_0} \rightarrow t_0.
\]

These three phases are almost completely omitted because the proof of Proposition 3.2.4 contains the details. Now comes the most crucial phase of the proof. First, we show that \(\delta_1 + t_0 \neq 0\) if \(\delta \frac{u_n^w}{t_n^w} \rightarrow \delta_0\), and if a finite-dimensional subspace \(F_n^w\) contains two specific elements and an arbitrary element \(w \in X\). Using this result, we can apply the \((S_1)\)-property of \(A\) to \((u_n^w)_{n=1}^\infty\) to obtain \(u_n^w \frac{t_n^w}{t_0} \rightarrow u_0\). Then the continuity properties of \(A\) and \(A'\) yield the equality \((E_0^0, w) = 0\). By varying \(w\), we show that \((u_0, t_0)\) is a zero of \(E_0^1\) in \(\partial B(0, r_2) \times [0,1]\).

**Proof of Lemma 4.4.9.** The proof of this claim is mostly the same as the proof of Proposition 3.2.4; steps 1–3 are exactly the same (some notational differences exist, of course) and are therefore omitted. We continue from the situation where we have proved that there exists an element \((u_0, t_0)\in \mathbb{X}\times \mathbb{R}\) such that

\[
(u_0, t_0) \in \bigcap_{F \in \mathbb{F}(X)} w-\text{cl}(G(F)).
\]

**Step 4.** We demonstrate that \(u_0\) belongs to \(\partial B(0, r_2)\) and satisfies \(E_{0}^1, u_0 = 0\). These results violate the conclusion of the Lemma 4.4.8, and thus constitute a desired contradiction.

We pick an arbitrary element \(w\in X\) and then let \(F_n^w \in F(X)\) be such that

\[
\frac{u_0, v', w}{F_n^w} \in B(0, 1) \subset X \text{ satisfies } ||A'w|| = ||A'u_0, v'||. \quad \text{According to Theorem 2.2.2, } v' \text{ really exists in } X. \quad \text{The superindex } w \text{ indicates that } F_n^w \text{ is related to } w. \quad \text{As in step 4 of the proof of Proposition 3.2.4, we obtain the existence of sequences } \left(\frac{E_n^0}{t_n^w}\right)_{n=1}^\infty \text{ in } F_1^w \text{ with } \frac{E_0^0}{u_0} = \frac{t_n^w}{t_0} \rightarrow u_0, \quad \frac{E_n^0}{v'} \rightarrow v',
\]

\[
(u_n^w, t_n^w) \in Z'(F_n^w, F_n^w), \quad \text{and } F_n^w \subset F_n^w. \quad \text{According to the definition of } Z'(F_n^w, F_n^w), \text{ we have } u_n^w \in \partial B(0, r_2) \cap F_n^w \text{ as well as } \left\langle E_n^0, u_n^w, v'\right\rangle \leq 0 \quad \text{and } \left\langle E_n^0, u_n^w, v\right\rangle \leq 0 \quad \text{for all } \nu \in F_n^w. \quad (4.12)
\]

Since \(A\) is bounded in \(B(0, r_2)\), also \(\delta\) is bounded in that set; thus, we may assume that \(\delta \frac{u_n^w}{t_n^w} \rightarrow \delta_0 \text{ as } n \rightarrow \infty\).

**Step 4.1.** We demonstrate that the case \(\delta_1 + t_0 = 0\) is impossible if \(u_0, v' \in F_n^w\).

As \(\delta_1 + t_0 = 0\) and both \(\delta_0\) and \(t_0\) are nonnegative, \(t_0 = \delta_0 = 0\). First we show that this implies \(u_0 = 0\), and then we show that this conclusion leads to a contradiction.
For every $v \in F^n_v$ we have
\[ \langle A^2w_0, v \rangle = \lim_{n \to 0} \langle A^2u_n^+, v \rangle = \lim_{n \to 0} \langle (E_{1,n}^+ v) - (\delta(u_n^+) + t_n^+)Au_n, v \rangle = 0 + 0 = 0 \]
because $A$ and $A'$ are bounded in $\overline{B(0, r_2)}$. Especially, for $v'$
\[ \|A^2w_0\| = \|A^2w_0, v'\| = 0. \]
Thus, $A^2w_0 = 0$ and the injectivity of $A'$ forces $w_0$ to be zero. So $u_n^+ \to u_0 = 0$ as $n \to \infty$.
Since $\delta(u_n^+) \to 0$ as $n \to \infty$, the definition of $\delta$ yields the existence of numbers $\tau_n^+ \in [0, 1], n = 1, \ldots, \infty$, such that
\[ \lim_{n \to 0} ((I - P^+)Au_n^+, u_n^+ - \tau_n^+ Tu_n^+) \leq 0. \]
Because $(\tau_n^+)^{n=1} \subset [0, 1], u_n^+ \overset{\text{strong}}{\to} u_0 = 0$, both $T$ and $P$ are compact, and $A$ is bounded in $\overline{B(0, r_2)}$, we get
\[ 0 \geq \lim_{j \to 0} ((I - P^+)Au_n^+, u_n^+ - \tau_n^+ Tu_n^+) = \lim_{j \to 0} ((Au_n^+, u_n^+ - \tau_n^+ Tu_n^+) = \lim_{j \to 0} (Au_n^+, u_n^+). \]
This and the condition $(S_\ast)$ of $A$ give $u_n^+ \overset{\text{a.e.}}{\to} u_0 = 0$, which is absurd because $u_n^+$ is in $\partial B(0, r_2)$ for every $n$.
Hence, the case $\delta_3 + t_0 = 0$ is always impossible when $u_0, v' \in F^\ast_v$.

Step 4.2. By using the result of the previous step, we show that also the case $\delta_0 + t_0 > 0$ is impossible when $u_0, v' \in F^\ast_v$.
Because $u_0 \in F^\ast_v, A$ is bounded in $B(0, r_2)$, and $A'$ is quasi-monotone, we can compute in the following way
\[ 0 \geq \lim_{j \to 0} ((E_{1,n}^+ u_0, u_n^+ - u_0) \text{ by (4.12)} = \lim_{n \to 0} ((\delta(u_n^+) + t_n^+)Au_n^+ + (1 - t_n^+)A'u_n^+, u_n^+ - u_0) \geq \lim_{n \to 0} (\delta(u_n^+) + t_n^+)Au_n^+ (u_n^+ - u_0) = (\delta_0 + t_0) \lim_{n \to 0} (Au_n^+, u_n^+ - u_0). \]
This and the $(S_\ast)$-property of $A$ cause $u_n^+$ to converge strongly to $u_0$ in $X$. Then $u_0$ is in $\partial B(0, r_2)$ because $\partial B(0, r_2)$ is closed. The continuity of $A'$ and $\delta$, and the demi-continuity of $A$ imply
\[ E_{1,n}^+ u_n^+ = (\delta(u_n^+) + t_n^+)Au_n^+ + (1 - t_n^+)A'u_n^+ \overset{\text{a.e.}}{\to} (\delta(u_0) + t_0)Au_0 + (1 - t_0)A'u_0 = E_{1,n}^+ u_0. \]
Especially,
\[ 0 = (E_{1,n}^+ u_n^+, w) \overset{\text{a.e.}}{\to} (E_{1,n}^+ u_0, w) \]
For every $v \in F^\ast_v$ we have
\[ \langle A^2w_0, v \rangle = \lim_{n \to 0} \langle A^2u_n^+, v \rangle = \lim_{n \to 0} \langle (E_{1,n}^+ v) - (\delta(u_n^+) + t_n^+)Au_n, v \rangle = 0 + 0 = 0 \]
because $A$ and $A'$ are bounded in $\overline{B(0, r_2)}$. Especially, for $v'$
\[ \|A^2w_0\| = \|A^2w_0, v'\| = 0. \]
Thus, $A^2w_0 = 0$ and the injectivity of $A'$ forces $w_0$ to be zero. So $u_n^+ \to u_0 = 0$ as $n \to \infty$.
Since $\delta(u_n^+) \to 0$ as $n \to \infty$, the definition of $\delta$ yields the existence of numbers $\tau_n^+ \in [0, 1], n = 1, \ldots, \infty$, such that
\[ \lim_{n \to 0} ((I - P^+)Au_n^+, u_n^+ - \tau_n^+ Tu_n^+) \leq 0. \]
Because $(\tau_n^+)^{n=1} \subset [0, 1], u_n^+ \overset{\text{strong}}{\to} u_0 = 0$, both $T$ and $P$ are compact, and $A$ is bounded in $\overline{B(0, r_2)}$, we get
\[ 0 \geq \lim_{j \to 0} ((I - P^+)Au_n^+, u_n^+ - \tau_n^+ Tu_n^+) = \lim_{j \to 0} ((Au_n^+, u_n^+ - \tau_n^+ Tu_n^+) = \lim_{j \to 0} (Au_n^+, u_n^+). \]
This and the condition $(S_\ast)$ of $A$ give $u_n^+ \overset{\text{a.e.}}{\to} u_0 = 0$, which is absurd because $u_n^+$ is in $\partial B(0, r_2)$ for every $n$.
Hence, the case $\delta_3 + t_0 = 0$ is always impossible when $u_0, v' \in F^\ast_v$.

Step 4.2. By using the result of the previous step, we show that also the case $\delta_0 + t_0 > 0$ is impossible when $u_0, v' \in F^\ast_v$.
Because $u_0 \in F^\ast_v, A$ is bounded in $B(0, r_2)$, and $A'$ is quasi-monotone, we can compute in the following way
\[ 0 \geq \lim_{j \to 0} ((E_{1,n}^+ u_n, u_n^+ - u_0) \text{ by (4.12)} = \lim_{n \to 0} ((\delta(u_n^+) + t_n^+)Au_n + (1 - t_n^+)A'u_n^+, u_n^+ - u_0) \geq \lim_{n \to 0} (\delta(u_n^+) + t_n^+)(Au_n^+, u_n^+ - u_0) = (\delta_0 + t_0) \lim_{n \to 0} (Au_n^+, u_n^+ - u_0). \]
This and the $(S_\ast)$-property of $A$ cause $u_n^+$ to converge strongly to $u_0$ in $X$. Then $u_0$ is in $\partial B(0, r_2)$ because $\partial B(0, r_2)$ is closed. The continuity of $A'$ and $\delta$, and the demi-continuity of $A$ imply
\[ E_{1,n}^+ u_n^+ = (\delta(u_n^+) + t_n^+)Au_n^+ + (1 - t_n^+)A'u_n^+ \overset{\text{a.e.}}{\to} (\delta(u_0) + t_0)Au_0 + (1 - t_0)A'u_0 = E_{1,n}^+ u_0. \]
Especially,
\[ 0 = (E_{1,n}^+ u_n^+, w) \overset{\text{a.e.}}{\to} (E_{1,n}^+ u_0, w) \]
because \( w \in F_0^n \) and \( u_0^n \in Z'(F_0^n, F_0^n) \) for each \( n \in \mathbb{Z}_+ \).

As \( w \in X \) was arbitrary and for any choice of \( w \) recall that the conclusion of the step 4.1 is that the case \( \delta_0 + t_0 = 0 \) does not occur when \( u_0, v \in F_0^n \) the achieved result is
\[
0 = (E^n_{t_0} u_0, w),
\]
the conclusion is that
\[
E^n_{t_0} u_0 = 0.
\]
Since \( u_0 \in \partial B(0, r_2) \), we have an outcome that violates the conclusions of Lemma 4.4.8 and the choice of \( r_2 \). This contradiction ends the proof. \( \square \)

4.4.5 Auxiliary results 5: the subspace \( F_2 \)

In this subsection, we assume that all the assumptions of Theorem 4.3.1 hold. Recall that \( r_1 > 0 \) is such that \( A(B(0, r_1)) \) is bounded (see Remark 4.4.7). Here the constant \( c \) in the definition of \( \delta \) (see Equation 4.11 on page 77) has the same value as in Subsection 4.4.4, and \( r_2 \in [0, r_1] \) is given by Lemma 4.4.8. So we have the inequality:
\[
c \leq \tau_1 := \frac{1}{2} \left( \frac{\varepsilon_1}{c_1} \right)^2 \left\| P \right\| \left\| I - P \right\| \left(1 + \| T \|\right)^{-1}
\]
where \( c_1 \) is the constant from Lemma 4.4.4 and \( c_2 = \sup \{ \| A u \| \mid u \in \overline{B(0, r_1)} \} \).

The main result of this subsection is the existence of a finite-dimensional subspace \( F_2 \subset X \) with the property: if
\[
E^n_{t} u = \delta(u)Au + A'u + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u,
\]
then
\[
Z'(F_2, F) := \{ (u, t) \in \partial B(0, r_2) \cap F \times [0, 1] \mid \langle E^n_{t} u, u \rangle \leq 0 \text{ and } \langle E^n_{t} u, v \rangle = 0 \text{ for all } v \in F_2 \}
\]
is empty for any finite-dimensional subspace \( F \supset F_2 \). This result is essential in the proofs of Lemma 4.4.18 and Lemma 4.4.19.

Again the auxiliary results of subsections 4.4.1, 4.4.2, and 4.4.3 are utilized.

**Lemma 4.4.11.** Assume that the assumptions of Theorem 4.3.1 hold, and that \( r_2 \) is given by Lemma 4.4.8. Then
\[
0 \neq E^n_{t} u := tA' + \delta(u)Au + (1-t)[\langle A' + \Gamma \rangle(I - \Pi)u - \langle A' + \Gamma \rangle\Pi u] = \delta(u)Au + A'u + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u
\]
for all \( t \in [0, 1] \) and \( 0 < \| u \| \leq r_2 \). In particular, \( E^n_{t} u \neq 0 \) for \( (t, u) \in [0, 1] \times \partial B(0, r) \) where \( 0 < r \leq r_2 \).

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then
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is empty for any finite-dimensional subspace \( F \supset F_2 \). This result is essential in the proofs of Lemma 4.4.18 and Lemma 4.4.19.

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\]
for all \( t \in [0, 1] \) and \( 0 < \| u \| \leq r_2 \). In particular, \( E^n_{t} u \neq 0 \) for \( (t, u) \in [0, 1] \times \partial B(0, r) \) where \( 0 < r \leq r_2 \).
In the proof, we suppose that the claim is not true and then derive a contradiction with other assumptions. We first derive an equality and then divide the proof into two separate cases. In the first case, we assume that $\delta(u_0) = 0$ and achieve a contradiction with the definition of $F_0$ and $R_0$. In the second case, $\delta(u_0) \neq 0$ and we derive a contradiction with the choice of $c$.

**Proof of Lemma 4.4.11.** Step 1. To prove the assertion, we suppose the contrary; that is, there are $u_0 \in X$ and $t_0 \in [0, 1]$ such that

$$E_0^2 u_0 = 0, \quad 0 < \|u_0\| \leq r_2.$$ 

We introduce the notations $f_0 = \Pi u_0$ and $r_0 = (I - \Pi)u_0$. Since

$$A' u_0 = (A' + \Gamma) u_0 - \Gamma u_0 = (A' + \Gamma)(I - T)u_0,$$

we have

$$0 = E_0^2 u_0 = \delta(u_0) A u_0 + t_0 A' u_0 + (1 - t_0) ((A' + \Gamma)(I - \Pi)u_0 - (A' + \Gamma)\Pi u_0)
= \delta(u_0) A u_0 + (A' + \Gamma)(t_0 (I - T)u_0 - (1 - t_0) f_0 + (1 - t_0) r_0)
= \delta(u_0) A u_0 + (A' + \Gamma)((2t_0 - 1) f_0 - t_0 T f_0 + r_0 - t_0 T r_0).$$

The case $t_0 = 1$ is impossible due to the conclusion of Lemma 4.4.8, the choice of $r_2$. So, hereafter in this proof, we assume that $t_0 \in [0, 1]$. Considering that $F_0$ and $R_0$ are invariant subspaces of $T$, the previous result implies that

$$(2t_0 - 1) f_0 - t_0 T f_0 \in F_0 \quad \text{and} \quad r_0 - t_0 T r_0 \in R_0.$$ 

We merely use the definitions of $R_0$ and $F_0$ and the projection $P^*$ of $X^*$ onto $F_0$ along $R_0$ to obtain

$$0 = \delta(u_0) P^* A u_0 + (A' + \Gamma)((2t_0 - 1) f_0 - t_0 T f_0),$$

$$0 = \delta(u_0) (I - P^*) A u_0 + (A' + \Gamma)(r_0 - t_0 T r_0).$$

(4.14)

**Step 2.** We first consider the case $\delta(u_0) = 0$. The equations in (4.14) are equivalent to

$$(A' + \Gamma)((2t_0 - 1) f_0 - t_0 T f_0) = 0 \quad \text{and} \quad (A' + \Gamma)(r_0 - t_0 T r_0) = 0.$$ 

Since $\langle (A' + \Gamma) u, u \rangle > 0$ whenever $u \neq 0$, the equations

$$(2t_0 - 1) f_0 - t_0 T f_0 = 0 \quad \text{and} \quad r_0 - t_0 T r_0 = 0$$

must hold. If $t_0 = 0$, then $u_0 = 0$, contradicting the assumption $u_0 \neq 0$. So $t_0$ must be in the interval $[0, 1]$. As $r_0 \in R_0$, it is not an eigenvector of $T$ corresponding to a characteristic value lying in the interval $[0, 1]$, and therefore the latter equation is valid if and only if $r_0 = 0$. Since $2t_0 - 1 < 1$ and $f_0 \in F_0$, the only candidate for $f_0$ is zero. So $u_0 = 0$, which is against our assumption $u_0 \neq 0$. Consequently, it is not possible that $\delta(u_0) = 0$.

