Numerical Calculations of Quasiparticle Dynamics in a Fermi Liquid

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Abstract

The problem of describing a system of many interacting particles is one of the most fundamental questions in physics. One of the central theories used in condensed matter physics to address the problem is the Fermi liquid theory developed by L. D. Landau in the 1956. The theory describes interacting fermions, and can be used to explain transport phenomena of electrons in metals and dynamics of helium three. Even when the theory is not directly applicable, it forms a basis against which other, more sophisticated theories can be compared.

In this thesis the Fermi liquid theory is applied to $^3$He-$^4$He-mixtures at temperatures where the bosonic $^4$He part is superfluid, and the mechanical properties of the system are largely determined by the $^3$He component, treated as a degenerate normal Fermi liquid. The dynamics of strongly interacting liquid $^3$He can be described as a collection of quasiparticles, elementary excitations of the system, which interact only weakly. In $^3$He-$^4$He mixtures the interactions can be continuously tuned by changing the temperature and the concentration of the mixture. The scattering time of quasiparticles depends on temperature, and thus the transition from the hydrodynamic limit of continuous collisions at higher temperatures to the collisionless ballistic limit at low temperatures can be studied. This gives invaluable information on the role of the interactions in the dynamics of the system.

In this work, by using the Fermi liquid theory and Boltzmann transport equation, the dynamics of helium mixture disturbed by a mechanical oscillator is described in the full temperature range. The solution necessarily is numeric, but new analytical results in the low temperature limit are obtained as well. The numerical approach enables one to study various boundary conditions thoroughly, and allows application of the theory to a specific geometry. It is shown that in order to explain the experimental observations, it is necessary to take into account the reflection of quasiparticles from the walls of the container. For suitable choice of oscillator frequency and container size, second sound resonances are observed at higher temperatures, while in the ballistic limit quasiparticle interference can be seen.

The numerical results are in quantitative agreement with experiments, thus attesting the accuracy of Fermi liquid theory. In particular, the previously observed decrease of inertia of a mechanical oscillator immersed in helium at low temperatures is reproduced in the calculations, and is explained by elasticity of the fluid due to Fermi liquid interactions.

Key words: Fermi liquid theory, Boltzmann Equation, $^3$He-$^4$He mixtures, Quantum fluids, Vibrating wire
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Oulu, March 2011       Timo Virtanen
LIST OF ORIGINAL PAPERS

The present thesis contains an introductory part and the following papers which will be referred in the text by their Roman numbers.


The author has had a central role in articles I, II, III, and V, and a minor role in the basic theory article, paper IV. He has developed the numerical machinery and done all the numerical simulations, developed the theory around the phenomena, written the initial drafts of articles I, II, and V, and contributed to writing of all the papers.
The figures in this thesis are available in color in the PDF version.

List of the most frequently used variables in the thesis is given below.

**General Terms**

- **T**: Temperature.
- **$k_B$**: Boltzmann constant, $k_B = 1.380 \times 10^{-23} \text{ J/K.}$
- **$\beta$**: $\beta = k_B T$, also a slip parameter, also angle between $\hat{p}$ and $\hat{x}$.
- **$\hbar$**: Planck’s constant, $\hbar = 1.0545 \times 10^{-34} \text{ Js.}$
- **$N_A$**: Avogadro’s constant, $N_A = 6.022 \times 10^{23}$.
- **$t$**: Time.
- **$\mathbf{r}$**: Position vector, in $xy$-plane $\mathbf{r} = r(\cos \theta \hat{x} + \sin \theta \hat{y})$.
- **$\hat{p}$**: Momentum direction, $\hat{p} = \sin \zeta (\cos \beta \hat{x} + \sin \beta \hat{y}) + \cos \zeta \hat{z}$.

**Fermi Liquid Theory**

- **$n_p$**: Distribution function (p. 8).
- **$\epsilon_p$**: Energy of a (quasi)particle (p. 8).
- **$\delta \epsilon_p$**: Energy correction term (p. 13).
- **$\psi \hat{p}$**: Transformed quasiparticle distribution (p. 14).
- **$\delta \bar{\epsilon}_p$**: Transformed energy correction term (p. 14).
- **$\mu, \mu_4$**: Chemical potential (of $^4\text{He}$) (p. 13).
- **$d\tau$**: $d\tau = 2d^3p/(2\pi \hbar)^3$ (p. 9).
- **$2N(0)$**: $2N(0) = m^* p_F/\pi^2 \hbar^3$.

- **$P_l(\hat{p} \cdot \hat{p}')$**: Legendre polynomial (p. 10).
- **$f(\hat{p} \cdot \hat{p}')$**: Interaction term (p. 10).
- **$F(\hat{p} \cdot \hat{p}')$**: $F(\hat{p} \cdot \hat{p}') = 2N(0) f(\hat{p} \cdot \hat{p}')$ (p. 10).
- **$F_1, F_0, F_1, F_2$**: Landau parameters (superscript $s$ is dropped) (p. 13).

- **$D$**: $D = 1 - m_3(1 + F_1/3)/m^*$, (also a constant) (p. 13).
- **$K$**: $K = m^*D/[(m_4(1 + F_1/3)] + 1 + \alpha$ (p. 14).

- **$T_F$**: Fermi temperature (p. 8).
- **$\epsilon_F$**: Fermi energy (p. 8).
- **$p_F$**: Fermi momentum (p. 8).
- **$v_F$**: Fermi velocity, $v_F = p_F/m^*$ (p. 10).
Numerical Averages
\[ c = \langle \psi_p \rangle_p \] Average of \( \psi_p \) (p. 17), also velocity of sound (p. 39).
\[ b = \langle \hat{p} \psi_p \rangle_p \] Weighted average of \( \psi_p \) (p. 17).
\[ d = \langle \hat{p} \hat{p} \psi_p \rangle_p \] Weighted average of \( \psi_p \) (p. 21).

Hydrodynamics
\( p \) Pressure.
\( \rho \) Density.
\( \eta \) Viscosity, \( \eta = p_F n_3 \ell / 5 \) for Fermi liquid.
\( \zeta \) Second viscosity, also slip length, also angle between \( \hat{p} \) and \( \hat{z} \).
\( \delta \) Viscous penetration depth, \( \delta = \sqrt{2 \eta / \rho \omega} \).
\( q = (1 + i) / \delta \) Viscosity parameter.
\( v \) Velocity (of fluid).
\( \chi \) Velocity potential, \( v = \nabla \times \chi \hat{z} \).
\( J \) Momentum flux.
\( \hat{\sigma} \) Stress tensor, \( \hat{\sigma} = -p \ 1 + \hat{\sigma}' \).
\( \hat{\sigma}' \) Viscous stress tensor.
\( \Pi \) Momentum flux density.
\( F \) Force exerted by the fluid on the wire.

Collision Term
\( \ell \) Mean free path, \( \ell = v_F \tau \).
\( \tau \) Mean free time.
\( I_p \) Collision term.

Helium Mixtures
\( N, N_3, N_4 \) Number of \( ^3\text{He}, ^4\text{He} \) particles.
\( n, n_3, n_4 \) Number density (of \( ^3\text{He}, ^4\text{He} \)), \( n_3 = p_F^3 / 2\pi^2 \hbar^3 \) (p. 8).
\( \alpha \) BBP parameter (p. 13, 71).
\( \alpha_1 \) \( \alpha_1 = (1 + \alpha)m_4 / m^* \) (p. 84).
\( x_3 \) Fraction of \( ^3\text{He} \) in the mixture.
\( s \) Sound velocity in pure \( ^4\text{He} \), also trajectory parameter.
\( u_2 \) Velocity of second sound (p. 84).
\( c_2 \) Reduced second sound velocity (p. 85).

Effective Mass
\( m_3 (m_4) \) Mass of \( ^3\text{He} (^4\text{He}) \) atom.
\( m^* \) Effective mass of a \( ^3\text{He} \) quasiparticle.
\( m_H, m_i \) Hydrodynamic mass, inertial mass (p. 73).
\( m_0^* \) Effective mass at \( x_3 \to 0 \) (p. 73).
Boundary Conditions

\( S \)  
Fraction of specular scattering.

\( g_w \)  
Boundary condition term, \( g_w = -\hat{n}_w \cdot \langle \hat{p}\psi\hat{p}(r_w) \rangle \hat{p} \) (p. 17).

\( g_c \)  
Boundary condition term, \( g_c = -\hat{n}_c \cdot \langle \hat{p}\psi\hat{p}(r_c) \rangle \hat{p} \) (p. 20).

Vibrating Wire

\( a \)  
Wire radius, typical value \( a = 62.5 \mu m \).

\( b \)  
Container radius.

\( 2h \)  
Width of slab container.

\( \rho_w \)  
Density of the wire, \( \rho_w = 16700 \text{kg/m}^3 \) for tantalum.

\( u = u_0 e^{-i\omega t} \hat{x} \)  
Velocity of the wire.

Harmonic Oscillator

\( \kappa \)  
Spring constant (of the wire).

\( f \)  
Resonant frequency.

\( f_{\text{vac}} \)  
Vacuum frequency.

\( \omega = 2\pi f \)  
Angular frequency.

\( \Delta f \)  
Frequency width.

\( \Omega = a\omega/v_F \)  
Dimensionless frequency parameter.

\( k, k' \)  
Stokes’ coefficients, also \( k = \omega/c \), also \( k = 1/\ell - i\omega/v_F \) (p. 28, 36).

\( Z = Z' + iZ'' \)  
Mechanical impedance (p. 23).

\( Z' \)  
Reduced impedance (divided by \( an_3p_F \)) (p. 75).

\( G = G' + iG'' \)  
Dynamic modulus, also geometry factor (p. 28).

Functions

\( H_n^{(1)}(z), H_n^{(2)}(z) \)  
Hankel functions of first and second type.

\( J_n(z), Y_n(z) \)  
Bessel functions of first and second type.

Notation: vectors are written in bold, \( \mathbf{a} \), unit vectors are identified by a hat, \( \hat{a} \), (second order) tensors are written as \( \leftrightarrow a \), the unit (identity) tensor is denoted by \( \leftrightarrow 1 \), and time derivatives are indicated by a point, \( \dot{a} \). Components of vectors and tensors are identified by subscripts, \( a_r, a_{xy} \). The notation \( \langle \ldots \rangle \hat{p} \) means average over unit sphere in momentum space. An equilibrium value is usually denoted by \( a^{(0)} \), and deviation from equilibrium by \( \delta a = a - a^{(0)} \). Subscripts 3 and 4 correspond to specific values for \( ^3\text{He} \) and \( ^4\text{He} \), respectively, while subscripts \( s \) and \( n \) refer to superfluid and normal components, and subscripts \( w \) and \( c \) refer to the wire and container, respectively.
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Chapter 1

Introduction

One of the basic problems in physics is the many-body problem; how to describe a system of several interacting particles. On one end, the problem can be treated statistically. For example, an ordinary fluid, consisting of a vast number of constantly interacting particles, can be remarkably well described by relatively simple hydrodynamic theory, based on the statistical averages of certain physical variables. On the other end of the scale, we can use Schrödinger equation to describe the physics of two or more interacting particles. This approach is limited by the exponentially increasing heaviness of the calculations, when the number of particles is increased.

Between these two ends, a number of approaches has been utilized. We will consider what is often referred to as the paradigm of many-body theories, namely Landau’s Fermi liquid theory [1]. Conventional statistical physics elegantly describes the behavior of a collection of non-interacting fermions, who obediently follow the Pauli exclusion principle of quantum mechanical nature, as an ideal Fermi gas. Landau’s great insight was that the essential properties of ideal Fermi gas are not significantly changed at low excitation energies, when the interactions are ”turned on”. By introducing the concept of quasiparticles, many properties of interacting fermion systems can be described, for example in metals and with some modifications in superconductors. The Fermi liquid theory applies particularly well to liquid $^3\text{He}$, in pure form as well as in $^3\text{He}^-^4\text{He}$ mixtures. These mixtures are the topic of this thesis.

The helium mixtures bring us to another theme addressed in the thesis, the transition from the kinetic theory to the hydrodynamic one. In the Fermi liquid theory of helium mixtures, the time scale of interactions is controlled by temperature. At higher temperatures the collision between quasiparticles occur continuously, while with decreasing temperature they become less frequent, and vanish when the absolute zero of temperature is approached. The mean free time between the collision is a measure of the nature of the system. If collisions occur continuously, they push the system into local equilibrium.
The system can then be described in terms of few averaged quantities, such as local fluid velocity, pressure, and viscosity, which are sufficient to represent the dynamics of the vast number of particles in a small volume of space, a fluid element. The theory of these averaged quantities is known as hydrodynamics. The hydrodynamic theory, despite its apparent simplicity, has beautifully explained various fluid phenomena in the macroscopic world.

When the mean free path is increased and there is no constant interaction between the particles, the system can be described in terms of the kinetic theory. It concerns non-equilibrium phenomena in dilute gases [2], i.e. in systems where the range of effective interactions between the gas particles is much less than the average distances between the particles, and the interactions between the particles can be considered to be due to instantaneous elastic collisions [3]. The theory gives a statistical description of the system, by providing average probabilities for finding a particle in a given phase space element at certain time. In the limit of infinite mean free path the collisions disappear, and the system behaves like a gas of ballistic particles, flying in straight trajectories and ignoring each other.

In helium mixtures at low temperatures we can now make a continuous transition from collisionless ballistic gas to a hydrodynamic fluid of continuous interactions, by using the mean free time, or corresponding mean free path, as a parameter. The uniqueness of $^3\text{He}-^4\text{He}$ mixtures at low temperatures as an experimental tool in many-body physics arises from the fact that the full scale of interactions can be probed by a single measuring device, just by changing temperature. In the same spirit, this thesis presents a single numerical tool that portrays the fluid with an interaction parameter $\ell$, the mean free path of particles, that can be varied across over 7 orders of magnitude. (In fact, with minor modifications we can reach the limit of infinite $\ell$.)

A word of explanation is needed for the experimental device mentioned above. The dance of quasiparticles is hidden from the naked eye, and a probe is needed to study their motion. The conditions for the experiment are very limiting: the accessible temperatures must be close to the lowest ever reached, and high sensitivity is required. A small, superconducting vibrating wire is a practical tool for the problem. The superconductivity ensures that the helium sample is not heated by the measuring device. Given that the volume of helium samples that can be cooled down to the necessary temperatures is very limited, the small size of the device is essential. Since the main interest lies in the bulk properties of the fluid, the measuring device that dictates the length scales of the measurable disturbances must be small enough compared to the whole volume. Otherwise, the surface effects may become dominant and hinder the interpretation of observations. It turns out that the surface effects, the behavior of the fluid near the wire and near the walls of sample container, still have a crucial role in understanding the experiments. This
is reasonable, since in the low temperature limit the mean free path of the quasiparticles clearly exceeds the length scales of the container.

Although the surface phenomena may be a nuisance in studying the bulk properties, they present a field of dynamics interesting in itself. In our case, the issue is the scattering process of quasiparticles from the wire and the container walls. The nature of scattering affects the fluid dynamics most distinctly in a layer close to the wire, and is therefore directly seen in the force on the wire, the main observable quantity. We will consider different types of scattering from the wire, and from the walls of helium container.

Luckily for the theorist, the measuring device is simple to model. The wire can be modeled as a harmonic oscillator with negligible oscillation amplitude. The geometry can be approximated by an infinite straight cylinder oscillating inside another, co-axial cylinder, to a first approximation. The different length scales, however, make the numerical problem challenging. On one end, the distribution of quasiparticles changes rapidly in a narrow transition layer near the wire, requiring extremely dense grid in the calculations. On the other hand, the grid must extend over the whole volume of the helium container, and the quasiparticle trajectories for even the longest mean free paths, $\ell \sim 10^4$ times the wire radius and beyond, need to be tracked.

Another feature that brought an interesting challenge to the numerical work is the central role of the confinement. In order to explain the observations, not only the size of the helium container must be taken into account, but also the specific shape. The slab geometry, where the wire oscillates perpendicularly between two plane walls, breaks the cylindrical symmetry, and raises the problem of fitting rectangular and cylindrical geometries together. As our initial approach was to build up the numerical machinery from scratch, and as we wanted to keep the calculations light enough to be computable on a (small cluster of) table top computer(s), this turned up to be somewhat tedious.

In this thesis we have reviewed and reanalyzed the Fermi liquid theory of helium mixtures, as described in Papers III and IV. The theory has been applied to vibrating wire resonator experiments, analytically in the ballistic limit, Papers I and II, and numerically for finite mean free path, Paper V. We have successfully reproduced the experimental results, and explained some new features. We have analyzed the role of various physical parameters for ranges covered in experiments and beyond. We have extensively studied the role of confinement of the helium sample, in terms of size and shape of the container, as well as in terms the boundary conditions.

The parameters appearing in the Fermi liquid theory, the effective mass and the interaction terms, have been a subject of special interest. Both their numerical values at various $^3$He concentrations and their general role in the dynamics of the mixtures have been examined. One of the most interesting conclusions in this work is related to the observed increase of resonant fre-
quency of the wire at lowest temperatures above the high temperature value. Part of this can be understood as decoupling of the normal component of the fluid from the wire’s motion in the ballistic limit. The observed shift cannot be explained by this alone, and it seems that the surrounding medium actually reduces the wire’s effective inertia at low temperatures, instead of increasing it!

We explain how the Fermi interactions lead to a new force, named *Landau force* in Paper III, which explains the increased frequency. The Fermi liquid can then be understood as an elastic medium, which increases the oscillation frequency. This may sound surprising, since elasticity is usually associated with solids, and while it is present in some viscoelastic liquids made of polymers, it is not expected to occur in a gas of ballistic particles.

The thesis is organized as follows. After this chapter, an introduction to the Fermi liquid theory is given in chapter 2. Then the limiting cases of hydrodynamic and ballistic limits are considered in chapter 3, with some notes on the experimental setup. After that, the numerical method is described in chapter 4, and some of the results in the Papers are reviewed and extended. A short summary is drawn in chapter 6.
Chapter 2

Fermi Liquid Theory

A principal problem in physics, especially in condensed matter physics, is to describe how a system of interacting particles behave. When the number of particles $N$ exceeds two, the problem becomes overwhelmingly difficult, and in condensed matter we often have $N \sim 10^{23}$, with hundreds of particles directly interacting with each other. It is clear that a brute force approach starting from Schrödinger equation is out of the question for any practical purpose, but there are approximations successfully explaining some limiting cases. Here, we consider the Fermi liquid theory.

Landau developed the Fermi liquid theory in 1956 originally for pure $^3$He\cite{1}. The Fermi liquid theory can be used to describe the low temperature properties of any interacting fermion system. It has been applied to the conduction electrons in metals, and its generalization can be used to describe the superconducting state in many metals. Fermi liquid theory is often considered a paradigm of many-body theorems, and $^3$He a paradigm of Fermi liquids. In this thesis we will concentrate on helium three.

