JOSEPHSON TRANSISTORS INTERACTING
WITH DISSIPATIVE ENVIRONMENT

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Abstract

The quantum-mechanical effects typical for single atoms or molecules can be reproduced in micrometer-scale electric devices. In these systems the essential component is a small Josephson junction (JJ) consisting of two superconductors separated by a thin insulator. The quantum phenomena can be controlled in real time by external signals and have a great potential for novel applications. However, their fragility on uncontrolled disturbance caused by typical nearby environments is a drawback for quantum information science, but a virtue for detector technology.

Motivated by this we have theoretically studied transistor kind of devices based on single-charge tunneling through small JJs. A common factor of the research is the analysis of the interplay between the coherent Cooper-pair (charge carriers in the superconducting state) tunneling and incoherent environmental processes. In the first work we calculate the current due to incoherent Cooper-pair tunneling through a voltage-biased small JJ in series with large JJs and compare the results with recent experiments. We are able to reproduce the main experimental features and interpret these as traces of energy levels and energy bands of the mesoscopic device. In the second work we analyze a similar circuit (asymmetric single-Cooper-pair transistor) but under the assumption that the Cooper-pair tunneling is mainly coherent. This predicts new resonant transport voltages in the circuit due to higher-order processes. However, no clear traces of most of them are seen in the experiments, and similar discrepancy is present also in the case of the symmetric circuit. We continue to study this problem by modeling the interplay between the coherent and incoherent processes more accurately using a density-matrix approach. By this we are able to demonstrate that in typical conditions most of these resonances are indeed washed out by strong decoherence caused by the environment. We also analyze the contribution of three typical weakly interacting dissipative environments: electromagnetic environment, spurious charge fluctuators in the nearby insulating materials, and quasiparticles. In the last work we model the dynamics of a current-biased JJ perturbed by a smaller JJ using a similar density-matrix approach. We demonstrate that the small JJ can be used also as a detector of the energy-band dynamics in a current biased JJ. The method is also used for modeling the charge transport in the Bloch-oscillating transistor.

Key words: Mesoscopic physics, Josephson junction, Cooper-pair tunneling, Decoherence
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Oulu, April 2009  Juha Leppäkangas
LIST OF ORIGINAL PAPERS

The present thesis contains an introductory part and supplements to the following papers which will be referred in the text by their Roman numbers.


The author has had a central role in all the articles presented in the thesis. He has developed the theory and computational tools around the phenomena, done all the simulations, and written the initial drafts of the articles.
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Chapter 1

Introduction

An interesting new field has emerged in the condensed matter physics within the last three decades. With the aid of advanced experimental techniques the quantum-mechanical effects typical for single atoms or molecules are reproduced in systems consisting of microfabricated electric-circuit elements. A central element in these devices is a Josephson junction (JJ) consisting of two superconductors separated by a thin insulator. These artificial entities can be controlled in a real-time fashion by electrical signals. The response of the device demonstrates quantum mechanics at appropriate conditions: sub-Kelvin temperatures and for micrometer-scale elements. This new field has opened up a unique possibility for exploring, testing and implementing quantum effects in microscale systems. Because the devices are easily integrable to other electrical entities, the novel applications proposed have a great potential in replacing the traditional computer and detector technology \[1\].

The triumph of science in the last century was the revealing of the physics that controls the dynamics of single atoms, namely the quantum mechanics. In short terms, it describes using a rigorous mathematical language that the matter interacts as it would consist of particles and travels as it would consist of waves. The theory seemingly contradicts with our observations in the everyday life, although it successfully explains the basic processes behind it. The true quantum mechanical properties, e.g., the wave nature of particles, are deep down in the condensed systems, but are directly detected only as small-scale single-particle effects in laboratories. The quantum mechanics in most cases reduces to the classical mechanics at the scale of the everyday life. But there are no fundamental obstacles for realising the quantum world at a much larger scale. Indeed, also systems with nonclassical behaviour at the macroscopic scale exists. An example of such a system is a superconductor, whose ultimate conductivity is explained by a collective quantum-mechanical state, becoming possible at very low temperatures. This thesis considers research done on microscale Josephson devices in which also other type of quantum effects can be brought a step closer to the scale of the everyday life.
The Josephson junction consists of two superconductors separated by a thin insulator. If the insulating barrier between the superconductors is of proper thickness, the collective quantum-mechanical state across the barrier results in special current-voltage characteristics. The effect is very sensitive to external influences, and its discovery about 50 years ago [2, 3] was followed by important applications in, for example, detection of weak magnetic fields [4]. The JJ can be described, in the simplest form, as a fictitious classical particle in a certain potential field. In the 1980s it was experimentally verified that the JJ has also a wave nature: in proper conditions the fictitious particle is able to quantum-mechanically tunnel across its potential field, suddenly switching the JJ to a different state [5, 6]. It was also shown that the energy of this particle is quantized [7]. Further, in the 1990s, the real-time controllability of the populations of these quantum states was experimentally demonstrated [8, 9]. Since then coupling to various other quantum systems, such as, nanoscale mechanical oscillators [10], photon cavities [11] and electromagnetic oscillators [12] have been actively investigated largely due to their great potential for nanotechnological applications. The most ambitious task is probably the building of a solid-state quantum-computer using the JJs as the elementary components of its quantum bits [13, 14]. Such a device would start a new age in the computer technology, but is not a subject of the nearby future. Before that the knowledge of the relevant physics needs to be greatly improved by the basic research in the field.

These “secondary” macroscopic quantum effects are characteristic for microscale JJs at low temperatures. The billions of particles involved suggests that the devices should be regarded macroscopic, but the length scale falls into the intermediate region between the macro and atomic worlds, namely to the mesoscopic region. Still, the fact that a mesoscopic device, enormous in the atomic scale, can behave collectively as a controllable single atom or molecule, is outstanding. Also, the nearby environment of such a device is large but behaves rather undeterministically causing a lot of disturbance, called noise. The noise can originate in, e.g., the electric circuitry connected to the system or in the nearby materials of the device. All the noise mechanisms in these devices are not clear yet. The quantum effects are very fragile to this kind of disturbance as they lead to loss of their “wave” coherence, to decoherence. Due to this the field aiming for processing information in quantum-mechanical states of the system has slightly lost ground, and is now fighting against this problem [15]. Another subfield has seen this property as a possibility for probing the mesoscopic physics, the small JJs can be made to work as excellent detectors of their environment [16, 17, 18, 19]. The work presented in this thesis can be seen to fall in both of these subfields. We study the effect of decoherence on the quantum-mechanical evolution of the systems that can be utilized whether in the quantum-information science or in the study of the mesoscopic physics.
The experimental study of macroscopic quantum systems in the last three decades has also revived the theoretical field of quantum decoherence [20], already discussed in the early days of quantum mechanics. It considers a (quantum) system interacting with a large environment, the very problem of the applications. In this approach the average properties of the system seen by the environment are calculated by finding an approximative solution of the Schrödinger equation written for the whole system. The environment causes decoherence when taking information out of the systems internal quantum evolution. Such a treatment is closely related to the fundamental idea of quantum measurement, that measuring a state collapses the original state. The conscious experimentalist is here replaced by environment, still gathering information out of the device. Indeed, the assumption that every interaction can be described by a Schrödinger equation actually leads to the probabilistic outcome of the process, as postulated in the usual interpretation of quantum mechanics. So is there a difference between ordinary many particle process and the thing called quantum measurement? An irritating difference is that in the former all the outcomes of the experiment occur, but do not interfere. In this picture the measurement causes only an appearance of the collapse: it does not occur but appears if looked in one of the possible lifelines. The problems of measurement, observation and sudden collapse still puzzle physicists but are now brought into new light as macroscopic quantum physics is a part of present-day experimental physics.

In this thesis we report theoretical studies done on transistor kind of devices build of small JJs. The operation of these Josephson transistors is based on controlling single-Cooper-pair tunneling through the JJs. We have calculated the current-voltage characteristics of the devices by simulating the dissipative quantum mechanics of the systems. The systems interact with their nearby environment, mainly consisting of an electromagnetic environment formed my the leads attached to the device, of spurious charge fluctuators in the nearby insulating materials, and of quasiparticles (unpaired electrons). They all can have drastic effect on the characteristics of the device, or the device can be based on their contribution. We study the interplay between coherent Cooper-pair tunneling and incoherent environmental processes using theoretical methods typical for problems dealing with quantum decoherence. We compare the results with recent experiments and try to improve the knowledge of the behaviour of such systems. We start in Chapter 2 by discussing the basic physics of Josephson junctions. In Chapter 3 we introduce models for dissipative quantum mechanics and discuss the properties of the relevant environments causing the dissipation. These chapters are meant to be introductions to the subjects and do not contain any new results. In Chapter 4 we introduce the studied devices, present the main results of Papers and extend some of the results. A summary is given in Chapter 5.
Chapter 2
Basics of Josephson junctions

The studies presented in this thesis consider characteristics of devices that are based on the Josephson effect. The effect occurs between two superconductors separated by a thin insulator. The superconducting state is that what makes the effect anomalous, as the collective quantum-mechanical ground state extends across the barrier leading to special current-voltage \((I - V)\) characteristics. The Josephson effect can be traced back to simple Josephson relations, but describing the behaviour of a more complicated many-body system. At low temperatures and for small JJs the effect acquires a new character as also “secondary” macroscopic quantum effects start to play an important role.

2.1 Josephson effect

Superconductivity can be understood as a quantum-mechanical state that extends to the macroscopic scale [21]. Despite their repulsive electric force, the conduction electrons in metal can have a small attractive net force due to higher-order interaction with phonons, i.e., lattice vibrations. If such an interaction exists, and the metal is cooled down so that the thermal energy \(k_B T\) is smaller than the pair binding energy \(\Delta\), the electrons form bound pairs, called the Cooper pairs [22, 4]. The pairs have no spin and behave as bosons. The electrons are not tightly bound, as the size of the pair is much larger than the typical distance between two nearby electrons. However, the pairs prefer the same quantum-mechanical state resulting in the ultimate conductivity of superconductors. It follows that the state of such a condensate can be described by the macroscopic wave function (of Cooper pairs)

\[
\Psi(t, \mathbf{r}) = \sqrt{\rho(t, \mathbf{r})} \exp[i\theta(t, \mathbf{r})],
\]

(2.1)

where \(\rho(t, \mathbf{r})\) is proportional to the Cooper-pair density and the changes of the phase \(\theta(t, \mathbf{r})\) can be related to the supercurrent.
CHAPTER 2. BASICS OF JOSEPHSON JUNCTIONS

Figure 2.1: Left: The Josephson junction consists of two superconductors separated by a thin insulator, across which Cooper pairs can quantum-mechanically tunnel. Right: The potential energy seen by the Cooper pairs.

When two superconductors are brought close together and separated just by a small layer of insulating material, the Cooper pairs have a finite amplitude of probability to tunnel across the effective potential barrier between the superconductors, as illustrated in Fig. 2.1. If the coupling is not too large, the Cooper-pair current across the barrier is simply

\[ I_J = I_C \sin \varphi, \quad (2.2) \]

where \( I_C \) is the critical current and \( \varphi = \theta_1 - \theta_2 \) the phase difference (of the macroscopic wave function) across the insulator. The critical current depends on the junction parameters and is usually between a microampere and a milliampere. In the semiclassical model the phase difference is treated as a classical variable satisfying

\[ \dot{\varphi} = \frac{2eV}{\hbar}, \quad (2.3) \]

where \( V \) is the voltage across the JJ. The time dependence reflects the quantum mechanical oscillation frequency of a pair hopping between two potentials with the energy difference \( 2eV \), see Fig. 2.1. The Josephson effect was predicted by Brian David Josephson in 1962 \[2\] and in 1973 he got the Nobel prize in physics for this discovery. The simple result (2.2) is obtained by a second-order perturbation theory \[23,24\] but holds well for low transparency junctions as it brilliantly describes numerous experiments in the field.

According to Eqs. (2.2)-(2.3) the constant voltage-biasing of the JJ, for which \( V \neq 0 \), results in oscillating supercurrent with the frequency \( 2eV/h \) and no net current occurs. To detect such oscillations as a net current one needs to bias the system also with an ac component \[4\]. If \( V = 0 \), a constant current flows having a value in between \([-I_C, I_C]\), depending on the constant value of \( \varphi \). The phenomena are called the Josephson ac and dc effects, respectively.
2.2. *Junction Dynamics*

2.1.1 Ambegaokar-Baratoff relation

When the conductors are in the normal (not superconducting) state, the tunnel junction shows a resistive behaviour with an ohmic law \( I \approx V/R_T \) due to electron tunneling through the barrier. As both the normal and Josephson current describe similar physics, they are not independent quantities. For zero temperature \( (T = 0) \), which is a good approximation in the studies considered, and general types of superconductors at different sides of the JJ, the so-called Ambegaokar-Baratoff relation \[26\] binds the critical current, the normal state resistance \( R_T \) and the superconducting energy gaps \( \Delta_i \) as

\[
I_C = \frac{\Delta_1}{eR_T} K \left[ \sqrt{1 - \left( \frac{\Delta_1}{\Delta_2} \right)^2} \right],
\]

(2.4)

where \( \Delta_1 \) is the smaller of the energy gaps and \( K(x) \) the complete elliptic integral of the first kind. The elliptic integral satisfies \( K(0) = \pi/2 \) and is an increasing function in the range considered. The nearby environment can also slightly contribute to this value \[27\]. Such corrections usually renormalize \( I_C \) but leave the functional dependence (2.2) unchanged.

2.2 Junction dynamics

The Josephson current is a ground state property and is dissipationless. The JJ stores energy which can be calculated to be

\[
\tilde{E}_J = \int dt I_J V = -E_J \cos(\varphi),
\]

(2.5)

where \( E_J = \hbar I_C/2e \) is the Josephson coupling energy. The junction region behaves also capacitively as charge gathers on the edges of the superconductors in the case of finite potential difference. The Coulomb energy of the JJ with capacitance \( C \) can be expressed as

\[
\tilde{E}_C = \frac{Q^2}{2C} = \frac{(CV)^2}{2C} = \frac{C}{2} \left( \frac{\hbar}{2e} \right)^2 \dot{\varphi}^2.
\]

(2.6)

The energies (2.5) and (2.6) can be interpreted as the potential and the kinetic energy, respectively, of a fictitious particle with mass \( C(h/2e)^2 \). The Lagrangian treatment with such an identification leads to the Hamiltonian function

\[
H = \frac{1}{2C} \left( \frac{2e}{\hbar} \right)^2 p^2 - E_J \cos(\varphi),
\]

(2.7)
Figure 2.2: Left: The dynamical model for a JJ consists of parallel capacitor $C$, resistor $R(V)$ and supercurrent $I_J$. Center: The supercurrent can also be seen as a nonlinear inductor. Right: The equivalent schematic drawing used in the thesis.

where $p = (\hbar/2e)Q$ is the conjugated variable of $\varphi$. The related Hamiltonian equations of motion describe the dissipationless dynamics of the JJ, i.e., charging of the capacitor due to the Josephson current and the evolution of the phase difference due to charge in the capacitor.