**Step 3.** Let us suppose that $\delta(u_0) \neq 0$ and again derive a contradiction.
Note that \( \delta(u_0) > 0 \). Using Lemma 4.4.4, we obtain
\[
\| f_0 \| \leq c_1 \min_{0 \leq s \leq 1} \| (A' + \Gamma)(2t - 1) f_0 - t T f_0 \| \tag{4.5}
\]
\[
\leq c_1 \| (A' + \Gamma)(2t_0 - 1) f_0 - t_0 T f_0 \|
\]
\[
= c_1 \| \delta(u_0) P'' A u_0 \| \tag{4.14}
\]
\[
\leq c_1 \| \delta(u_0) \| P'' \| A u_0 \|
\]
\[
\leq c_1 c_2 \| P'' \| \delta(u_0),
\]
where \( c_2 = \sup \{ \| A u \| : u \in \overline{B}(0, r_2) \} < \infty \). Since \( A' + \Gamma \) is strictly monotone, and since the second equality in (4.14) holds,
\[
(\langle I - P' \rangle A u_0, r_0 - t_0 T r_0) = -\frac{1}{\delta(u_0)} \langle (A' + \Gamma)(r_0 - t_0 T r_0), r_0 - t_0 T r_0 \rangle \leq 0.
\]

The definition of \( \delta(u) \) yields
\[
\delta(u_0) = \max \left\{ 0, c \min_{0 \leq s \leq 1} \langle (I - P') A u_0, (I - T) u_0 \rangle \right\}
\]
\[
\leq c \langle (I - P') A u_0, u_0 - t_0 T u_0 \rangle
\]
\[
= c \langle (I - P') A u_0, f_0 - t_0 T f_0 \rangle + \langle (I - P') A u_0, r_0 - t_0 T r_0 \rangle
\]
\[
\leq c \| (I - P') A u_0, f_0 - t_0 T f_0 \|
\]
\[
\leq c \| (I - P') \| \| A u_0 \| \| I - T \| \| f_0 \|
\]
\[
\leq c c_1 c_2 \| P'' \| \| I - P' \| \| (1 + \| T \|) \delta(u_0).\]

If we set \( c \leq \tau_1 \), where
\[
\tau_1 = \frac{1}{2} \left( c c_1 c_2 \| P'' \| \| I - P' \| \| (1 + \| T \|) \right)^{-1},
\]
we attain the inequality \( \delta(u_0) \leq 0 \), which is inconsistent with the assumption \( \delta(u_0) > 0 \). Hence, the case \( \delta(u_0) > 0 \) with \( u_0 \neq 0 \) is also impossible. The proof is now complete. \( \square \)

**Lemma 4.4.12.** Assume that the assumptions of Theorem 4.3.1 hold, and that \( r_2 \) is given by Lemma 4.4.8. If \( E^2 \) is the same mapping as in Lemma 4.4.11, that is,
\[
E^2 u := t \alpha u + \delta(u) \Pi u + (1 - t) \left[ (A' + \Gamma)(I - \Pi) u - (A' + \Gamma) \Pi u \right]
\]
\[
= \delta(u) \Pi u + (1 - t) \left[ (A' + \Gamma)(I - \Pi) u - (A' + \Gamma) \Pi u \right],
\]
then there exists \( F_2 \in F(X) \) such that
\[
Z^u(F_2, F) := \{ (u, t) \in \partial B(0, r_2) \cap F \times [0, 1] \mid \langle E^2 u, u \rangle \leq 0 \text{ and } \langle E^2 u, v \rangle = 0 \text{ for all } v \in F_2 \}
\]
is empty for any \( F \in F(X) \) with \( F_2 \subseteq F \).
Remark 4.4.13. Since $\delta A$, $A'$, and $\Gamma$ are quasimonotone, and since $A' + \Gamma$ is compact, $E_0^2$ does not necessarily possess the property $(S_\ast)$; therefore, the assertion of the Lemma 4.4.12 does not follow from Proposition 3.2.4.

Before going to the proof, we explain the plan a little bit. We suppose that the claim is not valid and construct a family of nonempty bounded sets whose weak closures have the finite intersection property. It then follows that the intersection of the closures have at least one element $(u_0, t_0)$ in common. Then we prove that for every finite-dimensional subspace $F^0$ of $X$ there exists a sequence $(u_n^0, t_n^0)_{n=1}^\infty$ that has certain desirable properties as well as

$$u_n^0 \xrightarrow{w} u_0 \quad \text{and} \quad t_n^0 \xrightarrow{w} t_0.$$  

These three steps are mostly omitted because the details are found in the proof of Proposition 3.2.4. Then comes the challenging part of the proof. We prove that

$$\delta (u_n^0) \xrightarrow{n} \delta t \neq 0$$

when $E_0^1$ is a subspace that contains three specific elements and an arbitrary element $w$ of $X$. In that case, we can apply the $(S_\ast)$-property of $A$ to $(u_n^0)_{n=1}^\infty$ to get $u_n^0 \xrightarrow{w} u_0$. Then the continuity properties of the mappings yield $(E_0^1 u_0, w) = 0$ and by varying $w$, we show that $(u_0, t_0)$ is a zero of $E_0^1$ in $\partial B(0, r_2) \times [0, 1]$.

Proof of Lemma 4.4.12. Much of the proof of this claim is similar to the proof of Proposition 3.2.4; steps 1–3 are omitted because they are exactly the same, except for some minor notational differences. We continue from the conclusion: there exists an element $(u_0, t_0) \in X \times \mathbb{R}$ such that

$$(u_0, t_0) \in \bigcap_{F \in F(X)} w-\text{cl}(G(F)).$$

Step 4. We show that $u_0$ belongs to $\partial B(0, r_2)$ and satisfies $E_0^1 u_0 = 0$. These results violate the conclusion of Lemma 4.4.11, which shows that the claim of this lemma is valid.

We pick an arbitrary element $w$ from the space $X$ and then let $F_0^w \in F(X)$ be such that $u_0, T u_0, \Pi u_0, w \in F_0^w$.

Since $(u_0, t_0)$ belongs to the bounded, weakly closed set $w-\text{cl}(G(F_0^w))$, we may proceed as in step 4 of the proof of Proposition 3.2.4 to find sequences $(F_n^w)_{n=1}^\infty \in F(X)$ and $(u_n^w, t_n^w)_{n=1}^\infty \in G(F_0^w)$ such that

$$u_n^w \xrightarrow{w} u_0, \quad t_n^w \xrightarrow{w} t_0,$$

$$(u_n^w, t_n^w) \in Z^w(F_0^w, F_n^w), \quad \text{and} \quad F_n^w \subset F_0^w.$$  

According to the definition of $Z^w(F_0^w, F_n^w)$, we have $u_n^w \in \partial B(0, r_2) \cap F_n^w$ as well as

$$\langle E_0^1 u_n^w, u_n^w \rangle \leq 0 \quad \text{and} \quad \langle E_0^1 u_n^w, v \rangle = 0 \quad \text{for all} \quad v \in F_0^w. \quad (4.16)$$

Remark 4.4.13. Since $\delta A$, $A'$, and $\Gamma$ are quasimonotone, and since $A' + \Gamma$ is compact, $E_0^2$ does not necessarily possess the property $(S_\ast)$; therefore, the assertion of the Lemma 4.4.12 does not follow from Proposition 3.2.4.

Before going to the proof, we explain the plan a little bit. We suppose that the claim is not valid and construct a family of nonempty bounded sets whose weak closures have the finite intersection property. It then follows that the intersection of the closures have at least one element $(u_0, t_0)$ in common. Then we prove that for every finite-dimensional subspace $F^0$ of $X$ there exists a sequence $(u_n^0, t_n^0)_{n=1}^\infty$ that has certain desirable properties as well as

$$u_n^0 \xrightarrow{w} u_0 \quad \text{and} \quad t_n^0 \xrightarrow{w} t_0.$$  

These three steps are mostly omitted because the details are found in the proof of Proposition 3.2.4. Then comes the challenging part of the proof. We prove that

$$\delta (u_n^0) \xrightarrow{n} \delta t \neq 0$$

when $E_0^1$ is a subspace that contains three specific elements and an arbitrary element $w$ of $X$. In that case, we can apply the $(S_\ast)$-property of $A$ to $(u_n^0)_{n=1}^\infty$ to get $u_n^0 \xrightarrow{w} u_0$. Then the continuity properties of the mappings yield $(E_0^1 u_0, w) = 0$ and by varying $w$, we show that $(u_0, t_0)$ is a zero of $E_0^1$ in $\partial B(0, r_2) \times [0, 1]$.

Proof of Lemma 4.4.12. Much of the proof of this claim is similar to the proof of Proposition 3.2.4; steps 1–3 are omitted because they are exactly the same, except for some minor notational differences. We continue from the conclusion: there exists an element $(u_0, t_0) \in X \times \mathbb{R}$ such that

$$(u_0, t_0) \in \bigcap_{F \in F(X)} w-\text{cl}(G(F)).$$

Step 4. We show that $u_0$ belongs to $\partial B(0, r_2)$ and satisfies $E_0^1 u_0 = 0$. These results violate the conclusion of Lemma 4.4.11, which shows that the claim of this lemma is valid.

We pick an arbitrary element $w$ from the space $X$ and then let $F_0^w \in F(X)$ be such that $u_0, T u_0, \Pi u_0, w \in F_0^w$.

Since $(u_0, t_0)$ belongs to the bounded, weakly closed set $w-\text{cl}(G(F_0^w))$, we may proceed as in step 4 of the proof of Proposition 3.2.4 to find sequences $(F_n^w)_{n=1}^\infty \in F(X)$ and $(u_n^w, t_n^w)_{n=1}^\infty \in G(F_0^w)$ such that

$$u_n^w \xrightarrow{w} u_0, \quad t_n^w \xrightarrow{w} t_0,$$

$$(u_n^w, t_n^w) \in Z^w(F_0^w, F_n^w), \quad \text{and} \quad F_n^w \subset F_0^w.$$  

According to the definition of $Z^w(F_0^w, F_n^w)$, we have $u_n^w \in \partial B(0, r_2) \cap F_n^w$ as well as

$$\langle E_0^1 u_n^w, u_n^w \rangle \leq 0 \quad \text{and} \quad \langle E_0^1 u_n^w, v \rangle = 0 \quad \text{for all} \quad v \in F_0^w. \quad (4.16)$$
Since $A$ is bounded in the set $B(0, r_2)$, also $\delta$ is bounded in that set. Therefore, we may assume that $\delta(u_t^\ast) \to \delta_0$ as $n \to \infty$.

**Step 4.1.** We show that $\delta_0 = 0$ cannot occur if $u_0, T u_0, \Pi u_0 \in F_0^\circ$.

We first show that $u_0 = 0$. The boundedness of $A$ in $B(0, r_2)$, the weak continuity of $\Gamma$, $\Pi$ and $A'$, and the relation $A' = (A' + \Gamma)(I - T)$ (remember that $T = (A' - \Gamma)^{-1} \Gamma$) imply

$$\begin{align*}
0 &= \langle \mathcal{E}_2^B u_t^\ast, v \rangle \\
&= \langle \delta(u_t^\ast) A u_t^\ast + A' u_t^\ast + (1 - \tau_0^\ast) \Gamma u_t^\ast - 2(1 - \tau_0^\ast)(A' + \Gamma)\Pi u_t^\ast, v \rangle \\
&\quad \Rightarrow (A' u_0 + (1 - t_0)\Gamma u_0 - 2(1 - t_0)(A' + \Gamma)\Pi u_0, v) \\
&= ((A' + \Gamma)(I - T)(t_0 u_0) + (A' + \Gamma)((1 - t_0) u_0 - 2(1 - t_0)\Pi u_0), v) \\
&= ((A' + \Gamma)(u_0 - t_0 T u_0 - 2(1 - t_0)\Pi u_0), v)
\end{align*}$$

for every $v \in F_0^\circ$. The fact that $u_0, T u_0, \Pi u_0 \in F_0^\circ$ implies

$$0 = \langle (A' + \Gamma)(u_0 - t_0 T u_0 - 2(1 - t_0)\Pi u_0), u_0 - t_0 T u_0 - 2(1 - t_0)\Pi u_0 \rangle$$

which is possible if and only if

$$u_0 - t_0 T u_0 - 2(1 - t_0)\Pi u_0 = 0. \quad (4.17)$$

The previous conclusion is due to the strict monotonicity of $A' + \Gamma$. Recall that $\Pi$ is the projection of $X$ onto $F_0$ along $R_0$ where $F_0$ and $R_0$ are invariant subspaces of $T$. Using the notations $f_0 = \Pi u_0$ and $(I - \Pi) u_0 = r_t$, (4.17) yields

$$0 = r_t - t_0 T r_t$$

and

$$0 = f_0 - t_0 T f_0 - 2(1 - t_0)f_0 = - (1 - 2t_0)f_0 - t_0 T f_0.$$

The first of these equations cannot be satisfied for nonzero $t_0$ according to the definition of $R_0$ and the restriction $t_0 \in [0, 1]$. Thus, $r_t = 0$. The situation $t_0 \in [0, 1)$, $f_0 \in F_0 \setminus \{0\}$ and $0 = (1 - 2t_0)f_0 + t_0 T f_0$ is also impossible because the inequality $- 1 \leq \frac{t_0}{1 - t_0} \leq 2$ has no solutions in $[0, 1]$. Hence $f_0 = 0$, and consequently $u_0 = 0$.

Recall that $\delta(u_t^\ast) \overset{d}{\to} \delta_0 = 0$. The definition of $\delta$ yields a sequence $(u_t^\ast)^n_{n=1} \subset [0, 1]$ with

$$\begin{align*}
0 &\geq \liminf_{n \to \infty} \langle (I - P)A u_t^\ast, u_t^\ast - t_0^n T u_t^\ast \rangle \\
&= \liminf_{n \to \infty} \langle (A u_t^\ast, u_t^\ast) - (A u_t^\ast, t_0^n T u_t^\ast) - (A u_t^\ast, P(u_t^\ast - t_0^n T u_t^\ast)) \rangle \\
&= \liminf_{n \to \infty} \langle A u_t^\ast, u_t^\ast \rangle
\end{align*}$$

As $(u_t^\ast)^n_{n=1} \subset [0, 1], u_t^\ast \overset{d}{\to} u_0 = 0$, $T$ and $P$ are compact, and $A$ is bounded in $B(0, r_2)$, we obtain

$$\begin{align*}
0 &\geq \liminf_{n \to \infty} \langle (I - P)A u_t^\ast, u_t^\ast - t_0^n T u_t^\ast \rangle \\
&= \liminf_{n \to \infty} \langle (A u_t^\ast, u_t^\ast) - (A u_t^\ast, t_0^n T u_t^\ast) - (A u_t^\ast, P(u_t^\ast - t_0^n T u_t^\ast)) \rangle \\
&= \liminf_{n \to \infty} \langle A u_t^\ast, u_t^\ast \rangle
\end{align*}$$

Since $A$ is bounded in the set $B(0, r_2)$, also $\delta$ is bounded in that set. Therefore, we may assume that $\delta(u_t^\ast) \to \delta_0$ as $n \to \infty$.

**Step 4.1.** We show that $\delta_0 = 0$ cannot occur if $u_0, T u_0, \Pi u_0 \in F_0^\circ$.

We first show that $u_0 = 0$. The boundedness of $A$ in $B(0, r_2)$, the weak continuity of $\Gamma$, $\Pi$ and $A'$, and the relation $A' = (A' + \Gamma)(I - T)$ (remember that $T = (A' - \Gamma)^{-1} \Gamma$) imply

$$\begin{align*}
0 &= \langle \mathcal{E}_2^B u_t^\ast, v \rangle \\
&= \langle \delta(u_t^\ast) A u_t^\ast + A' u_t^\ast + (1 - \tau_0^\ast) \Gamma u_t^\ast - 2(1 - \tau_0^\ast)(A' + \Gamma)\Pi u_t^\ast, v \rangle \\
&\quad \Rightarrow (A' u_0 + (1 - t_0)\Gamma u_0 - 2(1 - t_0)(A' + \Gamma)\Pi u_0, v) \\
&= ((A' + \Gamma)(I - T)(t_0 u_0) + (A' + \Gamma)((1 - t_0) u_0 - 2(1 - t_0)\Pi u_0), v) \\
&= ((A' + \Gamma)(u_0 - t_0 T u_0 - 2(1 - t_0)\Pi u_0), v)
\end{align*}$$

for every $v \in F_0^\circ$. The fact that $u_0, T u_0, \Pi u_0 \in F_0^\circ$ implies

$$0 = \langle (A' + \Gamma)(u_0 - t_0 T u_0 - 2(1 - t_0)\Pi u_0), u_0 - t_0 T u_0 - 2(1 - t_0)\Pi u_0 \rangle$$

which is possible if and only if

$$u_0 - t_0 T u_0 - 2(1 - t_0)\Pi u_0 = 0. \quad (4.17)$$

The previous conclusion is due to the strict monotonicity of $A' + \Gamma$. Recall that $\Pi$ is the projection of $X$ onto $F_0$ along $R_0$ where $F_0$ and $R_0$ are invariant subspaces of $T$. Using the notations $f_0 = \Pi u_0$ and $(I - \Pi) u_0 = r_t$, (4.17) yields

$$0 = r_t - t_0 T r_t$$

and

$$0 = f_0 - t_0 T f_0 - 2(1 - t_0)f_0 = - (1 - 2t_0)f_0 - t_0 T f_0.$$

The first of these equations cannot be satisfied for nonzero $t_0$ according to the definition of $R_0$ and the restriction $t_0 \in [0, 1]$. Thus, $r_t = 0$. The situation $t_0 \in [0, 1), f_0 \in F_0 \setminus \{0\}$ and $0 = (1 - 2t_0)f_0 + t_0 T f_0$ is also impossible because the inequality $- 1 \leq \frac{t_0}{1 - t_0} \leq 2$ has no solutions in $[0, 1]$. Hence $f_0 = 0$, and consequently $u_0 = 0$.

Recall that $\delta(u_t^\ast) \overset{d}{\to} \delta_0 = 0$. The definition of $\delta$ yields a sequence $(u_t^\ast)^n_{n=1} \subset [0, 1]$ with

$$\begin{align*}
0 &\geq \liminf_{n \to \infty} \langle (I - P)A u_t^\ast, u_t^\ast - t_0^n T u_t^\ast \rangle \\
&= \liminf_{n \to \infty} \langle (A u_t^\ast, u_t^\ast) - (A u_t^\ast, t_0^n T u_t^\ast) - (A u_t^\ast, P(u_t^\ast - t_0^n T u_t^\ast)) \rangle \\
&= \liminf_{n \to \infty} \langle A u_t^\ast, u_t^\ast \rangle
\end{align*}$$

As $(u_t^\ast)^n_{n=1} \subset [0, 1], u_t^\ast \overset{d}{\to} u_0 = 0$, $T$ and $P$ are compact, and $A$ is bounded in $B(0, r_2)$, we obtain

$$\begin{align*}
0 &\geq \liminf_{n \to \infty} \langle (I - P)A u_t^\ast, u_t^\ast - t_0^n T u_t^\ast \rangle \\
&= \liminf_{n \to \infty} \langle (A u_t^\ast, u_t^\ast) - (A u_t^\ast, t_0^n T u_t^\ast) - (A u_t^\ast, P(u_t^\ast - t_0^n T u_t^\ast)) \rangle \\
&= \liminf_{n \to \infty} \langle A u_t^\ast, u_t^\ast \rangle
\end{align*}$$
Step 4.2. Using the result of the previous step, we derive a contradiction.