The key point of the phenomenological theory is that even a strongly interacting system of fermions can be described in a manner similar to the ideal Fermi gas. At low temperatures the scattering process between $^3$He atoms is strongly constricted by the Fermi statistics: only particles close to the Fermi surface can be excited to different states via collisions. The scattering probability is proportional to the square of temperature, and at low temperature the system acts in many ways similar to the ideal non-interacting Fermi gas. Starting from the non-interacting Fermi gas and 'turning the interactions on' continuously gives one-to-one correspondence between the original $^3$He atoms and the new low energy excitations of the system, quasiparticles. These elementary excitations can be treated in many ways as particles, but instead of individual atoms, they consist of correlated motion of many particles.

In the following, after a quick general note on helium, we will briefly consider the case of ideal Fermi gas, to introduce the basic concepts. Then, we will consider interacting (normal) Fermi liquid, such as pure $^3$He. Finally,
we will study the case of Fermi-Bose mixtures, in particular mixtures of $^3$He in $^4$He. Detailed description of the Fermi liquid theory can be found in several text books, Pines & Nozières [4], Nozières [5], Baym & Pethick [6], Landau & Lifschitz [7].

2.1 Helium

Helium is the second lightest and the second most abundant element in the universe after hydrogen. The helium atom is smaller than any other element [8], and thus the attractive van der Waals forces between adjacent helium atoms are weaker than in any other element. The boiling point of helium (4.2 K) is therefore lowest of all elements, and the freezing point can be expected to be very low. In fact, the two stable isotopes of helium, $^3$He and $^4$He, are unique in that they remain liquid when temperature is reduced down to absolute zero at normal pressure.

The transition from liquid to solid state usually occurs when the attractive forces, tending to organize the atoms into an orderly lattice structure, overcomes the random thermal motion. In helium the attractive forces are extremely small, as is the thermal energy at low temperatures. Then the effect of quantum mechanical zero-point motion needs to be taken into account. The effect of zero point motion is to push the atoms apart, therefore resisting solidification. Because of their small mass, the zero point motion is strong for helium atoms, in such magnitude that the helium liquid fails to solidify even at the lowest temperatures. In this way quantum mechanical effects are visible at macroscopical scale at low temperatures in helium.

The failure of helium to solidify is not the only feature of quantum mechanical origin at low temperatures. For example, below a critical temperature helium can flow through narrow capillaries with no apparent viscosity, and the heat conductivity becomes so high that no temperature gradients can exist in the bulk fluid. These temperatures are 2.17 K for $^4$He and 2.7 mK for $^3$He (the actual values depend on pressure). Below these temperatures the isotopes are called superfluids, the physical properties of which differ greatly from those of ordinary fluids. On historical note, helium was first liquified by Kamerlingh Onnes in 1908, and first solidified by his student Keesom in 1926. The superfluidity of $^4$He was discovered in 1938 [9, 10], and that of $^3$He in 1972 [11, 12].

The $^3$He nucleus consists of two protons and a neutron, while that of $^4$He posses one more neutron. This is the fundamental reason for the different behavior of the chemically identical isotopes at low temperatures, since the former follows Fermi statistics and the latter Bose statistics. The mechanisms leading to superfluidity, for example, are different, but that is beyond the scope of this thesis.
2.1.1 $^3$He-$^4$He Mixtures

In the liquid phase the two isotopes $^3$He and $^4$He are miscible in all proportions at temperatures above 0.87 K, but at lower temperatures a phase separation takes place [13, 14]. As temperature decreases, the liquid separates into a $^4$He-rich phase with up to 6.5% $^3$He dissolved in it (at low temperatures and pressures), and to a $^3$He-rich phase with a rapidly decreasing portion of $^4$He. At higher pressures the maximum $^3$He concentration in $^4$He can be up to 9.5%. The $^3$He proportion $N_3/(N_3 + N_4)$, where $N_3$ and $N_4$ are the numbers of $^3$He and $^4$He atoms, respectively, is denoted by $x_3$.

As mentioned, pure $^4$He goes through superfluid transition at 2.17 K, but the transition temperature is reduced with increasing $x_3$. At temperatures below a concentration dependent value $T_F$, the Fermi temperature, the $^3$He part is said to be degenerate, i.e. it follows Fermi statistics with small deviation from the step function like behavior. The scope of this thesis covers mixtures of $^3$He in $^4$He at temperatures so low that the $^4$He part is clearly superfluid, and the $^3$He Fermi liquid is degenerate.

The mixtures of helium offer an interesting environment, where the validity of Fermi liquid theory can be studied. In a $^3$He-$^4$He solution the effective interactions between the $^3$He quasiparticles are weaker than in pure $^3$He, and thus the degenerate Fermi liquid picture is more accurate. In addition, the Fermi liquid parameters can be varied continuously by changing the $^3$He density of the solution.

The superfluid transition of the $^3$He part has been predicted theoretically, but has not been found experimentally so far. The transition is expected to take place around $\mu$K temperatures, see for example references [15] and [16]. For our purposes this is a huge advance, since by changing the temperature, the collision rate of the normal (not superfluid) Fermi liquid can be tuned. The experiments can span from the collisionless low temperature region to the hydrodynamic region of continuous collisions at higher temperatures. If we consider only the interactions via collisions, this allows us to describe a system of particles with continuously tunable interactions, thus giving valuable information on the many body problem.

2.2 Ideal Fermi Gas

The theory for ideal Fermi gas describes statistically the behavior of a collection of non-interacting fermions. Fermions have half-integral spin, and their wave function has to be anti-symmetric in exchange of any pairs of arguments. This leads to what is known as the Pauli exclusion principle: no two fermions can be found in the same quantum mechanical state. Without interactions, the energy of a single particle state is simply $\epsilon_p = p^2/2m$. The ground state of the system with $N$ particles is that all the states up to the
Fermi energy $\epsilon_F = \frac{p_F^2}{2m}$ are occupied, and the rest are empty. Here, $p_F$ is called the Fermi momentum, and it defines the Fermi surface in the momentum space: all states inside the Fermi surface ($p < p_F$) are full, and all the states outside are empty. An excited states consists of one or more of the fermions lying in higher energy states, leaving one or more empty states inside the Fermi surface. The fermions obey the Fermi-Dirac distribution

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1},$$

which gives the average occupation of a state with energy $\epsilon$ at temperature $T$. Here $\beta = 1/k_B T$, $k_B = 1.38 \times 10^{-23}$ J/K is Boltzmann’s constant, and $\mu$ is the chemical potential, specified by the condition $\sum_p n(\epsilon_p) = N$. The relation between the number density and Fermi momentum is

$$n = \frac{N}{V} = \frac{p_F^3}{3\pi^2\hbar^3},$$

where $V$ is the volume of the system and $N$ is the total number of particles.

Consider an excited state consisting of filled Fermi sphere plus one particle of momentum $p$, $p > p_F$. This excited state of the ideal gas corresponds to the excitation energy

$$\epsilon_p - \epsilon_F = \frac{p^2}{2m} - \epsilon_F \approx \frac{p_F}{m} (p - p_F).$$

This linear approximation holds if $p$ is close to the Fermi surface, $p - p_F \ll p_F$. We will see that similar energy momentum dispersion relation is valid even in the interacting Fermi liquid.

Experimental observations on $^3$He suggest that at low temperatures only particles close to the Fermi surface can be excited. This is evident in the measurements of specific heat $C$: At high temperatures the Fermi gas behaves like classical gas, where the energy per particle is $3k_B T/2$ and the specific heat $3k_B/2$ per article. At lower temperatures ($T \ll T_F$) the specific heat is reduced and depends linearly on $T$. This suggests that only the particles with energies close to the Fermi surface can be exited. Particles with energies more than $k_B T$ below the Fermi surface are not excited, and have no contribution to the specific heat. For $^3$He, the ideal Fermi gas model correctly describes the linearity of $C(T)$ at low temperatures, yet the observed slope is a factor 2.7 larger.

## 2.3 Fermi Liquid

When interactions are taken into account and have a crucial role in a system of fermions, it is referred to as a Fermi liquid. In this section we will consider
the interactions in liquid $^3$He at low temperatures. The interactions will show up in three ways: in the effective mass $m^*$, in the Landau parameters $F_l$ in the quasiparticle energy, and in the collision term in the kinetic equation.

The non-interacting ideal gas can be described as point-like particles flying in straight trajectories, never colliding with each other. The $^3$He atoms, however, have a finite size, and touch each other continuously. Thus for one particle to move, the others must give way. To first approximation, this increases the effective mass of a single particle, since part of the surrounding fluid has to move with it. On the quantum mechanical point of view, the elementary excitations of an interacting system are no longer excitations of the single atoms. Instead, we consider the low-energy excitations of the whole system, which are called quasiparticles. The excitations act like particles in that they have definite momentum and energy, and move in the volume occupied by the liquid. The number of quasiparticles is the same as that of the original $^4$He atoms. They cannot, however, be identified with any single atom, but consist of correlated motion of the whole liquid.

The basic idea in Fermi liquid theory is that the main features of the ideal Fermi gas remain the same for the low energy excitations of the interacting fermion system, when the role of the atoms is taken by quasiparticles. The low-energy part of the excitation spectrum remains qualitatively the same as in the non-interacting case. To be more specific, the momenta of the excitations are assumed to remain the same when the interactions are turned on, when the number of atoms is constant. The energies of the excitations can be shifted, but the equilibrium Fermi surface is assumed to remain unchanged. Also the quasiparticle energy is assumed to be linear in $p$ close to the Fermi surface, as in (2.3). Thus the excited states can still be specified by a quasiparticle distribution function $n_p$, and the ground state corresponds to the filled Fermi sphere. The Fermi momentum has not changed, equation (2.2) still gives the number density. In the following we will give the basic equations describing the interacting Fermi liquid.

In an ideal Fermi gas the total energy of the system is simply the sum of the quasiparticle energies (2.3), $E = \int \epsilon n_p \, d\tau$. For an interacting Fermi liquid this is no longer true, but $E$ depends on $n_p$ at all values of $p$, i.e. is a functional of $n_p$. The quasiparticle energy is defined by a functional derivative

$$\epsilon_p = \frac{\delta E}{\delta n_p}, \quad \delta E = \int \epsilon_p \delta n_p \, d\tau,$$

where $d\tau = 2d^3p/(2\pi\hbar)^3$, and $\delta E$ is the change of the total energy of the system, when the distribution is changed by a small amount $\delta n_p$. Note that we only consider spin-independent phenomena, and the spin only appears as a factor of 2 in $d\tau$. The equilibrium quasiparticle distribution is now

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}.$$
CHAPTER 2. FERMI LIQUID THEORY

This is of the same form as in the ideal Fermi gas, Eq. (2.1), but it must be noticed that $\epsilon$ depends on $n(\epsilon)$, so the equation is a complicated implicit expression. For low energy excitations the quasiparticle energy is linear in $p$ close to the Fermi surface

$$\epsilon_p - \epsilon_F \approx \frac{p_F}{m^*} (p - p_F).$$  

(2.6)

This resembles the case of ideal Fermi gas, but instead of the bare fermion mass $m$ we have a new quantity, the effective mass $m^*$. The Fermi velocity $v_F$ is now defined by $p_F = m^* v_F$. The effective mass is larger than the bare mass. As a first attempt to understand this, one can consider the excess mass as the hydrodynamic mass of a $^3$He atom moving in the background of other $^3$He atoms: as the atom moves, it drags along with it an amount of surrounding liquid, thereby increasing its inertial mass.

In more detail, let us consider a single $^3$He atom moving in stationary background of its fellow $^3$He atoms. In ideal Fermi gas it would behave like a point particle, moving in a straight trajectory without interacting with other atoms. But when the interactions are turned on, the moving particle has to push aside other atoms. If the original momentum of the particle is fixed, this means that part of the momentum is transferred by the surrounding liquid, and less is left for the original particle. The group velocity of quasiparticles from equation (2.6) is

$$v_{\text{group}} = \frac{dE_p}{dp} = \frac{p_F}{m^*} = v_F.$$  

(2.7)

The momentum carried by the original fermion is then $mv_F \hat{p} = (m/m^*)p$, and the rest, $(1 - m/m^*)p$ is carried by the surrounding liquid.

Now we must return to the quasiparticle energy expression (2.6). Landau discovered that Galilean invariance requires that interactions are added to this dispersion relation. We rewrite

$$\epsilon_p - \epsilon_F = \frac{p_F}{m^*} (p - p_F) + \frac{1}{V} \sum_{\sigma} \sum_{p} f(p, p') (n_p - n_p^{(0)}).$$  

(2.8)

Here, the function $f(p, p')$ describes the interaction energy between two quasiparticles having momenta $p$ and $p'$. We simplify $f(p, p')$ by limiting our consideration to isotropic liquids at energies close to the Fermi surface, so that $f$ depends only on the angle between $p$ and $p'$, i.e. $f(\hat{p} \cdot \hat{p}')$. We then write, using Legendre polynomials, $F(\hat{p} \cdot \hat{p}') = 2N(0) f(\hat{p} \cdot \hat{p}') = \sum_{l=0}^{\infty} F_l^p P_l(\hat{p} \cdot \hat{p}')$, where $N(0) = m^* p_F / 2\pi^2 \hbar^3$ is the density of states. Now the Galilean invariance gives

$$\frac{m^*}{m} = 1 + \frac{F_1}{3}.$$  

(2.9)
It is necessary to include the term $F_1^s$ to the quasiparticle energy in order to satisfy Galilean invariance. The other terms $F_l^s$ are kept for generality.

So far we have considered two consequences of the interactions between fermions: the effective mass $m^*$ and the weak interactions between the quasiparticles, described by the Landau parameters $F_l^s$. There is one more interactions mechanism to be considered, which dictates the temperature dependence of the Fermi liquid, namely collisions between quasiparticles. One of the simplest ways to take the collision into account is to use a relaxation time approximation for the collision term in the kinetic equation. In this approximation the rate of collision is described by the mean free time $\tau$, or a related quantity, the mean free path $\ell = v_F \tau$. (Here we consider only the case of single relaxation time, for brevity.) An important aspect of the Fermi liquid theory in the following chapters is that the mean free path of the quasiparticles behaves as $\ell \propto T^{-2}$. Let us look into this in a little more detail.

Consider a filled Fermi sphere plus one particle with energy $\epsilon_1 > \epsilon_F$. When this particle hits a particle inside the Fermi sphere, $\epsilon_2 < \epsilon_F$, the final state has two particles outside the Fermi sphere, $\epsilon_1', \epsilon_2' > \epsilon_F$, since the Pauli principle forbids all states inside. The number of final states becomes very limited, when the initial particle is close to the Fermi energy, namely both $\epsilon_1'$ and $\epsilon_2$ have to be chosen in an energy shell of thickness $\propto \epsilon_1 - \epsilon_F$. Then the final states are limited by factor $(\epsilon_1 - \epsilon_F)^2$. Thus the scattering of low energy particles is suppressed, and the behavior of Fermi liquid at low temperatures resembles that of ideal Fermi gas. In other words, the Fermi liquid theory is valid only when the distribution function $n_p$ differs from the step-function behavior only in a narrow region near the Fermi surface [7]. The width of this transition region is of order $T$, and the collisions must happen within this region. Thus we arrive at the relation

$$\tau \propto T^{-2}. \quad (2.10)$$

### 2.4 Fermi Liquid Theory in Mixtures

The dilute mixtures of $^3$He in $^4$He are unique in that the fermion number density can be continuously varied by varying the $^3$He concentration $x_3$. We are considering temperatures where the $^4$He part is superfluid, and the $^3$He part is degenerate Fermi liquid. The two fluid components are connected, the motion of $^4$He affects the dynamics of the $^3$He part and vice versa. The Fermi liquid theory was generalized to the mixtures by Khalatnikov [17], and discussion can be found in Refs. [18, 6]. We review the theory with some modifications in Paper IV; in the following we will present the basic equations.
The case of $^3$He in $^4$He differs from the case of pure $^3$He Fermi liquid in that part of the momentum of a quasiparticle is carried not only by the other fermions, but by the $^4$He part as well. The momentum $\mathbf{p}$ of a single quasiparticle travelling with the group velocity $\mathbf{v} = \mathbf{p}/m^*$ can be written as

$$\mathbf{p} = \left( m_3 + \frac{m_3 F_1}{3} + m^* D \right) \mathbf{v}, \quad (2.11)$$

where only the fraction $m_3/m^*$ of the momentum is carried by the bare particle, fraction $m_3 F_1/3m^*$ is carried by other fermions, and the fraction $D$ by the superfluid $^4$He, when measured in the frame where $\mathbf{v}_s = 0$. The parameter $D$ will be defined shortly in Eq. (2.18). The corresponding particle currents are given by the momenta divided by the mass of the particle, $\mathbf{p}/m^*$, $F_1 \mathbf{p}/3m^*$, and $D \mathbf{p}/m_4$, where $m_4$ is the mass of a bare $^4$He atom.

### 2.4.1 Basic Equations

We consider mixtures of $^3$He in $^4$He, and denote the bare particles masses by $m_3$ and $m_4$, and the number densities by $n_3$ and $n_4$. Let us first consider the case with no interactions between the particles. For the $^4$He part, the bosons are assumed to be condensed to a state with velocity $\mathbf{v}_4$ and all excited states of the boson system are neglected. The total momentum density of the helium mixture $\mathbf{J} = \mathbf{J}_3 + \mathbf{J}_4$ is then given by

$$\mathbf{J} = m_4 n_4 \mathbf{v}_4 + \int \mathbf{p} n_p d\tau, \quad (2.12)$$

Let us now consider, how the situation changes, when the interactions are turned on. The basic assumption of Landau’s theory is that when the interactions are turned on, the momenta of the excitations remain the same. The energies of the excitations are shifted, but the Fermi surface is assumed to remain unchanged, and the quasiparticle energies are linear close to the Fermi surface. As a consequence, the equation (2.12) has the same form for an interacting state

$$\mathbf{J} = m_4 n_4 \mathbf{v}_s + \int \mathbf{p} n_p d\tau, \quad (2.13)$$

but instead of the velocity of bosons it defines the superfluid velocity $\mathbf{v}_s$. The difference is, that in the interacting case the first term is not the momentum density of $^4$He atoms and the latter is not that of the $^3$He atoms. Because of the interactions, part of the viscous, ”normal”, flow is mediated by the $^4$He part of the fluid, and only a part of $^4$He flows as expected from an inviscid ideal fluid.
The quasiparticle energy is defined as

\[ \epsilon_p = \frac{\delta E({\{n_p\}, \mu_4, v_s})}{\delta n_p}, \]

where we use the "osmotic energy" form, i.e. we use \( \mu_4 \) as a variable instead of \( n_4 \) (see paper IV for details). Similar to the case of pure \(^3\)He, Eq. (2.3), the quasiparticle energy is linear near the Fermi surface,

\[ \epsilon_p = v_F (p - p_F) + \delta \epsilon_p. \]  

The second term \( \delta \epsilon_p \) in (2.15) can be written as

\[ \delta \epsilon_p = (1 + \alpha) \delta \mu_4 + D p \cdot v_s + \int f(p, p_1)(n_{p_1} - n_{p_1}^{(0)})d\tau_1, \]

where \( \delta \mu_4 = \mu_4 - \mu_4^{(0)} \) is the deviation of the \(^4\)He chemical potential \( \mu_4 \) from its equilibrium value \( \mu_4^{(0)} \). The second term in (2.16) comes from the flow velocity \( v_s \) of the superfluid \(^4\)He. In the case of pure \(^3\)He, \( D = 0 \). The third term in Eq. (2.16) is the standard Fermi-liquid interaction term with the average over the unit sphere.