The Josephson junction can also be treated as an inductance

$$L = \frac{V}{I_J} = \left(\frac{\hbar}{2e}\right)^2 \frac{1}{E_J \cos(\varphi)},$$

in parallel with the capacitor $C$. Since the inductance is not a constant, its response is nonlinear. This is an important property since the nonlinearity leads to nonconstant energy-level splitting, enabling the reduction of the system to two levels [24].

Also dissipative current can flow through the JJ via formation of quasiparticles, i.e., by breaking the Cooper pairs when $eV > 2\Delta$ [3]. This can be modeled as a parallel resistor $R(V)$. In the language of the fictitious particle this means that there is a friction force entering the system after the threshold velocity $4\Delta/\hbar$. The dynamical model of a Josephson junction is visualized in Fig. 2.2.

### 2.3 Interference and tunability

In the presence of a magnetic field the Josephson effect acquires interesting properties. The theory is as before, but the phase difference is replaced by its gauge invariant form

$$\gamma = \varphi - \frac{2\pi}{\Phi_0} \int A \cdot ds,$$

(2.9)
2.3. INTERFERENCE AND TUNABILITY

Figure 2.3: Left: The SQUID consists of two parallel JJs. The presence of a magnetic flux can lead to a demand of circulating supercurrent that totally inhibits the current across the device. Right: A long Josephson junction penetrated by a magnetic flux. The current across different parts of the JJ can cancel each other leading to a smaller effective critical current.

where $\Phi_0 = \hbar/2e$ is the flux quantum, $\mathbf{A}$ the vector potential and the integration path is across the insulator. For two parallel JJs the change in the phase $\varphi$ around the superconducting loop must be $2n\pi$, where $n$ is an arbitrary integer. Using this one obtains a relation between the phase differences $\gamma_i$, Fig 2.3, and the magnetic flux $\Phi$ through the loop [4],

$$\gamma_1 - \gamma_2 = \frac{2\pi \Phi}{\Phi_0} + n2\pi. \quad (2.10)$$

The total current across this superconducting quantum interference device (SQUID) can then be calculated to be

$$I_{\text{SQUID}} = I_{C1} \sin(\gamma_1) + I_{C2} \sin(\gamma_2) = (I_{C1} + I_{C2}) \sin(\varphi) \cos \left( \frac{\Phi}{\Phi_0} + n\pi \right)$$

$$+ |I_{C1} - I_{C2}| \cos(\varphi) \sin \left( \frac{\Phi}{\Phi_0} + n\pi \right), \quad (2.11)$$

where $\varphi = (\gamma_1 + \gamma_2)/2$ is the effective phase difference of the SQUID. The currents across the individual JJs can cancel each other leading to that the SQUID works effectively as a tunable single JJ. Especially if $I_{C1} = I_{C2}$ the currents (and the Josephson coupling energies) can cancel each other completely. The essential feature is the periodicity of the Josephson current as a function of the phase difference. Very sensitive magnetic field detectors are based on this effect [4]. The tunability of SQUIDs plays a central role also in the studies presented in this thesis.

Similarly, in a very long JJ whose electrodes are aligned with the applied magnetic field, Fig. 2.3, the currents across different positions can cancel...
each other. In the case of rectangular junctions the effective critical current can be put to the form

\[ I_C(\Phi) = I_C(0) \left| \frac{\sin(\pi \Phi/\Phi_0)}{\pi \Phi/\Phi_0} \right|, \quad (2.12) \]

which is called the Fraunhofer diffraction pattern in analogy to similar light diffraction across a rectangular slit.

### 2.4 Secondary macroscopic quantum effects

In the preceding treatment we assumed that \( \varphi \) is a classical variable. The phase difference can also have quantum fluctuations in proper conditions. As in all systems, this occurs if environmental noise can be made small when compared to the energy scale of the quantum effects. The *thermal noise* is lowered by going to very low temperatures and the *damping* is lowered by isolating the Josephson junction from its nearby environment. The energy scale of the quantum effects can be increased, for example, by lowering the capacitance \( C \), i.e., going to smaller JJs. If the Josephson coupling energy is kept large, the values \( C \sim 1 \text{ pF} \) are enough. If \( E_J \) is wanted to be smaller than the (single) charging energy \( E_C = e^2/2C \), capacitances in the fF range need to be obtained. The conditions are met for mesoscopic JJs that are cooled approximately to 0.1 K.

A straightforward way to find the Hamiltonian operator and the eigenstates of a JJ that is not connected to its environment (isolated JJ) would be the standard quantization of the Hamiltonian function (2.7). This leads to the commutation relation

\[ [\varphi, Q] = 2ei, \quad (2.13) \]

where \( \varphi \) and \( Q \) are now operators. However, the phase difference is a periodic variable and handling it requires more caution \[28\]. For an isolated JJ only \( 2\pi \)-periodic wave functions \( \psi(\varphi) \) are physical, corresponding to \( 2e \) quantization of the charge \[27\]. The problem can be formulated in this space by writing the Josephson coupling energy as

\[ \tilde{E}_J = -\frac{E_J}{2} (e^{i\varphi} + e^{-i\varphi}), \quad (2.14) \]

which is quantized in the charge basis \( |Q\rangle \) as

\[ \tilde{E}_J = -\frac{E_J}{2} \sum_Q (|Q + 2e\rangle\langle Q| + |Q - 2e\rangle\langle Q|). \quad (2.15) \]

We obtain that the quantized Josephson effect corresponds to coherent single-Cooper-pair tunneling of across the insulator. This operator form is valid as
2.4. SECONDARY MACROSCOPIC QUANTUM EFFECTS

Figure 2.4: The four lowest energy-levels of an isolated JJ as a function of the quasicharge $q$. The eigenstates located below the potential energy maximum $E_J$ have a weak dependence on $q$. This can be seen as a consequence of that the JJ has a weak tunneling amplitude across its potential energy $-E_J \cos(\varphi)$. For $E_J/E_C \gg 1$ the eigenstates are close to the harmonic ones (constant energy-level splitting), whereas for $E_J/E_C < 1$ they are close to the pure charge states with eigenenergies of the form $(q - n)^2/2C$, where $n$ is an integer.

long as there is always “enough” charge in the leads to be transmitted to the other side. This demand is automatically satisfied in the case of superconductors \cite{29}. More detailed considerations for the quantization of the Josephson effect can be made via the system’s many-particle wave function \cite{25}.

Together with the capacitive energy $Q^2/2C$ the total Hamiltonian operator can be represented in the charge basis as

$$H = \sum_Q \left[ \frac{(Q - q)^2}{2C} |Q\rangle \langle Q| - \frac{E_J}{2} (|Q + 2e\rangle \langle Q| + |Q - 2e\rangle \langle Q|) \right], \quad (2.16)$$

where for generality we have included a quasicharge $q$, which is an offset charge due to external doping or other environmental effects. This quantity can have continuous values as it describes an average behaviour of the charge fluid. The eigenstates and eigenenergies (energy bands) are $2e$-periodic in $q$ and determined by the ratio $E_J/E_C$, see Fig. 2.4. When the JJ is not isolated, the charge in the capacitor plates does not have to be quantized, as the charge can flow to the other parts of the circuit. Then also the wave functions do not have to be $2\pi$ periodic.

2.4.1 Semiclassical limit

We can represent the “classical” states as superpositions of the charge states. If the charging energy of a large JJ can be neglected ($C \rightarrow \infty$), the resulting Hamiltonian is analogous to the tight-binding model of an electron in
a periodic ion-lattice. The nearby electron sites can be visualized as series potential minimums between series barriers. The sites correspond to different number of transmitted pairs. The eigenstates of an isolated JJ are then of the form

$$|\varphi\rangle = N \sum_Q |Q\rangle e^{i\varphi_Q/2e},$$  \hspace{1cm} (2.17)

where $N$ is a normalization factor. The related eigenenergies are

$$E(\varphi) = -E_J \cos(\varphi).$$  \hspace{1cm} (2.18)

The different values of $\varphi$ correspond to different velocities of the fictitious electron (wave packet) in the ion lattice, or equivalently different values for the current across the JJ. The classical current is therefore

$$I = \frac{2e}{\hbar} \frac{\partial E(\varphi)}{\partial \varphi} = I_C \sin \varphi = I_J.$$  \hspace{1cm} (2.19)

Using the analogy with electron in the ion lattice, it seems natural that large JJs with small charging energies are driven to these states when interacting with classical surroundings. For large $C$ (not infinitely large) the different sites have now different energies and the eigenstates are more Gaussian-type distributions (eigenstates of a harmonic oscillator) but with quite dense energy-level spacing, which still allows an easy construction of states with almost localized $\varphi$. The environment “senses” average values of the current across the JJ rather than small fluctuations of the local voltage $Q/C$. The classical analysis fails for small $C$ as quantum-mechanical effects due to notable energy-level splittings start to dominate.
Chapter 3

Dissipative quantum mechanics

The quantum mechanics and the classical mechanics with dissipation seem to contradict each other. As the expectation value of energy in an isolated quantum system is a constant, the classical undriven damped oscillator will lose its energy in dissipation. For example, there seems to be no Lagrangian or Hamiltonian description for a velocity dependent friction of a damped oscillator. However, all the systems must follow the same laws of physics, and the dissipation in a damped oscillator originates in the coupling of the oscillator’s degrees of freedom to microscopic modes in the surroundings [30]. The oscillator’s energy is transferred to these modes and in the case of a “large” environment it does not return. The effect can be imitated in the Hamiltonian description by introducing an environmental Hamiltonian $H_{env}$ and an interaction term $H_{int}$ between the environment and the system $H_0$ [31]. It is then required that in the classical limit the equations of motion (for the main degree of freedom) reduce to the classical ones with the same frictional terms. As we do not have the calculational power to describe the surrounding universe exactly, always something has to be approximated. Typical approximations are a thermal equilibrium of the whole system, or some part of it, linear coupling between the environment and the subsystem, quadratic environmental Hamiltonian and so on. With the help of simplifying relations the quantities related to the subsystem can be calculated via, e.g., series expansion of the systems time-evolution operator. Such a treatment is very useful in problems dealing with macroscopic quantum coherence.

3.1 Reduced equation of motion

If there would be no interaction between the system and the environment, the Hamiltonian of the total system would be $H_0 + H_{env}$. The total density matrix has the product form $\rho_{\text{total}} = \rho \otimes \rho_{\text{env}}$ at all times, where $\rho$ is the system’s density matrix, if this is true at the initial time. Turning the interaction part
CHAPTER 3. DISSIPATIVE QUANTUM MECHANICS

Figure 3.1: The dissipation originates in the coupling of the main degree of freedom to microscopic modes in the environment. The energy in the total system is conserved but exchanged between the main oscillator and the smaller ones. The quantum limit can be studied by constructing a Hamiltonian function that produces correctly the classical equations of motion. In the figure the friction force felt by the main harmonic oscillator (a mass connected by a spring to the wall) is modeled by smaller oscillators connected to the mass of the main oscillator.

\[ H_{\text{int}} \text{ on at time } t_0, \text{ we define the elements of the reduced density matrix as} \]

\[ \rho_{mn}(t) = \sum_E \langle E|\langle m|\rho_{\text{total}}(t)|n\rangle|E\rangle, \quad (3.1) \]

where the states \( |m\rangle \) are eigenstates of \( H_0 \) and \( |E\rangle \) eigenstates of \( H_{\text{env}} \). Operators in the interaction picture are defined as

\[ A^I(t) = e^{i(H_0 + H_{\text{env}})(t-t_0)/\hbar} A(t) e^{-i(H_0 + H_{\text{env}})(t-t_0)/\hbar}, \quad (3.2) \]

where \( A(t) \) is the operator in the Schrödinger picture. Applying this for \( \rho_{\text{total}}(t) \) one can write

\[ \rho_{mn}(t) = \sum_E \langle E|\langle m|e^{-i(H_0 + H_{\text{env}})(t-t_0)/\hbar}\rho_{\text{total}}^I(t)e^{i(H_0 + H_{\text{env}})(t-t_0)/\hbar}|n\rangle|E\rangle \]

\[ = e^{i(E_n - E_m)(t-t_0)/\hbar} \sum_E \langle E|\langle m|\rho_{\text{total}}^I(t)|n\rangle|E\rangle \]

\[ = e^{i(E_n - E_m)(t-t_0)/\hbar} \sum_E \langle E|\langle m|U(t,t_0)\rho(t_0) \otimes \rho_{\text{env}}(t_0) U^\dagger(t,t_0)|n\rangle|E\rangle, \quad (3.3) \]

where \( U(t,t_0) \) is the time evolution operator in the interaction picture. Assuming that the environment is in thermal equilibrium at the initial time, \( \rho_{\text{total}}(t_0) = \rho(t_0) \otimes e^{-\beta H_{\text{env}}}/Z \), one obtains

\[ \rho_{mn}(t) = e^{i(E_n - E_m)(t-t_0)/\hbar} \sum_{a,b} \rho_{ab}(t_0) \text{Tr}_E \left( \langle b|U^\dagger|n\rangle \langle m|U|a\rangle e^{-\beta H_{\text{env}}} \right)/Z. \quad (3.4) \]

Eq. (3.4) describes how the element \( |a\rangle\langle b| \) of the initial density matrix contributes to the element \( |m\rangle\langle n| \) at time \( t \). Models for open quantum-systems are often based on a similar equation \[31\].
3.1. REDUCED EQUATION OF MOTION

3.1.1 Expanding the time-evolution operator

We proceed from Eq. (3.4) by determining $U(t, t_0)$. The time evolution in the interaction picture follows the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle^I = H^I_{\text{int}}(t) |\psi(t)\rangle^I.$$  \hspace{1cm} (3.5)

If the operator $H^I_{\text{int}}(t)$ commutes with itself at different times, the formal solution of Eq. (3.5) is

$$|\psi(t)\rangle^I = U(t, t_0) |\psi(t_0)\rangle^I = e^{-i \int_{t_0}^{t} H^I_{\text{int}}(t') dt' / \hbar} |\psi(t_0)\rangle^I.$$  \hspace{1cm} (3.6)

However, this assumption cannot be made in situations considered and we must use the most general solution

$$|\psi(t)\rangle^I = U(t, t_0) |\psi(t_0)\rangle^I = T e^{-i \int_{t_0}^{t} H^I_{\text{int}}(t') dt' / \hbar} |\psi(t_0)\rangle^I = \left[ 1 + \sum_{n=1}^{\infty} \left( \frac{-i}{\hbar} \right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H^I_{\text{int}}(t_1) \cdots H^I_{\text{int}}(t_n) \right] |\psi(t_0)\rangle^I.$$  \hspace{1cm} (3.7)

The special order of integration (defining the time-ordering operator $T^-$) assures that $t > t_1 > t_2 \cdots > t_n > t_0$. The time ordering of $U(t, t_0)$ is reversed. Insertion of Eq. (3.7) into Eq. (3.4) results in infinite summation of various terms describing all the physics that can occur at the intermediate times. Such summations can usually be rewritten more efficiently as diagrams, each of them representing a contribution in the sum [32, 33].