Assume that $u_0, T_{u_0}, \Pi_{u_0} \in F_{\Pi}^n$. Now we have $\delta = 0$. Because $u_0 \in F_{\Pi}^n$, $A$ is bounded, $A'$ and $\Gamma$ are quasi-monotone, and $(A' + \Gamma)\Pi$ is compact, we can estimate as follows:

$$0 \geq \limsup_{n \to \infty} (E_{u_0}^n u_0^n, u_0^n - u_0) \quad \text{by (4.16)}$$

$$= \limsup_{n \to \infty} \delta(u_0^n)\langle A(u_0^n) + A'\Pi u_0^n + (1 - t^n)\Gamma u_0^n - 2(1 - t^n)(A' + \Gamma)\Pi u_0^n, u_0^n - u_0 \rangle$$

$$\geq \limsup_{n \to \infty} \delta(u_0^n)\langle A(u_0^n) + A'\Pi u_0^n, u_0^n - u_0 \rangle.$$

By the $(S_\lambda$)-property of $A$, we have $u_0^n \to u_0$ in $X$. Then $u_0$ is in $\partial B(0, r_0)$ because $\partial B(0, r_0)$ is closed. Now, using the continuity properties of $A$, $A'$, $\Pi$, $\Gamma$, and $\delta$, we get

$$E_{u_0}^n u_0^n = \delta(u_0^n)\langle A(u_0^n) + A'\Pi u_0^n + (1 - t^n)\Gamma u_0^n - 2(1 - t^n)(A' + \Gamma)\Pi u_0^n, u_0^n \rangle$$

$$\to \delta(u_0)\langle A(u_0) + A'\Pi u_0 + (1 - t_0)\Gamma u_0 - 2(1 - t_0)(A' + \Gamma)\Pi u_0, u_0 \rangle = E_{u_0} u_0.$$

Especially,

$$0 = \limsup_{n \to \infty} (E_{u_0}^n u_0^n, w) \quad \text{by (4.16)}$$

$$= \limsup_{n \to \infty} \delta(u_0^n)\langle A(u_0^n) + A'\Pi u_0^n + (1 - t^n)\Gamma u_0^n - 2(1 - t^n)(A' + \Gamma)\Pi u_0^n, u_0^n - u_0 \rangle$$

$$\geq \limsup_{n \to \infty} \delta(u_0^n)\langle A(u_0^n) + A'\Pi u_0^n, u_0^n - u_0 \rangle.$$

For all possible choices of $w \in X$, the conclusion is $0 = \langle E_{u_0} u_0^n, w \rangle$. Thus, $E_{u_0} u_0 = 0$. Since $u_0 \in \partial B(0, r_0)$, we have an outcome that violates the conclusion of Lemma 4.4.11. The proof is now complete.

\[\Box\]

### 4.4.6 The proof of the index theorem

In this subsection, we assume that all the assumptions of Theorem 4.3.1 hold. Recall that $r_1 > 0$ is such that $A(B(0, r_1))$ is bounded (see Remark 4.4.7). Here the constant $c_1$ of the definition of $\delta$ (see Equation 4.11 on page 77) has the same value as in Subsection 4.4.4 and $r_2 \leq 0, r_1$ by Lemma 4.4.8. So we have the inequality

$$c \leq r_1 := \frac{1}{2} \left( c_1 r_2^2 \||P^*||r^* + \|1 + \|T\|\right)^{-1}$$

where $c_1$ is the constant from Lemma 4.4.4 and $c_2 = \sup\{\|Au\| | u \in B(0, r_1)\} < \infty$.

We define

$$F_3 := F_0 + F_2 + \text{span}\{w_1, \ldots, w_M\}$$

$$F_4 := F_0 + F_3 + \text{span}\{w_1, \ldots, w_M\},$$

where $F_1$ and $F_2$ are given by Lemma 4.4.9 and Lemma 4.4.12, $F_0 = \text{span}\{f_1, \ldots, f_M\}$, and $w_1, \ldots, w_M$ have the properties expressed in (4.10). Recall that $\text{span}\{w_1, \ldots, w_M\} = (\delta_0)^1$.

We let $Q$ be a continuous projection of $X$ onto $F_3$ and $Q'$ the adjoint of $Q$. Now both $Q$ and $Q'$ are compact, for they are continuous and have a finite-dimensional range. The
mapping \( J : X \to X^* \) is a duality mapping of \( X \) which is given by a locally uniformly convex norm of \( X \) which makes the corresponding norm of \( X^* \) locally uniformly convex. In this case, \( J \) satisfies the condition \( (S_+) \) and is demi-continuous, bijective, and locally bounded; see Theorem 2.4.11.

**Remark 4.4.14.** There are infinitely many candidates for \( Q \), but that does not matter because in the end of the proof of Theorem 4.3.1 we eliminate \( Q \) by taking an “approximation” of a mapping with respect to the space \( F_3 \). See Lemma 4.4.19 and the following outline for the “core” of the proof of Theorem 4.3.1.

We first prove that the index of \( A \) at zero is the degree of the mapping \( (1 + \delta(\cdot))A \) at zero with respect to \( \overline{B}(0,r_2) \). After that, we use two homotopies to arrive at a situation where we can prove that the index is equal to the degree (at zero with respect to \( \overline{B}(0,r_2) \)) of the “approximation” \( A^*_{\overline{B}(0,r_2)} \) of the mapping

\[ A^*_{\overline{B}(0,r_2)} := \delta(u)Au - (A' + \Gamma)\Pi u + (A' + \Gamma)(I - \Pi)u + (I - Q^*)J(I - Q)u. \]

Then we choose an appropriate basis \( v_1, \ldots, v_N \) for \( F_3 \) and show that the degree of \( A^*_{\overline{B}(0,r_2)} \) can be calculated as the degree of the linear mapping

\[ u \mapsto A^*_{\overline{B}(0,r_2)}u := \sum_{i=1}^{N} ((A' + \Gamma)(I - \Pi)u - (A' + \Gamma)\Pi u, v_i) v_i \]

at zero with respect to \( \overline{B}(0,r_2) \). We complete the proof by demonstrating that the degree of \( A^*_{\overline{B}(0,r_2)} \) at zero with respect to \( \overline{B}(0,r_2) \cap F_3 \) equals \((-1)^M\).

Admissible \( (S_-) \)-homotopies have a central role in the “core” of the proof of Theorem 4.3.1. The admissible \( (S_-) \)-homotopy is defined in Definition 2.4.7 on page 45.

**Lemma 4.4.15.** Assume that the assumptions of Theorem 4.3.1 are fulfilled, and that \( r_1 \) and \( c \) are as in the beginning of this subsection. Then

\[ \text{Ind}(A,0) = \text{Deg}(A,\overline{B}(0,r),0) = \text{Deg}((1+\delta(\cdot))A,\overline{B}(0,r),0) \]

for every \( 0 < r \leq r_2 \).

**Proof.** According to the conclusion of the Lemma 4.4.8, \((1+\delta(\cdot))A\) cannot have zeros in \( \partial B(0,r) \) for any \( 0 < r \leq r_2 \). Inasmuch as \( 1+\delta(u) \geq 1 \), \( A \) has no zeros in \( \partial B(0,r) \) for every \( 0 < r \leq r_2 \). Consequently,

\[ \text{Ind}(A,0) = \text{Deg}(A,\overline{B}(0,r),0) \]

for every \( 0 < r \leq r_2 \) and the affine homotopy

\[ (t,u) \mapsto (1-t)Au + t(1+\delta(u))Au = (1+\delta(u))Au \]

has no zeros in \([0,1] \times \partial B(0,r)\) for every \( 0 < r \leq r_2 \). Clearly, \((1+\delta(\cdot))A\) satisfies the condition \( (S_+) \) and is demi-continuous on \( \overline{B}(0,r_2) \). Thus, the affine homotopy is an admissible \( (S_+) \)-homotopy of \( A \) and \((1+\delta(\cdot))A\) (see Lemma 2.4.8). \( \square \)

mapping \( J : X \to X^* \) is a duality mapping of \( X \) which is given by a locally uniformly convex norm of \( X \) which makes the corresponding norm of \( X^* \) locally uniformly convex. In this case, \( J \) satisfies the condition \( (S_+) \) and is demi-continuous, bijective, and locally bounded; see Theorem 2.4.11.

**Remark 4.4.14.** There are infinitely many candidates for \( Q \), but that does not matter because in the end of the proof of Theorem 4.3.1 we eliminate \( Q \) by taking an “approximation” of a mapping with respect to the space \( F_3 \). See Lemma 4.4.19 and the following outline for the “core” of the proof of Theorem 4.3.1.

We first prove that the index of \( A \) at zero is the degree of the mapping \( (1 + \delta(\cdot))A \) at zero with respect to \( \overline{B}(0,r_2) \). After that, we use two homotopies to arrive at a situation where we can prove that the index is equal to the degree (at zero with respect to \( \overline{B}(0,r_2) \)) of the “approximation” \( A^*_{\overline{B}(0,r_2)} \) of the mapping

\[ A^*_{\overline{B}(0,r_2)} := \delta(u)Au - (A' + \Gamma)\Pi u + (A' + \Gamma)(I - \Pi)u + (I - Q^*)J(I - Q)u. \]

Then we choose an appropriate basis \( v_1, \ldots, v_N \) for \( F_3 \) and show that the degree of \( A^*_{\overline{B}(0,r_2)} \) can be calculated as the degree of the linear mapping

\[ u \mapsto A^*_{\overline{B}(0,r_2)}u := \sum_{i=1}^{N} ((A' + \Gamma)(I - \Pi)u - (A' + \Gamma)\Pi u, v_i) v_i \]

at zero with respect to \( \overline{B}(0,r_2) \). We complete the proof by demonstrating that the degree of \( A^*_{\overline{B}(0,r_2)} \) at zero with respect to \( \overline{B}(0,r_2) \cap F_3 \) equals \((-1)^M\).

Admissible \( (S_-) \)-homotopies have a central role in the “core” of the proof of Theorem 4.3.1. The admissible \( (S_-) \)-homotopy is defined in Definition 2.4.7 on page 45.

**Lemma 4.4.15.** Assume that the assumptions of Theorem 4.3.1 are fulfilled, and that \( r_1 \) and \( c \) are as in the beginning of this subsection. Then

\[ \text{Ind}(A,0) = \text{Deg}(A,\overline{B}(0,r),0) = \text{Deg}((1+\delta(\cdot))A,\overline{B}(0,r),0) \]

for every \( 0 < r \leq r_2 \).

**Proof.** According to the conclusion of the Lemma 4.4.8, \((1+\delta(\cdot))A\) cannot have zeros in \( \partial B(0,r) \) for any \( 0 < r \leq r_2 \). Inasmuch as \( 1+\delta(u) \geq 1 \), \( A \) has no zeros in \( \partial B(0,r) \) for every \( 0 < r \leq r_2 \). Consequently,

\[ \text{Ind}(A,0) = \text{Deg}(A,\overline{B}(0,r),0) \]

for every \( 0 < r \leq r_2 \) and the affine homotopy

\[ (t,u) \mapsto (1-t)Au + t(1+\delta(u))Au = (1+\delta(u))Au \]

has no zeros in \([0,1] \times \partial B(0,r)\) for every \( 0 < r \leq r_2 \). Clearly, \((1+\delta(\cdot))A\) satisfies the condition \( (S_+) \) and is demi-continuous on \( \overline{B}(0,r_2) \). Thus, the affine homotopy is an admissible \( (S_+) \)-homotopy of \( A \) and \((1+\delta(\cdot))A\) (see Lemma 2.4.8). \( \square \)
Lemma 4.4.16. Assume that $Q$ and $Q'$ are as in the beginning of this subsection. Then $(I - Q^*)J(I - Q) : X \to X^*$ is demi-continuous and satisfies the condition $(S_\ast)$.

Proof. Since $Q$ and $Q'$ are compact and $J$ is demi-continuous, $(I - Q^*)J(I - Q)$ is demi-continuous. Next, we prove that this mapping has the property $(S_\ast)$. If $u_n \to u_0$, then from the continuity and the linearity of $Q$, it follows that

$$(I - Q)u_n \to (I - Q)u_0.$$  

Thus, the computation

$$0 \geq \lim_{n \to \infty} ([I - Q^*)J(I - Q)u_n, u_n - u_0] = \lim_{n \to \infty} [J(I - Q)u_n, (I - Q)(u_n - u_0)]$$  

with the property $(S_\ast)$ of $J$ implies $(I - Q)u_n \to (I - Q)u_0$. Since $Q$ is completely continuous, this yields $u_n \to u_0$. Thus, the other claim has been proved. $\square$

Lemma 4.4.17. Assume that the assumptions of Theorem 4.3.1 hold and $r_1$ and $c$ are as in the beginning of this subsection. Then

$$A^1_t := E^1_t u + (1 - t)(I - Q^*)J(I - Q)u$$  

is an admissible $(S_\ast)$-homotopy on $\partial B(0, r_2)$. Moreover,

$$\text{Deg}(A^1_t, \overline{B}(0, r_2), 0) = \text{Deg}(A^1_0, \overline{B}(0, r_2), 0).$$

Proof. It suffices to prove the first assertion because the claim

$$\text{Deg}(A^1_t, \overline{B}(0, r_2), 0) = \text{Deg}(A^1_0, \overline{B}(0, r_2), 0)$$

follows from it.

Step 1. We show that $A^1_t$ has the desired continuity properties and satisfies the condition $(S_\ast)$ on $\overline{B}(0, r_2)$.

First, we note that $A^1_t$ can be written as an affine homotopy:

$$A^1_t u = (1 - t)[\delta(u)Au + A' u + (1 - t)(I - Q^*)J(I - Q)u] + t[\delta(u)Au + Au].$$

Here $J$ and $A$ are demi-continuous and bounded, and satisfy the condition $(S_\ast)$ on $\overline{B}(0, r_2)$. Since $Q$ and $Q'$ are compact, and $A'$ and $\delta$ are demi-continuous, quasi-monotone mappings

$3$, it follows from the perturbation results of Lemma 2.4.4 that $A^1_t$ is an affine homotopy of two demi-continuous $(S_\ast)$-mappings; consequently, Lemma 2.4.8 implies that $A^1_t$ satisfies the condition $(S_\ast)$ and has the desired continuity properties.

Step 2. We prove that $A^1_t$ has no zeros in $\partial B(0, r_2) \times [0, 1]$. Suppose that

$$0 = A^1_t u = (\delta(u) + t)Au + (1 - t)A' u + (1 - t)(I - Q^*)J(I - Q)u. \quad (4.18)$$

$3$ is quasi-monotone because $\delta$ is a nonnegative functional and $A$ is quasi-monotone.

Lemma 4.4.16. Assume that $Q$ and $Q'$ are as in the beginning of this subsection. Then $(I - Q^*)J(I - Q) : X \to X^*$ is demi-continuous and satisfies the condition $(S_\ast)$.

Proof. Since $Q$ and $Q'$ are compact and $J$ is demi-continuous, $(I - Q^*)J(I - Q)$ is demi-continuous. Next, we prove that this mapping has the property $(S_\ast)$. If $u_n \to u_0$, then from the continuity and the linearity of $Q$, it follows that

$$(I - Q)u_n \to (I - Q)u_0.$$  

Thus, the computation

$$0 \geq \lim_{n \to \infty} ([I - Q^*)J(I - Q)u_n, u_n - u_0] = \lim_{n \to \infty} [J(I - Q)u_n, (I - Q)(u_n - u_0)]$$  

with the property $(S_\ast)$ of $J$ implies $(I - Q)u_n \to (I - Q)u_0$. Since $Q$ is completely continuous, this yields $u_n \to u_0$. Thus, the other claim has been proved. $\square$

Lemma 4.4.17. Assume that the assumptions of Theorem 4.3.1 hold and $r_1$ and $c$ are as in the beginning of this subsection. Then

$$A^1_t := E^1_t u + (1 - t)(I - Q^*)J(I - Q)u$$  

is an admissible $(S_\ast)$-homotopy on $\partial B(0, r_2)$. Moreover,

$$\text{Deg}(A^1_t, \overline{B}(0, r_2), 0) = \text{Deg}(A^1_0, \overline{B}(0, r_2), 0).$$

Proof. It suffices to prove the first assertion because the claim

$$\text{Deg}(A^1_t, \overline{B}(0, r_2), 0) = \text{Deg}(A^1_0, \overline{B}(0, r_2), 0)$$

follows from it.

Step 1. We show that $A^1_t$ has the desired continuity properties and satisfies the condition $(S_\ast)$ on $\overline{B}(0, r_2)$.

First, we note that $A^1_t$ can be written as an affine homotopy:

$$A^1_t u = (1 - t)[\delta(u)Au + A' u + (1 - t)(I - Q^*)J(I - Q)u] + t[\delta(u)Au + Au].$$

Here $J$ and $A$ are demi-continuous and bounded, and satisfy the condition $(S_\ast)$ on $\overline{B}(0, r_2)$. Since $Q$ and $Q'$ are compact, and $A'$ and $\delta$ are demi-continuous, quasi-monotone mappings

$3$, it follows from the perturbation results of Lemma 2.4.4 that $A^1_t$ is an affine homotopy of two demi-continuous $(S_\ast)$-mappings; consequently, Lemma 2.4.8 implies that $A^1_t$ satisfies the condition $(S_\ast)$ and has the desired continuity properties.

Step 2. We prove that $A^1_t$ has no zeros in $\partial B(0, r_2) \times [0, 1]$. Suppose that

$$0 = A^1_t u = (\delta(u) + t)Au + (1 - t)A' u + (1 - t)(I - Q^*)J(I - Q)u. \quad (4.18)$$

$3$ is quasi-monotone because $\delta$ is a nonnegative functional and $A$ is quasi-monotone.
for some \((u, t) \in \partial B(0, r_2) \times [0, 1]\). A direct consequence is
\[
\langle E^u_1, u \rangle = ((\delta(u) + t)Au + (1 - t)A' u, u) \\
= -(1 - t)(J(I - Q)u, I - Q)u \\
= -(1 - t)\|I - Q)u\|^2 \leq 0.
\] (4.19)
Equation (4.18) yields also that \(0 = Q'[\delta(u) + t]Au + (1 - t)A' u]\). Hence,
\[
\langle E^u_1, v \rangle = ((\delta(u) + t)Au + (1 - t)A' u, v) \\
= ((\delta(u) + t)Au + (1 - t)A' u, Qv) \\
= \langle Q'[\delta(u) + t]Au + (1 - t)A' u, v \rangle = 0
\] (4.20)
for all \(v \in F_3\).