Following the discussion after Eq. (2.8), we can simplify equation (2.16) by assuming \( f(p, p_1) \approx f(\hat{p} \cdot \hat{p}_1) \), and using Legendre polynomials \( P_l(x) \)

\[ F(\hat{p} \cdot \hat{p}') \equiv 2N(0)f(\hat{p} \cdot \hat{p}') = \sum_{l=0}^{\infty} F_l P_l(\hat{p} \cdot \hat{p}'), \]

where \( N(0) \) is the density of states. Here the Landau parameters \( F_l \) are similar to those introduced in the previous section, but have different values in the mixtures. Here and in the following we drop the superscript \( s \) from the Landau parameters. Note that the definition of the parameters depends on our choice to use the osmotic energy. Khalatnikov[17] shows the difference between different definitions; only the first term \( F_0 \) changes. The present definition of \( F_1 \) is the same used by BBP [19] and Corruccini [20]. The parameters \( \alpha \) and \( D \) are constants not depending on \( p \). Parameter \( \alpha \) reduces to the BBP parameter used in Refs. [19, 6] in the limit of vanishing \(^3\)He concentration and \( D \) is related to \( m^* \), and \( F_1 \) by

\[ D = 1 - \frac{m_3}{m^*} \left( 1 + \frac{1}{3} F_1 \right), \]

as required by translational invariance.
Transformed Distribution

We see that $\delta\epsilon\hat{p}$ depends only on the momentum direction $\hat{p}$, not on the magnitude $p$, and we wish to define a new distribution function with the same property. In Paper IV we describe how a new field $\psi\hat{p}$ can be defined in a way that simplifies the equations; in particular it removes the coupling to the superfluid velocity. This field is used in the numerical calculations, and hence will frequently appear in the following chapters. Here we do not go into details of the derivation, but simply give the necessary equations expressed in terms of $\psi$. The new field is defined by

$$\psi\hat{p} = \int v_F dp (n_p - n_p^{(0)}) + \delta\epsilon\hat{p} + \left(\frac{m^*}{1 + F_1/3} - m_3\right) p_F \hat{p} \cdot \mathbf{v}_s, \quad (2.19)$$

where the term $\delta\epsilon\hat{p}$ is related to (2.15) by

$$\delta\epsilon\hat{p} = \delta\epsilon\hat{p} + \frac{Dm^*}{1 + F_1/3} \delta\mu_4 - \frac{Dp_F}{1 + F_1/3} \hat{p} \cdot \mathbf{v}_s. \quad (2.20)$$

This can be written as

$$\delta\epsilon\hat{p} = K \delta\mu_4 + \sum_{l=0}^{\infty} \frac{F_l}{1 + \frac{1}{2l+1}} \langle P_l (\hat{p} \cdot \hat{p}') \psi\hat{p}' \rangle \hat{p}', \quad (2.21)$$

where the coupling to the superfluid part $\delta\mu_4$ is parameterized by

$$K = \frac{m^* D}{m_4(1 + \frac{1}{3}F_1)} + 1 + \alpha. \quad (2.22)$$

It is useful to express densities and currents in terms of $\psi$. The change in fermion density is

$$\delta n_3 \equiv n_3 - n_3^{(0)} = \frac{2N(0)}{1 + F_0} \left(\langle \psi\hat{p}\rangle \hat{p} - K\delta\mu_4\right), \quad (2.23)$$

Here, the equilibrium number density $n_3^{(0)} = p_F^2/(3\pi^2\hbar^3)$. In the following we often drop the superscript (0); the theory is linear in small deviations such as $\delta n_3$, and it should be clear from the context what is meant by $n_3$. When the Fermi-Bose mixture is disturbed by a vibrating object, the quasiparticle number density $n_3$ deviates in space and time from its equilibrium value $n_3^{(0)}$. Naturally, when the number of $^3$He atoms is fixed, so is the number of quasiparticles, and on the average the number density is constant, $\int_V \delta n_3 dV = 0$. The changes in $^3$He density are related to the changes in the $^3$He density by

$$\delta n_4 \equiv n_4 - n_4^{(0)} = -(1 + \alpha)\delta n_3 + \frac{n_4}{m_4 s^2} \delta\mu_4. \quad (2.24)$$

In the limit $x_3 \to 0$ the parameter $s$ in (2.24) reduces to the velocity of sound.
2.4. FERMI LIQUID THEORY IN MIXTURES

In terms of $\psi$ the $^3$He mass current is

$$J_3 = \frac{3n_3m_3}{p_F} \langle \hat{p}\psi(\hat{p}) \rangle p, \quad (2.25)$$

and the total mass current (2.13) is

$$J = J_3 + J_4 = \rho_s v_s + \rho_n v_n, \quad (2.26)$$

where we have defined the normal fluid velocity $v_n = J_3/(m_3n_3)$. Although this velocity is by no means adequate to describes the dynamics of the quasiparticles, it is well defined in terms of the quasiparticle current $J_3$. Expression (2.26) also defines the densities of superfluid and normal components,

$$\rho_s = m_4n_4 - \frac{Dm^*n_3}{1 + F_1/3}, \quad \rho_n = \frac{m^*n_3}{1 + F_1/3}, \quad (2.27)$$

with the total density $\rho = \rho_n + \rho_s$. The momentum flux tensor is

$$\hat{\Pi} = P^{(0)} + 3n_3 \langle \hat{p}\hat{p}\psi(\hat{p}) \rangle p, \quad (2.28)$$

and the force on the wire can now be calculated from

$$\frac{dF}{da} = -\hat{n} \cdot \hat{\Pi} = -\frac{\rho_s}{m_4} \delta \mu_4 \hat{n} - 3n_3 \hat{n} \cdot \langle \hat{p}\hat{p}\psi(\hat{p}) \rangle p, \quad (2.29)$$

where we have neglected the equilibrium part, which should vanish in integration over the surface of the wire.

### 2.4.2 Kinetic Equation

In order to study the dynamics of quasiparticles we need an equation of motion, a kinetic equation. Kinetic theory describes statistically the distribution of particles in the phase space: the distribution function $n_p(r, t)$ gives the average number of particles in a phase-space element $d\mathbf{r}d\mathbf{p}/(2\pi\hbar)^3$. The kinetic equation is a continuity equation for the distribution function $n_p(r, t)$. It is obtained by equating the total time derivative $dn_p/dt$ with a collision term $I'_p \equiv (dn_p/dt)_{col}$. We get the Boltzmann equation

$$\frac{dn_p}{dt} = \frac{\partial n_p}{\partial t} + \nabla n_p \cdot \dot{r} + \frac{\partial n_p}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} = I'_p, \quad (2.30)$$

where $\dot{r}$ and $\dot{\mathbf{p}}$ are understood in terms of Hamilton equations $\dot{r} = \partial \epsilon_p / \partial \mathbf{p}$ and $\dot{\mathbf{p}} = -\nabla \epsilon_p$. The Landau-Boltzmann equation for the quasiparticle distribution then takes the form

$$\frac{dn_p}{dt} = \frac{\partial n_p}{\partial t} + \nabla n_p \cdot \frac{\partial \epsilon_p}{\partial \mathbf{p}} - \frac{\partial n_p}{\partial \mathbf{p}} \cdot \nabla \epsilon_p = I'_p. \quad (2.31)$$
The collision term takes into account the collisions between the particles, which can take particles away from the element of phase space under consideration, or vice versa. Assuming small deviation from equilibrium, we linearize the kinetic equation (2.31) and get

\[
\frac{\partial n_p}{\partial t} + v_F \hat{p} \cdot \nabla \left( n_p - \frac{\partial n_p^{(0)}}{\partial \epsilon_p} \delta \epsilon_p \right) = I_p'.
\]

(2.32)

In terms of \( \psi \) we can write the kinetic equation as

\[
\frac{\partial}{\partial t} [\psi \hat{p} - \delta \bar{\epsilon}_p] + v_F \hat{p} \cdot \nabla \psi \hat{p} = I \hat{p}.
\]

(2.33)

By analyzing this form, it can be seen that the contribution from the coupling term \( K \delta \mu_4 \) in (2.21) will be proportional to \( (a \omega/v_F)^2 \) (or \( ab(\omega/v_F)^2 \) where \( b \) describes confinement and can be large), and can thus be neglected in the limit \( a \omega/v_F \ll 1 \). The coupling is also weak for small concentration \( x_3 \), irrespective of frequency. In the following we will use the approximation

\[
\delta \bar{\epsilon}_p \approx \frac{F_0}{1 + F_0} \langle \psi' \rangle \hat{p}' + \frac{F_1}{1 + F_1/3} \hat{p} \cdot \langle \hat{p}' \psi' \rangle \hat{p}',
\]

(2.34)

and to simplify notation, we drop the bar and refer to the approximative form simply by \( \delta \epsilon_p \). The superfluid velocity \( v_s \) appearing in (2.16) is now decoupled from the normal part, and is assumed to follow ideal fluid behavior[18]: it is curl free \( \nabla \times v_s = 0 \), and obeys the ideal-fluid equations of motion,

\[
\frac{\partial v_s}{\partial t} + \frac{1}{m_4} \nabla \mu_4 = 0.
\]

(2.35)

**Relaxation Time Approximation**

We now have a linearized equation of motion (2.33) expressed in terms of \( \psi \), where the normal and superfluid components are decoupled. The task still remains to describe the collision term \( I \hat{p} \). For our purpose it is sufficient to use a relaxation time approximation

\[
I \hat{p} = -\frac{\psi \hat{p} - \psi^{le}}{\tau},
\]

(2.36)

where \( \tau \) is the relaxation time, i.e. the average time between collisions of the quasiparticles. The corresponding mean free path is \( \ell = \tau v_F \). We see that the collision term drives the distribution towards the local equilibrium: large deviation \( \psi - \psi^{le} \) corresponds to large collision term and thus to fast change of the distribution (towards the equilibrium). Similar reasoning explains the
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parameter $\tau$: a short relaxation time corresponds to large collision term, and thus to faster convergence to equilibrium, and vice versa.

The energy is conserved in collision between two quasiparticles. In addition, the collision term $I_\hat{p}$ must conserve quasiparticle number and momentum, i.e. $\langle I_\hat{p} \rangle_\hat{p} = 0$ and $\langle \hat{p} I_\hat{p} \rangle_\hat{p} = 0$. We can now express the local equilibrium distribution as

$$\psi_{\hat{p}}^{le} = \langle \psi_{\hat{p}'} \rangle_{\hat{p}'} + 3 \hat{p} \cdot \langle \hat{p}' \psi_{\hat{p}'} \rangle_{\hat{p}'} = c + \hat{p} \cdot b,$$

where we have defined the short hand notations

$$c = \langle \psi_{\hat{p}'} \rangle_{\hat{p}'} , \quad b = 3 \langle \hat{p}' \psi_{\hat{p}'} \rangle_{\hat{p}'} . \quad (2.37)$$

It is possible to introduce different relaxation times $\tau_l$ (with $l = 2, 3, \ldots$) corresponding to spherical harmonic decomposition of $\psi_{\hat{p}}$. We use a single relaxation time $\tau$, except in section 4.3, where we study the effect of a second relaxation time.

The mean free path $\ell$ is a central parameter throughout this thesis. Due to the nature of the Fermi distribution, it is inversely proportional to the square of temperature, as discussed in the previous sections. It describes interaction between quasiparticles via collisions, and can be used as a continuous parameter taking the Fermi system from the collisionless ballistic limit at low temperatures all the way to the hydrodynamic limit of continuously colliding particles at higher temperatures.

2.5 Application to Vibrating Wire

Our goal is to apply the theory to $^3$He-$^4$He mixtures disturbed by an infinite straight cylinder of radius $a$ moving slowly back and forth in the fluid. This cylinder is hereafter referred to as the wire. By slow movement we mean that the velocity $u$ of the wire is small compared to the Fermi velocity, $u \ll v_F$, and the oscillation frequency is low, $a \omega \ll v_F$. In the experiments that we will later consider, the amplitude of oscillations is of order nanometer, frequency is of order kilohertz, and the wire radius $\sim 10 \mu m$, so we expect $u \sim 10^{-5}$ m/s or less, and $a \omega \lesssim 0.1$ m/s, while $v_F \sim 20$ m/s. The aim is to calculate the mechanical impedance of the fluid, i.e. the response to the disturbance caused by the wire. For this end, we calculate the force exerted by the fluid on the wire, which allows comparison with experiments.

2.5.1 Solving the Kinetic Equation

First, we simplify Eq. (2.34) little further by neglecting $F_l$ for $l > 2$, to get a simple expression for the energy correction in terms of the averages (2.37),

$$\delta \epsilon = c \frac{F_0}{1 + F_0} + \frac{F_1}{3 + F_1} \hat{p} \cdot b. \quad (2.38)$$
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We want to solve the linearized Landau-Boltzmann equation with relaxation time approximation:

\[
\frac{\partial}{\partial t} [\hat{\psi}_p - \delta \epsilon \hat{p}] + v_F \hat{p} \cdot \nabla \psi_p = -\frac{\psi_p - \psi_p^{le}}{\tau}.
\]  

(2.39)

Integrating this, parameterized along a quasiparticle trajectory by 
\[ r(s) = r_0 + s \hat{p} \], and assuming time dependence \( \exp(-i\omega t) \) gives

\[
\left( \frac{1}{\ell} - \frac{i\omega}{v_F} \right) \psi_p(r(s)) + \frac{\partial \psi_p(r(s))}{\partial s} = \frac{\psi_p^{le}(r(s))}{\ell} - \frac{i\omega}{v_F} \delta \epsilon \hat{p}(r(s)).
\]  

(2.40)

Using integrating factor \( \exp(ks) \), where \( k = 1/\ell - i\omega/v_F \), and notation \( \psi(r(s)) = \psi(s) \), we get

\[
\psi_p(s) = \psi_p(s_0)e^{k(s_0-s)} + \int_{s_0}^{s} ds' \left[ \frac{1}{\ell} \psi_p^{le}(s') - \frac{i\omega}{v_F} \delta \epsilon \hat{p}(s') \right] e^{k(s'-s)}.
\]  

(2.41)

If we take \( s = 0 \), i.e. start from \( r = r(0) \), we get

\[
\psi_p(s) = \psi_p(s_0)e^{ks_0} + \int_{s_0}^{0} ds \left[ \frac{1}{\ell} \psi_p^{le}(s) - \frac{i\omega}{v_F} \delta \epsilon \hat{p}(s) \right] e^{ks}.
\]  

(2.42)

For convergence of the integral \( s_0 < 0 \), and the limit \( s_0 \to -\infty \) is approached for unlimited fluid. If the helium container is taken into account, \( s_0 \) is replaced by \( s_c \), the distance to the container wall along the trajectory, and the term \( \psi_p(r + s_0 \hat{p}) \) for outgoing trajectory on the container wall is obtained from the boundary condition. For small \( \ell \) the first term in (2.42) can be omitted, and in the numerical calculations, the integration is cut at \( s_0 = -Q\ell \), where \( Q \) is a constant of order \( 10^3 \).

The integral form (2.42) of the kinetic equation illustrates the method of relaxation time approximation. In order to obtain \( \psi_p \) at a given point \( r \) for given momentum direction \( \hat{p} \), we track the quasiparticle trajectory to the past. In other words, we consider where the quasiparticles are coming from, and take into account the conditions on their path before reaching to the point in question. The damping term \( \exp(s/\ell) \) takes the collisions into account: the trajectory direction is changed in a collision, and a straight trajectory no longer tells where the quasiparticle is coming from. With increasing \( |s| \) it becomes (exponentially) less probable that a quasiparticle with momentum direction \( \hat{p} \) ends up at \( r \). The mean free path controls the damping term, for small \( \ell \) the major contribution to the integral comes from close to point \( r \) only. The imaginary part in the exponent, \( i\omega s/v_F \), gives the effect of retardation: by the time a quasiparticle reaches \( r \) from distance \( |s| \), the oscillation of the wire (and the whole system) is in different phase, since the time \( |s|/v_F \) has passed.
2.5. APPLICATION TO VIBRATING WIRE

2.5.2 Boundary Conditions

To study the disturbance caused by the vibrating wire, we need to define boundary conditions on the wire surface. We use two kinds of scattering processes, specular and diffuse scattering. The scattering process depends on the properties of the surface (and the liquid) and we can use a parameter $S$, the fraction of specular scattering, to adjust the boundary condition to fit the experiments. For wire moving with velocity $u$, we use the boundary condition

$$\psi_\hat{p} = \psi_{\hat{p}-2\hat{n}(\hat{n} \cdot \hat{p})} + 2p_F(\hat{n} \cdot \hat{p})(\hat{n} \cdot u), \quad (2.43)$$

for specular scattering, while in the diffuse case we use the condition

$$\psi_{\hat{p}_{\text{out}}} = -2\langle \hat{n} \cdot \hat{p}_{\text{in}} \psi_{\hat{p}_{\text{in}}} \rangle \hat{p}_{\text{in}} + p_F(\hat{p}_{\text{out}} + \frac{2}{3}\hat{n}) \cdot u. \quad (2.44)$$

The outgoing quasiparticles are defined using the surface normal vector $\hat{n}$, pointed into the fluid, by $\hat{n} \cdot p_{\text{out}} > 0$. The average in (2.44) is taken over half of the unit sphere. In the following we make use of a shorthand notation for the average on the surface of the wire ($w$)

$$g_w = -2\langle \hat{n} \cdot \hat{p}_{\text{in}} \psi_{\hat{p}_{\text{in}}} \rangle \hat{p}_{\text{in}}. \quad (2.45)$$

We note that both specular and diffuse conditions satisfy $\hat{n} \cdot J_3 = n_3 m_3 \hat{n} \cdot u$, i.e. there is no flow through the surface. We require the same from the superfluid component, $\hat{n} \cdot v_s$.

We will briefly discuss the boundary conditions; a detailed derivation is given in Paper IV. Specular scattering means that a quasiparticle with momentum direction $\hat{p}$ is reflected to direction $p_R = p - 2\hat{n}(\hat{n} \cdot p)$. Strictly speaking this is true in the rest frame of the wire, but given the small velocity of the wire, the relation is adequate in the laboratory frame as well. The reflection means that in the rest frame $n_{p_R} = n_{p-2\hat{n}(\hat{n} \cdot p)}$, i.e. all quasiparticles coming in with momentum direction $\hat{p}$ are reflected to direction $\hat{p}_R$. It follows that in the rest frame $\psi_{\hat{p}'} - \psi_{\hat{p}_R'} = 0$, which is generalized to the laboratory frame by (2.43).

The specular scattering corresponds to a smooth wire, or in terms of hydrodynamic theory, to a perfect slip. It means, that the tangential fluid velocity on the wire surface is not limited by the boundary condition. Instead, it is required that the term $\sigma_{r\theta}$ in the momentum flux tensor is zero at the wire surface, i.e. there is no transfer of the tangential component of the momentum between the fluid and the wire. We see that (2.43) satisfies this condition, $\Pi_{r\theta} = 0$.