3.1.2 Route to the diagrammatic formulation

To illustrate where the diagrammatic formulation originates, let us consider the case the interaction part is of the (bi)linear form $H_{\text{int}} = Qq$, where $Q$ is related to the system and $q$ to the environment. In expansion (3.7) the constant 1 evidently describes a process where no interaction with the environment (transitions) takes place. The first-order term $-i \int_{t_0}^{t} dt_1 Q(t_1)q(t_1)/\hbar$ corresponds to a process where a single transition occurs and a summation (integration) over all the possible transition times is performed. The second-order term $-\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 Q(t_1)q(t_1)Q(t_2)q(t_2)/\hbar^2$ is a process where two transitions take place, the first at time $t_2$ and the second at time $t_1$, and so on. The operator $U(t, t_0)^\dagger$ includes the same terms but in a time reversed fashion, as the order of the Hamiltonians $H^I_{\text{int}}(t_i)$ is reversed.

The expansion of the two time-evolution operators in Eq. (3.4) shows that the final element is a sum of all the possible transitions at the intermediate times with certain amplitudes. The summation represents the fact that the
quantum-mechanical evolution can be understood as a sum of all the possible “paths” \(^{32}\). However, in the studied case we do not consider the environmental states separately, since in Eq. (3.4) we do a trace over the environmental degrees of freedom. This reduces the environmental operators to correlation functions such that \(\langle\langle q(t)q(t')q(t'')\rangle\rangle\), which are usually known quantities as they can be calculated from the initial thermal distribution. This is a general result for the initial product-state density-matrix. If the environmental Hamiltonian is also quadratic, the Wick’s theorem \(^{38}\) assures that the trace can be represented using only pair correlations \(\alpha(t-t') = \langle\langle q(t)q(t')\rangle\rangle\) and that odd correlations vanish. It follows that Eq. (3.4) is a sum of all the possible transitions with arbitrary pair correlations, Fig. 3.2. Also cases where the Wick’s theorem cannot be applied, and odd moments do not vanish, have attracted more attention recently \(^{39}\).

### 3.1.3 Master equation

Usually one represents Eq. (3.4) as a master equation by differentiating it with respect to time. The differentiation, when focusing on the operator \(U(t)\), annihilates (the constant 1 and) the integration over the variable \(t_1\) and leads to the substitution \(t_1 = t\). It results that the processes always end to a transition at time \(t\). One obtains formally

\[
\dot{\rho}_{mn}(t) = i(E_n - E_m)\rho_{mn}(t)/\hbar + \sum_{a,b} \Gamma_{b\rightarrow n}^{a\rightarrow m} \rho_{ab}(t_0),
\]  

(3.8)
where the first term in the r.h.s. describes the quantum-mechanical evolution in the case $H_{\text{int}} = 0$. The interaction with the environment induces transitions described by the transition-rate tensor $\Gamma$ including all diagrams starting at time $t_0$ and ending to a transition at time $t$.

Except that every diagram ends to a transition at time $t$, consider now a diagram which is also *irreducible* between times $t' > t_0$ and $t$, meaning that any vertical line at time $\tilde{t}$, $t' < \tilde{t} < t$ cuts at least one environmental line. For example, the fourth-order diagram in Fig. 3.2 is irreducible between the times $\tau_2$ and $t_1$. It cannot be cut to half in this region. Because there is generally a transition at time $t$, every diagram starting from $t_0$ ends to an irreducible part between certain times $t''$ and $t$. On the other hand, a fixed irreducible diagram between the times $t'$ and $t$ can have an arbitrary “starting” part between the times $t_0$ and $t'$. The formal summation of all the possible diagrams from $t_0$ to $t'$ gives the element $\rho_{a'b'}(t')$. Summing up (integrating) all the possible times $t'$ and intermediate states one obtains the equivalent form of Eq. (3.8) (by marking $a' = a$ and $b' = b$)

$$\dot{\rho}_{mn}(t) = i(E_n - E_m)\rho_{mn}(t)/\hbar + \sum_{a,b} \int_{t_0}^{t} dt' \tilde{\Sigma}_{a \rightarrow m}^{b \rightarrow n}(t - t')\rho_{ab}(t'),$$  \hspace{1cm} (3.9)

where the function (self-energy matrix) $\tilde{\Sigma}_{a \rightarrow m}^{b \rightarrow n}(t - t')$ is a sum of all irreducible diagrams starting from $t'$ and ending to $t$. This is the master equation in its most general form.

**Born approximation**

Often Eq. (3.9) needs to be simplified for practical calculations. If the interaction with the environment is weak, one can cut the diagrammatic summation to include at most certain order diagrams in $H_{\text{int}}$. The lowest-order approximation is called the *Born approximation*, where only the second-order diagrams are considered. The naming “weak” can lead to misconception of minor effects, as the interaction can still drastically interfere the quantum coherence in the system. By cutting Eq. (3.9) to include only few diagrams, the assumption of the product-state density-matrix at the initial time $t_0$ (Section 3.1) is effectively made for all times. This is actually a good approximation in situations where the macroscopic environment is only slightly perturbed by the interaction with the system.

**Markov approximation**

The master equation (3.9) is an integro-differential equation as both the differentiation and the integration with respect to time exists simultaneously. The *Markov approximation* aids in solving the equation by dropping the history out of the equation. This can be done by assuming that $\rho(t')$ on the
r. h. s. of Eq. (3.9) can be substituted by the interaction-picture (backward) time-evolution of \( \rho(t) \)

\[
\rho(t') \approx \exp[iH_0(t-t')/\hbar]\rho(t)\exp[-iH_0(t-t')/\hbar].
\] (3.10)

The assumption is valid if environmental memory loss is fast when compared to the timescale of transitions. Together with the Born approximation and the limit \( t_0 \to -\infty \) the master equation gets the Born-Markov (BM) form

\[
\dot{\rho}_{mn}(t) = i(E_n - E_m)\rho_{mn}(t)/\hbar + \sum_{a,b} \Gamma_{a \rightarrow m}^{b \rightarrow n} \rho_{ab}(t),
\] (3.11)

where \( \Gamma \) is the generalized transition-rate tensor in the golden-rule approximation [31].

Lindblad approximation

The truncation of the diagrammatic summation, and perhaps dropping out the history, are great simplifications for the master equation and in most cases vital for practical calculations. But they can also lead to problems, as the truncated equation of motion is rather phenomenological and there is no guarantee that the solution describes any physical system anymore. The BM equation preserves the trace of the density matrix but can, in the worst case, lead to large negative values for (some of) the populations. Such effects are to be expected, for example, in the region where the Born and/or Markov approximations are starting to fail. The most safe way to do master-equation simulations is to resort to the general form of the master equation that preserves the positivity of the density matrix, i.e., to the Lindblad form [40]

\[
\dot{\rho} = -i[H_0, \rho]/\hbar + \sum_i (L_i \rho L_i^\dagger - \frac{1}{2} L_i^\dagger L_i \rho - \frac{1}{2} \rho L_i^\dagger L_i),
\] (3.12)

where the operators \( L_i \) describe transitions between the eigenstates

\[
L_i = \sum_{mn} \gamma_i^{m \rightarrow n} |m\rangle \langle n|,
\] (3.13)

where \( \gamma_i^{m \rightarrow n} \) are constant numbers. Generally, the master equation in the Born-Markov approximation is not of Lindblad type. To dress it into this form one can either neglect some of the nondiagonal contributions or redefine their generalized transition rates. The choice is not unanimous. This “safe mode” of the master equation guarantees positivity of the density matrix, but is one further approximation to the original equation. Therefore a careful analysis is recommended before “choosing” the Lindblad form of the master equation, as a wrong Lindblad-type can loose important effects. Some master equations cannot be recast to the Lindblad form without loosing the essential physics they describe. Therefore also investigations of other reliable master equations are actively done [41, 42].
3.2 Environments

We will now study the Hamiltonian formalisms of the dissipative environments considered in the thesis. The first environment is the electromagnetic environment (EE) which is basically formed by the leads and external circuitry attached to the JJ. The second environment consists of quasiparticle degrees of freedom, i.e., unpaired electrons in the superconductors. The third environment describes spurious charge fluctuators (CF) in the nearby materials of the device. The first environment is bosonic whereas the second consists of fermions. The exact nature of the charge fluctuators is not specified, as this environment is the least known.

3.2.1 Electromagnetic environment

The electromagnetic environment is a linear environment between the bias and the system and can be characterized by an impedance $Z(\omega)$. It can be modeled as a set of harmonic oscillators coupled bilinearly to the system [31]. One has generally two choices for the modeling: whether to couple the environment to a phase-like variable or to a charge like. The methods are equivalent. The aim is to derive the correct Langevin equation for the system.

Let us consider the case when the environment is coupled to the phase variable of a Josephson junction. We write the Hamiltonian of the total system as

$$H = \frac{Q^2}{2C} - E_J \cos(\varphi) + \sum_{\alpha=1}^{N} \left[ \frac{q_\alpha^2}{2C_\alpha} + \left( \frac{\hbar}{2e} \right)^2 \left( \varphi - \varphi_\alpha \right)^2 \right], \quad (3.14)$$

where $[\varphi, Q] = 2ei$, $[\varphi_\alpha, q_\alpha] = 2ei$ and the cross commutators vanish. We identify the interaction

$$H_{\text{int}} = - \left( \frac{\hbar}{2e} \right)^2 \varphi \left( \sum_{\alpha=1}^{N} \frac{1}{L_\alpha} \varphi_\alpha \right), \quad (3.15)$$

a term often referred as the “renormalization” of $H_0$ [30]

$$H_{\text{rn}} = \left( \frac{\hbar}{2e} \right)^2 \varphi^2 \left( \sum_{\alpha=1}^{N} \frac{1}{2L_\alpha} \right), \quad (3.16)$$

and the environmental Hamiltonian

$$H_{\text{env}} = \sum_{\alpha=1}^{N} \left[ \frac{q_\alpha^2}{2C_\alpha} + \left( \frac{\hbar}{2e} \right)^2 \varphi_\alpha^2 \right]. \quad (3.17)$$
Figure 3.3: The Josephson junction in parallel with an impedance $Z(\omega)$ (left) and its equivalent circuit (right) where the impedance is constructed from an infinite number of parallel series-$LC$-oscillators.

Hamiltonian (3.14) describes a dissipationless Josephson junction in parallel with series-$LC$-oscillators. The aim is to show that the oscillators produce any desired frequency dependent impedance $Z(\omega)$, see Fig. 3.3.

The Heisenberg equations of motion for the total system read

$$\ddot{\varphi} = \frac{1}{C} \frac{2e}{\hbar} I_C \sin(\varphi) - \frac{1}{C} \sum_{\alpha=1}^{N} \left( \frac{\varphi - \varphi_\alpha}{L_\alpha} \right)$$  \hspace{1cm} (3.18)

$$\dot{\varphi}_\alpha = -\omega_\alpha^2 (\varphi_\alpha - \varphi).$$  \hspace{1cm} (3.19)

Regarding $\varphi(t)$ as a given operator in time, the formal solution for $\varphi_\alpha(t)$ can be written as

$$\varphi_\alpha(t) = \varphi_\alpha(t_0) \cos(\omega_\alpha t) + \frac{2e}{\hbar} \frac{1}{\omega_\alpha C_\alpha} q_\alpha(t_0) \sin(\omega_\alpha t) +$$

$$\omega_\alpha \int_{t_0}^{t} dt' \sin[\omega_\alpha(t - t')] \varphi(t').$$  \hspace{1cm} (3.20)

To obtain an expression where the “velocity” is present, we integrate the last term by parts and obtain

$$\varphi_\alpha(t) = \varphi_\alpha(t_0) \cos(\omega_\alpha t) + \frac{2e}{\hbar} \frac{1}{\omega_\alpha C_\alpha} q_\alpha(t_0) \sin(\omega_\alpha t)$$

$$+ \varphi(t) - \varphi(t_0) \cos(\omega_\alpha t) - \int_{t_0}^{t} dt' \cos[\omega_\alpha(t - t')] \dot{\varphi}(t').$$  \hspace{1cm} (3.21)
Then substituting this into the equation of motion for $\varphi$ we obtain

$$C \ddot{\varphi}(t) = \frac{2e}{\hbar}I_C \sin[\varphi(t)] - \int_{t_0}^{t} dt' \sum_{\alpha=1}^{N} \cos[\omega_\alpha(t-t')] \dot{\varphi}(t')$$

$$- \sum_{\alpha=1}^{N} \frac{\cos(\omega_\alpha t)}{L_\alpha} \varphi(t_0)$$

$$+ \sum_{\alpha=1}^{N} \varphi_\alpha(t_0) \frac{\cos(\omega_\alpha t)}{L_\alpha} + \frac{2e}{\hbar} \sum_{\alpha=1}^{N} \omega_\alpha q_\alpha(t_0) \sin(\omega_\alpha t).$$  \hspace{1cm} (3.22)

The first term on the r.h.s. of Eq. (3.22) describes the Josephson current across the JJ whereas the second term is a memory-friction term, related to the Fourier-transformed admittance $Y(\omega)$ in this situation. The last line represents fluctuations, i.e., the random force generated by the environment. The values (or in quantum case the operators) $\varphi_\alpha(0)$ and $q_\alpha(0)$ are taken from a proper statistical ensemble. The third term is spurious and depends on the initial value of $\varphi$. It has been shown that it is an artefact of the initial decoupling of the environment and the subsystem. However, it can be included in the random force by a careful definition of the statistical ensemble (and therefore we neglect it) [31]. One obtains the quantum mechanical version of the current balance (Langevin) equation for the Josephson junction in parallel with the impedance $Z(\omega)$

$$\ddot{\varphi}(t) + \frac{1}{C} \int_{t_0}^{t} dt' \tilde{Y}(t-t') \dot{\varphi}(t') - \frac{1}{C} \frac{2e}{\hbar} I_C \sin(\varphi) = \frac{2e}{\hbar} \frac{1}{C} I_n$$  \hspace{1cm} (3.23)

$$\tilde{Y}(t) = \sum_{\alpha=1}^{N} \frac{\cos(\omega_\alpha t)}{L_\alpha} \Theta(t)$$  \hspace{1cm} (3.24)

$$I_n = \frac{\hbar}{2e} \sum_{\alpha=1}^{N} \varphi_\alpha(t_0) \frac{\cos(\omega_\alpha t)}{L_\alpha} + \sum_{\alpha=1}^{N} \omega_\alpha q_\alpha(t_0) \sin(\omega_\alpha t),$$  \hspace{1cm} (3.25)

where $\Theta(t)$ is the step function. Clearly any kind of admittance can be modeled by the proper choice of parameters $L_\alpha$. By directly calculating the Fourier transform ($\lim_{s \to 0} \int dt e^{i\omega t - s|t|}$) of $\tilde{Y}(t)$ one obtains

$$Z(-\omega)^{-1} = Y(-\omega) = \lim_{s \to 0} \sum_{\alpha=1}^{N} \frac{1}{2L_\alpha} \left[ \frac{i}{\omega + \omega_\alpha + is} + \frac{i}{\omega - \omega_\alpha + is} \right],$$  \hspace{1cm} (3.26)

which defines the variables $L_\alpha$ and $C_\alpha$.