On the other hand, when \(E^u_1 = (\delta(u) + t)Au + (1 - t)A' u\), then Lemma 4.4.9 states that
\[
Z'(F_1, F) := \{(u, t) \in \partial B(0, r_2) \cap F \times [0, 1] | \langle E^u_1, u \rangle < 0 \text{ and } \langle E^u_1, v \rangle = 0 \text{ for all } v \in F_1\}
\]
is empty for any \(F \in F(X)\) with \(F_1 \subset F\). Since \(F_1 \subset F_3\), it follows from the definition of
\(Z'(F_1, F)\) that we should also have \(Z'(F_2, F) = \emptyset\) for any \(F \in F(X)\) with \(F \supset F_3\). But (4.19) and (4.20) show that \((u, t) \in Z'(F_2, F)\) with \(F = F_3 + \text{span}\{u\}\). So we have reached a contradiction, which ends the proof. \(\Box\)

**Lemma 4.4.18.** Assume that the assumptions of Theorem 4.3.1 hold, and \(r_1\) and \(c\) are as in the beginning of this subsection. Then
\[
A^u_1 := E^u_1 + (I - Q)J(I - Q)u \\
= \delta(u)Au + A' u + (1 - t)\Gamma u - 2(1 - t)(A' + \Gamma)Hu + (I - Q')J(I - Q)u
\]
is an admissible \((S_\gamma)\)-homotopy on \(\partial B(0, r_2)\). Moreover,
\[
\text{Deg}(A^u_1, B[0, r_2], 0) = \text{Deg}(A^u_1, B(0, r_2), 0) = \text{Deg}(A^u_1, B(0, r_2), 0).
\]
Proof. The equality \(\text{Deg}(A^u_1, B(0, r_2), 0) = \text{Deg}(A^u_1, B(0, r_2), 0)\) follows from the first assertion, and the claim \(\text{Deg}(A^u_1, B[0, r_2], 0) = \text{Deg}(A^u_1, B(0, r_2), 0)\) is certainly true because \(A^u_1 = A^u_1\). Thus, it is sufficient to prove the first claim.

**Step 1.** We show that \(A^u_1\) has the desired continuity properties and satisfies the condition \((S_\gamma)\), on \(B[0, r_2]\).
\(A^u_1\) is in fact an an affine homotopy:
\[
A^u_1u = (1 - t)\left[\delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)Hu + (I - Q')J(I - Q)u\right] \\
+ t\left[\delta(u)Au + A' u + (I - Q')J(I - Q)u\right].
\]
Note that $\delta A$, $A'$, and $\Gamma$ are bounded, demi-continuous, and quasi-monotone mappings while $(A' + \Gamma)\Pi$ is a compact mapping and $(I - Q')(I - Q)$ is a demi-continuous mapping fulfilling the condition $(S_+)$. Thus, by Lemma 2.4.4, $A'^2 u$ is an affine homotopy of two semi-continuous $(S_+)$-mappings. This, with Lemma 2.4.8, implies that $A'^2$ satisfies the condition $(S_+)$, and has the desired continuity properties.

Step 2. We demonstrate that $A'^2$ has no zeros in $\partial B(0, r_2) \times [0, 1]$. If

$$0 = A'^2 u = \delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u + (I - Q')(I - Q)u \quad (4.21)$$

for some $(u, t) \in \partial B(0, r_2) \times [0, 1]$, then

$$\langle E'^2 u, v \rangle = \langle \delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u, v \rangle$$

$$= - (1-t)||I - Q'||u||^2 \leq 0. \quad (4.22)$$

From (4.21), we infer that $0 = Q[\delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u]$, and consequently

$$\langle E'^2 u, v \rangle = \langle \delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u, v \rangle$$

$$= - (1-t)||I - Q'||u||^2 \leq 0. \quad (4.23)$$

for all $v \in F_2$.

According to Lemma 4.4.12, when

$$E'^2 u = \delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u,$$

then

$$Z''(F_2, F) := \{(u, t) \in \partial B(0, r_2) \cap F \times [0, 1] \mid \langle E'^2 u, v \rangle \leq 0 \text{ and } \langle E'^2 u, v \rangle = 0 \text{ for all } v \in F_2\}$$

is empty for any $F \in F(X)$ with $F \subset F$. Since $F_2 \subset F_2$, also $Z''(F_2, F)$ is empty whenever $F \supset F_2$. But Equation (4.22) and Equation (4.23) show that $(u, t)$ is in $Z''(F_2, F)$ with $F = F_2 + \text{span}\{u\}$. So we have two conclusions that are incompatible, which implies that the claim of the lemma is valid. \qed

**Lemma 4.4.19.** Assume that the conditions of Theorem 4.3.1 are satisfied. Then

$$\deg(\Lambda'^2, B(0, r_2), 0) = \deg(\Lambda'^2, B(0, r_2) \cap F_1, 0), \quad (4.24)$$

where $\Lambda'^2 = \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u + (I - Q')J(I - Q)u$ and the mapping $\Lambda'^2$ is an “approximation” of $A'^2$ with respect to the space $F_1$.

Note that $\delta A$, $A'$, and $\Gamma$ are bounded, demi-continuous, and quasi-monotone mappings while $(A' + \Gamma)\Pi$ is a compact mapping and $(I - Q')(I - Q)$ is a demi-continuous mapping fulfilling the condition $(S_+)$. Thus, by Lemma 2.4.4, $A'^2 u$ is an affine homotopy of two semi-continuous $(S_+)$-mappings. This, with Lemma 2.4.8, implies that $A'^2$ satisfies the condition $(S_+)$, and has the desired continuity properties.

Step 2. We demonstrate that $A'^2$ has no zeros in $\partial B(0, r_2) \times [0, 1]$. If

$$0 = A'^2 u = \delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u + (I - Q')(I - Q)u \quad (4.21)$$

for some $(u, t) \in \partial B(0, r_2) \times [0, 1]$, then

$$\langle E'^2 u, v \rangle = \langle \delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u, v \rangle$$

$$= - (1-t)||I - Q'||u||^2 \leq 0. \quad (4.22)$$

From (4.21), we infer that $0 = Q[\delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u]$, and consequently

$$\langle E'^2 u, v \rangle = \langle \delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u, v \rangle$$

$$= - (1-t)||I - Q'||u||^2 \leq 0. \quad (4.23)$$

for all $v \in F_2$.

According to Lemma 4.4.12, when

$$E'^2 u = \delta(u)Au + A' + (1-t)\Gamma u - 2(1-t)(A' + \Gamma)\Pi u,$$

then

$$Z''(F_2, F) := \{(u, t) \in \partial B(0, r_2) \cap F \times [0, 1] \mid \langle E'^2 u, v \rangle \leq 0 \text{ and } \langle E'^2 u, v \rangle = 0 \text{ for all } v \in F_2\}$$

is empty for any $F \in F(X)$ with $F \subset F$. Since $F_2 \subset F_2$, also $Z''(F_2, F)$ is empty whenever $F \supset F_2$. But Equation (4.22) and Equation (4.23) show that $(u, t)$ is in $Z''(F_2, F)$ with $F = F_2 + \text{span}\{u\}$. So we have two conclusions that are incompatible, which implies that the claim of the lemma is valid. \qed

**Lemma 4.4.19.** Assume that the conditions of Theorem 4.3.1 are satisfied. Then

$$\deg(\Lambda'^2, B(0, r_2), 0) = \deg(\Lambda'^2, B(0, r_2) \cap F_1, 0), \quad (4.24)$$

where $\Lambda'^2 = \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u + (I - Q')J(I - Q)u$ and the mapping $\Lambda'^2$ is an “approximation” of $A'^2$ with respect to the space $F_1$.\qed
Proof. According to the definition of the degree, we may calculate the \( \langle S_+ \rangle \)-degree of the demi-continuous \( (S_+) \)-mapping
\[
u \mapsto \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u + (I - Q')J(I - Q)u
\]
as the Brouwer degree of the “approximation” \( A_{F_3}^2F_3 \) of \( A_0^2 \) with respect to \( F_3 \) if for any finite-dimensional space \( F \supset F_3 \)
\[
\{u \in \partial B(0, r_2) \cap F \mid \langle A_{F_3}^2F_3, u \rangle \leq 0 \text{ and } \langle A_{F_3}^2F_3, v \rangle = 0 \text{ for all } v \in F_3 \} = \emptyset.
\]

Step 1. We recall a fact from an earlier lemma.
Lemma 4.4.12 states that when
\[
E_0^2u = \delta(u)Au + A' u + (1 - t)\Gamma u - 2(1 - t)(A' + \Gamma)\Pi u,
\]
then
\[
Z''(F_2, F) := \{(u, t) \in \partial B(0, r_2) \cap F \times [0, 1] \mid \langle E_0^2u, u \rangle \leq 0 \text{ and } \langle E_0^2u, v \rangle = 0 \text{ for all } v \in F_2 \}
\]
is empty for every \( F \in F(X) \) with \( F_2 \subset F \). Therefore, when we consider the mapping \( E_0^2u = \delta(u)Au + (A' + \Gamma)u - 2(\Lambda' + \Gamma)\Pi u \), then
\[
Z''(F_2, F) := \{u \in \partial B(0, r_2) \cap F \mid \langle E_0^2u, u \rangle \leq 0 \text{ and } \langle E_0^2u, v \rangle = 0 \text{ for all } v \in F_2 \} = \emptyset
\]
for every \( F \in F(X) \) with \( F \supset F_2 \). Since \( F_2 \subset F_3 \), we have \( Z''(F_3, F) = \emptyset \) when \( F \supset F_3 \).

Step 2. We prove that
\[
F'''(F_3, F) = \{u \in \partial B(0, r_2) \cap F \mid \langle A_{F_3}^2F_3, u \rangle \leq 0 \text{ and } \langle A_{F_3}^2F_3, v \rangle = 0 \text{ for all } v \in F_3 \} = \emptyset
\]
for every \( F \in F(X) \) with \( F \supset F_3 \).
For some \( u \in \partial B(0, r_2) \cap F \) and \( F \supset F_3 \) we have
\[
\langle A_{F_3}^2F_3, u \rangle = \langle \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u + (I - Q')J(I - Q)u, u \rangle \leq 0
\]
and
\[
\langle A_{F_3}^2F_3, v \rangle = \langle \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u + (I - Q')J(I - Q)u, v \rangle = 0
\]
for all \( v \in F_3 \), then
\[
\langle E_0^2u, u \rangle = \langle \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u, u \rangle \leq -\langle (I - Q)u, (I - Q)u \rangle \leq 0
\]
and
\[
\langle E_0^2u, v \rangle = \langle \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u, v \rangle = -\langle (I - Q)u, (I - Q)v \rangle = -\langle (I - Q)u, 0 \rangle = 0
\]
for all \( v \in F_3 \), then
\[
E_0^2u = \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u, u \leq 0
\]
and
\[
E_0^2u = \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u, v = -\langle (I - Q)u, (I - Q)v \rangle = 0
\]
for all \( v \in F_3 \), then
\[
E_0^2u = \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u, u \leq 0
\]
and
\[
E_0^2u = \delta(u)Au + (A' + \Gamma)u - 2(A' + \Gamma)\Pi u, v = -\langle (I - Q)u, (I - Q)v \rangle = 0
\]
for every \( v \in F_2 \). The equation \( (I - Q)v = 0 \) holds because \( Q \) is a projection of \( X \) onto \( F_2 \) and \( v \in F_2 \). These calculations imply that \( u \in \mathbb{Z}^m(F_1,F) \) with \( F = F_2 + \mathbb{H}(u) \). But this contradicts what we have expressed in step 1 of this proof; thus, \( \mathbb{H}_m(F_1,F) = \emptyset \) for every \( F \supseteq F_2 \).

\[ \square \]

**Lemma 4.4.20.** Let the vectors \( w_1, \ldots, w_M \) be from (4.10). Then

\( F_2 = F_0 + F_2 + \text{span}\{w_1, \ldots, w_M\} \)

has a basis \( v_1, \ldots, v_N \) such that

\( F_0 = \text{span}\{f_1, \ldots, f_M\} \subset \text{span}\{v_1, \ldots, v_{2M}\} \),

\( v_i = w_i \), for \( i = 1, \ldots, M \),

\( \langle (\mathcal{A} + \mathcal{B})f_i, v_k \rangle = \delta_{ik} \) for \( i, k = 1, \ldots, M \),

\( \langle (\mathcal{A} + \mathcal{B})v_i, v_k \rangle = 0 \) for \( i = 1, \ldots, M, r \in R_0 \),

and

\( \langle (\mathcal{A} + \mathcal{B})f_i, v_k \rangle = 0 \) for \( i = 1, \ldots, M \) and \( k = M + 1, \ldots, N \).

\[ \text{Proof.} \text{ It is evident that we can choose } v_i = w_i \text{ for } i = 1, \ldots, M. \text{ Then the vectors } v_i \text{ satisfy } \]

\( \langle (\mathcal{A} + \mathcal{B})f_i, v_k \rangle = \delta_{ik} \) for \( i, k = 1, \ldots, M \),

and

\( \langle (\mathcal{A} + \mathcal{B})v_i, v_k \rangle = 0 \) for \( i = 1, \ldots, M, r \in R_0 \). So the first \( M \) vectors are found.

Next, we show the existence of the other basis vectors. Since \( P \) is a projection of \( X \) onto \( \text{span}\{w_1, \ldots, w_M\} \) and \( v_1 = w_i \) for \( i = 1, \ldots, M \), we have

\( X = \mathcal{M}(P) \oplus \mathcal{M}(P) = \mathcal{M}(P) \oplus \text{span}\{w_1, \ldots, w_M\} \).

Thus, we may take the other basis vectors from \( \mathcal{M}(P) \). Because \( \{f_1, \ldots, f_M\} \) contains \( M \) vectors and \( X = \mathcal{M}(P) \oplus \mathcal{M}(P) \), we can choose \( v_i \), where \( i = M + 1, \ldots, 2M \), so that \( f_1, \ldots, f_M \in \text{span}\{v_1, \ldots, v_{2M}\} \). The inclusion

\( \{v_{M+1}, \ldots, v_N\} \subset \mathcal{M}(P) = (\mathcal{M}(P)^\perp)^\perp = \mathbb{F}_0^+ = (\mathcal{A} + \mathcal{B})F_0 \)^\perp

(see Theorem 2.2.21 and Subsection 4.4.3) imply the conditions

\( \langle (\mathcal{A} + \mathcal{B})f_i, v_k \rangle = 0 \) for \( i = 1, \ldots, M \) and \( k = M + 1, \ldots, N \).

Thus, the basis \( v_1, \ldots, v_N \) has the desired properties.

\[ \square \]

**Lemma 4.4.21.** Assume that

1. the conditions of Theorem 4.3.1 are satisfied.

for every \( v \in F_2 \). The equation \( (I - Q)v = 0 \) holds because \( Q \) is a projection of \( X \) onto \( F_2 \) and \( v \in F_2 \). These calculations imply that \( u \in \mathbb{Z}^m(F_1,F) \) with \( F = F_2 + \mathbb{H}(u) \). But this contradicts what we have expressed in step 1 of this proof; thus, \( \mathbb{H}_m(F_1,F) = \emptyset \) for every \( F \supseteq F_2 \).

\[ \square \]

**Lemma 4.4.21.** Let the vectors \( w_1, \ldots, w_M \) be from (4.10). Then

\( F_2 = F_0 + F_2 + \text{span}\{w_1, \ldots, w_M\} \)

has a basis \( v_1, \ldots, v_N \) such that

\( F_0 = \text{span}\{f_1, \ldots, f_M\} \subset \text{span}\{v_1, \ldots, v_{2M}\} \),

\( v_i = w_i \), for \( i = 1, \ldots, M \),

\( \langle (\mathcal{A} + \mathcal{B})f_i, v_k \rangle = \delta_{ik} \) for \( i, k = 1, \ldots, M \),

\( \langle (\mathcal{A} + \mathcal{B})v_i, v_k \rangle = 0 \) for \( i = 1, \ldots, M, r \in R_0 \),

and

\( \langle (\mathcal{A} + \mathcal{B})f_i, v_k \rangle = 0 \) for \( i = 1, \ldots, M \) and \( k = M + 1, \ldots, N \).

\[ \text{Proof.} \text{ It is evident that we can choose } v_i = w_i \text{ for } i = 1, \ldots, M. \text{ Then the vectors } v_i \text{ satisfy } \]

\( \langle (\mathcal{A} + \mathcal{B})f_i, v_k \rangle = \delta_{ik} \) for \( i, k = 1, \ldots, M \),

and

\( \langle (\mathcal{A} + \mathcal{B})v_i, v_k \rangle = 0 \) for \( i = 1, \ldots, M, r \in R_0 \). So the first \( M \) vectors are found.

Next, we show the existence of the other basis vectors. Since \( P \) is a projection of \( X \) onto \( \text{span}\{w_1, \ldots, w_M\} \) and \( v_1 = w_i \) for \( i = 1, \ldots, M \), we have

\( X = \mathcal{M}(P) \oplus \mathcal{M}(P) = \mathcal{M}(P) \oplus \text{span}\{w_1, \ldots, w_M\} \).

Thus, we may take the other basis vectors from \( \mathcal{M}(P) \). Because \( \{f_1, \ldots, f_M\} \) contains \( M \) vectors and \( X = \mathcal{M}(P) \oplus \mathcal{M}(P) \), we can choose \( v_i \), where \( i = M + 1, \ldots, 2M \), so that \( f_1, \ldots, f_M \in \text{span}\{v_1, \ldots, v_{2M}\} \). The inclusion

\( \{v_{M+1}, \ldots, v_N\} \subset \mathcal{M}(P) = (\mathcal{M}(P)^\perp)^\perp = \mathbb{F}_0^+ = (\mathcal{A} + \mathcal{B})F_0 \)^\perp

(see Theorem 2.2.21 and Subsection 4.4.3) imply the conditions

\( \langle (\mathcal{A} + \mathcal{B})f_i, v_k \rangle = 0 \) for \( i = 1, \ldots, M \) and \( k = M + 1, \ldots, N \).

Thus, the basis \( v_1, \ldots, v_N \) has the desired properties.

\[ \square \]
To begin with, we note that it suffices to show that $(u_l) = 0$. With this equality, we derive certain estimates for $\delta_1$ and $\cap \Pi_2$. Then the fact that $v_1, \ldots, v_N$ is a basis of $F_2$ with desirable properties to derive an equality concerning the term $t\delta(u)$. When $t\delta(u) = 0$, then the contradiction is achieved by showing that $u$ must be zero. In the case where $t\delta(u) \neq 0$, we obtain a contradiction with the condition $\delta(u) > 0$.