Let us consider the diffuse scattering first in the rest frame of the wire. In diffuse scattering the reflected quasiparticles are in equilibrium, and the
distribution of reflected quasiparticles $\psi_{\hat{p}_{\text{out}}}$ does not depend on the direction $\hat{p}_{\text{out}}$

$$\psi_{\hat{p}_{\text{out}}} = g.$$  \hfill (2.46)

The outgoing distribution is the same for all directions, and the constant $g$ is determined by the condition $\hat{n} \cdot J_3 = 0$, or $\hat{n} \cdot \langle \hat{p} \psi_{\hat{p}} \rangle \hat{p} = 0$. The average can be divided in two parts and we get $\hat{n} \cdot \langle \hat{p}_{\text{in}} \psi_{\hat{p}_{\text{in}}} \rangle \hat{p}_{\text{in}} = -\hat{n} \cdot \langle \hat{p}_{\text{out}} \psi_{\hat{p}_{\text{out}}} \rangle \hat{p}_{\text{out}} = -\frac{1}{2} g$, so that

$$\psi_{\hat{p}_{\text{out}}} = g = -2\hat{n} \cdot \langle \hat{p}_{\text{in}} \psi_{\hat{p}_{\text{in}}} \rangle \hat{p}_{\text{in}}.$$  \hfill (2.47)

This is generalized to the laboratory frame by (2.44).

The diffuse boundary condition (2.44) allows for a temperature dependent tangential slip on the surface. In the limit of small $\ell$ the slip vanishes, and (2.44) corresponds to the hydrodynamic no-slip condition, stating that the fluid velocity is zero at the boundary in the rest frame of the wall. The slip is discussed in more detail in section 3.2.3, where the boundary conditions are illustrated in Fig. 3.6. The specular boundary condition (2.43) corresponds to ”perfect slip” in hydrodynamic theory, as explained in chapter 3.

**Container**

So far we have considered the boundary condition on the wire surface. Another boundary condition is needed if the walls of the helium container are taken into account. The container is at rest in the laboratory frame, and for diffuse scattering from the container we can use the equation (2.47). Similar to (2.45), we define

$$g_c = -2\langle \hat{n}_c \cdot \hat{p}_{\text{in}} \psi_{\hat{p}_{\text{in}}} \rangle \hat{p}_{\text{in}},$$  \hfill (2.48)

$\hat{n}_c$ is the surface normal on the container wall, pointing into the fluid. The diffuse boundary condition can be used to model scattering from a rough but solid wall. In the experiments, however, the sample volume is often surrounded by sintered silver for cryotechnical reasons. Some of the quasiparticles are then more likely to be absorbed rather than scattered. We therefore introduce an alternative boundary condition, absorbing container walls. The distribution of the outgoing quasiparticles is then $\psi_{\hat{p}_{\text{out}}} = 0$, where $\hat{n}_c \cdot \hat{p}_{\text{out}} > 0$. In diffuse scattering the outgoing quasiparticle distribution has knowledge of the incoming distribution only on the average. In the case of fully absorbing walls, on the other hand, the outgoing distribution has no dependence on the incoming quasiparticles.

When the wire surface is hit, the solution (2.42) should be constructed piecewise, and the boundary conditions applied in between. For diffuse scattering from container walls and partially specular (with fraction $S$) scattering
from the wire, we can write the kinetic equation (2.42) as

\[
\psi_p(0) = -S2\hat{n}_c \cdot \langle \hat{p}_m \psi_p \rangle \hat{p}_m e^{ks} + \int_{s_w}^0 ds \left[ \frac{1}{\ell} \psi_p(s) - i \frac{\omega}{v_F} \delta \epsilon_p(s) \right] e^{ks}
\]

\[
+ S \int_{s_c}^{s_w} ds \left[ \frac{1}{\ell} \psi_p(s) - i \frac{\omega}{v_F} \delta \epsilon_p(s) \right] e^{ks} + S2p_F(\hat{n} \cdot \hat{p})(\hat{n} \cdot \hat{u}) e^{ks},
\]

(2.49)

if the trajectory is reflected from the wire (surface normal \( \hat{n} \)) at distance \( s_w \) along the trajectory, and hits the container (surface normal \( \hat{n}_c \)) at distance \( s_c \) from the starting point. In the latter integral the prime in direction \( \hat{p} \) stands for specularly reflected trajectory. For a trajectory with no collision with the wire, we get simply

\[
\psi_p(0) = -2\hat{n}_c \cdot \langle \hat{p}_m \psi_p \rangle \hat{p}_m e^{ks} + \int_{s_c}^0 ds \left[ \frac{1}{\ell} \psi_p(s) - i \frac{\omega}{v_F} \delta \epsilon_p(s) \right] e^{ks}.
\]

(2.50)

### 2.5.3 Symmetries

Since the vibrating wire is modeled by an infinite straight cylinder, we then have translational symmetry with respect to the \( z \)-coordinate, fixed along the cylinders axis. We fix the \( x \)-axis to the direction of wire’s motion \( \hat{u} = \hat{x} \), and consider further symmetries in the \( xy \)-plane. We use polar coordinates, distance from the wire axis \( r \) and the plane angle \( \theta \). Since the cross section of the wire is a perfect circle, and since we are limited to linear theory in \( \hat{u} \), the dependence on \( \theta \) of the quasiparticle distribution is simple. In terms of \( c \) and \( b \) we can write

\[
c(r) = c'(r)u \cos \theta, \quad b_r(r) = b'_r(r)u \cos \theta, \quad b_\theta(r) = b'_\theta u \sin \theta
\]

(2.51)

where the primes denote independence from the plane angle \( \theta \), and \( b = b_r \hat{r} + b_\theta \hat{\theta} \). We introduce a new short hand notation, \( \stackrel{\leftrightarrow}{d} = \langle \hat{p} \hat{p} \psi_p \rangle \hat{p} \), which can be used in calculating the force on the wire. The tensor \( \stackrel{\leftrightarrow}{d} \) obeys similar symmetry rules,

\[
\stackrel{\leftrightarrow}{d} \langle \hat{r} \hat{d}_r \hat{r} \rangle \hat{r} + \hat{\theta} \hat{\theta} \hat{d}_{\theta \theta} \hat{r} + (\hat{r} \hat{\theta} + \hat{\theta} \hat{r}) \hat{d}_{r \theta} \hat{r}
\]

\[
= \hat{r} \hat{u} \cos \theta \hat{d}_{r r}(r) + \hat{\theta} \hat{u} \cos \theta \hat{d}_{\theta \theta}(r) + (\hat{r} \hat{\theta} + \hat{\theta} \hat{r}) u \sin \theta \hat{d}_{r \theta}(r).
\]

(2.52)

These symmetries apply for wire in unlimited fluid, but the same symmetry applies if the helium container possesses the same symmetry as the wire, i.e. if the container is cylinder concentric with the wire. For the boundary condition terms (2.45) and (2.48) we write

\[
g_w = g'_w u \cos \theta, \quad g_c = g'_c u \cos \theta,
\]

(2.53)
where $g'_w$ and $g'_c$ are constants. We will consider more complicated geometries as well, such as wire oscillating between parallel plates. Although the cylindrical symmetry is broken, there are some other symmetries that can be used in numerical calculations, but are not very illustrative in terms of equations. One explicit consequence, however, is that the force on the wire due to the fluid is in the direction of the wire’s motion, $\mathbf{F} \parallel \mathbf{u}$.

### 2.5.4 Force on the Wire

The force exerted by the fluid on a surface element of the wire is calculated from (2.29)

$$\frac{dF}{da} = -\frac{\rho_s}{m_4} \delta \mu_4 \mathbf{n} - 3n_3 \mathbf{n} \cdot \langle \hat{p} \hat{p} \psi(\hat{p}) \rangle \hat{p}.$$  \hfill (2.54)

The total force (per unit length) is obtained by integrating over $\theta$

$$\mathbf{F} = i\omega a^2 \pi \rho_s G \mathbf{u} - 3an_3 \int_0^{2\pi} \langle (\mathbf{n} \cdot \hat{p})(\hat{p} \cdot \mathbf{u}) \psi(\hat{p}) \rangle \hat{p} d\theta.$$  \hfill (2.55)

The first term is due the superfluid part, and is described in the next chapter, Eq. (3.38). Assuming the symmetry $\mathbf{F} \parallel \mathbf{u}$, and we get a slightly simplified expression for the force

$$\mathbf{F} = i\omega a^2 \pi \rho_s G \mathbf{u} - 3an_3 \hat{u} \int_0^{2\pi} \langle (\mathbf{n} \cdot \hat{p})(\hat{p} \cdot \mathbf{u}) \psi(\hat{p}) \rangle \hat{p} d\theta,$$  \hfill (2.56)

or in terms of tensor $d$

$$\mathbf{F} = i\omega a^2 \pi \rho_s G \mathbf{u} - 3an_3 \hat{u} \int_0^{2\pi} d_{r\theta} d\theta.$$  \hfill (2.57)

Assuming cylindrical symmetry, we have $d_{r\theta} = d'_{rr} u \cos^2 \theta - d'_{r\theta} u \sin^2 \theta$, and

$$\mathbf{F} = i\omega a^2 \pi \rho_s G \mathbf{u} - 3n_3 a \pi (d'_{rr} - d'_{r\theta}) \mathbf{u}.$$  \hfill (2.58)

For fully specular scattering $d_{r\theta} = 0$ on the wire surface, and

$$\mathbf{F} = i\omega a^2 \pi \rho_s G \mathbf{u} - 3an_3 \hat{u} \cdot \int_0^{2\pi} \mathbf{n} d_{rr} d\theta.$$  \hfill (2.59)

We can calculate the contribution from the simple boundary condition terms (those involving $\mathbf{u}$ explicitly) in Eq. (2.49) separately. If we take $\psi^{\text{in}} = 0$, and

$$\psi^{\text{out}} = 2p_F(\hat{n} \cdot \hat{p})(\hat{n} \cdot \mathbf{u}), \quad \psi^{\text{diff}} = p_F(\hat{p}_{\text{out}} + \frac{2}{3} \mathbf{n}) \cdot \mathbf{u}$$  \hfill (2.60)
we get \( d_{rr} = p_F/4 \) and \( d_{r\theta} = 0 \) for the specular, and \( d_{rr} = 17p_F/72 \) and \( d_{r\theta} = -p_F/16 \) for the diffuse case. The resulting forces for purely specular and diffuse cases are, respectively,

\[
F_{\text{spec}} = -\frac{3\pi}{4} an_3p_F u, \quad F_{\text{diff}} = -\frac{43\pi}{48} an_3p_F u. \tag{2.61}
\]

Requiring linearity in \( u \), we can conveniently express the force exerted by the fluid on the wire in terms of the mechanical impedance of the fluid, \( Z \), as

\[
F = -Zu, \tag{2.62}
\]

where \( Z \) is a complex scalar, \( Z = Z' + iZ'' \). At high temperatures the whole fluid behaves like ideal fluid, and we have

\[
F_{\text{ideal}} = i\omega a^2 \pi \rho G u, \quad Z_{\text{ideal}} = -i\omega a^2 \pi \rho G, \quad Z''_{\text{ideal}} = -\omega a^2 \pi \rho G. \tag{2.63}
\]

For later use we define also reduced impedance by

\[
Z = Z_{\text{ideal}} + an_3p_F \tilde{Z}, \tag{2.64}
\]

which vanishes at high temperatures.
Chapter 3

Limiting Cases

In this chapter we describe the two limiting cases of the kinetic theory, the hydrodynamic and ballistic limits. In the hydrodynamic limit the quasiparticles are in continuous interaction with each other as the mean free path of a quasiparticle approaches zero. The system then behaves in many sense like a classical fluid, where it is sufficient to consider the averaged quantities such as local fluid velocity and viscosity. In the ballistic limit, on the other hand, there are no collision between the quasiparticles and the mean free path approaches infinity. The system of quasiparticles acts in some ways like a gas of ballistic particles, but the interactions remain in the equations through the Fermi liquid interaction terms $F$, and in the form of the effective mass $m^*$. The difference between the two limits is illustrated in Fig. 3.1. We see that for small mean free path the quasiparticle distribution is smooth, and can be described by the averaged terms $c$ and $b$, or their hydrodynamic counterparts $p$ and $v$, while for large $\ell$ the distribution is peaked, and kinetic theory is required.

Before we go into detailed calculations, it is necessary to consider the experiments used in studying the mechanical properties of helium mixtures. We will concentrate on the vibrating wire experiments, and in this chapter we will consider only the cylindrically symmetric case. Then the two limiting cases give analytically accessible points of comparison to the numerical calculations. After gaining confidence in the numerical calculations in the symmetric case, we can later study the more complicated geometries with the numerical treatment only.

3.1 Vibrating Wire Experiments

The mechanical properties of fluids often need to be measured in an indirect way by examining the damping force on an object moving in the fluid. Usually the properties of the bulk liquid are the primary targets of research, but the surface effects near the measuring devices may have strong contribution
Figure 3.1: A schematic diagram for the full distribution $\psi_{\hat{p}}$ (red lines) and the local equilibrium distribution $\psi_{\hat{p}}^{le} = c + \hat{p} \cdot b$ (dashed blue lines) for three different mean free paths, at three points close to the wire. For a given point $r$ the red surfaces reflect the value of (the real part of) $\psi_{\hat{p}}(r)$; a peak in the surface in direction $\hat{p}$ means larger $\psi_{\hat{p}}$ in that direction. The dashed blue surface is the same for local equilibrium distribution $\psi_{\hat{p}}^{le}$. The black line represents the wire, and the green arrows show the directions of vector $b$.

For $\ell = 0.1a$, $\psi_{\hat{p}}$ follows the local equilibrium distribution, but for larger $\ell$ we see strong peaks for trajectories coming from the wire. Close to the wire the peaks are wider, and narrow down when we go further from the wire. We have used parameters $S = 0, \Omega = 0.016, F_0 = -0.3, b = 10a$ for an absorbing cylindrical container.

to the measured force. This complicates the interpretation of experimental results, especially in the case of quantum liquids, where the length scales are in significant role. On the other hand, the surface phenomena can be of interest as such.

Viscosity measurements in liquid helium have been conducted by many types of oscillating devices, such as oscillating discs, other types of torsional oscillator, quartz forks and vibrating wires. This thesis concentrates on vibrating wire resonators, but the theory presented can be applied to other oscillators, as well as on ions in helium [21] (for example in particle image velocimetry), aerogel [22], and deformations of the liquid surface [23] (Wigner solid).

Typically, a vibrating wire resonator (VWR) experiment consists of a small chamber filled with liquid helium, in which a loop of superconducting wire is placed in a static homogenous magnetic field in the plane of the loop. The experiments are done at low temperature, the area of interest here is in the millikelvin range or below. The wires are made of material that is superconducting at the said temperature range, in order to avoid heating. Alternating current is led through the wire, which produces a Lorentz force that enforces the wire to vibrate. The vibration in turn induces an extra voltage into the wire. The change in electric impedance in the wire can be
measured, and associated with the mechanical impedance of the fluid. Using a Lorentzian fit the line width $\Delta f$ and resonant frequency $f$ of the oscillations are obtained, corresponding to energy dissipation and effective inertia of the wire.

Numerous vibrating wire experiments have been conducted to measure the viscosity of pure $^4$He and $^3$He, as well as the mixtures of these. For a review on the general properties of helium mixtures, see for example Edwards and Pettersen [24], and for surface phenomena Einzel and Parpia [25]. We concentrate on the VWR measurements on $^3$He-$^4$He mixtures at low temperatures [26, 27, 28, 29, 30], where the ballistic limit is approached. For our purposes the data published by Guénault et al. [27], Martikainen et al. [29], and Pentti et al. [31] are most convenient. In these experiments the results are presented as $(f, \Delta f)$-plots, where no interpretation based on the hydrodynamic theory or slip theory has been used. Another feature in such presentation is that an explicit temperature dependence is not needed.

Special attention is paid to the experiments made at the Low Temperature Laboratory in Otaniemi [29, 31], which were still rather recent when the present theoretical work was started. In these experiments, better resolution and lower temperatures than before were reached, and some new aspects were seen. In particular, the resonance frequency of the wire at the low temperature end became higher than that in the high temperature end, Figs. 3.2, 4.14. In other words, the hydrodynamic increase of inertial mass of the wire was less than that caused by the $^4$He back flow alone. This had not been seen in previous experiments, and there has been no satisfactory explanation to this observation.

The two vibrating wires in the experiments [29, 31] were made of tantalum, with density $\rho_w = 16700 \text{ kg/m}^3$ and radius $a = 62.5 \mu \text{m}$. The wires were bend in semicircular loops with diameters of about 8 mm. The dimensions of the two separate helium containers were approximately $10 \times 10 \times 1 \text{ mm}^3$, so that the distance to the nearest wall in the direction of the wire’s motion was only $h = 0.5 \text{ mm} = 8a$. The vacuum frequency for wire 1 was $f_{\text{vac}} = 1202.85$ Hz, and for wire 2 $f_{\text{vac}} = 1857.7$ Hz. Four different $^3$He concentrations $x_3$ were used (1.8%, 3.6%, 5.6% and 6.6%) at saturated vapor pressure, and two concentrations at (7.0% and 9.5%) at $p = 10$ atm. For wire 2 only the three largest concentrations were used.

### 3.1.1 Harmonic Oscillator

If we assume that the Lorentz force driving the wire is of the form $F_{\text{ext}} = F^{(0)} e^{-\omega t}$, the vibrating wire can be described as a driven harmonic oscillator in viscous medium. The equation of motion can be written as

$$m\ddot{x} = -\kappa x - c\dot{x} + F^{(0)} e^{-\omega t};$$

(3.1)
where \( m \) is the mass of the oscillator, and \( x \) is the deviation from the equilibrium position. In the absence of viscous medium and external force, the equation is \( m\ddot{x} = -\kappa x \), resulting in oscillations with natural frequency \( \omega_0 = \sqrt{\kappa/m} \). Here, \( \kappa \) is the spring constant, which describes the restoring force and depends on the mechanical properties of the wire. In a viscous medium there exists a dissipative force \(-c\dot{x}\), which is proportional to the velocity of the particle.

We will see that the viscous fluid surrounding the wire exerts a force of the form \( \mathbf{F}_{\text{fluid}} = \omega m'(ik - k') \mathbf{u} \) on the wire. In terms of equation (3.1), the real part of this force \(-\omega m'k'\dot{x}\) can be associated with the dissipative force \(-c\dot{x}\), while the imaginary part can be understood either as additional mass \(-i\omega m'k\dot{x} = m'k\dot{x}\) (if \( k > 0 \)), or as an increase in the spring constant \(-m'\kappa \omega^2\) (if \( k < 0 \)). The former choice is appropriate in the hydrodynamic limit, where \( k \) is positive and the additional mass is due to part of the fluid moving along with the wire. The latter interpretation can be used in the ballistic limit, where \( k \) is negative, and rather than talking about negative mass, the increase in \( \kappa \) can be understood as elasticity of the fluid due to the Fermi liquid interactions.