### 3.2.2 Quasiparticles

Quasiparticles in the superconductor are electrons that are not bound to Cooper pairs. They behave similarly as electrons in the normal metal and
also feel the presence of the condensate. The quasiparticles in the lead \( r \) (quasiparticle reservoir) can be described by the Hamiltonian

\[
H_{qp}^r = \sum_{k,i} E_{kr} \gamma_{ki}^\dagger \gamma_{ki},
\]

(3.27)

where the operators \( \gamma_{ki}^\dagger \) are excitation annihilation (creation) operators with momentum \( k \) and spin degree of freedom \( i \). They satisfy the fermionic anticommutation relations. The eigenenergies of the excitations are

\[
E_{kr} = (\epsilon_k^2 + \Delta_r^2)^{1/2},
\]

(3.28)

where \( \epsilon_k \) is the energy of the corresponding normal metal quasiparticle state measured from the Fermi level. It is assumed to depend only on the length of \( k \). The minimum energy for creating an excitation is \( \Delta_r \). Because there is one to one correspondence with the normal metal states, the density of states in the superconducting case can be shown to be

\[
N_{sc}^r(E) = \frac{E_{kr}}{\sqrt{E_{kr}^2 - \Delta_r^2}} N_0^r,
\]

(3.29)

where \( N_0^r \) is the normal metal density of states at the Fermi surface. In thermal equilibrium the excitations are present according to the probability (Fermi) distribution

\[
f(E_{kr}) = \frac{1}{1 + e^{\beta E_{kr}}},
\]

(3.30)

Two nearby leads with a tunnel junction in between interact by exchanging electrons and by an electric field described earlier via capacitance \( C \). In the case of low transparency junctions the electron tunneling can be modeled using the tunneling Hamiltonian formalism \[23\]. It describes an electron propagation across the insulating barrier as the interaction

\[
H_T = \sum_{k,l,i} [T_{kl} a_{ki}^\dagger a_{li} + h.c.],
\]

(3.31)

where \( a_{ki}^\dagger \) is the electron annihilation (creation) operator of the state \( k \) with the spin degree of freedom \( i \) and \( T_{kl} \) is the tunneling amplitude. If the motion of the particles is ballistic, one obtains for the transmission across a rectangular barrier with height \( U \) and width \( L \) \[23\]

\[
|T_{kl}|^2 = \frac{1}{4\pi^2} \frac{\delta_{ki} \delta_{li}}{\rho^2} \exp \left( -\frac{2L}{\hbar} \sqrt{2mU - k_{||}} \right),
\]

(3.32)

where \( \rho \) is the one dimensional density of states in metals perpendicular to the barrier. The transverse momentum \( (k_{||}) \) is conserved. According to the BCS
the quasiparticle operators have the relation $a_{k\uparrow} = u_k^\ast \gamma_{k0} + v_k \gamma_{k1}^\dagger$ and $a_{-k\downarrow} = -v_k^\ast \gamma_{k0}^\dagger + u_k^\ast \gamma_{k1}$, where $u_k$ and $v_k$ are superconducting coherence factors satisfying

$$|u_k|^2 = \frac{1}{2} \left( 1 + \frac{\epsilon_k}{E_{kr}} \right)$$  \hspace{1cm} \text{(3.33)}$$

$$|v_k|^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_k}{E_{kr}} \right) = 1 - |u_k|^2. \hspace{1cm} \text{(3.34)}$$

The effect of $H_T$ can be analyzed, for example, by using a master-equation approach similar as in Section 3.1. In the lowest-order calculation one obtains terms of the form

$$a_{k\uparrow} a_{-k\downarrow} = |u_k|^2 \gamma_{k0} \gamma_{k0}^\dagger + |v_k|^2 \gamma_{k1} \gamma_{k1}^\dagger,$$  \hspace{1cm} \text{(3.35)}$$

describing quasiparticle tunneling across the Josephson junction. Also terms like

$$a_{k\uparrow} a_{k\downarrow} = -u_k^\ast v_k \gamma_{k0}^\dagger \gamma_{k0} + v_k u_k^\ast \gamma_{k1} \gamma_{k1}^\dagger,$$  \hspace{1cm} \text{(3.36)}$$

exist describing the Josephson effect. The terms (3.36) are already taken into account as the potential energy $-E_J \cos(\phi)$ and therefore their contribution is neglected. The expression (3.35) includes two terms with superconducting coherence factors $u_k$ and $v_k$. However, for every $k$ there is a momentum $k'$ for which $E_k = E_{k'}$ but $k \neq k'$ and $|u(k')|^2 = |v(k)|^2$. Since $|u(k)|^2 + |v(k)|^2 = 1$ the factors $u$ and $v$ vanish from the final expressions (in the lowest-order treatment). Further, the first term on the r.h.s. of Eq. (3.35) contributes if the state is occupied, i.e., by the factor $f(E_k)$ and the second one if the state is empty, i.e., by the factor $1 - f(E_k)$. Since $1 - f(E_k) = f(-E_k)$ the contribution of the second term can be included to the first one in the equivalent semiconductor picture \textsuperscript{[4]}, where the energies (3.28) of the states whose momentum lies below the Fermi surface are reversed. In such a model no coherence factors exists and by the special energy spectrum it is assured that the same elements are not summed twice.

Linking the factors $T_{k\downarrow}$ to the normal state behaviour \textsuperscript{[23]} one obtains

$$I_{ss}(V) = \frac{1}{eR_T} \int_{-\infty}^{\infty} \frac{|E|}{\sqrt{E^2 - \Delta_1^2}} \frac{|E + eV|}{\sqrt{(E + eV)^2 - \Delta_2^2}} [f(E) - f(E + eV)]dE,$$  \hspace{1cm} \text{(3.37)}$$

where the integration excludes the values such that $|E| < \Delta_1$ and $|E + eV| < \Delta_2$. Generally, for a given temperature this have to be calculated numerically, but behaves approximately as $I_{ss}(V) \sim \Theta(|eV| - \Delta_1 - \Delta_2)V/R_T$.  

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Figure 3.4: The Cooper pair breaks into two quasiparticles as a single electron tunnels across the Josephson junction. The threshold energy for such a process is $\Delta_1 + \Delta_2$ which in this case is obtained from the potential difference.

The current is approximately the normal state one after the energy $\Delta_1 + \Delta_2$ needed for creating excitations on both sides of the JJ is obtained from the electrostatic potential difference, Fig. 3.4. The rest of the energy goes to the kinetic degrees of freedom. The current (3.37) is a sum of two contributions corresponding to different tunneling directions. To separate the individual transition rates we write

$$
\Gamma_{ss}^+ = \frac{I_{ss}(V)}{e} \frac{1}{1 - e^{-\beta eV}}
$$

$$
\Gamma_{ss}^- = \frac{I_{ss}(V)}{e} \frac{1}{e^{\beta eV} - 1}.
$$

(3.38)

Because we are interested in the case $\Delta \gg k_B T$ we can assume that only one of the rates is nonvanishing at a time (the negative direction part in the case $-eV > \Delta_1 + \Delta_2$). For later use we define an effective damping function

$$
D_q^\pm(\omega) = \hbar^2 \frac{\Gamma_{ss}^\pm(\hbar\omega/e)}{2},
$$

(3.39)

where $\hbar\omega$ is the energy release in a process induced by quasiparticle tunneling.

3.2.3 Charge fluctuators

The insulating materials nearby the JJs can in an idealized situation be considered only as potential walls with no internal degrees of freedom. However, experimental studies [13, 14, 45, 46] have shown that typical environments are not frozen but include many degrees of freedom that couple to the state of the JJs. The physical origin of this noise is partly unclear and seems not to be dominated by ordinary fluctuating impurities in the vicinity of the system. If such an environment can couple to the degrees of freedom of the quantum device in an uncontrolled way, it evidently causes the system to decohere. At the moment the deep understanding of (and eventually the
3.2. ENVIRONMENTS

Figure 3.5: Charge fluctuators in the nearby materials of the JJs can couple to the device and cause it to decohere. Small islands (left) suffer from island charge fluctuations and large JJs (right) from fluctuations of the critical current.

shielding from) this noise is a necessary task before these systems can be used in the quantum-information processing.

We will discriminate two types of charge noise; whether it entangles to the charge of the JJ or to its critical current, Fig. 3.5. In experiments one often studies small islands rather than single JJs. For example, a system consisting two JJs in series works effectively as an isolated JJ (see Sections 4.1 and 4.2). In such a system nearby fluctuating charges will polarize the charge in the island, effectively leading to its fluctuations also. It seems that this noise does not originate in “classical” charge-fluctuations in the environment \[46\]. For example, coherent two-level systems \[47\] and Andreev fluctuators \[48\] have been considered as the origin. The noise is of \(1/f\)-type at low frequencies and has \(f\)-type continuation at higher frequencies \[44\], pointing to warm and frozen dynamics of the same states, correspondingly. Their effect can be analyzed by similar methods as considered earlier, i.e., by introducing an environmental Hamiltonian describing the fluctuators and an interaction term describing the coupling to the studied system. The interaction can be assumed to be of the form

\[
H_{\text{int}} = Qq \tag{3.40}
\]

where \(Q\) is the charge in the island between the JJs and \(q\) its polarization due to the nearby charges. If the environment can be assumed to be only weakly perturbed and independent on the state of the system, its effect reduces to correlation functions of the type \(\langle q(t)q(t') \rangle\) (see Section 3.1). However, the noise is not generally of Gaussian type and cannot be fully described by a set of harmonic oscillators.

A two-level fluctuator in the tunneling barrier rather affects to the Josephson coupling energy and causes it to fluctuate \[49\]. Such fluctuators seem to disturb large-area JJs (high capacitances) as the probability for their existence in fF-capacitance JJs is very small. The environment is of discrete nature as usually only few two-level states exists in a typical large JJ \[45\ 50\].
3.3 Fluctuations in a linear system

3.3.1 Fluctuation-dissipation theorem

A dissipative system in thermal equilibrium performs random fluctuations that are a counterpart of the damping. They can be viewed as random forces kicking the system out of its ground state into which the damping is trying to drive it. The system’s relaxation to the equilibrium after external forcing is described by the dynamical susceptibility which can be related to the equilibrium fluctuations via the famous fluctuation-dissipation theorem (FDT). The FDT has a central role in many studies that consider the effect of a noisy environment to the system under study.

Consider voltage fluctuations in a circuit that consists of a capacitor $C$ and an impedance $Z(\omega)$. From Section 3.1 we know that the Hamiltonian of the total system is quadratic and the equations of motion linear. For a linear system the FDT holds exactly and due to Ehrenfest theorem it is sufficient to consider only classical equations of motion for determining the dynamical susceptibility. Let us choose the arbitrary force $f(t)$ to be a delta function at an infinitesimal time after the initial time, before which the system is in the equilibrium $Q_e = 0$. Then for arbitrary times we have the voltage-balance equation

$$\frac{Q(t)}{C} + \int_{0}^{t} \tilde{Z}(t-t')I(t')dt' = Q_0\delta^+(t), \quad (3.41)$$

where the Fourier transform of the kernel $\tilde{Z}(t)$ is the impedance $Z(-\omega)$ and $I = \dot{Q}$. By Laplace transforming Eq. (3.41) one obtains

$$\frac{Q(s)}{C} + Z(-is)(sQ(s) - Q_e) = Q_0, \quad (3.42)$$

which can be recast to the form (using $Q_e = 0$)

$$Q(s) = \frac{Q_0}{\frac{1}{C} + sZ(-is)}, \quad (3.43)$$

The dynamical susceptibility $S(t)$ is defined as the linear response $Q(t) = Q_0S(t)$ and one obtains

$$\int_{0}^{\infty} S(t)e^{i\omega t}dt = \frac{1}{\frac{1}{C} - i\omega Z(-\omega)} = Z_Q(\omega). \quad (3.44)$$

The quantum FDT relates the equilibrium fluctuations of $V$ to $S(t)$ as

$$\int_{-\infty}^{\infty} e^{i\omega t}\langle V(t)V(0)\rangle dt = \frac{2\hbar}{1 - e^{-\hbar\omega}} \frac{\text{Im}[Z_Q(\omega)]}{C^2}, \quad (3.45)$$
3.3. FLUCTUATIONS IN A LINEAR SYSTEM

where we have used that \( Q = CV \). For \( Z(\omega) = R \) one obtains an important result

\[
\int_{-\infty}^{\infty} e^{i\omega t} \langle V(t)V(0) \rangle dt = \frac{2\hbar \omega}{1 - e^{-\hbar \omega \beta}} \frac{R}{1 + \left( \frac{\omega}{\omega_c} \right)^2},
\]

(3.46)

where \( \omega_c = 1/RC \) is the cut-off frequency of the voltage fluctuations.