Proof of Lemma 4.4.21. To begin with, we note that $A_{ij}^{(0)}(F_2)$ is continuous as a mapping from $\mathcal{D} \cap F_1 \times [0, 1]$ to $F_2$ because $A$ is bounded and semi-continuous in $B(0, r_2)$ and $A', \Gamma, \Pi,$ and $\delta$ are continuous. Next, we prove the other claims. To prove the equality

$$\deg(A_{ij}^{(0)}, B(0, r_2) \cap F_2, 0) = \deg(A_{ij}^{(0)}, B(0, r_2) \cap F_2, 0),$$

it suffices to show that $A_{ij}^{(0)}$ has no zeros in $\partial B(0, r_2) \cap F_2 \times [0, 1]$. We argue this claim by contradiction.

Clearly, $A_{ij}^{(0)}(u) = 0$ if and only if

$$\langle t\delta(u)Au - (A' + \Gamma)\Pi u + (A' + \Gamma)(I - \Pi)u, v_i \rangle = 0 \quad \text{for all} \quad i = 1, \ldots, N.$$ 

Let $f = \Pi u$ and $r = (I - \Pi)u$. Then the fact $w_i = v_i$ for $i = 1, \ldots, M$ implies

\begin{align*}
0 &= \sum_{i=1}^{M} \langle t\delta(u)Au - (A' + \Gamma)f + (A' + \Gamma)r, v_i \rangle (A + \Gamma)f_i \\
&= P' \langle t\delta(u)Au - (A' + \Gamma)f + (A' + \Gamma)r, v_i \rangle \\
&= \delta(u)P^*Au - P'(A' + \Gamma)f + P'(A' + \Gamma)r \\
&= \delta(u)P^*Au - (A' + \Gamma)f,
\end{align*}

that is,

$$t\delta(u)P^*Au = (A' + \Gamma)f. \quad (4.25)$$

With this equality, we derive certain estimates for $u, f,$ and $r$.

**Step 1.** We derive a certain estimate for $f$ and show that $t\delta(u) = 0$ is not possible. By taking into account that $v_1, \ldots, v_N$ is a basis of $F_1$, we get that

$$\langle t\delta(u)(I - P')Au + (A' + \Gamma)r, v \rangle = (t\delta(u)Au - t\delta(u)P^*Au + (A' + \Gamma)r, v) = (t\delta(u)Au - (A' + \Gamma)f + (A' + \Gamma)r, v) = 0$$

**Step 1.** We derive a certain estimate for $f$ and show that $t\delta(u) = 0$ is not possible. By taking into account that $v_1, \ldots, v_N$ is a basis of $F_3$, we get that

$$\langle t\delta(u)(I - P')Au + (A' + \Gamma)r, v \rangle = (t\delta(u)Au - (A' + \Gamma)f + (A' + \Gamma)r, v) = 0$$
for all $v \in F_2$. As $f, u \in F_2$; we have $r = u - f \in F_1$ also. Hence,
\[ \langle r \delta(u)(I - P^*)Au, r \rangle = \langle (A' + \Gamma)r, r \rangle \leq 0. \]  
(4.26)

This is a kind of estimate for $r$ that can be employed to show that $r = 0$ if the left hand side is zero. Next, we derive a certain estimate for $f$. Lemma 4.4.4 implies the estimate
\[ \|f\| \leq c_1 \min_{0 \leq i \leq i \Gamma} \| (A' + \Gamma)(2s - 1) f - iT f \| \leq c_1 \| (A' + \Gamma)f \| \]
(4.27)

\[ = c_1 c_2 \| A' \| \| \delta(u) \|. \]

If $t \delta(u) = 0$, then (4.27) and (4.26) yield $f = r = 0$ which is impossible, for our assumption is $u \neq 0$. Hence, $t \delta(u) \neq 0$.

Step 2. We consider the case $t \delta(u) \neq 0$.

First, we note that the definition of $\delta$ implies
\[ 0 < \delta(u) \leq \| I - P^* \| (I - rT)u \leq c \| (I - P^*)Au, u \|. \]  
(4.28)

In the definition of $\delta$, we have chosen $c \leq \tau_1 = \frac{1}{2} (c_1 c_2 \| P^* \| (I - P^*)^{-1})$. So
\[ \langle (I - P^*)Au,u \rangle = \langle (I - P^*)Au, f \rangle + \langle (I - P^*)Au, r \rangle \]
\[ \leq \langle (I - P^*)Au, f \rangle \]
\[ \leq \| I - P^* \| \| Au \| \| f \| \]
\[ = \| I - P^* \| c_1 c_2 \| P^* \| \| \delta(u) \| \]
\[ \leq \| I - P^* \| c_1 c_2 \| P^* \| \| \delta(u) \| \]

and a contradiction follows because this result and Equation (4.28) imply that
\[ 0 < \delta(u) \leq c \| I - P^* \| c_1 c_2 \| P^* \| \delta(u) \leq \frac{1}{2} \delta(u), \]

which in turn imply that $\delta(u) \leq 0$.

Step 3. According to the previous steps, $\lambda_{j_0}^{i, L}$ has no zeros in $B(0, r_2) \cap F_3 \times [0, 1]$ and consequently
\[ \deg(\lambda_{j_0}^{i, L}, B(0, r_2) \cap F_3, 0) = \deg(\lambda_{j_0}^{i, L}, B(0, r_2) \cap F_3, 0) \]
as required.

\[ \square \]

Lemma 4.4.22. Assume that
1. the conditions of Theorem 4.3.1 are satisfied,
2. the basis $v_1, \ldots, v_s$ of $F_1$ is given by Lemma 4.4.20,
3. and $r_1$ and $c$ are as in the beginning of this subsection.

for all $v \in F_2$. As $f, u \in F_2$; we have $r = u - f \in F_1$ also. Hence,
\[ \langle r \delta(u)(I - P^*)Au, r \rangle = \langle (A' + \Gamma)r, r \rangle \leq 0. \]  
(4.26)

This is a kind of estimate for $r$ that can be employed to show that $r = 0$ if the left hand side is zero. Next, we derive a certain estimate for $f$. Lemma 4.4.4 implies the estimate
\[ \|f\| \leq c_1 \min_{0 \leq i \leq i \Gamma} \| (A' + \Gamma)(2s - 1) f - iT f \| \leq c_1 \| (A' + \Gamma)f \| \]
(4.27)

\[ = c_1 c_2 \| A' \| \| \delta(u) \|. \]

If $t \delta(u) = 0$, then (4.27) and (4.26) yield $f = r = 0$ which is impossible, for our assumption is $u \neq 0$. Hence, $t \delta(u) \neq 0$.

Step 2. We consider the case $t \delta(u) \neq 0$.

First, we note that the definition of $\delta$ implies
\[ 0 < \delta(u) = c \min_{0 \leq i \leq i \Gamma} \langle (I - P^*)Au, (I - rT)u \rangle \leq c \langle (I - P^*)Au, u \|. \]  
(4.28)

In the definition of $\delta$, we have chosen $c \leq \tau_1 = \frac{1}{2} (c_1 c_2 \| P^* \| (I - P^*)^{-1})$. So
\[ \langle (I - P^*)Au,u \rangle = \langle (I - P^*)Au, f \rangle + \langle (I - P^*)Au, r \rangle \]
\[ \leq \langle (I - P^*)Au, f \rangle \]
\[ \leq \| I - P^* \| \| Au \| \| f \| \]
\[ = \| I - P^* \| c_1 c_2 \| P^* \| \| \delta(u) \| \]
\[ \leq \| I - P^* \| c_1 c_2 \| P^* \| \| \delta(u) \| \]

and a contradiction follows because this result and Equation (4.28) imply that
\[ 0 < \delta(u) \leq c \| I - P^* \| c_1 c_2 \| P^* \| \delta(u) \leq \frac{1}{2} \delta(u), \]

which in turn imply that $\delta(u) \leq 0$.

Step 3. According to the previous steps, $\lambda_{j_0}^{i, L}$ has no zeros in $B(0, r_2) \cap F_3 \times [0, 1]$ and consequently
\[ \deg(\lambda_{j_0}^{i, L}, B(0, r_2) \cap F_3, 0) = \deg(\lambda_{j_0}^{i, L}, B(0, r_2) \cap F_3, 0) \]
as required.

\[ \square \]

Lemma 4.4.22. Assume that
1. the conditions of Theorem 4.3.1 are satisfied,
2. the basis $v_1, \ldots, v_s$ of $F_1$ is given by Lemma 4.4.20,
3. and $r_1$ and $c$ are as in the beginning of this subsection.
Then the mapping $A_{0,F_1}^3 : \overline{B(0,r_2)} \cap F_3 \to F_1$ defined by

$$A_{0,F_1}^3 u = \sum_{i=1}^N (\langle A' + \Gamma \rangle (I - \Pi) u - (A' + \Gamma) \Pi u, v_i) v_i,$$

satisfies

$$\deg(A_{0,F_1}^3)_{\overline{B(0,r_2)} \cap F_3,0} = (-1)^M,$$

where $M$ is the sum of the algebraic multiplicities of the eigenvalues $\lambda \in [1, \infty)$ of $T$.

**Proof.** Step 1. We derive a more useful representation of $A_{0,F_1}^3$.

Because $\langle (A' + \Gamma) \gamma, v_i \rangle = 0$ when $i = 1, \ldots, M$ and $r \in \mathbb{R}$, we have

$$\langle (A' + \Gamma)(I - \Pi)u, v_i \rangle = 0 \quad \text{for} \quad i = 1, \ldots, M.$$

Equivalently,

$$\langle (A' + \Gamma)u, v_i \rangle = \langle (A' + \Gamma)\Pi u, v_i \rangle \quad \text{for} \quad i = 1, \ldots, M.$$

Since $\langle (A' + \Gamma)f_j, v_i \rangle = 0$ for $j = 1, \ldots, M$ and $i = M + 1, \ldots N$ and $\Pi$ is the continuous projection of $X$ onto $F_3$ along $R_0$, we have

$$\langle (A' + \Gamma)\Pi u, v_i \rangle = 0$$

when $i = M + 1, \ldots, N$. Especially,

$$\langle (A' + \Gamma)(I - \Pi)u, v_i \rangle = \langle (A' + \Gamma)u, v_i \rangle \quad \text{for} \quad i = M + 1, \ldots, N.$$

So we may write

$$A_{0,F_1}^3 u = \sum_{i=1}^N \langle (A' + \Gamma)(I - \Pi)u - (A' + \Gamma)\Pi u, v_i \rangle v_i$$

$$= \sum_{i=M+1}^N \langle (A' + \Gamma)(I - \Pi)u, v_i \rangle v_i - \sum_{i=1}^M \langle (A' + \Gamma)\Pi u, v_i \rangle v_i$$

$$= \sum_{i=M+1}^N \langle (A' + \Gamma)u, v_i \rangle v_i - \sum_{i=1}^M \langle (A' + \Gamma)u, v_i \rangle v_i.$$

Step 2. We calculate the degree of $A_{0,F_1}^3$ at zero with respect to $\partial B(0,r_2) \cap F_3$.

Let $v_1^*, \ldots, v_N^*$ be the dual basis of $v_1, \ldots, v_N$, that is, $\langle v_j^*, v_i \rangle = \delta_{ij}$ for $i, j = 1, \ldots, N$. Observe that

$$u = \sum_{i=1}^N \langle v_j^*, u \rangle v_i$$

for all $u \in F_3$. We consider the homotopy

$$A_{0,F_1}^3 u = t \sum_{i=M+1}^N \langle (A' + \Gamma)u, v_i \rangle v_i - t \sum_{i=1}^M \langle (A' + \Gamma)u, v_i \rangle v_i$$

$$+ (1 - t) \sum_{i=M+1}^N \langle v_j^*, u \rangle v_i - (1 - t) \sum_{i=1}^M \langle v_j^*, u \rangle v_i,$$
Clearly, this is continuous as a mapping $F_3 \times [0, 1] \to F_3$. If $A_{1,F_1}^t u = 0$ with some $t \in [0,1]$ and $u \in \partial B(0,r_2) \cap F_3$, then

$$t \langle (A' + \Gamma)u, v_i \rangle = -(1 - t) \langle v_i, u \rangle$$

for each $i = 1, \ldots, N$. Consequently,

$$0 \leq t \langle (A' + \Gamma)u, u \rangle = t \sum_{i=1}^{N} \langle v_i, u \rangle \langle (A' + \Gamma)u, v_i \rangle = -(1 - t) \sum_{i=1}^{N} \langle v_i, u \rangle^2 \leq 0$$

and the conclusion is $u = 0$, which is absurd because $u \in \partial B(0,r_2) \cap F_3$. Thus,

$$\text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0).$$

If $E$ is the matrix that corresponds to $A_{1,F_1}^t$ with respect to the basis $v_1, \ldots, v_N$, then $E$ is the diagonal matrix whose first $M$ diagonal elements are $-1$ and the other diagonal entries are $1$. Thus, we conclude that

$$\text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \det E = (-1)^M.$$ 

This ends the proof. \qed

**Remark 4.4.23.** In his books [167, Section 3.4], [169, Section 1.4], and [170, Section 2.5], Skrypnik does not emphasize that the basis ought to be specific if we want to deduce in the way we did in the proof of Lemma 4.4.21. However, as we have seen in the previous proof, the basis has to be carefully chosen. For this reason, it would be useful to know that the Brouwer degree of the "approximation" of a demi-continuous $(S_\epsilon)$-mapping is independent of the choice of the basis and the space $F_3$. If the degree depends on the choice of the basis and the subspace, then the previous results applies only to the basis and the subspace that we have chosen, and Theorem 4.3.1 would be difficult to apply.

By taking into account Lemmas 4.4.17, 4.4.18, 4.4.19, 4.4.21, and 4.4.22, we achieve the following equality to which we have aspired:

$$\text{Ind}(A, 0) = (-1)^M,$$

where $M$ is the sum of the algebraic multiplicities of the eigenvalues $\lambda \in [1, \infty]$ of $T$.  

Clearly, this is continuous as a mapping $F_3 \times [0, 1] \to F_3$. If $A_{1,F_1}^t u = 0$ with some $t \in [0,1]$ and $u \in \partial B(0,r_2) \cap F_3$, then

$$t \langle (A' + \Gamma)u, v_i \rangle = -(1 - t) \langle v_i, u \rangle$$

for each $i = 1, \ldots, N$. Consequently,

$$0 \leq t \langle (A' + \Gamma)u, u \rangle = t \sum_{i=1}^{N} \langle v_i, u \rangle \langle (A' + \Gamma)u, v_i \rangle = -(1 - t) \sum_{i=1}^{N} \langle v_i, u \rangle^2 \leq 0$$

and the conclusion is $u = 0$, which is absurd because $u \in \partial B(0,r_2) \cap F_3$. Thus,

$$\text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0).$$

If $E$ is the matrix that corresponds to $A_{1,F_1}^t$ with respect to the basis $v_1, \ldots, v_N$, then $E$ is the diagonal matrix whose first $M$ diagonal elements are $-1$ and the other diagonal entries are $1$. Thus, we conclude that

$$\text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \det E = (-1)^M.$$ 

This ends the proof. \qed

**Remark 4.4.23.** In his books [167, Section 3.4], [169, Section 1.4], and [170, Section 2.5], Skrypnik does not emphasize that the basis ought to be specific if we want to deduce in the way we did in the proof of Lemma 4.4.21. However, as we have seen in the previous proof, the basis has to be carefully chosen. For this reason, it would be useful to know that the Brouwer degree of the "approximation" of a demi-continuous $(S_\epsilon)$-mapping is independent of the choice of the basis and the space $F_3$. If the degree depends on the choice of the basis and the subspace, then the previous results applies only to the basis and the subspace that we have chosen, and Theorem 4.3.1 would be difficult to apply.

By taking into account Lemmas 4.4.17, 4.4.18, 4.4.19, 4.4.21, and 4.4.22, we achieve the following equality to which we have aspired:

$$\text{Ind}(A, 0) = (-1)^M,$$

where $M$ is the sum of the algebraic multiplicities of the eigenvalues $\lambda \in [1, \infty]$ of $T$.  

Clearly, this is continuous as a mapping $F_3 \times [0, 1] \to F_3$. If $A_{1,F_1}^t u = 0$ with some $t \in [0,1]$ and $u \in \partial B(0,r_2) \cap F_3$, then

$$t \langle (A' + \Gamma)u, v_i \rangle = -(1 - t) \langle v_i, u \rangle$$

for each $i = 1, \ldots, N$. Consequently,

$$0 \leq t \langle (A' + \Gamma)u, u \rangle = t \sum_{i=1}^{N} \langle v_i, u \rangle \langle (A' + \Gamma)u, v_i \rangle = -(1 - t) \sum_{i=1}^{N} \langle v_i, u \rangle^2 \leq 0$$

and the conclusion is $u = 0$, which is absurd because $u \in \partial B(0,r_2) \cap F_3$. Thus,

$$\text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0).$$

If $E$ is the matrix that corresponds to $A_{1,F_1}^t$ with respect to the basis $v_1, \ldots, v_N$, then $E$ is the diagonal matrix whose first $M$ diagonal elements are $-1$ and the other diagonal entries are $1$. Thus, we conclude that

$$\text{deg}(A_{1,F_1}^t, B(0,r_2) \cap F_3, 0) = \det E = (-1)^M.$$ 

This ends the proof. \qed
5 Two counter-examples

In this section, we scrutinize two examples concerning the assumptions of the index theorem. We consider mappings that act between \( \ell^p \) spaces where \( \infty > p > 2 \); in the latter example, \( p \) is also an integer. To begin with, we list some properties of these spaces when \( 1 < p < \infty \) although some of the results hold in more general setting. The interested reader may study book [178] by Taylor & Lay.