In terms of theory of viscoelastic materials, the response of the fluid can be expressed in terms of dynamic modulus \( G = G' + iG'' \), the ratio of stress to strain in vibrating motion [32]. The real part \( G' \) is known as the storage modulus, while the imaginary part \( G'' \) is known as the loss modulus. In pure elastic materials the stress and strain are in phase, while in purely viscous materials there is a \( \pi/2 \) phase difference. As equations, we can write

\[
\begin{align*}
 m\ddot{x} &= -\kappa x + F_{\text{fluid}} + F_{\text{ext}} \\
 (m + m'k)\dot{x} &= -\kappa x - m'\kappa \omega'k\dot{x} + F_{\text{ext}} \\
 m\dot{x} &= -(\kappa - m'\kappa \omega^2)x - m'\kappa \omega'k\dot{x} + F_{\text{ext}}.
\end{align*}
\]

The increase of effective inertia is seen as decrease in the resonant frequency \( f \) of the wire (compared to the vacuum frequency \( f_{\text{vac}} = \omega_0/2\pi \)), while the increase in the dissipative part \( k' \) is seen as increase of the line width \( \Delta f \) of the oscillation. The resonant frequency and line width are related to the components of the force \( F_{\text{fluid}} \) by [33]

\[
\frac{f_{\text{vac}} - f}{f} = \frac{\rho_n k' + \rho_s}{2\rho_w}, \quad \text{and} \quad \frac{\Delta f}{f} = \frac{\rho_n}{\rho_w k'},
\]

where \( \rho_w, \rho_n, \) and \( \rho_s \) are the densities of the wire, and the normal and superfluid components of the fluid, respectively. In the following sections we will describe how the force exerted by the fluid on the wire can be calculated in the two limiting cases of the kinetic theory: the hydrodynamic limit at high temperatures and the ballistic limit at low temperatures.
3.1.2 Simple Physical Picture

Before going to the details, it is useful to give a simple physical picture of the results. In Fig. 3.2 typical plots of numerical results are shown in terms of resonant frequency $f$ and line width $\Delta f$. Panel a shows a $(f, \Delta f)$-curve as a parametric plot in temperature, while in panel b $f$ and $\Delta f$ are plotted separately against the mean free path $\ell$, which is used as a parameter in the calculations ($\ell \propto T^{-2}$). The lower part of the graph in panel a is at higher temperatures (still below 1 K), while the upper part is at lower temperatures (100 $\mu$K or below). Starting from the high temperature end, we see that as temperature starts to decrease, dissipation starts to increase and resonant frequency starts to decrease. In the hydrodynamic region both can be explained by increasing viscosity [$\eta \propto T^{-2}$, Eq. (3.7)]: increasing viscosity means increasing friction, and thus increasing dissipation. Also the viscous penetration depth increases [$\delta \propto 1/T$, Eq. (3.17)] so that more and more fluid moves with the wire, increasing its effective mass and thus decreasing the resonant frequency.

As the temperature is further reduced, the mean free path of quasiparticles exceeds the radius of the wire, and the hydrodynamic picture breaks

![Figure 3.2: Typical results for vibrating wire resonator. a): Line width $\Delta f$ against resonant frequency $f$ as a parametric plot in temperature. The lower part of the curve is at higher temperatures, while the upper end of the curve is at low temperatures. The vertical dashed line shows the high temperature resonant frequency. The horizontal dashed line gives the maximum dissipation in the case of unlimited fluid, and the dashed-dotted line gives the corresponding limit for a cylindrical container of radius $b = 8a$. b): The line width $\Delta f$ and resonant frequency plotted separately as functions of the mean free path $\ell$. The left vertical axis shows $\Delta f$ (upper part of the graph) and the right vertical axis shows $f$ (lower part of the graph). Increasing $\Delta f$ corresponds to increasing dissipation, while increasing $f$ means decreasing inertia. The dashed lines are the same as in panel a.](image-url)
Figure 3.3: Simple illustration of the results. Far from the wire we expect the quasiparticle distribution to be in equilibrium, \( \psi = 0 \).

\( \text{a) } T=0 \quad \psi_\text{out} = \psi_\text{in} + 2p_F(n \cdot u)(n \cdot p) \)

\( \psi_\text{in} = \psi_0 = 0 \)

\( \psi_0 = 0 \)

\( \text{b) } T>0 \quad \ell \)

Let us first consider the simple case where the confinement and the Fermi liquid interactions are neglected. Far from the wire we expect the quasiparticle distribution to be in equilibrium, \( \psi = 0 \). In the collisionless ballistic limit, if the Fermi interactions and confinement are neglected, we then have \( \psi_\text{in} = 0 \). We then have \( \Delta E \propto p_F u \) for the energy transferred from the wire to the fluid, i.e. the force on the wire is dissipational. The reflected quasiparticles carry the energy to the bulk liquid far from the wire. At higher temperatures the reflected quasiparticles collide with other quasiparticles, and part of the energy is returned to the wire.

This is clearly seen in Fig. 3.2b around \( \ell \sim a \), as the hydrodynamic calculations start to deviate from the full numerical calculations. After reaching its minimum at \( \ell \sim a \) the resonant frequency starts to increase, until it converges to a maximum value at low temperatures. This maximum frequency is higher than the "ideal fluid" value at high temperatures. Part of this increase can be understood as decoupling of the quasiparticles from the motion of the wire, part of the increase is due to confinement of the fluid to a finite geometry, and part is explained by the "Landau Force" due to Fermi liquid interactions, as will be explained later. With decreasing temperature the dissipation keeps increasing, until it reaches its maximum (shown by the horizontal lines in Fig. 3.2) at the lowest temperatures. This maximum dissipation as well as the decoupling of the ballistic quasiparticle gas from the wire can be understood in terms of simple kinetic arguments, illustrated in Fig. 3.3.
3.1. VIBRATING WIRE EXPERIMENTS

\[ \text{Equilibrium} \]
\[ \delta \epsilon \neq 0 \]
\[ \delta \epsilon = \epsilon_0 + 2 \epsilon_p(n \cdot \mathbf{u})(n \cdot \mathbf{p}) \]

\( \psi_{\text{in}} \neq 0 \)

\( \psi_{\text{out}} = \psi_{\text{in}} + 2 \epsilon_p(n \cdot \mathbf{u})(n \cdot \mathbf{p}) \)

\( \psi_0 = 0 \)

Figure 3.4: a) Simple illustration of the Landau Force. The beam of quasiparticles alters the distribution of quasiparticles approaching the wire so that \( \psi_{\text{in}} \neq 0 \). The effect of \( \delta \epsilon \) happens to be such that the dissipation is practically unaffected, but there is a decrease in the effective inertia of the wire. b) Illustration of the confinement. The effect of the wire is seen on the container walls, \( \psi_{\text{wall}} \neq 0 \), and thus the force on the wire changes. In the ballistic limit interferences are observed between the incoming and outgoing quasiparticles.

obtained from the wire to the bulk liquid far from the wire.

At higher temperatures the reflected quasiparticles collide with other quasiparticles immediately after scattering from the wire. In the series of collisions, part of the energy stolen from the wire is returned to it, and the dissipation is less than the maximum value at ballistic limit. The collisions between incoming and outgoing quasiparticles change the distribution \( \psi_{\text{in}} \), and due to time delay the complex phase of \( \psi_{\text{in}} \) changes. This is seen as a change in the effective inertia of the wire. The collisions push the quasiparticle distribution towards equilibrium, which in the high temperature limit is such that the fluid follows the motion of the wire: there is no dissipation, but a moderate increase in effective mass of the oscillator.

Next, we consider how the situation changes, when the Fermi liquid interactions are accounted for in the ballistic limit. Because of the interactions, quasiparticles feel the presence of other quasiparticles, even in the absence of collisions, through the energy correction term \( \delta \epsilon_p(r, t) \), which can be interpreted as a potential. Consider a simplified model illustrated in Fig. 3.4. We assume that \( \delta \epsilon = 0 \), except in a beam of quasiparticles (which originates from the wire, unlike in the illustration). Now the incoming quasiparticle distribution, which is originally \( \psi_{\text{in}} = 0 \), is altered by the potential \( \delta \epsilon \) in the beam, as discussed in Paper III. In the case of helium mixtures, the energy correction term \( \delta \epsilon_p(r, t) \) is such that it increases the oscillation of the wire. This can be interpreted as a decrease of the mass of the oscillator, or as elasticity of the fluid.

The confinement of helium to a container of finite size has a considerable effect on the results at lower temperatures. This is reasonable, since the mean free path of the quasiparticles easily exceeds the size of the container.
at the lowest temperatures. In simple terms, the reason for this is that the quasiparticles that have scattered from the wire are reflected at the container wall, and may collide with the wire again as shown in Fig. 3.4. In terms of the previous discussion, the distribution of the incoming quasiparticles is not in global equilibrium to start with, i.e. $\psi_{p_n} \neq 0$. Naturally, there is a retardation effect associated with this: the relative phase between the wire and $\psi_{p_n}$ depends on oscillation frequency $\omega$, size of the container $b$, and Fermi velocity $v_F$. These effects are discussed in more detail in section 3.3.

3.2 Hydrodynamic Limit

The kinetic theory of previous chapter is a statistical approach to describe the collective behavior of a vast number of interacting particles. Although the interactions are not taken into account precisely, they are clearly visible in the equations via the Landau parameters in the quasiparticle energy expressions, and via the collision term. In this chapter, we take one step further away from the exact quantum mechanical description of interacting particles, in order to obtain a more intuitive picture of the behavior of the liquid as a whole. In hydrodynamic theory, we no longer consider the dynamics of single atoms (or quasiparticles), but the local values of such averaged terms as fluid velocity, viscosity, pressure and density. These continuous quantities are associated with what is known as the fluid element, a volume in space which is so small that the hydrodynamic variables can be considered constants inside it, but so large that it contains such a number of atoms that only their averaged properties are visible.

The hydrodynamic theory can be derived from the kinetic theory; integrating the Boltzmann equation in the relaxation time approximation gives Navier-Stokes equations (in the limit small gradient and time derivative terms in $\psi$). The hydrodynamic limit at high temperatures corresponds to vanishing mean free path of the quasiparticles, since $\ell \propto T^{-2}$. In other words, in hydrodynamics the atoms (or quasiparticles) are in continuous interaction with each other.

At high temperatures (still below 1 K) the basic hydrodynamic theory describes well the response of the $^3$He-$^4$He mixture to the motion of a vibrating wire immersed in the liquid. The theory of fluid flow around an oscillating infinite cylinder is due to Stokes, already in the nineteenth century [34]. The $^4$He part is treated as an inviscid superfluid, described by the ideal fluid Euler equation. The $^3$He part, in turn, is described as an incompressible viscous fluid, with viscosity proportional to the mean free path, $\eta \propto \ell \propto T^{-2}$.

In this section, we will first briefly describe how the hydrodynamic equations follow from the kinetic theory. Then we will write the basic hydrodynamic theory for the fluid flow around an infinite cylinder oscillating in right
angle to its axis. (Using the dimensions for VWR given in the end of section 3.1, we see that the radius of the loop is of order \(10^3\) larger than the radius of the wire, so the wire can be treated effectively as an infinite straight cylinder.) We will consider the no-slip and perfect-slip boundary conditions, and study the case of ideal fluid, that describes the superfluid \(^4\)He. We will also shortly consider the slip theory, extension of the hydrodynamics to rarified gas, or in our case, to lower temperatures.

### 3.2.1 Kinetic Theory in the Hydrodynamic Limit

In this section we study the kinetic theory in the limit of continuous collisions \(\tau \to 0\). The short relaxation time implies that the quasiparticle distribution \(\hat{\psi}_p\) is close to the local equilibrium distribution \(\psi^le = c + \hat{p} \cdot b\). The averaged quantities \(c\) and \(b\) can be identified with more familiar hydrodynamic variables \(v_n\) and \(p\). We identify the normal component fluid velocity \(v_n\) with the fermion particle current by

\[
v_n = \frac{J_3}{n_3 m_3} = \frac{3}{p_F} \langle \hat{p} \hat{\psi}_p \rangle \hat{p} = \frac{b}{p_F}. \tag{3.6}
\]

The local fluid velocity is a hydrodynamic concept, and strictly speaking is defined only in the hydrodynamic limit \(\ell \to 0\), but the above definition in terms of the fermion current expands the concept to lower temperatures as well. The fluid velocity term \(b\) carries information about the distribution on the average, but for large \(\ell\) it is not sufficient to describe the whole dynamics, unlike in the hydrodynamic theory.

In appendix A it is shown, by solving the kinetic equation in the hydrodynamic limit, that the viscosity \(\eta\) and pressure \(p\) can be written as

\[
\eta = \frac{p_F n_3}{5 k} = \frac{1}{5} p_F n_3 \ell, \quad p = p_0 + n_3 \langle \hat{p} \hat{\psi}_p \rangle \hat{p} + \rho_s \delta \mu_4 / m_4, \tag{3.7}
\]

where the pressure components can be written as \(p_n = n_3 \langle \hat{p} \hat{\psi}_p \rangle \hat{p}\), \(p_s = \rho_s \delta \mu_4 / m_4\). For a cylinder oscillating with small frequencies in infinite fluid the chemical potential \(\delta \mu_4\) and superfluid pressure \(p_s\) take the forms

\[
\delta \mu_4 = -i \omega a^2 \hat{r} \cdot \mathbf{u}, \quad p_s = -i \omega \rho_s a^2 \hat{r} \cdot \mathbf{u}. \tag{3.8}
\]

Using the expression for pressure above, we can rewrite the kinetic momentum flux tensor as

\[
\hat{\Pi} = n_3 \langle (3 \hat{p} \hat{p} - \hat{1}) \hat{\psi}_p \rangle \hat{p} + p \hat{1}. \tag{3.9}
\]

Comparing this to the hydrodynamic equation \(\hat{\Pi} = p \hat{1} + \rho \nu \nu - \hat{\sigma}'\) and neglecting the second order term in velocity, we get for the viscous stress tensor

\[
\hat{\sigma}' = n_3 \langle (\hat{1} - 3 \hat{p} \hat{p}) \hat{\psi}_p \rangle \hat{p} = n_3 c \hat{1} - 3 n_3 \hat{d}. \tag{3.10}
\]
In conclusion, we have correspondence between the kinetic and hydrodynamic variables

\[ c = \frac{p_n}{n_3}, \quad b = 3p_F v_n, \quad d = \langle \hat{p} \hat{p} \psi(\hat{p}) \rangle_p = \frac{c}{3} \frac{\sigma'}{3n_3}, \] (3.11)

and the quasiparticle distribution can be approximated as (A.16)

\[ \psi(\hat{p}) = \frac{p_n}{n_3} + p_F \hat{p} \cdot v_n - \frac{5}{2} \hat{p} \cdot \hat{p}' \cdot \hat{p}. \] (3.12)

### 3.2.2 Navier-Stokes Equation

The fluid consist of two components, the normal component with density \( \rho_n \) and the superfluid component with density \( \rho_s \) given in Eq. (2.27). The superfluid part can be treated as an irrotational, incompressible ideal fluid, and the normal part as a viscous fluid. In the following we will discuss the normal part in terms of the hydrodynamic theory, and refer to the superfluid part as a special case with \( \eta = 0 \). The hydrodynamic equation of motion for viscous fluids in general is the Navier-Stokes equation

\[ \rho_n \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \eta \nabla^2 \mathbf{v} + (\zeta + \frac{1}{3} \eta) \nabla (\nabla \cdot \mathbf{v}), \] (3.13)

where it is assumed that the first and second viscosities are constant throughout the fluid. The equation can be simplified by assuming that the fluid is incompressible, \( \nabla \cdot \mathbf{v} = 0 \). This is the usual assumption for classical fluids, and is obeyed by the quasiparticle system in the limit \( \ell \to 0 \), assuming that the velocities are much smaller than sound velocities in the fluid. Next, we assume that the wire velocity \( \mathbf{u} \) remains small, \( u \ll v_F \) and \( u \ll a \omega \). (The relative magnitudes of the remaining four terms in Eq. (3.13) are, from left to right, \( a \omega, u, a \omega, v_F \ell / a \).) In the experiments the amplitude of the wire’s oscillation is of order nanometers, \( 10^4 \) times smaller than the wire radius, so the condition is easily satisfied. Thus, we can ignore the nonlinear velocity term, and concentrate on laminar flow. Then, the linearized Navier-Stokes equation reads

\[ \nabla p = \eta \nabla^2 \mathbf{v} - \rho \frac{\partial \mathbf{v}}{\partial t}. \] (3.14)

Using \( \mathbf{v} = \nabla \times \chi \hat{z} \), where \( \chi = \chi_1 + \chi_2 \), the equations of motion become

\[ \nabla^2 \chi_1 = 0, \quad (\nabla^2 + \frac{i \rho \omega}{\eta}) \chi_2 = 0, \] (3.15)

the general solutions of which are

\[ \chi_1(r, \theta, t) = u_0 e^{-i \omega t} \sin \theta \left( A_1 r + B_1 r \right), \]
\[ \chi_2(r, \theta, t) = u_0 e^{-i \omega t} \sin \theta \left( A_2 H_1^{(1)}(qr) + B_2 H_1^{(2)}(qr) \right), \] (3.16)
where the constants $A_1$, $A_2$, $B_1$ and $B_2$ are determined from boundary conditions, $q = (1 + i)/\delta$ and the viscous penetration depth $\delta$ is

$$\delta = \sqrt{\frac{2\eta}{\rho n \omega}} = \sqrt{\frac{2\nu F}{5(1 + F_1^2/3)\omega}}. \quad (3.17)$$

For an infinite medium (asymptotically vanishing velocity) the constants $B_1$ and $B_2$ are zero, but for $v$ to be zero at a finite distance $b$, they must be nonzero. For velocity we get the expression

$$v = [-\frac{A_1}{r^2} + B_1 + A_2 qH_1^{(1)'}(qr) + B_2 qH_1^{(2)'}(qr)]u + [\frac{2A_1}{r^2} + A_2 qH_2^{(1)}(qr) + B_2 qH_2^{(2)}(qr)](\hat{r} \cdot u)\hat{r} \quad (3.18)$$

The pressure is found from

$$p = p_0 - i\omega \rho (\frac{A_1}{r} - B_1 r)(\hat{r} \cdot u). \quad (3.20)$$

The force on the wire is found by using [36]

$$\sigma_{rr} = -p + 2\eta \frac{\partial v_r}{\partial r}, \quad \sigma_{r\theta} = \eta(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}), \quad \frac{dF}{da} = \hat{n} \cdot \sigma \quad (3.21)$$

and the total force, integrated over the wire surface, is

$$\mathbf{F} = i\rho \omega \pi a^2 \left[\frac{A_1}{a^2} - B_1 - \frac{A_2}{a} H_1^{(1)}(qa) - \frac{B_2}{a} H_1^{(2)}(qa)\right]u. \quad (3.22)$$

We will consider several boundary conditions: on the wire surface we use either the no-slip condition $v_n = u$, or the specular condition $\hat{n} \cdot v_n = \hat{n} \cdot u$ and $\sigma_{r\theta} = 0$; far from the wire we assume either vanishing velocity at infinity or at radius $b$, or specular scattering at $r = b$.