Similarly, we can always define a phase-difference operator related to the voltage \( V \) as

\[
\dot{\phi} = \frac{2eV}{\hbar} = \frac{2eQ}{\hbar C},
\]

(3.47)

described by the classical current balance equation

\[
C \frac{\hbar}{2e} \dot{\phi}(t) + \int_{-\infty}^{t} dt' \dot{Y}(t-t') \frac{\hbar}{2e} \dot{\phi}(t') = 0,
\]

(3.48)

where \( \dot{Y}(t) \) is related to the admittance \( Y(\omega) = 1/Z(\omega) \). This is the Langevin equation of a quantum Brownian motion. Via a similar calculation as we did for the voltage fluctuations one obtains

\[
\int_{-\infty}^{\infty} e^{i\omega t} \langle \phi(t)\phi(0) \rangle dt = \frac{2\hbar}{1 - e^{-\hbar \omega \beta}} \text{Im}[Z(\phi(\omega))] \]

(3.49)

\[
Z(\phi(\omega)) = -\frac{1}{C\omega^2 + i\omega Y(-\omega)}.
\]

(3.50)

Again for the ohmic resistor

\[
\int_{-\infty}^{\infty} e^{i\omega t} \langle \phi(t)\phi(0) \rangle dt = \frac{1}{\omega} \frac{2\hbar}{1 - e^{-\hbar \omega \beta}} \frac{R}{1 + \left( \frac{\omega}{\omega_c} \right)^2}.
\]

(3.51)

The spectrum diverges for \( \omega \to 0 \) and \( \langle \phi(t)\phi(0) \rangle \) diverges for all times. This is natural since Brownian motion of a free particle is not restricted to any part of the space.

The proper correlation function for considering the phase fluctuations is

\[
J(t) = \langle (\phi(t) - \phi(0))\phi(0) \rangle,
\]

(3.52)

where the diverging part of Brownian motion is subtracted off. By considering the element \( \langle \phi(t)\phi(0) \rangle \) one can derive the result

\[
J(t) = 2 \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{\text{Re}[Z(\omega)]}{R_Q} \left\{ \coth \left( \frac{1}{2} \beta \hbar \omega \right) [\cos(\omega t) - 1] - i \sin(\omega t) \right\},
\]

(3.53)

where \( R_Q = \hbar/4e^2 \) is the resistance quantum and \( Z(\omega) \) the tunneling impedance, related to the result (3.50) as

\[
Z(\omega) = \frac{1}{i\omega C + \frac{1}{Z(\omega)}} = i\omega Z(\omega)(-\omega).
\]

(3.54)
3.3.2 Incoherent Cooper-pair tunneling

When the capacitor $C$ is a JJ with a small critical current, the Cooper-pair tunneling can be treated as a perturbation to the equilibrium dynamics considered in Section 3.3.1. In contrast to the case of ordinary coherent Cooper-pair tunneling, the process is called incoherent or inelastic. This is because one effectively assumes that the (coherent) Cooper-pair tunneling is soon interrupted by an incoherent environmental process. If the JJ and the impedance $Z(\omega)$ are in series with a voltage source $V$, the process dissipates the energy $\sim 2eV$ released in each of the single-Cooper-pair tunneling events. The tunneling perturbs the environment, but if it can be assumed to relax to its equilibrium before the next tunneling occurs, the current can be calculated using the golden-rule treatment, called the $P(E)$-theory [16]. This important theory applies for arbitrary $Z(\omega)$ and states that the net current across the JJ is

$$I = 2e\Gamma^+ - 2e\Gamma^- = 2e\frac{\pi}{2\hbar}E_J^2[P(2eV) - P(-2eV)],$$  \hspace{1cm} (3.55)

where

$$P(E) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp \left[ 4J(t) + \frac{i}{\hbar}Et \right],$$  \hspace{1cm} (3.56)

is interpreted as the probability for the energy $E$ to be absorbed/emitted by the dissipative environment in the tunneling process. For $|V| \neq 0$ a net current can flow only if the energy $2eV$ (released in the Cooper-pair tunneling) can be dissipated by the environment. It results that the energy-level structure of the environment is seen as peaks in the $I-V$ characteristics. For this reason the voltage-biased small JJ is an excellent tool for probing various properties of its nearby environment [52, 53, 18, 19].
Chapter 4

Applications

In this Chapter we go through the basics of the devices studied in Papers I–IV. We also discuss the main results of the Papers and extend some subjects not covered. The circuits studied are often not single JJs but rather islands, in order to avoid a direct contact between the device and the leads attached to the system. At high frequencies the leads act as transmission lines causing unwanted decoherence and dissipation. In general, all the noise sources (Section 3.2) should be shielded as well as possible, if the experiment is not studying them.

4.1 Cooper-pair box

The simplest device that acts analogously to an isolated JJ, but is not in a direct contact with the leads, is probably the Cooper-pair box (CPB), depicted in Fig. 4.1. The mesoscopic island is connected to the outside world via a JJ and a gate capacitor. The Cooper pairs can tunnel coherently through the JJ to the island and vice versa. The size of the island is of mesoscopic scale in order to minimize its capacitance to ground and to make the lumped circuit model valid. Then the charging energy of the island is determined by the capacitance $C_\Sigma = C + C_g$. The name “box” originates in that if $E_J \sim E_C$, usually only a few excess Cooper pairs, as compared to the background charge, populate the island. The decoherring effect of voltage fluctuations due to the impedance $Z(\omega)$ is reduced by a factor $(C_g/C_\Sigma)^2$ which can be seen as the JJ’s fraction of the total voltage fluctuations across the CPB to the second power (if the CPB is treated only as two capacitors in series). The factor can be made very small by going to small gate capacitances while the polarisation charge $Q_0 = -C_g V$ is kept a constant by increasing $V$. The first demonstrations on the real-time controllability of the JJ’s quantum states were performed in this kind of circuit [8, 9]. In these studies also an ac-signal or additional voltage pulse was applied to $V$ in order to induce transitions between the eigenstates of the CPB.
Figure 4.1: Left: The Cooper-pair box consists of a small island connected to a JJ and a “gate” capacitor $C_g$. The leads are voltage biased and the dynamic variable in the circuit is the island charge $Q$. Right: In Paper I we study the situation where also a small JJ is connected to the CPB.

The Hamiltonian of the CPB is a sum of the Josephson coupling energy, the island’s charging energy and of the work term related to the voltage source. It can be represented in the island-charge basis $|Q\rangle$ as

$$H_{CPB} = \sum_Q \left[ \frac{(Q + Q_0)^2}{2C_{\Sigma}} |Q\rangle \langle Q| - \frac{E_J}{2} (|Q + 2e\rangle \langle Q| + |Q - 2e\rangle \langle Q|) \right]. \quad (4.1)$$

The energy-level structure is similar as in Fig. 2.4 but $Q_0$ playing the role of the quasicharge $q$. As for most such systems, the Hamiltonian can be derived either by starting from a Lagrangian treatment for the current balance in the nodes \[54\] or by directly summing the individual electrical energies as a function of the branch variables and using the Kirchhoff rules for considering only independent variables \[55\]. It is notable that the work done by the voltage source is also a quantum variable, as it depends on the charge tunneled to the island. Therefore it should not be understood as the charge supplied by the actual classical voltage source at the end of the leads, but rather a collective behaviour of the nearby electrical components acting effectively as the bias. At the relevant frequencies (above the GHz range) the CPB “sees” only objects in its immediate vicinity due to finite propagation speed of the electromagnetic waves \[56\]. At high frequencies the leads act as transmission lines with characteristic impedance $Z \approx 100 \Omega$.

The first experiments \[8,9\] for this system focused on the regime $E_J/E_C < 1$ where the quasicharge dependence, and therefore also the external controllability, of the energy levels is strong. However, the limit turned out also to be fragile to spurious charge fluctuations in the environment. In Eq. (4.1) they could be included by redefining $Q_0 = -C_gV + Q_{00}$ where $Q_{00}$ is an effective offset charge due to other nearby charges (having its own dynamics). Due to this drawback the limit $E_J \gg E_C$, where the quasicharge dependence of the energy-levels is weak, has obtained more attention in studies motivated by the quantum information processing \[57\].
4.1. ENERGY-LEVEL PROBING

Paper I analyzes a situation where the energy levels of the CPB are probed by incoherent Cooper-pair tunneling through an additional small JJ (probe) attached to the island, Fig. 4.1. A net probe-current is possible if its environment is able to absorb the energy $2eV$ released in each of the single-Cooper-pair tunneling processes, similarly as in the case of Section 3.3.2. The relevant environment consists of the CPB, of nearby electric-circuitry (EE), and of spurious charge-fluctuators (CF). The study is done in the eye of the experiment \cite{58} where the larger JJ was realized as a SQUID and the system could be studied \textit{in situ} through a wide range of the ratio $E_{J1}/E_C$ by applying different magnetic fields to the system.

In the theoretical part of Paper I the current across the probe is calculated perturbatively. The (unperturbed) Hamiltonian of the CPB is similar as in Eq. (4.1) with the changes $Q_0 \approx C_0U - C_2V$ and $C_\Sigma = C_0 + C_1 + C_2$. We assume that $C_0 \ll C_1, C_2$ so fluctuations of $U$ can be neglected. If $E_{J1}/E_C \gg 1$ the lowest energy-eigenfunctions are vanishingly small at the potential maxima in the $\varphi$-space and the related eigenenergies have no quasicharge dependence, see Fig. 2.4 (left). In this regime the CPB can be treated as a harmonic oscillator and only the second-order term in the expansion

$$-E_{J1}\cos(\varphi) = E_{J1}\left(-1 + \frac{\varphi^2}{2!} - \frac{\varphi^4}{4!} + \ldots\right),$$

is important. For lower values of the ratio $E_{J1}/E_C$ also the anharmonic terms start to play an important role and finally the quasicharge dependence emerges as an effect of considerable tunneling across the Josephson potential.

We calculate the probe current as in the $P(E)$-theory \cite{16}, which is a model for arbitrary linear environments (Section 3.3), but extend the theory by taking into account also the anharmonicity and the band structure of the CPB. This is done at the cost that the method is restricted to the case of small $\text{Re}[Z(\omega)] \ll R_Q$ or small $C_2$. If considering the limit $E_{J1}/E_C \gg 1$, the total tunneling-impedance (3.54) can be usually represented as a sum

$$Z_t \approx Z_{\text{CPB}} + Z_{\text{EE}},$$

where $Z_{\text{CPB}}$ is the impedance of a damped harmonic-oscillator (CPB) and $Z_{\text{EE}}$ a “background” impedance due to the EE. In such a case the total $P(E)$-function is the convolution of the two sub-environments \cite{53}

$$P(E) \approx \int_{-\infty}^{\infty} dE' P_{\text{CPB}}(E')P_{\text{EE}}(E - E'),$$

and the energy absorption/release due to the broadened CPB states and the fluctuations of the EE can be treated separately.
We apply the relation (4.4) for arbitrary \( E_{J1}/E_C \) but determine \( P_{CPB}(E) \) using a more direct method. We can calculate the damping of the CPB due to the coupling to the EE (or similarly to the CF) perturbatively by modeling the total system as \( H = H_{CPB} + H_{int} + H_{env} \) and identifying

\[
H_{int} = -\frac{QV_f C_2}{C_C} \equiv Q_{int} V_f,
\]

where \( V_f \) is the fluctuating part of the voltage across the CPB. In the Born approximation the dynamics of \( V_f \) are determined by \( H_{env} \) describing the equilibrium fluctuations of \( Z(\omega) \) connected to \( C_{12} = (1/C_1 + 1/C_2)^{-1} \) (Section 3.2.1). The transition rates between the energy eigenstates of the CPB are in the second-order approximation

\[
\gamma_{f\rightarrow i} = \frac{\langle V_f(t)V_f(0) \rangle_\omega |\langle f|Q_{int}|i \rangle|^2}{\hbar^2},
\]

where \( \hbar \omega = E_{if} = E_i - E_f \). The spectrum \( \langle V_f(t)V_f(0) \rangle_\omega \) can be determined by the quantum fluctuation-dissipation theorem (3.45). It follows that the first-order result (4.6) is small whether \( \text{Re}[Z(\omega)] \approx R \ll R_Q \) or \( C_2 \ll C_C \). The transition rates (4.6) lead to finite lifetimes of the eigenstates \( |i \rangle \) and to the broadenings \( \Delta_i \) of their eigenenergies \( E_i \). The rate for Cooper-pair tunneling across the probe is then written as

\[
\Gamma^{\pm}_{f\rightarrow i} = \frac{E_J^2}{\hbar} |\langle f|e^{\mp i\phi}|i \rangle|^2 \frac{\Delta_b}{4(E_f - E_i \mp 2eV)^2 + \Delta_b^2} = \frac{\pi}{2\hbar} P_{CPB}(\pm 2eV),
\]

where \( \langle f|e^{\mp i\phi}|i \rangle \) is the matrix element describing positive/negative direction Cooper-pair tunneling and \( \Delta_b = \Delta_i + \Delta_f \) the sum of the two broadenings. The current is determined by the relation (3.55). Such a treatment is in accordance with the \( P(E) \)-theory in the limit \( E_J/E_C \gg 1 \) and can also be applied to the case of arbitrary \( E_J/E_C \).

In the experimental part of Paper I we apply this model when analysing the experimental findings of the research reported partly in Ref. [58]. We are able to reproduce the main features that show dependence on the magnetic field and identify current peaks due to excitations of two coupled CPBs, see Fig. 4.2. We see also additional behaviour which we interpret as extra resonators in the vicinity of the CPB. One of them showed Fraunhofer pattern (Section 2.3) as a function of the magnetic field, hinting that it originates in additional large JJs produced in the used manufacturing process. Traces of the CF are seen as increased downward transition rates. Interestingly, no gate dependence for the main resonances were detected, in contrary to the theory. Instead, averaging the theoretical \( I-V \) curves produced similar

\[\text{(4.7)}\]

\[\text{In Papers I and II we give the opposite (false) definition, which however was not used in the numerical simulations.}\]
4.1. COOPER-PAIR BOX

Figure 4.2: Left: The experimental current as a function of the transport voltage \( V \) and the magnetic flux \( \Phi \) through two SQUID loops. Right: The equivalent model for the first half-period in \( \Phi \). We identify the resonances as the energy levels of two coupled CPBs and spurious external resonators. The coloring of the resonance corresponds to widening of the energy levels due to the quasicharge dependence. For more details see Paper I.

I – V curves. From these we could identify the width of the bands that were in accordance with the theory. Recently, a clear detection of the quasicharge dependence in the higher energy-levels was done in a similar circuit \[57\].