Remember that the (real) space \( \ell^p \), where \( \infty > p \geq 1 \), consists of those real sequences \( (a_n)_{n=1}^{\infty} \) whose \( \ell^p \)-norm is finite, that is,

\[
\| (a_n)_{n=1}^{\infty} \|_p = \sum_{n=1}^{\infty} |a_n|^p < \infty.
\]

If \( \infty > p > 1 \), we define \( p' \) by \( \frac{1}{p} + \frac{1}{p'} = 1 \). An important inequality is Hölder’s inequality:

\[
\sum_{n=1}^{\infty} a_n b_n \leq \| (a_n)_{n=1}^{\infty} \|_p \| (b_n)_{n=1}^{\infty} \|_p \quad \text{where} \quad (a_n)_{n=1}^{\infty} \in \ell^p \quad \text{and} \quad (b_n)_{n=1}^{\infty} \in \ell^p.
\]

The triangle inequality and Hölder’s inequality are proved in book [84, Sections 2.8 and 2.11] by Hardy, Littlewood & Polya. The space \( \ell^p \) is separable ([178, Example 2, p. 57]) and reflexive when \( \infty > p > 1 \). It is proved in Taylor & Lay’s book [178, Theorem 5.2, p. 143] that the dual space of \( \ell^p \), \( \infty > p > 1 \), can be identified with \( \ell^{p'} \) and we may consider the \( \ell^{p'} \)-norm as the dual norm. In fact, it is shown that for each \( x^* \in (\ell^p)^* \) there is unique element \( (d_n)_{n=1}^{\infty} \in \ell^{p'} \) so that

\[
\langle x^*, (u_n)_{n=1}^{\infty} \rangle = \sum_{n=1}^{\infty} c_n d_n, \quad \text{for all} \quad (c_n)_{n=1}^{\infty} \in \ell^p.
\]

Moreover, \( \| x^* \|_{(\ell^p)^*} = \| (d_n)_{n=1}^{\infty} \|_{\ell^{p'}} \).

The duality pairing of \( \ell^p \) and \( \ell^{p'} \) is defined by

\[
\langle v, u \rangle = \sum_{n=1}^{\infty} v_n u_n \quad \text{where} \quad v = (v_n)_{n=1}^{\infty} \in \ell^p, \quad u = (u_n)_{n=1}^{\infty} \in \ell^{p'}.
\]

Moreover, \( \| x \|_{\ell^p} = \| (d_n)_{n=1}^{\infty} \|_{\ell^p} \).

The duality pairing of \( \ell^p \) and \( \ell^{p'} \) is defined by

\[
\langle v, u \rangle = \sum_{n=1}^{\infty} v_n u_n \quad \text{where} \quad v = (v_n)_{n=1}^{\infty} \in \ell^p, \quad u = (u_n)_{n=1}^{\infty} \in \ell^{p'}.
\]
Observe that after such identifications we are no longer studying mappings \( \ell^p \to (\ell^p)^* \), where \((\ell^p)^* \) denotes the topological dual of \( \ell^p \), but mappings \( \ell^p \to \ell^q \). Nevertheless, we use the isometric isomorphism described above and identify \((\ell^p)^* \) with \( \ell^q \).

We also use the following inequality
\[
|a - b|^p \leq (|a| + |b|)^p \leq \left(2 \max\{|a|, |b|\}\right)^p \leq 2^p(2|a|^p + |b|^p)
\]
where \( a \) and \( b \) are any real numbers and \( p > 0 \). A useful result concerning the weak convergence in \( \ell^p \) is the following: if \( x^p = (x^n)^{n=1} \in \ell^p \) and \( x = (x^n)^{n=1} \in \ell^p \), where \( 1 < p < \infty \), then
\[
x_n \xrightarrow{n \to \infty} x \quad \text{in} \quad \ell^p
\]
if and only if
\[
\sup_{n \in \mathbb{Z}_+} \|x_n\|_p < \infty \quad \text{and} \quad \lim_{n \to \infty} x_n = x_i \quad \text{for every} \quad i \in \mathbb{Z}_+.
\]

This result is stated in Taylor & Lay’s book [178, Problem 5, p. 180].

5.1 The necessity of the continuity of \( \Pi(A' + \Gamma)^{-1} \)

This example a detailed version of the example outlined by Kartatsos & Skrypnik in the article [110, Example 3.4]. Kartatsos & Skrypnik outline the example in one page.

**Example 5.1.1.** Here \( p > 2 \). We define the linear mappings \( A': \ell^p \to \ell^{p'} \) and \( \Gamma: \ell^p \to \ell^{p'} \) by (we shall later prove that these are linear mappings \( \ell^p \to \ell^{p'} \))
\[
A'u = \left(-c_1 + \sum_{n=2}^\infty c_n \frac{c_0}{n^2} \frac{c_1}{n^3} \frac{c_2}{n^4} \frac{c_3}{n^5} \cdots \right)
\]
and
\[
\Gamma u = (2c_1, 0, 0, 0, \ldots),
\]

where \( u = (c_n)^{n=1} \in \ell^p \). We claim that

1. \( A' \) is everywhere defined, bounded, injective, and quasi-monotone;
2. \( A' + \Gamma \) is strictly monotone, and \( \Pi \) and \( T = (A' + \Gamma)^{-1} \Gamma \) are compact;
3. 2 is the only eigenvalue of \( T \) in the interval \( [1, \infty) \), and the algebraic multiplicity of the eigenvalue 2 is 1;
4. \( F_0 = \mathcal{A}'(2I - T) = \text{span}\{1, 0, 0, \ldots\} \) and \( R_0 = \partial \mathcal{A}'(2I - T) = \{(c_n)^{n=1} \in \ell^p | c_1 = 0\} \);
5. \( \Pi(A' + \Gamma)^{-1}: (A' + \Gamma)(\ell^p) \subset \ell^{p'} \to \ell^{p'} \) is unbounded, where \( \Pi \) is the projection of \( \ell^p \) onto \( F_0 \) along \( R_0 \);
6. \( F_0 := (A' + \Gamma)F_0 \neq \{0\}, \ R_0 := (A' + \Gamma)R_0 = \ell^{p'} \), and the assertion \( \ell^{p'} = F_0 \oplus R_0 \) is false.

Observe that after such identifications we are no longer studying mappings \( \ell^p \to (\ell^p)^* \), where \((\ell^p)^* \) denotes the topological dual of \( \ell^p \), but mappings \( \ell^p \to \ell^q \). Nevertheless, we use the isometric isomorphism described above and identify \((\ell^p)^* \) with \( \ell^q \).

We also use the following inequality
\[
|a - b|^p \leq (|a| + |b|)^p \leq \left(2 \max\{|a|, |b|\}\right)^p \leq 2^p(2|a|^p + |b|^p)
\]
where \( a \) and \( b \) are any real numbers and \( p > 0 \). A useful result concerning the weak convergence in \( \ell^p \) is the following: if \( x^p = (x^n)^{n=1} \in \ell^p \) and \( x = (x^n)^{n=1} \in \ell^p \), where \( 1 < p < \infty \), then
\[
x_n \xrightarrow{n \to \infty} x \quad \text{in} \quad \ell^p
\]
if and only if
\[
\sup_{n \in \mathbb{Z}_+} \|x_n\|_p < \infty \quad \text{and} \quad \lim_{n \to \infty} x_n = x_i \quad \text{for every} \quad i \in \mathbb{Z}_+.
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5.1 The necessity of the continuity of \( \Pi(A' + \Gamma)^{-1} \)

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**Example 5.1.1.** Here \( p > 2 \). We define the linear mappings \( A': \ell^p \to \ell^{p'} \) and \( \Gamma: \ell^p \to \ell^{p'} \) by (we shall later prove that these are linear mappings \( \ell^p \to \ell^{p'} \))
\[
A'u = \left(-c_1 + \sum_{n=2}^\infty c_n \frac{c_0}{n^2} \frac{c_1}{n^3} \frac{c_2}{n^4} \frac{c_3}{n^5} \cdots \right)
\]
and
\[
\Gamma u = (2c_1, 0, 0, 0, \ldots),
\]

where \( u = (c_n)^{n=1} \in \ell^p \). We claim that

1. \( A' \) is everywhere defined, bounded, injective, and quasi-monotone;
2. \( A' + \Gamma \) is strictly monotone, and \( \Pi \) and \( T = (A' + \Gamma)^{-1} \Gamma \) are compact;
3. 2 is the only eigenvalue of \( T \) in the interval \( [1, \infty) \), and the algebraic multiplicity of the eigenvalue 2 is 1;
4. \( F_0 = \mathcal{A}'(2I - T) = \text{span}\{1, 0, 0, \ldots\} \) and \( R_0 = \partial \mathcal{A}'(2I - T) = \{(c_n)^{n=1} \in \ell^p | c_1 = 0\} \);
5. \( \Pi(A' + \Gamma)^{-1}: (A' + \Gamma)(\ell^p) \subset \ell^{p'} \to \ell^{p'} \) is unbounded, where \( \Pi \) is the projection of \( \ell^p \) onto \( F_0 \) along \( R_0 \);
6. \( F_0 := (A' + \Gamma)F_0 \neq \{0\}, \ R_0 := (A' + \Gamma)R_0 = \ell^{p'} \), and the assertion \( \ell^{p'} = F_0 \oplus R_0 \) is false.
Step 1. We demonstrate that $A'$ is everywhere defined, bounded, injective, linear mapping $\ell^p \rightarrow \ell^p$, and that $\Gamma$ is everywhere defined, bounded, linear mapping $\ell^p \rightarrow \ell^p$.

The linearity of $A'$ and $\Gamma$ is evident. From the calculation

$$
\|\Gamma(c_n)\|_{\ell^p} = 2|c_1| \leq 2\|c_n\|_{\ell^p}
$$

we see that $\Gamma$ is everywhere defined and bounded mapping $\ell^p \rightarrow \ell^p$. Note that $p > 0$, $p' = \frac{p}{p-1}$, and $\frac{1}{p} + \frac{1}{p'} = 1$. Since

$$
\|A'u\|_{\ell^p'} = \left( -c_1 + \sum_{n=2}^{\infty} \frac{c_n}{n} \right)^{1/p'}
$$

we get

$$
\begin{align*}
\|A'u\|_{\ell^p'} &\leq 2^{1/p'}\left| -c_1 + \sum_{n=2}^{\infty} \frac{c_n}{n} \right|^{1/p'} \\
&\leq 2^{1/p'}\left| -c_1 + \sum_{n=2}^{\infty} \frac{c_n}{n} \right|^{1/p'} \\
&\leq 2^{1/p'}\left( \sum_{n=2}^{\infty} \left| c_n \right| n^{-2} \right)^{1/p'} \\
&\leq 2^{1/p'}\left( \sum_{n=2}^{\infty} \left| c_n \right| n^{-2} \right)^{1/p'} < \infty.
\end{align*}
$$

$A'$ is bounded and defined on the whole space $\ell^p$. Next, we demonstrate that $A'$ is injective. If $A'u = 0$, then $c_1 - \sum_{n=2}^{\infty} c_n n^{-2} = 0$ and $c_n n^{-2} = 0$ when $n \geq 2$. So the only possibility is that $c_n = 0$ for all $n \geq 1$, that is, $u = 0$. Therefore, $A'$ is injective.

Step 2. We show that $\Gamma$ and $T = (A' + \Gamma)^{-1} \Gamma : \ell^p \rightarrow \ell^p$ are compact, $A' : \ell^p \rightarrow \ell^p$ is quasi-monotone, and $A' + \Gamma : \ell^p \rightarrow \ell^p$ is strictly monotone.

Since $\Gamma$ is continuous and has a one-dimensional range, it is compact. As also $A'$ is everywhere defined, $A' + \Gamma$ is defined on the whole space $\ell^p$. Clearly,

$$
(A' + \Gamma)u = \left( \sum_{n=1}^{\infty} c_n n^{-2}, c_2 2^{-2}, c_3 3^{-2}, \ldots \right).
$$

We show that $A' + \Gamma$ is strictly monotone. We begin with the computation

$$
\langle (A' + \Gamma)u, u \rangle = c_1^2 + \sum_{n=2}^{\infty} c_n n^{-2} + \sum_{n=2}^{\infty} c_n^2 n^{-2}
$$

$$
= c_1^2 - 2^{-2} c_1^2 \sum_{n=2}^{\infty} n^{-2} + \sum_{n=2}^{\infty} (c_2 2^{-2} + c_3 3^{-2}) n^{-2}
$$

$$
= c_1^2 \left( 1 - \frac{\pi^2}{6} \right) + \sum_{n=2}^{\infty} (c_2 2^{-2} + c_3 3^{-2}) n^{-2},
$$

that holds for each $u = (c_n)_{n=1}^{\infty}$. As $5/4 - \pi^2/24 > 0$, it is evident that $A' + \Gamma$ is strictly monotone.

The quasi-monotonicity of $A'$ is a consequence of the strict monotonicity of $A' + \Gamma$ and the compactness of $\Gamma$. We have

$$
\lim_{n \to \infty} \langle (A' + \Gamma)(u_n - u_0), u_n - u_0 \rangle = \lim_{n \to \infty} \langle (A' + \Gamma)(u_n - u_0), u_n - u_0 \rangle
$$

$$
+ \lim_{n \to \infty} \langle (A' + \Gamma)u_0, u_n - u_0 \rangle \geq 0.
$$

Step 1. We demonstrate that $A'$ is everywhere defined, bounded, injective, linear mapping $\ell^p \rightarrow \ell^p$, and that $\Gamma$ is everywhere defined, bounded, linear mapping $\ell^p \rightarrow \ell^p$.

The linearity of $A'$ and $\Gamma$ is evident. From the calculation

$$
\|\Gamma(c_n)\|_{\ell^p} = 2|c_1| \leq 2\|c_n\|_{\ell^p}
$$

we see that $\Gamma$ is everywhere defined and bounded mapping $\ell^p \rightarrow \ell^p$. Note that $p > 0$, $p' = \frac{p}{p-1}$, and $\frac{1}{p} + \frac{1}{p'} = 1$. Since

$$
\|A'u\|_{\ell^p'} = \left( -c_1 + \sum_{n=2}^{\infty} \frac{c_n}{n} \right)^{1/p'}
$$

we get

$$
\begin{align*}
\|A'u\|_{\ell^p'} &\leq 2^{1/p'}\left| -c_1 + \sum_{n=2}^{\infty} \frac{c_n}{n} \right|^{1/p'} \\
&\leq 2^{1/p'}\left| -c_1 + \sum_{n=2}^{\infty} \frac{c_n}{n} \right|^{1/p'} \\
&\leq 2^{1/p'}\left( \sum_{n=2}^{\infty} \left| c_n \right| n^{-2} \right)^{1/p'} \\
&\leq 2^{1/p'}\left( \sum_{n=2}^{\infty} \left| c_n \right| n^{-2} \right)^{1/p'} < \infty.
\end{align*}
$$

$A'$ is bounded and defined on the whole space $\ell^p$. Next, we demonstrate that $A'$ is injective. If $A'u = 0$, then $c_1 - \sum_{n=2}^{\infty} c_n n^{-2} = 0$ and $c_n n^{-2} = 0$ when $n \geq 2$. So the only possibility is that $c_n = 0$ for all $n \geq 1$, that is, $u = 0$. Therefore, $A'$ is injective.

Step 2. We show that $\Gamma$ and $T = (A' + \Gamma)^{-1} \Gamma : \ell^p \rightarrow \ell^p$ are compact, $A' : \ell^p \rightarrow \ell^p$ is quasi-monotone, and $A' + \Gamma : \ell^p \rightarrow \ell^p$ is strictly monotone.

Since $\Gamma$ is continuous and has a one-dimensional range, it is compact. As also $A'$ is everywhere defined, $A' + \Gamma$ is defined on the whole space $\ell^p$. Clearly,

$$
(A' + \Gamma)u = \left( \sum_{n=1}^{\infty} c_n n^{-2}, c_2 2^{-2}, c_3 3^{-2}, \ldots \right).
$$

We show that $A' + \Gamma$ is strictly monotone. We begin with the computation

$$
\langle (A' + \Gamma)u, u \rangle = c_1^2 + \sum_{n=2}^{\infty} c_n n^{-2} + \sum_{n=2}^{\infty} c_n^2 n^{-2}
$$

$$
= c_1^2 - 2^{-2} c_1^2 \sum_{n=2}^{\infty} n^{-2} + \sum_{n=2}^{\infty} (c_2 2^{-2} + c_3 3^{-2}) n^{-2}
$$

$$
= c_1^2 \left( 1 - \frac{\pi^2}{6} \right) + \sum_{n=2}^{\infty} (c_2 2^{-2} + c_3 3^{-2}) n^{-2},
$$

that holds for each $u = (c_n)_{n=1}^{\infty}$. As $5/4 - \pi^2/24 > 0$, it is evident that $A' + \Gamma$ is strictly monotone.
Since
\[ (A' + \Gamma)^{-1} = \left( d_1 - \sum_{n=2}^{\infty} d_n, 2^2d_2, 3^2d_3, \ldots \right), \]
where \( v = (d_n)_{n=1}^{\infty} \in (A' + \Gamma)(\ell^p) \subset \ell^p', \)
the effect of \( T = (A' + \Gamma)^{-1} \Gamma \colon \ell^p \to \ell^p \) is given by
\[ Tu = (\alpha_1, 0, 0, \ldots) \quad \text{for} \quad u = (\alpha_n)_{n=1}^{\infty}. \]

Obviously, \( T \) is compact, the vector \((1, 0, 0, \ldots)\) is the only eigenvector\(^1\) of \( T \) corresponding to the eigenvalue \( 2 \), and \( T \) has no other eigenvalues lying in the interval \([1, \infty[\). Note that \( (2I - T)^{-1}u = 0 \) if and only if
\[ (2I - T)^{-1}u \in \mathcal{N}(2I - T) = \text{span}\{(1, 0, 0, \ldots)\}, \]
and that the inclusion \( (2I - T)(\alpha_n)_{n=1}^{\infty} \in \mathcal{N}(2I - T) \) holds if and only if \( c_1 \in \mathbb{R} \) and \( c_n = 0 \) for all \( n \geq 2 \). Hence, we deduce that for each \( k \geq 1 \) the equality \( (2I - T)^{-1}u = 0 \) holds if and only if \( u \in \text{span}\{(1, 0, 0, \ldots)\} \). Thus, if \( \mu \) is the algebraic multiplicity of the eigenvalue \( 2 \), then
\[ \mu = \dim \bigcup_{n=1}^{\infty} \mathcal{N}(2I - T)^n = \dim \mathcal{N}(2I - T) = \dim \text{span}\{(1, 0, 0, \ldots)\} = 1. \]

Step 3. We show that \( \Pi(A' + \Gamma)^{-1} \) is unbounded, discover the invariant subspaces \( F_0 \) and \( R_0 \) of \( T \), and show that the sum of \( R_0 \) and \( F_0 \) is not direct.

Since the algebraic multiplicity of the eigenvalue \( 2 \) is \( 1 \) and \( T \) has no other eigenvalues lying in \([1, \infty[\), \( F_0 \) and \( R_0 \) satisfy the conditions
\[ F_0 = \mathcal{N}(2I - T) = \{ u = (\alpha_n)_{n=1}^{\infty} \in \ell^p \mid \alpha_n = 0, \text{ for } n \geq 2 \} \]
\[ R_0 = \delta(2I - T) = \{ u = (\alpha_n)_{n=1}^{\infty} \in \ell^p \mid \alpha_1 = 0 \}, \]
and the decomposition \( \ell^p = F_0 \oplus R_0 \) holds.