The equations above are based on cylindrical symmetry: we assumed the simple form $\chi \propto \sin \theta$, and the Hankel functions are known to describe solutions in cylindrical geometry. We note that the expressions for pressure, velocity, and force follow particularly simple dependence on the plane angle $\theta$, namely $p \propto \cos \theta$, $v_r \propto \cos \theta$, and $v_\theta \propto \sin \theta$. Similar symmetries are extensively used in the numerical calculations using the kinetic theory. We will find exquisite agreement between hydrodynamics and the kinetic calculations in the limit $\ell \to 0$, and will not attempt to formulate hydrodynamic equations for more complicated geometries.
CHAPTER 3. LIMITING CASES

Unlimited Fluid

If the fluid is at rest at infinity, we have $B_1 = B_2 = 0$. The equations for velocity, pressure and force are then simplified to

$$v = \left[ -\frac{A_1}{r^2} + A_2 q H_1^{(1)'}(qr) \right] u + \left[ \frac{2A_1}{r^2} + A_2 q H_2^{(1)'}(qr) \right] (\hat{r} \cdot u) \hat{r}, \quad (3.23)$$

$$p = p_0 - i\omega \rho \frac{A_1}{r} (\hat{r} \cdot u), \quad F = i\rho \omega \pi a^2 \left[ \frac{A_1}{a^2} - \frac{A_2}{a} H_1^{(1)}(qa) \right] u. \quad (3.24)$$

The velocity components calculated with different boundary conditions are shown in Fig. 3.5. For the no-slip boundary condition we find

$$A_1 = a^2 + \frac{2a H_0^{(1)'}(qa)}{q H_0^{(1)}(qa)}, \quad A_2 = \frac{2}{q H_0^{(1)}(qa)}, \quad (3.25)$$

(we have used properties of the Hankel functions: $H_1^{(1)'}(z) = H_0^{(1)}(z) + H_0^{(1)'}(z)/z$) while for the specular case we have

$$A_1 = a^2[1 + \frac{4H_0^{(1)'}(qa)}{2aqH_0^{(1)}(qa) - a^2q^2H_0^{(1)'}(qa)}], \quad A_2 = \frac{4}{2qH_0^{(1)}(qa) - a^2q^2H_0^{(1)'}(qa)}. \quad (3.26)$$

The corresponding force expressions are

$$F_{\text{no-slip}} = i\pi \rho_n \omega a^2[1 + \frac{4 H_0^{(1)'}(qa)}{aq H_0^{(1)}(qa)}] u, \quad (3.27)$$

$$F_{\text{spec}} = i\pi \rho_n \omega a^2[1 + \frac{8 H_0^{(1)'}(qa)}{2aqH_0^{(1)}(qa) - a^2q^2H_0^{(1)'}(qa)}] u. \quad (3.28)$$

The former is the original Stokes’ solution [34], and it defines the ”original Stokes’ coefficients” $k_0$ and $k_0'$, as

$$k_0 + ik_0' = 1 + \frac{4}{aq H_0^{(1)}(qa)}. \quad (3.29)$$

The subscripts are used because any force on the wire can be expressed in terms of $k$ and $k'$, as $F = i\pi \rho \omega a^2(k + ik')u$, even outside the hydrodynamic region. In the limit of small viscosity it is illustrative to express the coefficients in terms of the viscous penetration depth $\delta$ as $k_0 = 1 + 2\delta/a$, and $k_0' = 2\delta/a + \delta^2/a^2$. The specular result can be written in terms of the original Stokes’ coefficients (3.29), with notation $K_0 = k_0 - 1 + k_0'$

$$k_{\text{spec}} - 1 + ik_{\text{spec}}' = \frac{K_0}{1 - \frac{1}{8}a^2q^2K_0}. \quad (3.30)$$

Using this we see that our expression for the force in the specular case $F_{\text{spec}}$ is equivalent to the force given by Bowley and Owers-Bradley [39] in the case of infinite slip. Different force expressions are compared in Fig. 3.9 in terms of the mechanical impedance $Z = -F/u$. 
3.2. HYDRODYNAMIC LIMIT

Figure 3.5: The radial and tangential components of velocity as functions of \( r \) (real parts only). Two mean free path are considered, \( \ell = 0.01a \) (solid) and \( \ell = 0.1a \) (dashed). Five cases are given (not all are visible due to overlap): ideal fluid (3.33) (orange), viscous fluid formula (3.23) with no-slip condition (green), and with specular condition (blue), kinetic calculations with diffuse wire (red), and with specular wire (violet). We see that there is excellent correspondence between the kinetic and hydrodynamic calculations for small \( \ell \). For specular boundary condition the hydrodynamic and kinetic results are indistinguishable, but the difference between the diffuse and the no-slip cases is visible already at \( \ell = 0.1a \). For pressure the differences are (relatively) larger (not shown).

Finite Container

Next, we consider the fluid oscillating in a cylindrical container of radius \( b \), concentric with the wire. From the four possible configurations of no-slip and specular conditions we only consider two, where the same conditions are applied on both surfaces. First, we apply the no-slip condition on both surfaces, \( \mathbf{v}(a) = \mathbf{u} \), and \( \mathbf{v}(b) = 0 \). The constants are then [26]

\[
\begin{align*}
A_1 &= a^2b^2[J_2(aq)Y_2(bq) - J_2(bq)Y_2(aq)]D, \\
A_2 &= i\frac{q}{a^2}[a^2H_2^{(2)}(aq) - b^2H_2^{(2)}(bq)]D, \\
B_1 &= -\frac{2i\delta^2}{\pi} + a^2(J_2(aq)Y_0(bq) - J_0(bq)Y_2(aq))]D, \\
B_2 &= i\frac{q}{a^2}[-a^2H_2^{(1)}(aq) + b^2H_2^{(1)}(bq)]D,
\end{align*}
\]

where

\[
D = \left[ \frac{4\delta^2i}{\pi} + a^2(J_2(aq)Y_0(bq) - J_0(bq)Y_2(aq)) \\
+ b^2(J_2(bq)Y_0(aq) - J_0(aq)Y_2(bq)) \right]^{-1}.
\]

The case of specular wire oscillating in a specular, cylindrical, stationary container is described by the boundary conditions \( \hat{n} \cdot \mathbf{v}(a) = \hat{n} \cdot \mathbf{u} , \hat{n} \cdot \mathbf{v}(b) = 0 \),
and $\sigma_{\tau\theta}(a) = \sigma_{\tau\theta}(b) = 0$. We then find

\[
A_1 = \frac{b^2(-2 + a^2 q^2)}{(b^2 - a^2)q^2}, \quad A_2 = \frac{-iH_1^{(2)}(bq)}{aq^2[J_1(bq)Y_1(aq) - J_1(aq)Y_1(bq)]},
\]

\[
B_1 = \frac{-2 + a^2 q^2}{(b^2 - a^2)q^2}, \quad B_2 = \frac{iH_1^{(1)}(bq)}{aq^2[J_1(bq)Y_1(aq) - J_1(aq)Y_1(bq)]}.
\]

The force expressions are obtained using Eq. (3.22).

### Ideal Fluid

If the viscosity vanishes, $\eta = 0$, we are left with the Euler equation

\[
\nabla^2 \chi = \nabla^2 \chi_1 = 0, \quad (3.31)
\]

or in the terms of the previous treatment, $\chi_2 \equiv 0$ i.e. $A_2 = B_2 = 0$. We then have

\[
v = [-\frac{A_1}{r^2} + B_1]u + \frac{2A_1}{r^2} (\hat{r} \cdot u)\hat{r}. \quad (3.32)
\]

If we assume the fluid velocity to vanish at infinity, we have $B_1 = 0$, and from the condition that the fluid does not penetrate the wire, $\hat{n} \cdot v(a) = \hat{n} \cdot u$, we get $A_1 = a^2$, leading to [36]

\[
v_s = \frac{a^2}{r^2}[2(\hat{r} \cdot u)\hat{r} - u]. \quad (3.33)
\]

Here, the subscript $s$ refers to superfluid, which is described by this ideal fluid equation. If, on the other hand, there is a stationary cylindrical wall at radius $b$, we get the condition $\hat{r} \cdot v(b) = 0$, in addition to $\hat{n} \cdot v(a) = \hat{n} \cdot u$, and then

\[
A_1 = \frac{a^2 b^2}{b^2 - a^2} = \frac{a^2}{1 - \frac{a^2}{b^2}}, \quad \text{and} \quad B_1 = -\frac{a^2}{b^2 - a^2}. \quad (3.34)
\]

The ideal fluid velocity is then

\[
v_s = \frac{a^2}{r^2} \frac{b^2}{b^2 - a^2}[2(\hat{r} \cdot u)\hat{r} - u] - \frac{a^2}{b^2 - a^2}u. \quad (3.35)
\]

The force on the wire for ideal fluid is

\[
F_s = i\pi \rho_s \omega a^2 \frac{b^2 + a^2}{b^2 - a^2}u. \quad (3.36)
\]

We note that the ideal fluid does not stick to the surface, i.e. does not obey the no-slip condition. The tangential component of the ideal fluid velocity on the wire surface,

\[
\hat{\theta} \cdot v_s(a) = -\frac{b^2}{b^2 - a^2}(\hat{\theta} \cdot u) - \frac{a^2}{b^2 - a^2}(\hat{\theta} \cdot u) = -(\hat{\theta} \cdot u), \quad (3.37)
\]
which is the opposite of the no-slip condition: $v(a) = u$, see Fig. 3.5. It is worth emphasizing that the no-slip condition does not lead to ideal fluid motion in the limit of no viscosity. In the no-slip case there is a boundary layer surrounding the wire, where the relative tangential velocity $v_\theta - u_\theta$ goes to zero, while the velocity profile outside the layer resembles that of the ideal fluid. The thickness of the boundary layer (depicted by $\delta$) approaches zero asymptotically when $\ell \to 0$. However, the force on the wire in the no-slip case in the limit of no viscosity is the same as that for the ideal fluid. Therefore in the high temperature limit the total force is always

$$F_{\text{ideal}} = i\pi \rho \omega a^2 \frac{b^2 + a^2}{b^2 - a^2} u,$$  \hspace{1cm} (3.38)

with $\rho = \rho_s + \rho_n$. This is the same as Eq. (2.63), with $G = (b^2 + a^2)/(b^2 - a^2)$. The factor $G$ takes the geometry of the container into account, for infinite fluid we have $G = 1$, and for wire oscillating between two parallel walls with distance $h$ from the wire we approximate $G \approx 1 + \pi^2 a^2/12h^2$ [35].

**Resonances**

For ideal fluid, it is possible to take the compressibility into account. The equation of motion to be solved is then the wave equation

$$\omega^2 \chi + c^2 \nabla^2 \chi = 0,$$  \hspace{1cm} (3.39)

where $c = \sqrt{\partial p/\partial \rho}$ is the velocity of sound. We consider a cylinder oscillating in a coaxial stationary cylinder of radius $b$, for which the solution is of the form

$$\chi = [A J_1(kr) + B Y_1(kr)] \cos \theta,$$  \hspace{1cm} (3.40)

where $k = \omega/c$. As boundary conditions, we require that the radial velocity

$$v_r = \frac{\partial \phi}{\partial r} = k[A J'_1(kr) + B Y'_1(kr)] \cos \theta$$  \hspace{1cm} (3.41)

has to equal $\hat{r} \cdot u$ at $r = a$ and vanish at $r = b$. We thus obtain the conditions

$$J_1(ka) A + Y_1(ka) B = u/k, \quad \text{and} \quad J_1(kb) A + Y_1(kb) B = 0.$$  \hspace{1cm} (3.42)

From these we can solve $A$ and $B$,

$$A = \frac{Y'_1(kb)}{J'_1(ka)Y_1'(kb) - Y'_1(ka)J'_1(kb)} \frac{u}{k},$$  \hspace{1cm} (3.43)

$$B = \frac{-J'_1(kb)}{J'_1(ka)Y_1'(kb) - Y'_1(ka)J'_1(kb)} \frac{u}{k}.$$  \hspace{1cm} (3.44)
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Using (3.39) we get $p = p_0 + i\omega \rho_s \chi$, and thus the force on the cylinder is

$$F = -i\pi a \rho_s \omega [AJ_1(ka) + BY_1(ka)].$$

(3.45)

Inserting $A$ and $B$ from (3.43) gives

$$F = -\frac{i\pi a \rho_s \omega u J_1(ka)Y_1'(kb) - Y_1(ka)J_1'(kb)}{k} = \frac{J_1(ka)Y_1'(kb) - Y_1(ka)J_1'(kb)}{k}.$$

(3.46)

In the limit $k \to 0$ this reduces to (3.38), but at higher $k$ it gives a series of resonances at $k_n$ corresponding to the zeros of denominator in (3.46).

3.2.3 Slip Theory

The hydrodynamic theory has been extended to lower temperatures by the slip theory. The hydrodynamic no-slip boundary condition states that the fluid velocity on the wire surface is the same as that of the surface, i.e. in the rest frame of the wire the tangential fluid velocity vanishes. It was discovered already in the nineteenth century that the no-slip boundary condition is not valid for rarified gases (see discussion in Ref. [25]), and in the present case the increasing mean free path makes the helium mixture behave as rarified gas. Therefore, one has to allow for slippage of the tangential component of the velocity on the wire surface. In the slip theory, the boundary conditions of the hydrodynamic theory are changed to allow slip, the magnitude of which is described by the slip length $\zeta$. The slip length for degenerate fermions was calculated from the Boltzmann equation for plane surface by Jensen $et \ al.$ [37]. The generalization to curved surfaces was discussed by Einzel $et \ al.$ [38], and by Bowley and Owers-Bradley [39] in collaboration with Perisanu and Vermeulen [30, 40]. Although the present work does not employ the slip length as a parameter, but is based on direct numerical solution of the Boltzmann equation, it is useful to study the concept of slip in connection to the present theory.

First we note that although the diffuse scattering (2.44) resembles the hydrodynamic no-slip boundary condition in that they both corresponds to scattering from a rough wire, there is a fundamental difference between the two conditions. In the diffuse scattering, the outgoing distribution has zero net tangential momentum in the rest frame of the wire, but this is not necessarily so for the incoming distribution. In the limit $\ell \to 0$ the collisions push tangential component of the incoming distribution to zero, leading to indistinguishability of the two conditions in the hydrodynamic limit. When the mean free path increases, however, this is no longer true, and there is slippage of the tangential component of the velocity on the wire surface.

Figure 3.6 shows an attempt to illustrate the boundary conditions in the laboratory frame. We have chosen a point $r$ on the wire surface, and present
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Figure 3.6: A schematic diagram for the boundary conditions on the wire surface for specular (blue) and diffuse (red) scattering at three values of mean free path $\ell$. The distributions are calculated using the numerical method described in chapter 4. The wire surface is shown as the dashed black line, and it separates the incoming (below the dashed line) and outgoing momentum directions. The diagram is drawn in the laboratory frame, the wire is moving to the right (black arrow). The length of the thin blue and red arrows pointing to discrete directions $\hat{p}$ show the relative magnitudes of $\psi_{\hat{p}}$ for specular and diffuse wire, respectively (see text). The thick blue and red arrows give the direction of averaged vector $b$. The real part of numerically calculated $\psi_{\hat{p}}(r)$ for various directions $\hat{p}$ (in $xy$-plane) by arrows pointing to these directions. The length of each arrow is given by $L_{\hat{p}} = 1 + \text{Re}(\psi_{\hat{p}})/uF$; since $-1 \lesssim \text{Re}(\psi_{\hat{p}}) \lesssim 1$ we have added 1 to avoid negative lengths. The averaged quantity $b$ is shown by the thick, shorter arrows. In the hydrodynamic limit $\ell = 0.01a$ we see that the diffuse boundary condition (2.44) leads to fluid moving with the wire, $\hat{b} = \hat{u}$ on the wire surface. For the specular boundary condition (2.43) we see that $b$ is nearly in the direction of $\hat{n}$. For $\ell = 10a$ the quasiparticles are decoupled from the wire: the incoming distribution is nearly isotropic in the laboratory frame, not in the rest frame of the wire.

In Fig. 3.7 we show the numerically calculated tangential velocity component $v_\theta$ and the component $\sigma_{r\theta}$ of the momentum flux on the wire surface as functions of $\ell$. Naturally, the local fluid velocity is defined only in the hydrodynamic region, but we generalized the concept to all values of $\ell$ in the sense of definition (3.6). An approximation for the slip length, calculated from two grid points near the wire, is shown in Fig. 3.8. We will consider the symmetric case where $v_\theta = v'_\theta u \sin \theta$. The slip length is defined as the distance behind the wire surface, at which the extrapolated relative tangential velocity reaches zero. Using the terms defined in equation (2.51), the no-slip condition can be written as $b'_r(a) = p_F$ and $b'_\theta(a) = -p_F$. We see that even for fully diffuse wire the no-slip condition is obeyed only in the hydrodynamic region.

We can derive some analytic limits for the tangential velocity and the transverse impedance in the ballistic limit. In the simplest case in the ballistic
Figure 3.7: a): The tangential velocity \( v'_\theta \) on the surface of the wire, as function of \( \ell \) and \( S \). In the hydrodynamic limit the diffuse curve \( S = 0 \) obeys the condition \( v'_\theta = -1 \) at \( \ell \to 0 \). The limiting values for \( v'_\theta \) in the ballistic limit are somewhat larger than predicted by the theory for case \( \omega = 0 \). In the ballistic limit \( v'_\theta \) should approach zero for the specular case, and \(-1/2\) for the diffusive case. b): The off-diagonal part \( d'_{r\theta}(a) \) of the momentum flux tensor (3.11). For specular wire, \( d'_{r\theta} \equiv 0 \), and for diffuse wire we expect in the ballistic limit \( d'_{r\theta} = -p_F/16 \).

Figure 3.8: a): An approximation of the slip length \( \zeta \) is obtained by linear extrapolation from two grid points closest to the wire (solid lines). The dashed lines give the slip length calculated from \( d'_{r\theta} \) and \( b'_\theta \) (3.52). The two results agree in the hydrodynamic limit. The black, short-dashed line shows slip calculated from Eq. (3.55). b): The parameter \( \beta \) as function of \( \ell \) calculated from the various proposed formulas (3.56), (3.57), (3.58), and (3.59), in the case \( S = 0 \). The solid lines give \( \beta \) calculated from our numerical results for slip length (panel a), using formula \( \beta = \zeta/(1+\zeta) \) (violet) or \( \beta = \zeta/(1+2\zeta) \) (red), for the diffuse case (\( S = 0 \)).

limit (\( \omega = 0 \), no chamber walls) we have \( \psi_{p_{\text{in}}} = 0 \), \( \psi_{p_{\text{out}}} = 2p_F(\hat{\mathbf{n}} \cdot \hat{\mathbf{p}}_{\text{out}})(\hat{\mathbf{n}} \cdot \mathbf{u}) \) for specular, and \( \psi_{p_{\text{out}}} = p_F(\hat{\mathbf{p}}_{\text{out}} + 2/3\hat{\mathbf{n}} \cdot \mathbf{u}) \) for diffuse case. Using these we can calculate the velocity on the wire surface in the ballistic limit

\[
\mathbf{v} = \frac{3}{p_F} \langle \hat{\mathbf{p}} \psi_{\mathbf{p}} \rangle \mathbf{p} = \frac{3}{2p_F} \langle \hat{\mathbf{p}}_{\text{out}} \psi_{p_{\text{out}}} \rangle \mathbf{p}_{\text{out}},
\]

which gives \( v'_\theta(a) = 0 \) for the specular, and \( v'_\theta(a) = -1/2 \) for the diffuse case.
We see that the combination of these,

\[ v'_\theta(a) = -\frac{1}{2}(1 - S), \]  

(3.48)

to a specular scattering fraction \( S \), is in reasonable agreement with the numerical results in Fig. 3.7 a.