To supplement Paper I we discuss the effect of quasiparticle tunneling. The tunneling across the probe gives the dominant contribution when compared to the tunneling across the large JJs, as it is energetically favourable. Above the threshold \( eV = 2\Delta \) it leads to linear \( I-V \) characteristics \( \sim V/R_T^2 \). At the studied subgap region it can contribute through two indirect mechanisms. First, it can enhance relaxation (damping) of the excited states. The simultaneous quasiparticle tunneling (in the positive direction) and energy-level transition can be included by a contribution similar to Eq. (4.6) but changing \( Q_{\text{int}} \) to \( e^{-i\phi/2} \), and the voltage-fluctuation spectrum to the quasiparticle current \( 2D_q^{+}(\omega) \), Eq. (3.39), where \( \hbar\omega = E_i - E_f + eV \). The treatment is described in Paper III. The process turns out to be weaker than the damping due to coupling to the EE and CF, and does not contribute to the overall current significantly. The second mechanism is a direct quasiparticle tunneling due to a higher-order process. Again, the amplitudes for such processes remain small. However, a speculative process is the Andreev tunneling enhanced in a diffusive environment \[59\]. In this process a Cooper pair breaks into two quasiparticles via two subsequent tunneling of electrons, with the threshold \( eV = \Delta \). Such a process can produce a current that is proportional to the density of states of the quasiparticles, Eq. (3.29). This can be viewed as that the system probes also the quasiparticle density of...
states. Similar feature emerges in the experimental data, above which (some of) the resonances were identified. However, its observed magnitude is about 50 times greater than in the best fit with a realistic choice of parameters.

### 4.2 Single-Cooper-pair transistor

The single-Cooper-pair transistor (SCPT) consists of a mesoscopic island connected to two JJs and a gate capacitor, Fig. 4.3. The circuit is otherwise the same as studied in Section 4.1.1 except that necessarily no bias between the “source” and “drain” is applied. The naming originates in the analogy with the field-effect transistor whose source-drain current is controlled by a potential field induced by the gate. Here it is the polarization charge induced by the gate capacitor that affects the current across the system. Although nicknamed as a "transistor", its conventional use is in the implementations of quantum coherence [60, 61, 17].

The SCPT differs from the CPB in the net current across the system. Usually the state between the source and drain can be treated classically. Then the SCPT acts as a classical JJ whose Josephson coupling energy is determined by the quantum-mechanical state of the island, to which also the gate voltage contributes. The relevant Hamiltonian is in this case

\[
H_{\text{SCPT}}(\theta) = \sum_Q \left[ \frac{(Q + Q_0)^2}{2C_\Sigma} |Q\rangle \langle Q| - \frac{E_J(\theta)}{2} (|Q + 2e\rangle \langle Q| + |Q - 2e\rangle \langle Q|) \right],
\]

(4.8)

where \(Q_0 = (C_1 - C_2)V/2 + C_0 U\) and \(E_J(\theta) = 2E_{J1} |\cos(\theta/2)|\) follows from the assumption \(E_{J1} = E_{J2}\) and a similar mathematical manipulation that was done in Section 2.3. The phase difference across the SCPT \(\theta\) satisfies the Josephson relation (2.3). If the island populates the eigenstate \(|i\rangle\) with the eigenenergy \(E_i(\theta)\), the current across the device is

\[
I(\theta) = \frac{2e}{h} \frac{\partial E_i(\theta)}{\partial \theta}.
\]

(4.9)

For a high voltage-bias, or generally for fast processes, the charge tunneled through the SCPT is also treated as a quantum variable. Expressing \(E_i(\theta)\) as an operator and including the automatic work term, the relevant Hamiltonian
4.2. SINGLE-COOPER-PAIR TRANSISTOR

Figure 4.3: Left: The voltage-biased single-Cooper-pair transistor. The small island is connected to the outer world via two JJs, through which a net current can flow, and via a gate capacitor. Right: The eigenstates of the voltage-biased SCPT form a WS ladder. In the case of an asymmetric SCPT the charge tunneled across the probe $Q_2 = 2em$ provides a convenient basis for the ladder (for the exact definition of the states see the text). A dissipative interaction with the environment can induce transitions between the eigenstates (visualized as arrows and the operators $Q$ and $Q_2$ mediating the fluctuations) and cause the SCPT to slide down in the WS ladder. This corresponds to a net current across the system (from the source to the drain).

becomes \[ H_{SCPT} = \sum_Q \left[ \frac{(Q + Q_0)^2}{2C_{\Sigma}} |Q\rangle \langle Q| - \frac{1}{2} \bar{Q} V |\bar{Q}\rangle \langle \bar{Q}| \right] \]

\[ -\frac{E_j}{2} \sum_{Q,Q'} [ (|\bar{Q} + 2e\rangle \langle \bar{Q}| + |\bar{Q} - 2e\rangle \langle \bar{Q}|) (|Q + 2e\rangle \langle Q| + |Q - 2e\rangle \langle Q|)] , \]

(4.10)

where $\bar{Q} = Q_1 + Q_2$, $Q_i$ being the charge tunneled across the $i$:th JJ. In this case a net current can flow only if there is a mechanism for dissipation. As the ordinary Cooper-pair tunneling does not provide it, the net current can flow only via energy exchange between the SCPT and its environment. At low voltages the resonant Cooper-pair tunneling is the essential conduction mechanism \[62\] \[63\] \[64\] \[55\]. The naming “resonant” originates in that the tunneling is drastically enhanced at certain values of the transport voltage. In the resonance the feed energy due to the bias equals the energy change in (the state of) the island. At least one of the Josephson coupling energies has to be small, if compared to other elements in the Hamiltonian, so that speaking of single-tunneling processes is sound.
4.2.1 Resonant Cooper-pair tunneling

Papers II and III consider resonant tunneling of Cooper pairs across a voltage-biased SCPT. In a resonant process the island is excited as charge is transported across the system, and vice versa. Irreversible processes can relax the excited state to a lower-energy one but in a way the charge originally transported is not returned. Therefore the resonant tunneling leads to a net current with $I - V$ peaks that can be related to the system’s energy-level structure, similarly as in Paper I. However, in Paper II we assumed that the Cooper-pair tunneling across the small probe is incoherent due to fast interruption by the irreversible processes. In such a case the perturbative treatment of the probe is well valid. In Paper III we start from the opposite assumption and treat all the tunneling nonperturbatively and coherently. The decoherence is included perturbatively by using the Pauli-master-equation [31], where only diagonal elements (populations) of the density matrix are modeled. Paper III then gives a treatment that applies for arbitrary decohering times. It also analyzes how different environments contribute to the net current.

The nonperturbative calculation includes arbitrary-order Cooper-pair tunneling across the JJs. The theoretical treatment can in many situations be reduced to a simple two-state system extended to a Wannier-Stark (WS) ladder [65, 66]. Which two eigenstates are selected in a single step depends on $V$ and $U$. In the case of asymmetric SCPT the relevant states in a resonant situation between the CPB’s ground and the first excited-state due to single-Cooper-pair tunneling across the probe are $|E_1(Q_0) - E_0(Q_0)| = 2eV$, where $E_i(Q_0)$ are eigenenergies of the CPB

\begin{align}
|+, m⟩ &= \frac{1}{\sqrt{2}}(|0⟩|m⟩ + |1⟩|m + 1⟩) \\
|−, m⟩ &= \frac{1}{\sqrt{2}}(|0⟩|m⟩ - |1⟩|m + 1⟩),
\end{align}

where the states are expressed in the product basis of the CPB’s eigenstates and the charge tunneled through the probe $Q_2 (2em)$. The corresponding eigenenergies are

\begin{align}
E_{+m} &= -\frac{E_J}{2}|⟨1|e^{-iϕ}|0⟩| - 2eV m \\
E_{−m} &= +\frac{E_J}{2}|⟨1|e^{-iϕ}|0⟩| - 2eV m.
\end{align}

The eigenstates form a WS ladder with respect to $m$, i.e., a new eigenstate is obtained after a translation $m \rightarrow m \pm 1$ with eigenenergy $E \rightarrow E \pm 2eV$. If the system is closed, the states are steady and describe coherent superpositions of the CPB’s eigenstates induced by coherent Cooper-pair tunneling across the probe.
4.2. SINGLE-COOPER-PAIR TRANSISTOR

The operator mediating interaction with the EE (CF and quasiparticles are treated analogously) is identified to be

\[ H_{\text{int}}^{\text{EE}} = \left( \frac{C_2}{C_{\Sigma}} Q + Q_2 \right) V_f, \] (4.15)

which differs from the expression (4.5) by the presence of the operator \( Q_2 V_f \). The transition rates between the diagonal elements of the density matrix are determined by Eq. (4.6). This second-order result is sufficient if \( \text{Re}[Z(\omega)] \ll R_Q \). The interaction induces transitions that favor a decrease of the SCPT’s energy causing it to slide down in the WS ladder. The net current is proportional to the average speed of this sliding. The effect of \( Q_2 V_f \) is similar to the effect of \( P_{\text{EE}}(E) \) in Paper I, i.e., it broadens the resonances via low frequency fluctuations. The operator \( C_2 QV_f/C_{\Sigma} \) then causes transitions between the different steps of the WS ladder by entangling states such as \( |1\rangle|m + 1\rangle \) and \( |0\rangle|m + 1\rangle \), Fig. 4.3.

The general relation for a resonance between the CPB eigenstates \( |a\rangle \) and \( |b\rangle \) with eigenenergies \( E_a(Q_0) \) and \( E_b(Q_0) \) in a tunneling of \( n \) Cooper pairs across the probe is

\[ 2eV = \frac{E_a(Q_0) - E_b(Q_0)}{n}. \] (4.16)

This implies a huge amount of possible resonances (actually infinitely many). In order for the resonance to be detected as an \( I-V \) peak, the resonant states need to be populated also. In Paper II we analyze the \( I-V \) characteristics of the asymmetric SCPT obtained by calculating the steady-state current due to transitions between the SCPT’s eigenstates. The same calculation for a symmetric SCPT is done in the standard paper of the field, Ref. [62]. We call the treatment as the coherent tunneling model (CTM) as one implicitly assumes low transition rates if compared to the splittings of the SCPT’s eigenstates.

4.2.2 Corrections using the density-matrix-approach

In experiments [64, 55] the subgap features seem to be much smoother than what is obtained in Ref. [62]. Also in Ref. [58] and Paper II no traces of sharp higher-order (several-Cooper-pair tunneling) resonances, predicted by Paper II, are seen. This is something that might be expected, as there are many technical doubts related to accuracy and stability of the bias. But what it comes to the problem of fluctuations in the bias line, the treatment should describe them in the limit of the approximations made. Most of the resonances obtained by the CTM are actually artefacts that do not exist in a more accurate solution of the problem. Such a solution can be found, for example, using a density matrix approach (DMA) shown in Paper III.
Figure 4.4: Left: The net current through a symmetric SCPT (parameters given in Paper III) using the CTM. At low voltages one can distinguish two set of resonant lines corresponding to the third and fifth-order resonances. Right: The current in the same system but now obtained by the DMA. At lower voltages the fifth-order resonance has vanished due to thermal fluctuations of the EE. At voltages above 150 $\mu$eV also the third-order resonance weakens significantly due to the quantum Zeno effect caused by intensive quasiparticle tunneling. For more details see Paper III.

The difference we make in Paper III is that we model also the nondiagonal elements of the density matrix (not just populations), but still restrict to the lowest-order calculation in the interaction with the environment(s). However, this enables quantum Zeno-type effects [67, 68] in the charge transport, where strong decoherence drives, or continuously “measures”, the SCPT into superpositions of eigenstates that are more stable under the decoherence. In these situations the density matrix can still be made diagonal, but in a different basis. In the considered case (Section 4.2.1) the quantum Zeno effect corresponds to the possibility that a fast relaxation of the CPB’s state $|1\rangle$ inhibits the natural quantum evolution of the eigenstates (4.11) and (4.12) trying to drive the system to the superposition $(|+\rangle + |\rangle - \rangle) / \sqrt{2} = |0\rangle |m\rangle$. The Rabi oscillation to the state $(|+\rangle - |\rangle - \rangle) / \sqrt{2} = |1\rangle |m\rangle$ is too slow to occur and it is mostly washed out due to fast decoherence. The transition can still occur but by a much lower rate, decreasing the corresponding $I - V$ peak in comparison to the results obtained by the CTM. Thermal fluctuations can also lead to similar effects, as the treatment describes their contribution more accurately also, Fig. 4.4.

The quantum Zeno effect is analogous to the Zeno arrow paradox originally presented by pre-Socratic Greek philosopher Zeno of Elea (lived $\sim$ 490 – 430 BC) in trying to convince that movement is only an illusion. At any one instant of time a flying arrow moves to where it is or to where it is not. It cannot move to where it is not, because this is one instant in time. Moving to where it already is, is not a movement. Therefore the flying arrow cannot be moving. In quantum mechanics such an effect is possible. At every moment a continuous observer measures the system to a state in which it
4.2. SINGLE-COOPER-PAIR TRANSISTOR

Figure 4.5: Left: The effect of a resonance in the bias line can be modeled quantitatively by replacing it by an LC oscillator. Right: The $I - V$ characteristics obtained by the DMA for the same SCPT as in Fig. 4.4 but now in series with an LC oscillator with the resonance frequency $\hbar \omega_p = 80 \, \mu eV$ and $C = 0.5 \, pF$. Instead of using an island-charge noise characterized by $R_I = 3 \, \Omega$ (see Paper III), we couple a similar noise to the LC oscillator ($R_f = 1 \, \Omega$) describing its losses. New resonant lines appear due to simultaneous Cooper-pair tunneling and excitation of the mode. For example, the horizontal line $V = 40 \, \mu V$ corresponds to simultaneous single-Cooper-pair tunneling across both the JJs and single-photon emission to the mode, whereas the third-order resonance (lines below $\sim 100 \, \mu V$ in Fig. 4.4) has now a dim replica shifted by $80/3 \, \mu V$ corresponding to (tunneling of two Cooper pairs across one JJ and single Cooper pair across the other JJ and) simultaneous single-photon emission to the mode.

already was (infinitesimal time before) or to where it was not. It practically cannot be measured to the state where it was not since the amplitude for this is very small. Therefore the state is not changing under continuous measuring.

4.2.3 External oscillator

The net current across a voltage-biased SCPT characterizes also its environment. The treatment done in Paper III can be adapted for any weakly interacting “large” environment. Also, as a first approximation, the model can be applied to the case of occasional high environmental impedances. Such a treatment is adequate, for example, for the case of a resonant transmission line [52]. Since this lowest-order treatment neglects all higher-order effects, it does not produce effects such as splitting of eigenstates or higher order-order resonances, but is computationally very effective.