The effect of \( \Pi(A' + \Gamma)^{-1} \) is given by the formula
\[ \Pi(A' + \Gamma)^{-1}h = \left( \sum_{n=1} h_n, 0, 0, 0, \ldots \right) \]
for \( h = (h_n)_{n=1}^{\infty} \in (A' + \Gamma)p. \) We embed the vector \((n^{-1}m)_{m=1}^{\infty} \) in \( \ell^p \) in the usual way:
\[ (n^{-1}m)_{m=1}^{\infty} \rightleftharpoons (1^{-1}, 2^{-1}, \ldots, m^{-1}, 0, 0, 0, \ldots). \]
Note that \( (A' + \Gamma)(v_m)_{m=1}^{\infty} = (n^{-1}m)_{m=1}^{\infty}, \) where \( (v_m)_{m=1}^{\infty} \in \ell^p \) is such that
\[ \begin{align*}
  v_m &= 1 - \sum_{i=1}^{m-1} i^{-1}, \quad n = 1, \\
  v_m &= n, \quad n = 2, \ldots, m, \\
  v_m &= 0, \quad n > m.
\end{align*} \]

\[ \text{Here we exclude the vectors that are achieved by multiplying this vector by a scalar.} \]

Since
\[ (A' + \Gamma)^{-1}v = \left( d_1 - \sum_{n=2}^{\infty} d_n, 2^2d_2, 3^2d_3, \ldots \right), \]
where \( v = (d_n)_{n=1}^{\infty} \in (A' + \Gamma)(\ell^p) \subset \ell^p', \)
the effect of \( T = (A' + \Gamma)^{-1} \Gamma \colon \ell^p \to \ell^p \) is given by
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Obviously, \( T \) is compact, the vector \((1, 0, 0, \ldots)\) is the only eigenvector\(^1\) of \( T \) corresponding to the eigenvalue \( 2 \), and \( T \) has no other eigenvalues lying in the interval \([1, \infty[\). Note that \( (2I - T)^{-1}u = 0 \) if and only if
\[ (2I - T)^{-1}u \in \mathcal{N}(2I - T) = \text{span}\{(1, 0, 0, \ldots)\}, \]
and that the inclusion \( (2I - T)(\alpha_n)_{n=1}^{\infty} \in \mathcal{N}(2I - T) \) holds if and only if \( c_1 \in \mathbb{R} \) and \( c_n = 0 \) for all \( n \geq 2 \). Hence, we deduce that for each \( k \geq 1 \) the equality \( (2I - T)^{-1}u = 0 \) holds if and only if \( u \in \text{span}\{(1, 0, 0, \ldots)\} \). Thus, if \( \mu \) is the algebraic multiplicity of the eigenvalue \( 2 \), then
\[ \mu = \dim \bigcup_{n=1}^{\infty} \mathcal{N}(2I - T)^n = \dim \mathcal{N}(2I - T) = \dim \text{span}\{(1, 0, 0, \ldots)\} = 1. \]

Step 3. We show that \( \Pi(A' + \Gamma)^{-1} \) is unbounded, discover the invariant subspaces \( F_0 \) and \( R_0 \) of \( T \), and show that the sum of \( R_0 \) and \( F_0 \) is not direct.

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\[ F_0 = \mathcal{N}(2I - T) = \{ u = (\alpha_n)_{n=1}^{\infty} \in \ell^p \mid \alpha_n = 0, \text{ for } n \geq 2 \} \]
\[ R_0 = \delta(2I - T) = \{ u = (\alpha_n)_{n=1}^{\infty} \in \ell^p \mid \alpha_1 = 0 \}, \]
and the decomposition \( \ell^p = F_0 \oplus R_0 \) holds.

The effect of \( \Pi(A' + \Gamma)^{-1} \) is given by the formula
\[ \Pi(A' + \Gamma)^{-1}h = \left( \sum_{n=1} h_n, 0, 0, 0, \ldots \right) \]
for \( h = (h_n)_{n=1}^{\infty} \in (A' + \Gamma)p. \) We embed the vector \((n^{-1}m)_{m=1}^{\infty} \) in \( \ell^p \) in the usual way:
\[ (n^{-1}m)_{m=1}^{\infty} \rightleftharpoons (1^{-1}, 2^{-1}, \ldots, m^{-1}, 0, 0, 0, \ldots). \]
Note that \( (A' + \Gamma)(v_m)_{m=1}^{\infty} = (n^{-1}m)_{m=1}^{\infty}, \) where \( (v_m)_{m=1}^{\infty} \in \ell^p \) is such that
\[ \begin{align*}
  v_m &= 1 - \sum_{i=1}^{m-1} i^{-1}, \quad n = 1, \\
  v_m &= n, \quad n = 2, \ldots, m, \\
  v_m &= 0, \quad n > m.
\end{align*} \]

\[ \text{Here we exclude the vectors that are achieved by multiplying this vector by a scalar.} \]

---

1. Here we exclude the vectors that are achieved by multiplying this vector by a scalar.
Since \( p' > 1 \), we have \( (n^{-1})_{n=1}^{m} \to (m^{-1})_{m=1}^{n} \) in \( \ell^p \). From the calculation

\[
\| \Pi(A' + \Gamma)^{-1} ((n^{-1})_{n=1}^{m}) \|_p = \left| 1 - \sum_{n=2}^{n-1} \frac{m-n}{m} \right|
\]

we infer that \( \Pi(A' + \Gamma)^{-1} \) is unbounded.

According to the results of earlier steps, the inclusion \( (A' + \Gamma)\mathcal{R}_0 \subset \ell^p \) holds. We shall prove that \( \ell^p = \mathcal{R}_0 = (A' + \Gamma)\mathcal{R}_0 \). If this is not the case, then the Hahn-Banach theorem implies that there exists \( w = (w_n)_{n=1}^{\infty} \subset \ell^p \) such that \( w \neq 0 \) and

\[
\langle r^*, w \rangle = 0 \quad \text{for all} \quad r^* \in \mathcal{R}_0.
\]

Recall that \( \mathcal{R}_0 = (A' + \Gamma)\mathcal{R}_0 \) and that \( r^* = 0 \) if \( r = (r_n)_{n=1}^{\infty} \in \mathcal{R}_0 \). Thus,

\[
0 = \left( \langle (A' + \Gamma) r, w \rangle = \sum_{n=2}^{\infty} r_n w_1 + \sum_{n=2}^{\infty} \frac{w_n}{n} \right) \text{ for all } r = (r_n)_{n=1}^{\infty} \in \mathcal{R}_0.
\]

Since \( (r^*_1)_{i=1}^{\infty} \in \mathcal{R}_0 \) for \( i = 2, 3, 4, \ldots \), we obtain from the previous equality that

\[
w_1 + w_n = 0 \quad \text{for all} \quad n = 2, 3, 4, \ldots
\]

If \( w_1 = 0 \), then we have \( w = 0 \), which contradicts the assumption \( w \neq 0 \). If \( w_1 \neq 0 \), then \( w_n = -w_1 \) for all \( n \), and hence \( \| w \|_p = \infty \), which is a contradiction. Thus, \( \ell^p = (A' + \Gamma)\mathcal{R}_0 \).

Since \( \ell^p = (A' + \Gamma)\mathcal{R}_0 \), the sum

\[
(A' + \Gamma)\mathcal{R}_0 = \mathcal{R}_0 + \mathcal{R}_0
\]

is not direct. This finishes the example.

\section{5.2 The necessity of \( 0 \notin \text{w-cl}(\sigma_0) \)}

Our next example demonstrates that it is necessary to assume that the weak closure of \( \sigma_0 \) does not contain zero. It is a detailed form of an example outlined by Skrypnik. Skrypnik uses only one page for the sketch of this example; see [167, p. 135].

**Example 5.2.1.** Let \( p > 2 \) be an integer. We define the operator \( A : \ell^p \to \ell^{p'} \) by

\[
A u = (c_n^{-1} + c_n a^{2-p} - 2n^{1-p} f(n a_n)^{p-n})_{n=1}^{m}
\]

where \( u = (c_n)_{n=1}^{m} \in \ell^p \) and \( f : \mathbb{R} \to \mathbb{R} \) is the piecewise linear function defined by

\[
f(t) =
\begin{cases}
0, & \text{when } |t| > \frac{1}{2p} \\
2p(t-1) + 1, & \text{when } 1 - \frac{1}{2p} \leq t \leq 1 \\
1 - 2p(t-1), & \text{when } 1 \leq t \leq 1 + \frac{1}{2p}
\end{cases}
\]

Our claims are

\[
\text{Since } p' > 1, \text{ we have } (n^{-1})_{n=1}^{m} \to (m^{-1})_{m=1}^{n} \text{ in } \ell^{p'}. \text{ From the calculation}
\]

\[
\| \Pi(A' + \Gamma)^{-1} ((n^{-1})_{n=1}^{m}) \|_p = \left| 1 - \sum_{n=2}^{n-1} \frac{m-n}{m} \right|
\]

we infer that \( \Pi(A' + \Gamma)^{-1} \) is unbounded.

According to the results of earlier steps, the inclusion \( (A' + \Gamma)\mathcal{R}_0 \subset \ell^{p'} \) holds. We shall prove that \( \ell^{p'} = \mathcal{R}_0 = (A' + \Gamma)\mathcal{R}_0 \). If this is not the case, then the Hahn-Banach theorem implies that there exists \( w = (w_n)_{n=1}^{\infty} \subset \ell^{p'} \) such that \( w \neq 0 \) and

\[
\langle r^*, w \rangle = 0 \quad \text{for all} \quad r^* \in \mathcal{R}_0.
\]

Recall that \( \mathcal{R}_0 = (A' + \Gamma)\mathcal{R}_0 \) and that \( r^* = 0 \) if \( r = (r_n)_{n=1}^{\infty} \in \mathcal{R}_0 \). Thus,

\[
0 = \left( \langle (A' + \Gamma) r, w \rangle = \sum_{n=2}^{\infty} r_n w_1 + \sum_{n=2}^{\infty} \frac{w_n}{n} \right) \text{ for all } r = (r_n)_{n=1}^{\infty} \in \mathcal{R}_0.
\]

Since \( (r^*_1)_{i=1}^{\infty} \in \mathcal{R}_0 \) for \( i = 2, 3, 4, \ldots \), we obtain from the previous equality that

\[
w_1 + w_n = 0 \quad \text{for all} \quad n = 2, 3, 4, \ldots
\]

If \( w_1 = 0 \), then we have \( w = 0 \), which contradicts the assumption \( w \neq 0 \). If \( w_1 \neq 0 \), then \( w_n = -w_1 \) for all \( n \), and hence \( \| w \|_p = \infty \), which is a contradiction. Thus, \( \ell^{p'} = (A' + \Gamma)\mathcal{R}_0 \).

Since \( \ell^{p'} = (A' + \Gamma)\mathcal{R}_0 \), the sum

\[
(A' + \Gamma)\mathcal{R}_0 = \mathcal{R}_0 + \mathcal{R}_0
\]

is not direct. This finishes the example.
we conclude that the Frechet derivative of $B$ is compact and $A' + \Gamma$ is strictly monotone;  
3. $T$ has no eigenvalues in the interval $[1, \infty)$, $F_0 = \{0\}$, and $R_0 = T$;  
4. $\Pi(A' + \Gamma)^{-1} \cdot (A' + \Gamma)(t^p) \subset l^p \to l^p$ is continuous where $\Pi$ is the projection of $l^p$ onto $F_0$ along $R_0$;  
5. $0 \in \varepsilon(c_{\ell})$ for all $\varepsilon > 0$;  
6. 0 is not an isolated critical point.

**Step 1.** We give an outline for this example. Claims 2–6 are proved in step 5. Since

$$A = B_1 + B_2 - B_3,$$

where

$$B_1 u = (e_n^{p-1})_{n=1}^{\infty}, \quad B_2 u = (e_n^{p-2})_{n=1}^{\infty}, \quad B_3 u = (2n^{-1-p} f(nc_n))_{n=1}^{\infty} \quad \text{for} \quad u = (c_n)_{n=1}^{\infty},$$

we study $B_1$, $B_2$, and $B_3$ individually. Claim 1 is argued by showing that

1. $B_1$ is everywhere defined, bounded, continuous mapping $\ell^p \to \ell^p$ that has the property $(S_{\ell})$ and whose Frechet derivative at origin is the zero mapping;  
2. $B_2$ is everywhere defined, linear, bounded, quasi-monotone mapping $\ell^p \to \ell^p$;  
3. $B_3$ is everywhere defined, bounded, completely continuous mapping $\ell^p \to \ell^p$ whose Frechet derivative at origin is the zero mapping.

These assertions are proved in steps 2–4; $B_1$ is dealt with in step 2, $B_2$ in step 3, and $B_3$ in step 4. Claim 1 then follows from the perturbation results concerning Frechet differentiability, continuity, and the property $(S_{\ell})$.

Note that $p' = \frac{p}{p-1}$ and $\frac{1}{p-1} + \frac{1}{p'} = 1$.

**Step 2.** We consider first $B_1$. Recall that $p > 2$ is an integer. The computation

$$\|B_1 u\|_{p'} = \|e_n^{p-1}\|_{p'} = \|u\|_{p}^{p-1}\|

shows that $B_1$ is a bounded, everywhere defined mapping $\ell^p \to \ell^p$. Since

$$\|h\|_{p}^{-1} \|B_1(0 + h) - B_1(0) - 0\|_{p'} = \|h\|_{p}^{-1} \sum_{n=1}^{\infty} (b_n^{p-1} - 0 - 0) p' \frac{1}{p} = \|h\|_{p}^{p-2} h - 0,$$

we conclude that the Frechet derivative of $B_1$ at origin is zero.

**Step 2.1.** We show that $B_1$ is continuous. Assume that $u_j = (c_{nj})_{n=1}^{\infty} \xrightarrow{\text{w}} u = (c_n)_{n=1}^{\infty}$ in $\ell^p$. Then

$$\|B_1 u_j\|_{p'} = \|(c_{nj})^{p-1}_{n=1}\|_{p'} = \|u\|_{p}^{p-1} \xrightarrow{\text{w}} \|u\|_{p}^{p-1} = \|(c_n)^{p-1}_{n=1}\|_{p'} = \|B_1 u\|_{p'}.$$

where

$$B_1 u = (e_n^{p-1})_{n=1}^{\infty}, \quad B_2 u = (e_n^{p-2})_{n=1}^{\infty}, \quad B_3 u = (2n^{-1-p} f(nc_n))_{n=1}^{\infty} \quad \text{for} \quad u = (c_n)_{n=1}^{\infty},$$

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These assertions are proved in steps 2–4; $B_1$ is dealt with in step 2, $B_2$ in step 3, and $B_3$ in step 4. Claim 1 then follows from the perturbation results concerning Frechet differentiability, continuity, and the property $(S_{\ell})$.

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we conclude that the Frechet derivative of $B_1$ at origin is zero.

**Step 2.1.** We show that $B_1$ is continuous. Assume that $u_j = (c_{nj})_{n=1}^{\infty} \xrightarrow{\text{w}} u = (c_n)_{n=1}^{\infty}$ in $\ell^p$. Then

$$\|B_1 u_j\|_{p'} = \|(c_{nj})^{p-1}_{n=1}\|_{p'} = \|u\|_{p}^{p-1} \xrightarrow{\text{w}} \|u\|_{p}^{p-1} = \|(c_n)^{p-1}_{n=1}\|_{p'} = \|B_1 u\|_{p'}.$$
It follows from the assumption \( u_j \xrightarrow{\ell^p} u \) in \( \ell^p \) that \((e_{n+1}^{p-1})_{n=1}^\infty \) is in \( \ell^p \). \((e_{n+1}^{p-1})_{n=1}^\infty \) is bounded in \( \ell^p \), and
\[
 c_{j,n}^{p-1} \xrightarrow{\ell^p} c_n^{p-1}.
\]
These in turn imply
\[
 B_1u_j \xrightarrow{\ell^p} B_1u.
\]
Hence, the Kadec property of \( \ell^p \) yield \( B_1u_j \xrightarrow{\ell^p} B_1u \) (see Theorem 2.2.13) showing that \( B_1 \) is continuous.

Step 2.2. We prove that \( B_1 \) possesses the property \((S_\infty)\).
Assume that
\[
u_j = (c_{j,n})_{n=1}^\infty u = (c_n)_{n=1}^\infty \text{ in } \ell^p \text{ and } 0 \geq \lim_{j \to \infty} (B_1u_j, u_j - u).
\]
Then \((c_{j,n}^{p-1})_{n=1}^\infty \) is bounded in \( \ell^p \). \((c_n^{p-1})_{n=1}^\infty \) is in \( \ell^p \), and \( c_{j,n}^{p-1} \xrightarrow{\ell^p} c_n^{p-1} \). Thus,
\[
 (c_{j,n}^{p-1})_{n=1}^\infty \text{ (or } (c_n^{p-1})_{n=1}^\infty \text{ in } \ell^p
\]
Lemma 2.2.5 states that
\[
 ||(c_n)_{n=1}^\infty||_p \leq \lim_{j \to \infty} ||(c_{j,n})_{n=1}^\infty||_p,
\]
and our assumption implies that
\[
 0 \geq \lim_{j \to \infty} (B_1u_j, u_j - u) = \lim_{j \to \infty} (B_1u_j, u) - (B_1u, u)
\]
\[
 0 = \lim_{j \to \infty} \sum_{n=1}^\infty c_{j,n}^{p-1} c_n - \sum_{n=1}^\infty e_{n}^{p-1} c_n
\]
\[
 = \lim_{j \to \infty} \sum_{n=1}^\infty (c_{j,n})_{n=1}^\infty p - ||(c_n)_{n=1}^\infty||_p.
\]
So the only possibility is that
\[
 ||u||_p = ||(c_{j,n})_{n=1}^\infty||_p \xrightarrow{\ell^p} ||(c_n)_{n=1}^\infty||_p = ||u||_p.
\]
Now the Kadec property of \( \ell^p \) (see Theorem 2.2.13) yields
\[
u_j \xrightarrow{\ell^p} u \text{ in } \ell^p.
\]
showing that \( B_1 \) has the property \((S_\infty)\).