Next, we consider the transverse mechanical impedance in terms of the momentum flux tensor component \( \Pi_{r\theta} \). We use the short hand notation \( \hat{d} = \langle \hat{p}\hat{p}\hat{\psi}_r \rangle \hat{\rho} \), and from Eq. (A.14) we find \( \Pi_{r\theta} = 3n_3d_{r\theta} \). Again, we consider the symmetric case, where \( d_{r\theta} = d'_{r\theta}u\sin\theta \). Now, using the same approximation as before in the ballistic limit, we find for the diffuse case

\[ d'_{r\theta} = \frac{p_F}{16}\hat{\theta} \cdot \hat{u} = -\frac{p_F}{16}u\sin\theta, \]  

(3.49)

and for the specular case \( d_{r\theta} \equiv 0 \), which were already calculated in section 2.5.4. Similarly, we found \( d_{rr} = p_F\hat{n} \cdot \hat{u}/4 \) for specular wire, and \( d_{rr} = 17p_F\hat{n} \cdot \hat{u}/72 \) for diffuse wire, which lead to the force expressions (2.61).

We see in Fig. 3.7 that the combined expression \( d'_{r\theta}(a) = (S - 1)p_F/16 \) is in agreement with the numerical results in the ballistic limit.

We notice that it is possible to derive an expression for the slip length \( \zeta \) using the values of \( v_\theta(a) \) and \( d_{r\theta}(a) \) in the hydrodynamic region. Using the hydrodynamic definition from (3.21)

\[ \frac{\sigma_{r\theta}}{\eta} = \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}, \]  

(3.50)

and assuming the symmetries (2.51), we get on the wire surface for the derivative

\[ \left. \frac{\partial v'_\theta}{\partial r} \right|_{r=a} = \frac{\sigma'_{r\theta}(a)}{\eta} + \frac{1 + v'_\theta(a)}{a}, \]  

(3.51)

where \( v'_\theta = v_\theta/(u\sin\theta) \), \( \sigma'_{r\theta} = \sigma_{r\theta}/(u\sin\theta) = -3n_3d'_{r\theta} \), and we have inserted \( v'_\theta(a) = 1 \). Using linear extrapolation \( v'_\theta(r) = v'_\theta(a) + (r - a)\partial v'_\theta/\partial r \) and demanding \( v'_\theta(a - \zeta) = -1 \) we get for the slip length

\[ \zeta = \frac{1 + v'_\theta(a)}{\partial v'_\theta/\partial r} = \frac{1 + v'_\theta(a)}{\sigma'_{r\theta}/\eta + (1 + v'_\theta(a))/a}, \]  

(3.52)

which was suggested by Einzel et al.[38]. The equation (3.50) is valid only in the hydrodynamic region, but we can study what happens to \( \zeta \) if we continue to use it for arbitrary \( \ell \). In the ballistic limit \( \sigma_{r\theta}/\eta \) vanishes and the expression for \( \zeta \) is reduced to

\[ \zeta = a\frac{1 + v'_\theta(a)}{1 + v'_\theta(a)} = a. \]  

(3.53)
Slip length calculated from \( (3.52) \) agrees with the numerical results using the extrapolation from two grid points near the wire, as shown in Fig. 3.7. For larger \( \ell \), equation \( (3.50) \) is not valid (the corresponding kinetic equation is more complicated), and approximation \( (3.52) \) fails. On the other hand, the numerical approximation depends to some degree on the grid properties, and the points chosen for the extrapolation (although the results seem to converge with increasing grid density).

**Slip Corrections**

In the slip theory of Jensen et al. [37] the equation for \( k \) and \( k' \) is

\[
k + ik' = 1 + \frac{4}{qa} \frac{\gamma(qa)}{1 - \beta qa \gamma(qa)} = 1 + \frac{1}{(k_0 - 1 + ik'_0)^{-1} - i \beta a^2/(2 \delta^2)},
\]

where \( \gamma(z) = -H_1^{(1)}(z)/H_0^{(1)}(z) \), \( k_0 \) and \( k'_0 \) are the original (no-slip) Stokes’ coefficients, and \( \beta \) can be defined in several ways. For the no-slip solution \( \beta = 0 \), while for the specular case, corresponding to infinite slip, we have \( \beta = 1/2 \). The other propositions for \( \beta \) depend on the slip length \( \zeta \), which is usually approximated by

\[
\zeta = 0.582 \frac{1 + S}{1 - S \ell}.
\]

The value 0.582 is from Einzel et al. [41], while Jensen et al. [37] originally suggested 0.579. For large \( \ell \), this definition is in clear contradiction with numerical results shown in Fig. 3.7, but can still be used as a parameter for the quantity \( \beta \) in the hydrodynamic theory. Jensen et al. gives

\[
\beta = \frac{\zeta}{\zeta + a},
\]

while Carless, Hall and Hook (CHH) [26] have proposed an extrapolation with slightly different form for \( \beta \),

\[
\beta = \frac{0.579 \lambda}{a} \left( \frac{1 + \alpha \lambda/a}{1 + \lambda/a} \right),
\]

where \( \alpha \) is a numerical constant of order 2. Guénault et al. [27] have proposed an extrapolation for \( \beta \), based on a fit to experiments,

\[
\beta = \frac{0.579 \lambda}{a} \sqrt{\frac{1 + 10 \alpha^2 \lambda/a}{1 + 10 \lambda/a}}.
\]

(Note that the factors of 10 are missing from the published formula [27, 31].) Einzel et al. [38] give for curved surface

\[
\beta = \frac{\zeta}{2 \zeta + a}.
\]
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which lead to the specular case $\beta = 1/2$ in the limit $\ell \to \infty$. A more complicated expression, taking into account the curvature of the wire surface, is given by Perisanu and Vermeulen [40] and Bowley and Owers-Bradley [39]. Although our results for $\zeta$ at large $\ell$ differ radically from (3.55), we can calculate $\beta = \zeta/(\zeta + a)$, which differs only slightly from $\beta$ calculated by using (3.59) (Fig. 3.8b). The impedance calculated with different slip-approximation are shown in Fig. 3.9, with comparison to our kinetic calculations and experiments of Martikainen et al. [29].

3.3 Ballistic Limit

When the absolute zero temperature is approached, the mean free path of $^3$He quasiparticles becomes infinite, and the collision term in the kinetic equation (2.42) vanishes. We are left with

$$\psi_{\hat{p}} = \psi_{\hat{p}}(s_0) e^{k s_0} + \int_{s_0}^{0} k \delta \epsilon_{\hat{p}} e^{k s} ds,$$

(3.60)

where the factor in exponent takes the form $k = -i \omega / v_F$ in the ballistic limit. The interactions between the quasiparticles are now visible only through the effective mass $m^*$, and the Fermi liquid interactions, parameterized by $F_l$. If the Fermi interactions are turned off, i.e. $F_l$ are set to zero, there remains no contribution to the inertia of the wire, and the system behaves as a gas of ballistic particles. When these interactions are taken into account, however, it is realized that in the collisionless limit the system is not an ordinary gas or a liquid, but a viscoelastic medium with unique properties.

The general case including time dependence and Fermi liquid interactions will be studied with the full numerical machinery only. Some semi-analytic results in the ballistic limit can be derived, but they require non-trivial numerical solution as well, and do not provide significant insight to the problem. In this section we will consider the limit $\omega = 0$, where the Fermi liquid interactions have no effect. We will also consider non-zero $\omega$ in the case that the interactions are neglected, and we will make comparison to the full numerical calculations that include the interactions.

3.3.1 Slow Oscillations

Let us first consider the limit of slow oscillations, $\omega \to 0$. This results in significant simplification, since the $\delta \epsilon$ term drops out, and the Landau-Boltzmann equation becomes (Paper II)

$$\hat{p} \cdot \nabla \psi_{\hat{p}}(r) = 0.$$

(3.61)
The equation implies that $\psi$ is constant along trajectories, and changes only when trajectory hits a wall. In the case of unlimited fluid, any incoming trajectory has $\psi_{in} = 0$. The distribution for outgoing trajectories is then obtained from the boundary condition (2.43) or (2.44). The resulting force was calculated in (2.61), and the corresponding impedance of the fluid is

$$Z'_{\text{spec}} = \frac{3\pi}{4}an_3p_F, \quad Z'_{\text{diff}} = \frac{43\pi}{48}an_3p_F.$$  

(3.62)
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These expressions give the maximum dissipation of the oscillation of the wire in this approximation, due to the normal fluid component \( \rho_n \). The reactive part of the impedance is zero if \( \omega = 0 \), but to first order in \( \omega \) the change in inertia is given by the superfluid component \( Z_s = -i\pi a^2 \rho_s \omega \), when interactions are neglected. Comparing to the high temperature limit, the difference in inertia comes from decoupling of the normal component of the fluid from the wire. In terms of the reactive part of impedance \( Z'' \), using (2.63), the difference to the high temperature limit is

\[
\tilde{Z}'' = \frac{Z''_s - Z''_\text{ideal}}{an_3p_F} = \frac{\pi}{1 + F_1/3} \frac{a\omega}{v_F}.
\]

(3.63)

We see that the Fermi liquid interactions have no explicit contribution to the expressions (3.62). The effect of interactions becomes visible when the container walls are taken into account, or when the frequency dependence is accounted for. These cases will be considered in the following.

3.3.2 Confinement

Next, we study confinement of the helium mixture by assuming a diffuse cylindrical container of radius \( b \), concentric with the wire. In addition to being a more realistic model of the experimental setup, the confinement allows to study the details of the Fermi liquid theory in the analytically more accessible ballistic limit. We consider only the case of diffuse container; in the ballistic absorbing walls correspond effectively to unlimited fluid.

First, we consider diffusive wire. We still assume \( \omega \) to be negligible, so the equation of motion is (3.61), with boundary conditions (2.44) and (2.47). Contrary to the unlimited case, we can not assume \( \psi_{p_n} = 0 \) on the wire surface, but need to take the container walls into account. Using the fact that \( \psi_p \) is now constant along trajectories, we find for the incoming distribution on the wire surface, at \( r_w \),

\[
\psi_p^{\text{in}}(r_w) = \psi_p^{\text{out}}(r_c),
\]

(3.64)

where \( r_c \) is the point on the container wall, where the trajectory comes from. Then we can use the boundary condition (2.47) i.e. \( \psi_p^{\text{out}}(r_c) = g_c(r_c) \) on the container wall. Using similar arguments and taking the geometry of the system into account, we derive expressions for the diffuse boundary condition terms \( g'_c \) and \( g'_w \) (2.53) in Appendix B,

\[
g'_c = A_i g'_c + B_i (g'_w + 2 \frac{2}{3} p_F) + \frac{2}{3} C_i p_F, \quad g'_w = D_i g'_c.
\]

(3.65)

The integral expressions \( A_i, B_i, C_i \) and \( D_i \) are defined in (B.9), (B.14), (B.19) and (B.20). These integrals are functions of \( a/b \), and give the corrections
due to the container walls. Integrals $A$ and $B$ can be expressed in terms of elementary functions, but for $B$ and $D$ numerical solution or expansion in the limit $a \ll b$ is needed. For specular wire we obtain by similar treatment

$$g'_c = p_F E_i + g'_c(F_i + A_i), \quad (3.66)$$

with integrals $E_i$ and $F_i$ defined in (B.30) and (B.32). In the same manner we obtain the force due to quasiparticles reflected back to the wire from the container walls,

$$F_g^{\text{diff}} = -a n_3 g'_c \left[ \frac{\pi}{2} D_i + \frac{3\pi}{4} \frac{b}{a} C_i \right] u, \quad F_g^{\text{spec}} = -a n_3 \pi g'_c G_i u, \quad (3.67)$$

where the integral $G_i$ is defined in (B.35). The total force on the wire in diffusive cylindrical container of radius $b$ in the ballistic limit, assuming $F_i = 0$ and neglecting $\omega$ is then, to first order in $a/b$,

$$F_{\text{spec}} = -a n_3 p_F \left( \frac{3\pi}{4} + 64 \frac{a}{9\pi b} \right) u, \quad (3.68)$$

$$F_{\text{diff}} = -a n_3 p_F \left[ \frac{43\pi}{48} + \frac{256 + 32\pi^2 + \pi^4 a}{64\pi} \frac{4}{b} \right] u. \quad (3.69)$$

The first terms in these expressions are due to the boundary conditions on the wire surface, Eq. (3.62), and the second terms are corrections due to container walls. The main conclusion is that the dissipation can be considerably increased by decreasing the chamber radius $b$, see Fig. 3.10, while the inertia is not affected in this approximation. In Fig. 3.11 we show the results from the full numerical calculations in the ballistic limit with varying container radius $b$, where the frequency and Fermi interactions are taken into account. Fig. 3.11 b) also shows the effect of the Landau force, introduced in Paper V, in particular its dependence on $b$ and $F_0$. This force describes the elasticity of the Fermi liquid, which leads to increased resonant frequency in the ballistic limit. We see that decreasing the container size can lead to considerable increase in the resonant frequency in the ballistic limit.

### 3.3.3 Non-Interacting Case

As a third approximation, we can take $\omega \neq 0$, but set the Fermi liquid terms $F_i$ to zero, so that $\delta \epsilon = 0$. In this case the kinetic equation takes the form

$$-i\omega \psi_{\hat{p}} + \hat{p} \cdot \nabla \psi_{\hat{p}}(r) = 0, \quad (3.70)$$

the solution of which is $\psi_{\hat{p}} = \psi_{\hat{p}}(s) e^{iks}$, where $s$ is the distance along a trajectory, and $k = -i\omega/v_F$. We see that the magnitude of $\psi$ is constant along a trajectory, but the complex phase changes. In particular, the contribution
3.3. BALLISTIC LIMIT

Figure 3.10: \textbf{a):} The dissipative part of impedance in the ballistic limit with diffusive chamber walls at distance \( b \). The dots present the analytic calculations to third order in \( a/b \) (for \( \omega = 0 \)), and the solid lines present the full numerical calculations (with \( F_i \)) for specular (violet) and diffusive (green) wire. Dissipation depends on \( S \) and \( b \), but is not sensitive to \( F_0 \) or \( F_1 \), or \( \Omega \): results from (3.71) or (3.72) would be indistinguishable from the solid lines. \textbf{b):} The reactive part of the impedance (resonant frequency) in the ballistic limit with diffusive chamber walls at distance \( b \) for specular (violet) and diffusive (green) wires. The lowest line (aqua) gives the case of non-interacting Fermi gas with the same frequency \( \Omega = a\omega/v_F = 0.017 \). The dots present the integral expressions (3.72). We see that the resonant frequency is sensitive to the Fermi interaction terms, in addition to \( S \) and \( b \).

of the quasiparticles reflected from the container walls, described by \( g_c \), experience phase modulation: for example \( \psi_{\mathbf{p}}(\mathbf{r}_w) = g'_c \mathbf{r}_c \cdot \mathbf{u} \exp(ks_c) \), where \( s_c \) is the distance along the trajectory to the container wall. In the case of unlimited fluid this approximation gives the same force on the wire as the first one (3.62), since we still have \( \psi_{\mathbf{p}}(\mathbf{r}_w) = 0 \), but in the case of chamber walls the results differ.

The treatment in section 3.3.2 has been generalized in Appendix B to arbitrary \( \omega \) by defining new integrals \( A_2, B_2, C_2, D_2, E_2, F_2 \) and \( G_2 \). Using these functions the impedance \( Z \) can be written as

\[
Z_{\text{spec}} = -i\pi a\omega^2 G\rho_s + \pi an_3p_F \left( \frac{3}{4} + \frac{E_2G_2}{1 - F_2 - A_2} \right), \tag{3.71}
\]

\[
Z_{\text{diff}} = -i\pi a\omega^2 G\rho_s + \pi an_3p_F \left( \frac{43}{48} + \frac{a}{3b} \left( D_2 + \frac{b}{a} C_2 \right)^2 \right), \tag{3.72}
\]

where we have used (3.38) to include the superfluid component. The results obtained from these semi-analytic integrals can be used as a point of comparison for the full numerical calculations, presented in chapter 4. In Fig. 3.10 \textbf{b)} we show the excellent agreement between the two approaches, when
Figure 3.11: a): Dissipation versus resonant frequency in the ballistic limit. The solid lines give the results for full scale of mean free path for three cases: non-interacting fluid with $F_0 = 0$ for diffuse wire (green), $F_0 = -0.33$, $F_1 = 0.27$ for diffuse wire (orange) and for specular wire (blue). The frequency is given by $\Omega = 0.16$, and the chamber radius $b = 10a$ for all three curves. The lines with long dashes give the corresponding results, but with absorbing container. The dashed lines give the corresponding results in the ballistic limit, with varying $b$. As $b$ is increased (toward $100a$), the curves converge to a constant value of $Z'$ (and $Z''$, eventually). b): The effect of $F_0$ on the curves, for varying chamber size $b$. The effect of $F_0$ is enhanced by restricted geometry.

$F_1$ are set to zero in the full numerical calculations.

The expressions (3.71,3.72) can be used to study the response of the fluid to the oscillation of the wire at arbitrary frequencies. It must be emphasized these expressions are valid to accuracy $(\omega a/v_F)^2$ only, since the coupling to the superfluid component $\delta \mu_4$ in $\delta \epsilon$ was neglected based on the smallness of $(\omega a/v_F)^2$. However, the integrals $A_2-G_2$ have interesting properties as such, which have been studied in Fig. 3.12. We consider the diffusive wire, and use an alternate reduced impedance

$$\tilde{Z} = \frac{Z - Z_s - Z_0}{a n_3 p_F}, \quad (3.73)$$

where we have subtracted the contribution from the superfluid part $Z_s = -i\pi a^2 \rho_s \omega$, and the simple boundary condition term (3.62) $Z_0 = (43\pi/48)an_3 p_F$. Note that this differs from the reduced impedance $\tilde{Z}$ defined in (2.64). Two interesting observations are made. First, we notice that the real and imaginary parts of $\tilde{Z}$ oscillate with period $\pi a/(b-a)$. Second, we see that $\tilde{Z}$ vanishes when $\omega \to \infty$. Both observations can be explained in a simple way by considering the form of the integrals $A_2-D_2$. As explained in Appendix B, the integrals have an exponential factor of the form $\exp(i\omega s/v_F) = \exp(i\Omega s/a)$, where $s$ is distance along trajectory. For the vanishing amplitude of $\tilde{Z}$ for
ω → ∞, we note that the complex phase given by the exponential term oscillates faster, until the whole integral cancels out.

When oscillations and the Fermi liquid interactions are accounted for, the integral equation (3.60) needs to be calculated numerically, the only difference to the solution of equation (2.42) being that \( 1/\ell = 0 \). This differs from the previous approximations already in the case of unlimited fluid.