Alternatively a resonance in the line can be treated quantitatively as an LC-oscillator in series with the SCPT [56], Fig. 4.5. In this case it is convenient to do a similar change of variables as in the case of asymmetric SCPT. The role of the probe is played by the SCPT. By summing up the
CHAPTER 4. APPLICATIONS

circuit’s electrical energies and using the Kirchhoff rules, the Hamiltonian can be represented in the form \( H = H_{LC} + H_{SCPT} + H_{CC} \) where the individual Hamiltonians of the SCPT and the LC oscillator are separated and then entangled capacitively by the term \( H_{CC} \). Under the assumption \( C \gg C_\Sigma = C_1 + C_2 + C_0 \), which is very reasonable since usually \( C_1 \sim 1 \text{ fF} \) and \( C \sim 1 \text{ pF} \), the operators are

\[
\begin{align*}
H_{LC} &= \frac{(Q_l - C_{12}V)^2}{2C} + \frac{\varphi_l^2}{2L} \\
H_{SCPT} &= \frac{(Q_r + UC_0)^2}{2C_\Sigma} - \frac{V}{2}\bar{Q} \\
-H_{J1} \cos(-\varphi_l + \varphi_r + \bar{\varphi}) - H_{J2} \cos(-\varphi_r + \bar{\varphi}) \\
H_{CC} &= \frac{Q_l(Q_r + UC_0)}{2C},
\end{align*}
\]

where \( C_{12} = C_1/2 \) and we use the assumption \( C_1 = C_2 \), \( Q_l \) (\( Q_r \)) is the left (right) island charge, a conjugated variable to \( \varphi_l \) (\( \varphi_r \)), and the sum of the charges tunneled across the JJs \( \bar{Q} \) is a conjugated variable to \( \bar{\varphi} \). The Hamiltonian matrix is most efficiently written in the basis \( |l\rangle|Q_r\rangle|\bar{Q}\rangle \) where \( |l\rangle \) is an eigenstate of \( H_{LC} \) with eigenenergy \( \hbar \omega_p = \hbar/\sqrt{LC} \). The excited states of the oscillator correspond to different number of photons in the mode. The matrix elements \( \langle l|e^{i\varphi_l}|n\rangle \) \( (l,n = 0, 1, 2 \ldots) \) can be calculated analytically and describe excitation of the oscillator when tunneled charge flows into it.

Referring to the last section of Paper III, the resonant structure in this case follows the lines

\[
eV = \frac{4EC}{l+r} \left[ \frac{(l-r)^2}{2} + \frac{Q_0}{e} + n \right] + \frac{2\hbar \omega_p}{l+r},
\]

some of which can be identified in Fig. 4.5

4.3 Current-biased Josephson junction

The current-biased JJ is the device in which the macroscopic quantum-effects of small JJs were first verified [5, 7, 6]. The phase difference across the JJ was shown to be able to quantum-mechanically tunnel across its potential field causing a voltage to appear across the JJ. This switching rate had a minimum value which the lowering of the temperature did not anymore affect. Today similar current-biased JJs are also good candidates for qubits [69] and mesoscopic amplifiers [70]. Their benefit is the immunity to the “island” charge fluctuations, but in larger JJs (qubits) also discrete fluctuators in the insulating barriers start to exist (Section 3.2.3).

In the current-biased case the Hamiltonian of the JJ acquires an extra term \(-\hbar I \varphi/2e\), which is the energy supplied by the current source \( I \). The
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Figure 4.6: The relevant eigenstates for a current biased JJ depend on the strength of the parallel resistor. For $R \ll R_Q$ (and large $E_J/E_C$) the metastable states in the local minima of the washboard potential are a good basis. For $R \gg R_Q$ the quasicharge is localized and system performs Bloch oscillations in the energy bands of the isolated JJ.

Hamiltonian is therefore

$$H = \frac{Q^2}{2C} - E_J \cos(\varphi) - \frac{\hbar}{2e} I \varphi. \quad (4.21)$$

The effective potential energy $-E_J \cos(\varphi) - \hbar I \varphi/2e$ is called the tilted washboard potential [4], due to its shape in the $\varphi - E$ space. Depending on the strength of $I$, the potential can have local minima occurring with the spacing of $2\pi$. However, the potential is not periodic, as the Josephson coupling energy, but describes interaction with the current source which provides continuous charge flow to the JJ. The $2\pi$-periodicity of the JJ’s wave function is not anymore guaranteed and depends on the fluctuations of $I$. If there are none, or they can be regarded classical, the effect of $I$ is to change the quasicharge to the momentary value $q = \int I dt$. However, in experiments the leads attached to the JJ behave effectively as a parallel resistor $R \approx 100 \, \Omega$ which makes the quantum fluctuations of $q$ important [71, 27]. This can be avoided by placing a large resistor $R$ in the immediate vicinity of the JJ and applying a series voltage-bias (equivalent to a current bias $V/R$ and a resistor $R$ in parallel with the JJ). In the case $R \gg R_Q$ the fluctuations of the quasicharge are small and a good basis is the one with (momentary) discrete values of $Q$ (wave functions periodic in $\varphi$). In such a situation one usually studies the case $E_J \sim E_C$ where the band structure of the JJs is evident. If $R \ll R_Q$ the metastable states obtained by treating $Q$ as a continuous variable is a good basis. But for them to exist the ratio $E_J/E_C$ has to be also large so that the damping would be small [4], Fig. 4.6. In both regimes the (remaining) effect of the resistor can be calculated perturbatively by
decomposing $I = I_{\text{static}} + \hat{I}_f$ and identifying

$$H_{\text{int}} = -\frac{\hbar}{2e} \hat{I}_f \varphi,$$

as the term mediating energy exchange between the current-biased JJ and the dissipative environment. The environment is harmonic and can be represented similarly as in the expressions (3.14-3.17).

### 4.3.1 Calculation scheme

In paper [IV] we consider a system where the current bias is provided by a voltage bias and a nearby resistor $R \gg R_Q$. This is equivalent to a current bias $V/R$ and a parallel $R$. Therefore the eigenstates of an isolated JJ are the basis for our analysis. The effect of the parallel resistor is taken into account perturbatively by using the density-matrix equation of motion in the BM approximation (Section 3.1.3). For such a calculation we define our basis as $|n, k\rangle$ where $n$ is a discrete parameter referring to the energy band and $k = q/2e$ a continuous parameter referring to the quasimomentum [71, 72]. The states correspond to the eigenstates of the JJ’s Hamiltonian (with quasicharge $q$) and are normalized as

$$\langle n, k|n', k'\rangle = \delta(k' - k)\delta_{n', n}.$$  

(4.23)

An arbitrary state is represented as $|\psi\rangle = \sum_n \int dk \psi^n(k)|n, k\rangle$ and the density matrix as

$$\rho = \sum_{n_1, n_2} \int dk_1 \int dk_2 \rho^{n_1 n_2}(k_1, k_2)|n_1, k_1\rangle \langle n_2, k_2|.$$  

(4.24)

When solving the density matrix’s equation of motion one needs to calculate matrix elements of $\varphi$ which can be represented as

$$\varphi_{kk'}^{nn'} = i \frac{\partial}{\partial k} \delta(k - k')\delta_{nn'} + \varphi_{kk'}^{nn'}\delta(k - k')(1 - \delta_{nn'})$$  

(4.25)

$$\varphi_{kk'}^{nn'} = -i \left\langle n, k \left| \frac{\partial}{\partial k} \right| n', k \right\rangle.$$  

(4.26)

The first term in the r.h.s. of equation (4.25) describes intraband transitions whereas the second term interband transitions. The element (4.26) is calculated between the $2\pi$-periodic parts of the wave functions and can be recast to the form

$$\varphi_{kk'}^{nn'} = -i \frac{\left\langle n, k \left| \frac{2eQ}{C_S} \right| n', k \right\rangle}{E_n(k) - E_{n'}(k)}.$$  

(4.27)
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\[ E_C = 100 \mu eV \]
\[ T = 20 \text{ mK} \]
\[ R = 50 \text{ k}\Omega \]

\[ E_J/E_C = 0.1 \]

Figure 4.7: The voltage (left) and probability distribution (right) of a current biased JJ in a single-band approximation. For low \( I \) the quasicharge settles nearby a point in which the current across the parallel resistor \( \langle V \rangle / R \) cancels the bias current \( I \). Above certain threshold (depends on \( E_J \)) the quasicharge is in a constant motion at the lowest band and the bias current is cancelled by periodic transfer of Cooper pairs across the JJ (Bloch oscillations).

where \( n' \neq n \). This is obtained by calculating the partial derivative of the Schrödinger equation \( H_{2\pi}(k)|n'\rangle = E_{n'}(k)|n'\rangle \) with respect to \( k \) and taking matrix elements with respect to \( \langle n \rangle \). The Hamiltonian function is

\[
H_{2\pi}(k) = \frac{(Q - 2ek)^2}{2C} - E_J \cos \varphi, \tag{4.28}
\]

and is solved with \( 2\pi \)-periodic boundary conditions.

**Voltage across the Josephson junction**

The model presented in Paper [IV] simulates dynamics of the quasicharge, but the preceding notation is an excellent mathematical tool in determining the relevant equations of motion. As an example let us determine the voltage across the JJ. The voltage is expressed as \( V = Q/C \) and can be calculated on the grounds of the relation \( V = \Phi_0 \dot{\varphi}/2\pi = i/(2e) \times [H, \varphi] \). Consider the element

\[
\langle n', k'|V|\psi \rangle = \frac{i}{2e} \sum_n \int dk \langle n', k'|(H\varphi - \varphi H)\psi^n(k)|n, k \rangle =
\]

\[
\frac{i}{2e} \sum_n \int dk \left[ E^{n'}(k')\langle n', k'|\varphi|n, k \rangle - E^n(k)\langle n', k'|\varphi|n, k \rangle \right] \psi^n(k) =
\]

\[
-\frac{1}{2e} \sum_n \delta_{nn'} \int dk \left[ E^{n'}(k') \frac{\partial}{\partial k'} \delta(k' - k) - E^n(k) \frac{\partial}{\partial k} \delta(k - k') \right] \psi^n(k) +
\]

\[
\frac{i}{2e} \sum_n (1 - \delta_{nn'}) \int dk \left[ E^{n'}(k') \Gamma'^{n}(k) \delta(k - k') - E^n(k) \Gamma^{n}(k) \delta(k - k') \right] \psi^n(k).
\tag{4.29}
\]
For the next expression we use the relation \( \frac{\partial}{\partial k'} \delta(k' - k) = -\frac{\partial}{\partial k} \delta(k' - k) \), do integration by parts with respect to \( k \), use the fact that the substitutional terms vanish (periodic system) and obtain

\[
\langle n', k' | V | n, k \rangle = \frac{\delta_{mn'}}{2e} E'(k') \psi^{n'}(k') + \frac{i}{2e} \sum_{n \neq n'} \left\{ [E^{n'}(k') - E^n(k')] \Gamma^{n'n}(k') \psi^{n'}(k') \right\}.
\]

(4.30)

In the case the original state is a pure Bloch state \(|m, k_0 \rangle = \int dk \delta(k - k_0) |n, k \rangle\) one obtains

\[
\langle n', k' | V | n, k_0 \rangle = \frac{\delta_{mn'}}{2e} E'(k') \delta(k' - k_0)
+ \frac{i}{2e} (1 - \delta_{nn'}) [E^{n'}(k') - E^n(k')] \Gamma^{n'n}(k') \delta(k' - k_0).
\]

(4.31)

The elements are diagonal with respect to quasimomentum. If we can restrict to considering only such diagonal elements, the normalisation \( \delta(k' - k_0) \) can be neglected (cancels out in the equations) and one obtains by marking \( k_0 = k \) that

\[
\langle n', k | V | n, k \rangle = \frac{\delta_{mn'}}{2e} E'(k) + \frac{i}{2e} (1 - \delta_{nn'}) [E^{n'}(k) - E^n(k)] \Gamma^{n'n}(k).
\]

(4.32)

One sees that the Bloch state’s expectation value for the voltage is proportional to \( E'(k) \) and that superpositions produce terms proportional to the Zener-tunneling amplitude. In Fig. 4.7 we show results for the steady state voltage and probability distribution (in \( k \)-space) as a function of the bias current \( I \). The simulations are based on Eq. (4.32) and the equations found in Paper IV.

### 4.3.2 Probing energy-band dynamics

Paper IV considers probing of energy-band dynamics in a current-biased JJ using a small JJ. The circuit can be also seen as the Bloch-oscillating transistor \([70, 73, 74]\) (BOT) with JJ at the base. The BOT is analogous to the bipolar transistor in which the base current controls the current between the emitter and the collector, Fig. 4.8. Alternatively, the system can also be seen as a SCPT in which the gate capacitor is replaced by a large resistor \( R \) and therefore by a dynamic polarisation charge \( q \). The idea in Paper IV is that for certain voltages \( V_b \) and \( V \) the small base JJ induces (extra) transitions between the system’s energy bands and enhances the net current across the base. Therefore the base current probes the system’s energy-band structure and quasicharge dynamics.

The equivalent current-biased circuit is shown in Fig. 4.8 (right). Here we have placed the bias in parallel with the emitter JJ, which is assumed to
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Figure 4.8: Left: The Bloch-oscillating transistor consists of the emitter JJ, the collector resistor and a base junction which is in this case also a JJ. In the operation \( V_b < 0 \) the charge fed by the collector to the island can be discharged, after several Bloch oscillations and subsequent Zener tunneling of the emitter JJ (changing the island to a Coulomb-blockaded higher-voltage state), to the base lead. It results that single-charge tunneling through the base controls tunneling of many Cooper pairs across the emitter. Right: The equivalent model for the BOT. The emitter JJ can be seen to be current \( (I = V/R) \) biased and perturbed by the small base JJ. Actually it is the island charge \( Q \) whose dynamics are truly modeled.

be much larger than the base JJ. In this special system the voltage-biased SCPT (or CPB probed by the small JJ) and the current-biased JJ are present simultaneously. The convenient Hamiltonian function for this case is

\[
H = \frac{(Q + Q_0)^2}{2C_\Sigma} - E_{J1} \cos(\varphi) - \frac{\hbar}{2e} (I + \dot{I}_f) \varphi + Q_b V_b - E_{J2} \cos(\varphi - \varphi_b),
\]

(4.33)

where \( Q_0 = C_2 V_b \) and the fluctuations across the resistor \( R \) are described by \( \dot{I}_f \). Note that the voltage across the emitter and the base (the “SCPT”) is \( 0 - V_b = -V_b \). The current-bias acts similarly as the emitter JJ but effectively causes “continuous” tunneling of charge to the island. Therefore the related phase difference and the island charge \( Q \) are conjugated variables. The two last terms in Hamiltonian (4.33) describe tunneling across the probe and are treated perturbatively, similarly as in Paper I (Section 4.1.1).