Step 3. We examine \( B_2 \).
Evidently, \( B_2 \) is linear, and the calculation
\[
 ||B_2u||_\rho = ||(\alpha^{\frac{p}{p-1}} c_n)_{n=1}^\infty||_\rho = \left( ||(\alpha^{\frac{p}{p-1}} c_n)_{n=1}^\infty||_1 \right)^{\frac{p}{p-1}}
\]
\[
 \leq \left( ||(c_n)_{n=1}^\infty||_1 \right)^{\frac{p}{p-1}} \left( ||(\alpha^{\frac{p}{p-1}})_{n=1}^\infty||_1 \right)^{\frac{p-1}{p}}
\]
\[
 \leq \left( ||u||_\rho \cdot ||(\alpha^{\frac{p}{p-1}})_{n=1}^\infty||_1 \right)^{\frac{p-1}{p}} \leq ||u||_\rho ||(\alpha^{\frac{p}{p-1}})_{n=1}^\infty||_p^{\frac{p-1}{p}}
\]
It follows from the assumption \( u_j \xrightarrow{\ell^p} u \) in \( \ell^p \) that \((e_{n+1}^{p-1})_{n=1}^\infty \) is in \( \ell^p \). \((e_{n+1}^{p-1})_{n=1}^\infty \) is bounded in \( \ell^p \), and
\[
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Hence, the Kadec property of \( \ell^p \) yield \( B_1u_j \xrightarrow{\ell^p} B_1u \) (see Theorem 2.2.13) showing that \( B_1 \) is continuous.

Step 2.2. We prove that \( B_2 \) possesses the property \((S_\infty)\).
Assume that
\[
u_j = (c_{j,n})_{n=1}^\infty u = (c_n)_{n=1}^\infty \text{ in } \ell^p \text{ and } 0 \geq \lim_{j \to \infty} (B_1u_j, u_j - u).
\]
Then \((c_{j,n}^{p-1})_{n=1}^\infty \) is bounded in \( \ell^p \). \((c_n^{p-1})_{n=1}^\infty \) is in \( \ell^p \), and \( c_{j,n}^{p-1} \xrightarrow{\ell^p} c_n^{p-1} \). Thus,
\[
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\]
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\[
 0 \geq \lim_{j \to \infty} (B_1u_j, u_j - u) = \lim_{j \to \infty} (B_1u_j, u) - (B_1u, u)
\]
\[
 0 = \lim_{j \to \infty} \sum_{n=1}^\infty c_{j,n}^{p-1} c_n - \sum_{n=1}^\infty c_n^{p-1} c_n
\]
\[
 = \lim_{j \to \infty} \sum_{n=1}^\infty (c_{j,n})_{n=1}^\infty p - ||(c_n)_{n=1}^\infty||_p.
\]
So the only possibility is that
\[
 ||u||_p = ||(c_{j,n})_{n=1}^\infty||_p \xrightarrow{\ell^p} ||(c_n)_{n=1}^\infty||_p = ||u||_p.
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Now the Kadec property of \( \ell^p \) (see Theorem 2.2.13) yields
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\]
\[
 \leq \left( ||(c_n)_{n=1}^\infty||_1 \right)^{\frac{p}{p-1}} \left( ||(\alpha^{\frac{p}{p-1}})_{n=1}^\infty||_1 \right)^{\frac{p-1}{p}}
\]
\[
 \leq \left( ||u||_\rho \cdot ||(\alpha^{\frac{p}{p-1}})_{n=1}^\infty||_1 \right)^{\frac{p-1}{p}} \leq ||u||_\rho ||(\alpha^{\frac{p}{p-1}})_{n=1}^\infty||_p^{\frac{p-1}{p}}
\]
shows that \( B_2 \) is bounded, everywhere defined mapping \( \ell^p \to \ell^r \). The term \( \|(n^{-1})_n^{-1}\|_p \) is finite because the series \( \sum_{n=1}^{\infty} n^{-1} \) converges when \( s > 1 \). Especially, \( B_2 \) is continuous. Since \( B_2 \) is a linear, continuous, everywhere defined mapping, it is its Fréchet derivative at origin.

We prove that \( B_2 \) is quasi-monotone. Suppose that

\[ u_j = (c_{j,n})_{n=1}^\infty \overset{\text{in} \ \ell^p}{\longrightarrow} u = (c_n)_{n=1}^\infty. \]

Then

\[ \lim_{j \to \infty} B_1(u_j,u_j-u) = \lim_{j \to \infty} B_1(u_j-u,u_j-u) = \lim_{j \to \infty} \sum_{n=1}^{\infty} n^{-r}(c_{j,n} - c_n)(c_{j,n} - c_n) \geq 0, \]
as required.

**Step 4.** We study \( B_3 \).

Since \( |f(t)| \leq 1 \) for all \( t \in \mathbb{R} \) and \( (n^{-1})_n^{-1} \in \ell^p \),

\[ \|B_3u\| \leq \|(2n^{-1}p f(nc_n))\|_p \leq 2(n^{-1}p)_n^{-1}, \]
demonstrating that \( B_3 \) is bounded and defined on the whole space \( \ell^p \).

**Step 4.1.** We show that \( B_3 \) is completely continuous. Suppose that \( \varepsilon > 0 \) and

\[ u_j = (c_{j,n})_{n=1}^\infty \overset{\text{in} \ \ell^p}{\longrightarrow} u = (c_n)_{n=1}^\infty \]
in \( \ell^p \).

Because \( 0 \leq f(t) \leq 1 \) for all \( t \in \mathbb{R} \) and \( (n^{-1}p)_n^{-1} \in \ell^p \), there is an integer \( k \) such that

\[ \sum_{n=k+1}^{\infty} n^{-1}p(f(nc_n) - f(nc_n)) \leq \frac{\varepsilon^p}{2}. \]

Note that \( k \) is independent of \( j \in \mathbb{Z}_+ \). We fix \( k \) for the rest of this step. The assumption

\[ u_j = (c_{j,n})_{n=1}^\infty \overset{\text{in} \ \ell^p}{\longrightarrow} u = (c_n)_{n=1}^\infty \]

implies that

\[ c_{j,n} \overset{\text{in} \ \ell^p}{\longrightarrow} c_n \]

for every \( n \in \mathbb{Z}_+ \). By the continuity of \( f \), we have

\[ f(nc_{j,n}) \overset{\text{in} \ \ell^p}{\longrightarrow} f(nc_n) \]

for each \( n \in \mathbb{Z}_+ \), and therefore there is an integer \( N_1 \) such that

\[ \|f(nc_n) - f(nc_{j,n})\|_p < \frac{\varepsilon^p}{2k}. \]

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\[ u_j = (c_{j,n})_{n=1}^\infty \overset{\text{in} \ \ell^p}{\longrightarrow} u = (c_n)_{n=1}^\infty. \]

Then

\[ \lim_{j \to \infty} B_1(u_j,u_j-u) = \lim_{j \to \infty} B_1(u_j-u,u_j-u) = \lim_{j \to \infty} \sum_{n=1}^{\infty} n^{-r}(c_{j,n} - c_n)(c_{j,n} - c_n) \geq 0, \]
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\[ \|f(nc_n) - f(nc_{j,n})\|_p < \frac{\varepsilon^p}{2k}. \]
when \( j \geq N_1 \) and \( n = 1, 2, \ldots, k \). Consequently,

\[
\frac{\|B_{3u}-B_{u}\|_{\mathfrak{P}}}{2^p} = \sum_{n=1}^{\infty} |n^{-p}(f(nc_n)-f(nc_{n,1}))|^{\mathfrak{P}}
\]

\[
= \frac{1}{2^p} \left[ \sum_{n=1}^{\infty} |n^{-p}(f(nc_n)-f(nc_{n,1}))|^{\mathfrak{P}} + \sum_{n=1}^{\infty} |n^{-p}(f(nc_n)-f(nc_{n,1}))|^{\mathfrak{P}} \right]
\]

\[
< k \frac{e^{\mathfrak{P}}}{2^p} + \frac{e^{\mathfrak{P}}}{2} = e^{\mathfrak{P}}
\]

whenever \( j \geq N_1 \). This shows that \( B_{3u} \xrightarrow{\mathfrak{P}} B_{u} \) in \( \mathfrak{P} \) whenever \( u \xrightarrow{\mathfrak{P}} u \) in \( \mathfrak{P} \). Thus, \( B_{3} \) is completely continuous.

**Step 4.2.** Our next task is to show that the Fréchet derivative of \( B_{3} \) at zero is the zero mapping.

We demonstrate that

\[
\|h\|_{\mathfrak{P}}^{-1} \|B_{3}h\|_{\mathfrak{P}} = \|h\|_{\mathfrak{P}}^{-1} \left[ \sum_{n=1}^{\infty} |2f(nh_n)|^{\mathfrak{P}} n^{-p} \right]^{\mathfrak{P}} \|h\|_{\mathfrak{P}}^{-0} \to 0
\]

where \( j \to \infty \). This implies that at least for a subsequence, the inequality

\[
\|h\|_{\mathfrak{P}}^{\mathfrak{P}} \sum_{n=1}^{\infty} |2f(nh_n)|^{\mathfrak{P}} n^{-p} \geq a > 0
\]

holds for all \( j \in \mathbb{Z}_+ \) and for a constant \( a \) independent of \( j \). So, for each \( j \in \mathbb{Z}_+ \) there is at least one positive integer \( n(j) \) for which

\[
f(n(j)h_{n(j)}) \neq 0,
\]

that is,

\[
\frac{2p-1}{n(j)2p} < h_{n(j)} < \frac{2p+1}{n(j)2p} \quad \text{(5.2)}
\]

(see the definition of \( f \)). For each \( j \in \mathbb{Z}_+ \), we denote by \( I_j \) the set of all integers \( n = n(j) \) with the property (5.2):

\[
I_j = \left\{ n \in \mathbb{Z}_+ \mid |nh_n-1| < \frac{1}{2p} \right\}
\]

when \( j \geq N_1 \) and \( n = 1, 2, \ldots, k \). Consequently,

\[
\frac{\|B_{3u}-B_{u}\|_{\mathfrak{P}}}{2^p} = \sum_{n=1}^{\infty} |n^{-p}(f(nc_n)-f(nc_{n,1}))|^{\mathfrak{P}}
\]

\[
= \sum_{n=1}^{\infty} |n^{-p}(f(nc_n)-f(nc_{n,1}))|^{\mathfrak{P}} - \sum_{n=1}^{\infty} |n^{-p}(f(nc_n)-f(nc_{n,1}))|^{\mathfrak{P}} \to 0
\]

\[
< k \frac{e^{\mathfrak{P}}}{2^p} + \frac{e^{\mathfrak{P}}}{2} = e^{\mathfrak{P}}
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whenever \( j \geq N_1 \). This shows that \( B_{3u} \xrightarrow{\mathfrak{P}} B_{u} \) in \( \mathfrak{P} \) whenever \( u \xrightarrow{\mathfrak{P}} u \) in \( \mathfrak{P} \). Thus, \( B_{3} \) is completely continuous.

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\]

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\[
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\]

(see the definition of \( f \)). For each \( j \in \mathbb{Z}_+ \), we denote by \( I_j \) the set of all integers \( n = n(j) \) with the property (5.2):

\[
I_j = \left\{ n \in \mathbb{Z}_+ \mid |nh_n-1| < \frac{1}{2p} \right\}
\]
Note that $I_j$ may be different for different $j$. Evidently,

$$
\|B_k n\|_{p'}^p = \sum_{n \in I_j} 2|f(nh_k)|^p n^{-p} = \sum_{n \in I_j} 2|f(nh_k)|^p n^{-p} \leq \sum_{n \in I_j} 2n^{-p}.
$$

Equation (5.2) implies the estimate

$$
\left(\frac{2p-1}{2p}\right)^p \sum_{n \in I_j} n^{-p} < \sum_{n \in I_j} |h_n|^p \leq \sum_{n \in I_j} |h_n|^p = \|h\|_{p'}^p.
$$

Accordingly,

$$
0 < a^{1/p} \leq \|h\|_{p'}^{-1} \|B_k n\|_{p'}
\leq \|h\|_{p'}^{-1} \|B_k n\|_{p'} \left(\sum_{n \in I_j} n^{-p}\right)^{1/p'}
\leq \|h\|_{p'}^{-1} \|B_k n\|_{p'} \left(\sum_{n \in I_j} n^{-p}\right)^{1/p'}
\leq \|h\|_{p'}^{-1} \left(\frac{2p-1}{2p}\right)^{-p/p'} \|h\|_{p'}^p
= 2^{1/p'} \left(\frac{2p-1}{2p}\right)^{-p/p'} \|h\|_{p'} ^{p-2} J_{m-1} 0
$$

which is a contradiction. Therefore the claim is true.

**Step 5.** We prove the claims concerning $\Gamma$, and show that zero is a nonisolated critical point.

Note that

$$
(A')^{-1} v = (n^{p-2} d_v)_{n=1}^\infty \quad \text{for} \quad v = (d_v)_{n=1}^\infty.
$$

The mapping $(A')^{-1}$ is unbounded, but this causes no problems if we choose $\Gamma = 0$ because then the mappings $\Gamma$ and $T = (A' + 0)^{-1} = 0$ are compact and $T$ has no eigenvalues in the interval $[1, \infty]$. As $F_0$ contains only zero, $\Pi$ is zero everywhere and $\Pi(A' + \Gamma)^{-1}$ is continuous. Moreover,

$$
\langle A' u, v \rangle = \sum_{n=1}^\infty c_n^2 n^{2-p} > 0
$$

when $\nu \neq 0$, that is, $A' + \Gamma = A'$ is strictly monotone. So all the assumptions of Theorem 4.3.1 except “zero does not belong to the weak closure of $\sigma_d$” hold. Now, zero is a nonisolated critical point of $A$. To see this, we consider the sequence $(v_k)_{k=1}^\infty$ where

$$
v_k = \left(\frac{1}{n} \delta_{d_k}\right)_{n=1}^\infty, \quad \delta_{d_k} = \begin{cases} 0, & \text{when } n \neq k \\ 1, & \text{when } n = k. \end{cases}
$$

Clearly, $v_k \in \ell^p$ for every $k \in \mathbb{Z}_+$ and $v_k \to 0$ as $k \to \infty$. Since

$$
A v_k = \sum_{n=1}^\infty \left(n^{1-\rho} \delta_{d_k} n^{-1} + n^{1-\rho} \delta_{d_k} - 2 f(\delta_{d_k}) n^{1-\rho}\right) = 0,
$$

Note that $I_j$ may be different for different $j$. Evidently,

$$
\|B_k n\|_{p'}^p = \sum_{n \in I_j} 2|f(nh_k)|^p n^{-p} = \sum_{n \in I_j} 2|f(nh_k)|^p n^{-p} \leq \sum_{n \in I_j} 2n^{-p}.
$$

Equation (5.2) implies the estimate

$$
\left(\frac{2p-1}{2p}\right)^p \sum_{n \in I_j} n^{-p} < \sum_{n \in I_j} |h_n|^p \leq \sum_{n \in I_j} |h_n|^p = \|h\|_{p'}^p.
$$

Accordingly,

$$
0 < a^{1/p} \leq \|h\|_{p'}^{-1} \|B_k n\|_{p'}
\leq \|h\|_{p'}^{-1} \|B_k n\|_{p'} \left(\sum_{n \in I_j} n^{-p}\right)^{1/p'}
\leq \|h\|_{p'}^{-1} \|B_k n\|_{p'} \left(\sum_{n \in I_j} n^{-p}\right)^{1/p'}
\leq \|h\|_{p'}^{-1} \left(\frac{2p-1}{2p}\right)^{-p/p'} \|h\|_{p'}^p
= 2^{1/p'} \left(\frac{2p-1}{2p}\right)^{-p/p'} \|h\|_{p'} ^{p-2} J_{m-1} 0
$$

which is a contradiction. Therefore the claim is true.

**Step 5.** We prove the claims concerning $\Gamma$, and show that zero is a nonisolated critical point.

Note that

$$
(A')^{-1} v = (n^{p-2} d_v)_{n=1}^\infty \quad \text{for} \quad v = (d_v)_{n=1}^\infty.
$$

The mapping $(A')^{-1}$ is unbounded, but this causes no problems if we choose $\Gamma = 0$ because then the mappings $\Gamma$ and $T = (A' + 0)^{-1} = 0$ are compact and $T$ has no eigenvalues in the interval $[1, \infty]$. As $F_0$ contains only zero, $\Pi$ is zero everywhere and $\Pi(A' + \Gamma)^{-1}$ is continuous. Moreover,

$$
\langle A' u, v \rangle = \sum_{n=1}^\infty c_n^2 n^{2-p} > 0
$$

when $\nu \neq 0$, that is, $A' + \Gamma = A'$ is strictly monotone. So all the assumptions of Theorem 4.3.1 except “zero does not belong to the weak closure of $\sigma_d$” hold. Now, zero is a nonisolated critical point of $A$. To see this, we consider the sequence $(v_k)_{k=1}^\infty$ where

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v_k = \left(\frac{1}{n} \delta_{d_k}\right)_{n=1}^\infty, \quad \delta_{d_k} = \begin{cases} 0, & \text{when } n \neq k \\ 1, & \text{when } n = k. \end{cases}
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$$
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$$

Note that $I_j$ may be different for different $j$. Evidently,

$$
\|B_k n\|_{p'}^p = \sum_{n \in I_j} 2|f(nh_k)|^p n^{-p} = \sum_{n \in I_j} 2|f(nh_k)|^p n^{-p} \leq \sum_{n \in I_j} 2n^{-p}.
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Equation (5.2) implies the estimate

$$
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$$

Accordingly,

$$
0 < a^{1/p} \leq \|h\|_{p'}^{-1} \|B_k n\|_{p'}
\leq \|h\|_{p'}^{-1} \|B_k n\|_{p'} \left(\sum_{n \in I_j} n^{-p}\right)^{1/p'}
\leq \|h\|_{p'}^{-1} \|B_k n\|_{p'} \left(\sum_{n \in I_j} n^{-p}\right)^{1/p'}
\leq \|h\|_{p'}^{-1} \left(\frac{2p-1}{2p}\right)^{-p/p'} \|h\|_{p'}^p
= 2^{1/p'} \left(\frac{2p-1}{2p}\right)^{-p/p'} \|h\|_{p'} ^{p-2} J_{m-1} 0
$$

which is a contradiction. Therefore the claim is true.
v_k is a critical point of A. Thus, \( w_k = v_k / \|v_k\| = (\delta_{nk})_{n=1}^{\infty} \in \sigma_e \). The claim \( 0 \in \text{w-cl}(\sigma_e) \) follows if we show that \( w_k \to 0 \). Since \( (w_k)_{k=1}^{\infty} \) is bounded and

\[
\langle e_k, w_j \rangle \to 0 \quad \text{for every} \quad e_k = (\delta_{nk})_{n=1}^{\infty}
\]

the assertion \( w_k \to 0 \) is certainly true. This completes the final example.
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