In the absence of the current bias and base tunneling the island charge wants to settle nearby the minimum value of the charging energy \( (Q + Q_0)^2/2C_\Sigma \), which corresponds to the case where no voltage appears across the emitter JJ. Therefore it is convenient to work in the basis where one has done a shift \( Q' = Q + Q_0 \equiv Q \). In this basis the expectation value of \( Q/C_\Sigma \) is actually the expectation value of \(-Q_1^{\text{capacitor}}/C_1\), i.e., the voltage across the emitter JJ. The effect of finite \(-I\varphi\) is to cause Bloch oscillations and Zener tunneling in the energy bands. The term \(-I_f\varphi\) describes the current across the resistor. Its thermal noise broadens the quasicharge distribution.
Figure 4.9: The time evolution of initially localized quasicharge distribution in three energy-bands (figures a-c), the distribution in the first band with different coloring (d) and visualization of the dynamics in the energy-band picture (e). The voltage $V_b$ is chosen to be such (250 $\mu$V) that the system tends to be excited from the first band (a) to the third band (b) at the initial position. The band splitting between the second and third band is small and the system makes easily a downward Zener tunneling to the second band (c) with no change in the direction of the flow. The splitting between the first and second band is larger and causes the distribution to split into two when returning to the lowest band (d). The two paths correspond to the charges $Q = \pm e$ at the Zener-tunneling region. The steady state distribution is obtained in the limit $t \to \infty$. The parameters are $E_C = (5/3)E_J = 5E_{J2} = 100 \mu$eV, $R = 180$ k$\Omega$, $T = 100$ mK and $\Delta_b = 25 \mu$eV.
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and quantum noise results in incoherent current $\sim Q/C_\Sigma R$.

The presence of base tunneling can excite the system to higher bands from which the system returns to lower bands by “sliding” down the energy bands and making downward Zener transitions nearby the points of minimum splitting. The sliding correspond to charging of the island via the collector current and its speed is proportional to the voltage across the resistor. Such an effect is visualized in Fig. 4.9 where the real-time dynamics are plotted starting from a narrow Gaussian distribution at $k \sim 0.2$. Such resonant processes contribute as peaks in the $I_b - V_b$ characteristics.

Numerical modeling

The steady-state distribution of all the quasicharge dynamics is a solution of the discretized matrix-equation, derived in Paper IV. The equation has been greatly simplified by considering only diagonal states in $q$ (the approximation holds in the case $R \gg R_Q$). Still, it is important to discretize the quasicharge space much denser than typical changes occur in the corresponding energy-eigenstates. For small band-splittings this becomes a problem as changes occur rapidly nearby $k = 0$ or $k = 1/2$, see Fig. 2.4. Therefore, in many cases the calculation is still quite challenging and needs to be solved with an appropriate method.

A numerically effective algorithm takes into account that only transitions between the nearby states of the quasicharge exist. This results in a block-tridiagonal matrix-equation for the steady state of the system, which has then its own effective inversion algorithm (Thomas algorithm). The upper-right and the lower-left corners of the matrix equation have still nonvanishing elements, which are not present in the block-tridiagonal form, but their effect can be included by using the Woodbury matrix identity. When using general algorithms the computational limits occurred nearby $n \sim 500$ in three-band simulations, whereas in the case of the block-tridiagonal algorithm the limit was nearby $n \sim 30000$. After that it was the the numerical error that made the algorithm out of use. Indeed, Fig. 2 in Paper IV includes an effect of too wide discretization, as the Zener-tunneling region is separated into two different regions. The origin of this separation is the Bloch oscillations in the second band, which reduce the base current and result in the observed dip between the pointed regions. Such oscillations was later shown to be an artefact of too wide discretization. In the more accurate simulations the Zener-tunneling region is uniform as the dip is enhanced to the level of the two maxima.

Quasiparticle tunneling

To model quasiparticle tunneling across the JJs we proceed similarly as in Paper II. Assuming that the interaction with quasiparticles can be modeled
by the lowest-order expansion of the density-matrix’s equation of motion \( \langle R_T > R_Q \rangle \), one obtains that the relevant operator describing the transitions is

\[
\hat{T} = \sum_Q |Q - e\rangle \langle Q|, \tag{4.34}
\]

and that quasiparticles are traced into the effective damping function \( D_q^\dagger(\omega) \equiv D_q(\omega) \), Eq. (3.39). In the case of the base JJ the quasiparticle tunneling and simultaneous transition \(|n_i, k_i\rangle \rightarrow |n, k_f\rangle\) releases/absorbs the energy \( \hbar \omega^{n_i}(k_i) - \hbar \omega^{n_f}(k_f) - eV_b \). Using the notation

\[
\langle n', k'|T|n, k\rangle = T_{nn'}^{n}(k')\delta(k' - k - \frac{1}{2}) \tag{4.35}
\]

\[
\langle n', k'|T^\dagger|n, k\rangle = T_{nn'}^{n}(k')\delta(k' - k + \frac{1}{2}), \tag{4.36}
\]

one can calculate the terms originating in the positive-direction tunneling (\( \hat{T} \)-operator)

\[
\dot{\rho}_{qp}^{n'}(k) = \frac{1}{\hbar^2} \sum_{n_i, n_i'} \rho_{nn'}^{n'} \left( k + \frac{1}{2}, k + \frac{1}{2} \right) T_{nn'}(k)T_{nn'}^{\ast}(k) \times \\
\left\{ D_q \left[ \omega^{n_i} \left( k + \frac{1}{2} \right) - \omega^{n_f}(k) - \frac{eV_b}{\hbar} \right] + D_q \left[ \omega^{n_i'} \left( k + \frac{1}{2} \right) - \omega^{n_f'}(k) - \frac{eV_b}{\hbar} \right] \right\}, \tag{4.37}
\]

which can also be seen as the “type 1” diagrams in the language of the Appendix in Paper III. The type 2 diagrams give the contribution

\[
\dot{\rho}_{qp}^{n'}(k) = -\frac{1}{\hbar^2} \sum_{n_i, n_i'} \rho_{nn'}^{n'} \left( k + \frac{1}{2} \right) T_{nn'}^{\ast}(k) \times \\
D_q \left[ \omega^{n_i'} \left( k + \frac{1}{2} \right) - \omega^{n_f} \left( k + \frac{1}{2} \right) - \frac{eV_b}{\hbar} \right], \tag{4.38}
\]

and the mirror diagram

\[
\dot{\rho}_{qp}^{n'}(k) = -\frac{1}{\hbar^2} \sum_{n_i', n_i} \rho_{nn'}^{n'} \left( k + \frac{1}{2} \right) T_{nn'}^{\ast}(k) \times \\
D_q \left[ \omega^{n_i'} \left( k + \frac{1}{2} \right) - \omega^{n_f} \left( k + \frac{1}{2} \right) - \frac{eV_b}{\hbar} \right]. \tag{4.39}
\]

The opposite direction tunneling is obtained by the change \( T \rightarrow T^\dagger \) and \( eV_b \rightarrow -eV_b \). For the tunneling across the emitter JJ the automatic work term \( \pm eV_b \) is set to zero and in Eq. (4.39) one uses the emitter quasiparticle-current \( I_{ss}(\hbar \omega/e) \) (instead of the base current).
4.3. CURRENT-BIASED JOSEPHSON JUNCTION

Improvements for the Cooper-pair tunneling rates

The Cooper-pair tunneling across the emitter is treated exactly but across the base perturbatively. In Paper [11] we introduce the transition rates between the eigenstates due to the base tunneling as a Lorentzian function (4.7). This is a sufficient treatment for small $E_{J2}$. For a qualitative description of moderate $E_{J2}$ we should improve this approximation by two ways. First, the probability that the environment emits (rather than absorbs) the energy $E$ during the Cooper-pair tunneling is damped by $\exp(-E/k_B T)$ [16]. Such an effect brings changes to the supercurrent region, i.e., for processes where charge is transported across the system with no change in the state of the island. The “supercurrent” occurs here via incoherent Cooper-pair tunneling and leads to the phase-diffusion branch in the low-voltage part of the $I-V$ characteristics [53]. The second improvement we can make is to take into account the possible nondiagonality of the island’s density matrix. In analogy to what is done in the DMA in Paper [11] we can model also transitions between arbitrary nondiagonal elements of the density matrix. As the related damping function (describing the effect of the base’s environment) we use the transition rate function (4.7) and consider only diagonal states in the tunneled charge $Q_2$. Further, Lindblad approximation of the perturbation (by redefining some of the nondiagonal transition rates) can be taken for large values of $E_{J2}$ in order to ensure the positivity of the density matrix, see Paper [11]. In this case the current across the base (in the second-order approximation) is a sum of the contributions

$$I = 2e \sum_{n_i, n'_i, n_k} \text{Re} \left[ \rho_{n_i n'_i}^{\text{ss}}(k) \Gamma_{n_i n'_i \rightarrow n_k} \left( k \right) \right], \quad (4.40)$$

where $\rho^{\text{ss}}(k)$ is the steady-state density-matrix and $\Gamma$ the generalized transition-rate tensor due to Cooper-pair tunneling across the base JJ. Positive direction terms are summed with positive signs and the negative direction terms with negative signs. The quasiparticle current across the base JJ is calculated similarly. The average current from the collector to the island is deduced to be [71, 72]

$$I_c = I + \langle \frac{Q_1 \text{capacitor}}{C_1} \rangle = \frac{V}{R} - \frac{\langle Q/C_{\Sigma} \rangle}{R}, \quad (4.41)$$

and the emitter current follows from the relation $I_E = I_R + I_C$.

4.3.3 Model for the Bloch-oscillating transistor

The model presented in Paper [11] with the supplements given in Section 4.3.2 can be used for modeling the steady-state properties of the BOT. The BOT is designed to work as an amplifier for the base current and therefore in
Figure 4.10: The BOT collector current with $I_b = 0$ for two values of $E_{J2}$ as a function of the collector voltage $V$ swept from the left to the right. The curves from the bottom to the top correspond to the values $E_{J1} = 10 \rightarrow 100 \ \mu$eV with the steps of $10 \ \mu$eV, respectively. For small $E_{J2}$ (left figure) the characteristics evolve from the $P(E)$-theory type centering nearby the point $eV = 2E_C$ to the regime of linear behaviour $I = V/R_C$. These correspond to incoherent Cooper-pair tunneling across a voltage-biased small JJ in series with a resistor $R \gg R_Q$ [16] and the Bloch oscillations in the lowest energy band, respectively. In the case of a moderate $E_{J2}$ (right figure) the supercurrent between the base and emitter starts to contribute and in some cases makes the characteristics even discontinuous and hysteretic. The effect of the supercurrent seems to be similar with the case $I_b \neq 0$ when the base is an NIS junction [73]. The parameters are $T = 100 \ \text{mK}$, $R = 180 \ \text{k}\Omega$, $\Delta_b = 40 \ \mu$eV, $\Delta = 200 \ \mu$eV, $R_{T1} = 12.8 \ \text{k}\Omega$ and $E_C = 100 \ \mu$eV. We use the A-B relation [24] when determining the normal state resistance of the base $R_{T2}$, whereas the choice for $R_{T1}$ corresponds to $E_{J1} = 50 \ \mu$eV. The incoherent Cooper-pair tunneling across the base is treated as in Section 4.3.2.

continuous variation of the base voltage. However, the base lead practically provides quite a stable voltage-bias due to its large capacitance to ground. Therefore the base voltage can be assumed to be a constant during the single-charging processes.

In Fig. 4.10 we show results for two BOT simulations with the base current $I_b = 0$. For a realistic model of the system we have first sought the voltage $V_b$ that matches to the incoming current $I_b$ and is stable under small fluctuations. The value is searched in a way that if the initial value chosen produces too high/low current into the base, the voltage provided by the leads (coupled capacitively to the ground) is increased/decreased slightly. In a stable point small fluctuations of voltage are compensated by an opposite direction current, i. e., the region with “negative differential resistance” is not allowed. As compared to the $P(E)$-theory simulations [73, 74], the model provides a smooth transition from the “$P(E)$-peak” behaviour at low $E_{J1}/E_C$
to almost linear behaviour at large $E_{J1}/E_C$. We also observe hysteresis, even for $I_b = 0$, due to the presence of the supercurrent. Both the observations are in qualitative agreement with recent experimental studies for the BOT with SIS junction at the base [75]. More detailed investigation is going to be published elsewhere.
Chapter 5

Summary

This thesis presents a theoretical study of charge transport across small transistor-type Josephson devices. The simulations rely on the quantum-mechanical treatment of the Cooper-pair tunneling through mesoscopic JJs perturbed by dissipative environmental processes. The study is motivated by several experimental works whose previous understanding was only qualitative, and generally by the basic research of the macroscopic quantum physics in small JJs.

In Paper I we analyze how incoherent Cooper-pair tunneling across a small JJ probes the energy levels of one or more large JJs. By using the developed model we show that a detailed understanding of recent experimental data can be achieved. In particular, the multiphoton processes between different mesoscopic elements of the system and external resonators can be identified.

In Paper II we study a similar system as in Paper I (a voltage-biased asymmetric SCPT) but now presuming that the Cooper-pair tunneling across the small JJ is mainly coherent. The approach predicts new resonances in the circuit, corresponding to higher-order effects in the Cooper-pair tunneling across the small JJ. However, the results are not very consistent with experiments, where most of such effects seem to be missing. Similar discrepancy exists also in the more studied case of the symmetric SCPT.

In Paper III we study this discrepancy by more accurate modeling of the interplay between the coherent Cooper-pair tunneling and incoherent environmental processes, by using the density-matrix approach. We show that most of the higher-order resonances tend to be washed out by the quantum or thermal noise caused by the environment. This explains why only a few of them have been detected in the experiments. We also analyze the effect of three typical dissipative environments present in mesoscopic superconducting devices, and how they manifest themselves in the $I - V$ characteristics.

In Paper IV we study the dynamics of a current-biased JJ probed by a small JJ. The circuit differs from the SCPT by a “dynamical quasicharge”,...
driven by the resistor that replaces the gate capacitor. We consider how the current through the small JJ reveals information, for example, of the Bloch oscillations and Zener tunneling in the energy bands. With the supplements given in Section 4.3, the model covers also the operation of the Bloch-oscillating transistor.

In summary, our analysis has increased quantitative understanding of the studied systems and has made a good groundwork for a later research. For example, the charge transport across SCPTs in more general environments, e.g., photon cavities and nanomechanical oscillators, are of great interest. Also theoretical analysis of the Bloch-oscillating transistor’s operation in the high $E_J/E_C$-limit can be now done. The purpose of the thesis has not been the study of direct applicability of the phenomena, but rather the better understanding of the physics around them. The field will undoubtedly produce novel electronics in the future, but at which magnitude, will be determined by the later work in the field. In any case the quantum effects in small JJs offer a great working field for the basic research of the macroscopic quantum physics.
Bibliography


[75] L. Korhonen (private communication).