![Figure 3.12](image.png)

Figure 3.12: (a): Using the integral expressions for \( A_2, \ldots, D_2 \) we have studied the real and imaginary parts of the reduced impedance \( \tilde{Z} \) as function of \( \Omega = a\omega/v_F \) for diffuse wire in diffuse container of radius \( b = 10a \). The situation resembles a damped harmonic oscillator: the reduced components \( \tilde{Z}' \) and \( \tilde{Z}'' \) oscillate around zero with period \( \pi a/(b - a) \), a phase difference \( \pi/2 \) between the real and imaginary parts, and a damping factor of approximately \( \exp(-\Omega) \). (b): The same data plotted in \( \tilde{Z}' \) versus \( \tilde{Z}'' \) diagram gives a spiral that approaches origin, which corresponds to the unlimited fluid solution, as \( \Omega \to 0 \). Here we have also plotted the case of interacting Fermi liquid (red dotted line), with \( F_0 = -0.44 \) and \( F_1 = 0.449 \). For small \( \Omega \) the differences to the noninteracting case are small, but for larger values we see that the interactions reduce the dissipation (\( \tilde{Z}' \)) and increase the resonant frequency (\( \tilde{Z}'' \)). Figure adapted from Paper V.

From equation (3.60) it can be immediately seen that the Fermi liquid interactions are mediated by the field \( \delta\epsilon_B(r, t) \), as already discussed in section 3.1.2. If the Fermi liquid interactions are turned off, \( F_l = 0 \), the integral term in (3.60) vanishes along with \( \delta\epsilon \). The field \( \delta\epsilon \) describes a peculiar wave propagation in the limit \( \omega\tau \gg 1 \), known as the zero sound [6, 7].

The origin of the Landau force was briefly discussed in section 3.1.2, where \( \delta\epsilon \) was described as a beam of quasiparticles. Figure 3.12 shows the numerically calculated \( \delta\epsilon \) at \( \ell = 500a \) near the wire. We see that rather than a beam, \( \delta\epsilon \) forms a cloud around the wire, with a shape described roughly by \( \delta\epsilon \propto -\cos \theta \). The real part of \( \delta\epsilon \) is an order of magnitude larger than
the imaginary part, so it is the dominating term. In Eq. (3.60) the factor $k = -i\omega/v_F$ in the integrand shows that in the force due to $\delta\epsilon$, the imaginary part is the dominating term, so that the force is reactional rather than dissipational (see Fig. 4.7). In other words, the Landau force is proportional to $\delta\dot{\epsilon}$ rather than $\delta\epsilon$, and the effect will be larger for large $\omega$, as indicated in Fig. 3.12. The shape of the field $\delta\epsilon$ and the negative sign of $F_0$ in mixtures determines the direction of the Landau force; a positive $F_0$ would lead to decrease in the resonant frequency, as discussed in Paper V.

![Image](image_url)

**Figure 3.13:** The real (a) and imaginary (b) parts of the energy correction term $\delta\epsilon\hat{p}(r,t)$ close to the wire (moving to the right). We have used $\hat{p} = -\hat{r}$ for the direction-dependent part of $\delta\epsilon$. The density term $c$ is dominating in $\delta\epsilon$, and since $F_0 < 0$ in mixtures, $\delta\epsilon$ is negative in front of the wire. Calculations are made in diffuse cylindrical container, $b = 10a$, $\Omega = 0.16$, $F_0 = -0.33$, $F_1 = 0.27$, $S = 0$. 
Chapter 4

Numerical Calculations

In this chapter we will describe the methods used in the numerical solution of the kinetic equation. The equation for $\psi_{\hat{p}}$ is solved in a lattice of discrete grid points $r_n$ around the wire, for discrete momentum directions $\hat{p}_{jk}$, by integrating along quasiparticle trajectories. In calculating the integral, the angular averages $c = \langle \psi_{\hat{p}} \rangle_{\hat{p}}$ and $b = 3 \langle \hat{p} \psi_{\hat{p}} \rangle_{\hat{p}}$ stored at each lattice point are used, with interpolation between the lattice points. By repeating the integration at lattice point $r_n$ for each momentum direction, new numerical angular averages $c_n$ and $b_n$ are calculated. This iterative process can then be repeated until the solution converges. For large $\ell$ convergence can be reached after only a few dozen iterations, but for small $\ell$ millions of iteration cycles may be needed. We therefore prefer to use a slightly different approach, the matrix method, which exploits the linearity of the problem.

Now, instead of calculating the values for $c_n$ and $b_n$ from the previous values (starting from an initial guess $c_0$ and $b_0$), we calculate the weights with which $c_n$ and $b_n$ depend on the values $c_m$ and $b_m$ at all lattice points $m$. After repeating this for all lattice points $n$, we can write a matrix equation for $c$ and $b$, where the boundary conditions appear as inhomogeneous term. The matrix equation can be solved using numerical matrix inversion.

We will consider specifically the cylindrically symmetric case, where the dependence on the plane angle $\theta$ is of particularly simple form. This makes the problem effectively one dimensional, and makes it easier to illustrate the various variables used in calculations. Another principal geometry used in calculations is the slab geometry, where wire oscillates between two parallel plates, with direction of motion perpendicular to the walls. This geometry corresponds to the experiments of Martikainen et al., and was used to calculate our main results in Paper V. The basic geometries are illustrated in Fig. 4.1. Other, more asymmetric geometries are discussed in section 4.2.
Figure 4.1: A schematic figure for lattices used in calculations for a cylindrical container with radius $b = 6a$ and a slab container with thickness $2h = 12a$. We also show some trajectories starting from an arbitrary point, two of the shown trajectories are reflected specularly from the wire. For the polar lattice the density of lattice point decreases exponentially with distance from the wire; more precision is needed close to the wire where the gradients are larger. In the slab geometry we use two different lattices, rectangular and cylindrical, which overlap in a transition region.

### 4.1 Numerical Method

The kinetic equation (2.49) to be solved can be written as,

$$
\psi(p) = Sg_c(r_e)e^{ks_c} + \int_{s_w}^{0} dsX_p(s)e^{ks} + S \int_{s_c}^{s_w} dsX_p'(s)e^{ks} + \left\{ S2pF(\hat{n} \cdot \hat{p})(\hat{n} \cdot u) + (1 - S)[pF(\hat{p} + 2/3\hat{n}) \cdot u + g_w(r_w)] \right\} e^{ks_w},
$$

where the position is parameterized as $r = r_n + s\hat{p}$, $s < 0$, and the integrand is

$$
X_p(s) = \frac{1}{\ell}\psi_p^\dagger(s) - \frac{i}{\nu_F}\delta\epsilon_p(s).
$$

Here we have used the boundary condition terms (2.45) and (2.48), and assumed that the trajectory is reflected from the wire at $r_w$ with distance $s_w$ along the trajectory, and hits the container at $r_c$ with distance $s_c$ from the starting point. In the latter integral the prime in direction $\hat{p}'$ stands for specularly reflected trajectory. For a trajectory with no collision with the
wire, we get simply (2.50)

\[ \psi^\rho(r_n) = g_c(r_c)e^{ksc} + \int_{s_c}^0 ds \left[ \frac{1}{\ell} \psi^le(s) - \frac{i}{v_F} \delta \epsilon^\rho(s) \right] e^{ks}. \tag{4.3} \]

Using the expression for the local equilibrium distribution \( \psi^le = c + \hat{p} \cdot b \) and the simplified energy correction term (2.38)

\[ \delta \epsilon^\rho = c \frac{F_0}{1 + F_0} + \frac{F_1}{3 + F_1} \hat{p} \cdot b, \tag{4.4} \]

the integrand \( X^\rho \) (4.2) can now be written as

\[ X^\rho = \frac{1}{\ell} \psi^le - \frac{i}{v_F} \delta \epsilon^\rho = c(\frac{1}{\ell} - \frac{i}{v_F} \frac{F_0}{1 + F_0}) + \hat{p} \cdot b(\frac{1}{\ell} - \frac{i}{v_F} \frac{F_1}{3 + F_1}). \tag{4.5} \]

It is numerically favorable to separate equation (4.1) into parts by

\[ \psi^\rho(r_n) = I^\rho(r_n) + B^\rho(r_n) + Q^\rho(r_n) + H^\rho(r_n), \tag{4.6} \]

where we have defined

\[ I^\rho(r_n) = \int_{s_w}^0 ds X^\rho(s)e^{ks}, \tag{4.7} \]

\[ B^\rho(r_n) = [S2pF(\hat{n} \cdot \hat{p})(\hat{n} \cdot \mathbf{u}) + (1 - S)p_F(\hat{p} + \frac{2}{3} \hat{n} \cdot \mathbf{u})]e^{ks_w}, \tag{4.8} \]

\[ Q^\rho(r_n) = = (1 - S)g_w e^{ks_w}, \quad H^\rho(r_n) = S g_c e^{ksc}. \tag{4.9} \]

Here we have separated the integral part \( I^\rho \), the \( \psi \)-independent part of the boundary condition \( B^\rho \), and the diffuse boundary conditions terms \( g_c \) and \( g_w \) that depend on the incoming distribution \( \psi^\rho_{in} \). The term \( Q^\rho \) takes into account the diffuse boundary condition term \( g_w (2.45) \) on the wire surface, and similarly \( H^\rho \) takes into account \( g_c (2.48) \) on the container wall. All of these quantities depend on \( t \) as \( \exp(-i\omega t) \), and on the position \( r_n \) where \( \psi^\rho \) is being calculated, in addition to \( \hat{p}_{jk} \). If there is no collision with the wire, we have

\[ I^\rho = \int_{s_c}^0 ds X^\rho(s)e^{ks}, \quad B^\rho = 0, \quad Q^\rho = 0, \quad H^\rho = g_ce^{ksc}. \tag{4.10} \]

The separation of the terms makes it easier to study the effects of boundary conditions on the quasiparticle distribution separately, but it also provides some numerical advantage. For example, the boundary collision term \( B^\rho \) is a crucial driving term, and due to its simple form and the separation of the terms (4.6), we can calculate it with more precision.

In order to use numerical methods we must discretize the volume occupied by the fluid, and the momentum directions. The quasiparticle distribution is
then solved at discrete lattice points \( r \) for discrete momentum directions \( \mathbf{p} \).

The momentum direction \( \mathbf{p}_{jk} = \sin \zeta_j (\cos \beta_k \mathbf{x} + \sin \beta_k \mathbf{y}) + \cos \zeta_j \mathbf{z} \) is parameterized by angles \( \beta_k \) and \( \zeta_j \), where coordinate \( z \) is fixed along the wire axis, and \( \mathbf{x} = \mathbf{u} \). As explained in section 2.5.3, we can limit the calculations to the \( xy \)-plane, and the location \( r_n = \mathbf{x}_i \rho_r \cos \theta_i + \mathbf{y}_j \rho_r \sin \theta_i \) can be parameterized with cylindrical coordinates \( \rho_r \) (distance from the axis), \( \theta_i \) (plane angle), or alternately we can use cartesian coordinates, \( r_n = \mathbf{x}_i \rho_r + \mathbf{y}_j \rho_r \).

4.1.1 Symmetric Case

To keep the discussion on a simple level, we will first describe the numerical method in the cylindrically symmetric case, where the dependence on the plane angle \( \theta \) is of a simple form (2.51). We have \( c = c'u \cos \theta, b_r = b'_r u \cos \theta, b_\theta = b'_\theta u \sin \theta \) for the local equilibrium distribution terms, and for the boundary condition terms we have \( g_c = g'_c \hat{n}_c \cdot u \) and \( g_w = g'_w \hat{n}_w \cdot u \). The values of \( c'_n = c'(r_n), b'_{r,n} = b'_r (r_n), \) and \( b'_{\theta,n} = b'_\theta (r_n) \) will be calculated and tabulated at each lattice point, and interpolations is used in calculating the integral between the lattice points. The quasiparticle distribution \( \psi_{njk} = \psi_{\mathbf{p}_{jk}}(r_n) \) for each lattice point \( n \) and for each discrete trajectory direction can be written, using Eq. (4.6), as the sum

\[
\psi_{njk} = I_{njk} + B_{njk} + Q_{njk} g'_w + H_{njk} g'_c,
\]

(4.11)

where the coefficients will be defined shortly, using (4.7-4.9). The terms \( B_{njk} \) do not depend on \( \psi \), but can be directly calculated as seen from Eq. (4.8).

The other terms depend on \( \psi \), i.e. on the values of \( \psi_{njk}^{le} \). We will make use of the short hand notation \( \varphi^{(1)}_m = c'_m, \varphi^{(2)}_m = b'_{r,m}, \) and \( \varphi^{(3)}_m = b'_{\theta,m} \). In appendix C it is shown that the integral (4.7) at a given lattice point \( r_n \) can be written as a sum over the values of \( c_m \) and \( b_m \),

\[
I_{njk} = \sum_{h=1}^{3} \sum_m a_{njm}^{(h)} \varphi^{(h)}_m,
\]

(4.12)

where \( a_{njm}^{(h)} \) are numerically calculated coefficients and where the sum is over all lattice points \( m \). Similarly, we define \( H_{njk} = S \hat{n}_c \cdot u e^{ks_c} \) (with \( S \) replaced by 1 if there is no collision with the wire) and \( Q_{njk} = (1 - S) \hat{n}_w \cdot u e^{ks_w} \).

Naturally, the distances \( s_c, s_w \) and the normal vectors \( \hat{n}_c \) and \( \hat{n}_w \) depend on the starting point \( r_n \) and the direction \( \mathbf{p}_{jk} \), as well as on the geometry of the system, and need to be numerically calculated.
4.1. NUMERICAL METHOD

Figure 4.2: The real parts of coefficients $Q_m^{(h)}$ and $H_m^{(h)}$ for three different mean free paths: $\ell = a$ (green), $\ell = 10a$ (blue), and $\ell = 100a$ (red). The horizontal axis is distance from the origin, corresponding to lattice points $r_m$. The solid lines give the dependence on $c'$ ($h = 1$), the dashed lines correspond to $b'_c$ ($h = 2$) and the short-dashed lines correspond to $b'_g$ ($h = 3$). From panel a) we see that the values of $\varphi_m^{(h)}$ close to the wire have stronger dependence on $g'_w$ than those further away, but for larger $\ell$ the dependence spreads further to the fluid. We also see that the dependence on $g'_w$ is strongest for the density component $c'$, and weakest for the tangential velocity component $b'_g$. In panel b) we see how $\varphi_m^{(h)}$ depends on $g'_c$: for small $\ell$ only the lattice points close to the container see the wall, but for larger $\ell$ $g'_c$ affects all the lattice points. Close to the wire the effect of container is screened by the shadow of the wire.

Ultimately we are interested in the averages of these terms, so that we can calculate the components $c$ and $b$ of the local equilibrium distribution $\psi^\Delta_p$. We write, for $i = 1, 2, 3$,

$$
\varphi_n^{(i)} = \sum_{m, h} d_{nm}^{i h} \varphi_m^{(h)} + H_n^{(i)} g'_c + Q_n^{(i)} g'_w + B_n^{(i)},
$$

(4.13)

where the ($\theta$-independent) averaged coefficient are defined as

$$
d_{nm}^{i h} = \frac{\langle a_{nmjk} \rangle \hat{p}}{u \cos \theta_n}, \quad d_{nm}^{2 h} = \frac{3 \hat{r}_n \cdot \langle \hat{p} a_{nmjk} \rangle \hat{p}}{u \cos \theta_n}, \quad d_{nm}^{3 h} = \frac{3 \hat{\theta}_n \cdot \langle \hat{p} a_{nmjk} \rangle \hat{p}}{u \sin \theta_n}
$$

(4.14)

for the integral terms, while for the boundary condition terms we have

$$
B_n^{(1)} = \frac{\langle B_{njk} \rangle \hat{p}}{u \cos \theta_n}, \quad B_n^{(2)} = \frac{3 \hat{r}_n \cdot \langle \hat{p} B_{njk} \rangle \hat{p}}{u \cos \theta_n}, \quad B_n^{(3)} = \frac{3 \hat{\theta}_n \cdot \langle \hat{p} B_{njk} \rangle \hat{p}}{u \sin \theta_n},
$$

(4.15)

for the container wall terms we write

$$
H_n^{(1)} = \frac{\langle H_{njk} \rangle \hat{p}}{u \cos \theta_n}, \quad H_n^{(2)} = \frac{3 \hat{r}_n \cdot \langle H_{njk} \rangle \hat{p}}{u \cos \theta_n}, \quad H_n^{(3)} = \frac{3 \hat{\theta}_n \cdot \langle H_{njk} \rangle \hat{p}}{u \sin \theta_n},
$$

(4.16)
Figure 4.3: The real parts of coefficients $g_{c,m}^{(h)}$ and $g_{w,m}^{(h)}$, with the same notation as in Fig. 4.2. In panel a) we see how the boundary conditions terms $g'_w$ on the wire surface depend on $\varphi_m^{(h)}$. For small $\ell$, the nearest lattice points have the largest influence, the $c'$ and $b'_r$ contribute with roughly the same magnitude, but with opposite signs. For larger $\ell$ the contribution is distributed evenly among the lattice points, with decreasing magnitude. In panel b) we see that $g'_c$ depends mostly on the closest points, particularly so for small $\ell$ (inset).

and for the wire surface terms we get

$$Q_n^{(1)} = \frac{\langle Q_{njk} \rangle}{u \cos \theta_n}, \quad Q_n^{(2)} = \frac{3\vec{r}_n \cdot \langle Q_{njk} \rangle}{u \cos \theta_n}, \quad Q_n^{(3)} = \frac{3\vec{\theta}_n \cdot \langle Q_{njk} \rangle}{u \sin \theta_n}. \quad (4.17)$$

To be precise, these angular averages are calculated at the lattice point $r_n$. The averages over momentum directions $\vec{p}$ are calculated numerically as

$$\langle f_{jk} \rangle_{\vec{p}} = \frac{1}{4\pi} \int_0^\pi d\zeta \int_0^{2\pi} d\beta \sin \zeta f(\zeta, \beta) = \sum_j \sum_k w_j f_{jk}, \quad (4.18)$$

for a function of type $f_{jk} = f(\vec{p}_{jk}) = f(\zeta_j, \beta_k)$. The weight factor $w_j$ contains normalization, and the appropriate weights for Gauss' integration formula, used for $\zeta$. For $\beta$ we use simply sum over equidistant angles with equal weights.

To have a complete set of equations, we still need to write the expression for $g'_c$ and $g'_w$. We have

$$g'_c = -\frac{2\vec{n}_c \cdot \langle \hat{p}_{\psi_m^c} \hat{p}_{m} \rangle}{\vec{n}_c \cdot u} = \sum_{m,h} g_{c,m}^{(h)} \varphi_m^{(h)} + g'_c H_c + g'_w Q_c + B_c, \quad (4.19)$$

$$g'_w = -\frac{2\vec{n}_w \cdot \langle \hat{p}_{\psi_m^w} \hat{p}_{m} \rangle}{\vec{n}_w \cdot u} = \sum_{m,h} g_{w,m}^{(h)} \varphi_m^{(h)} + g'_c H_w, \quad (4.20)$$

where the averaged, $\theta$-independent coefficients are similar to those in Eq. (4.13), but the averages are now calculated only over the incoming trajectories on the surfaces. (The terms $g_{c,m}^{(h)}, g_{w,m}^{(h)}, H_c, Q_c, B_c$ and $H_w$ can actually
be written in terms of the general coefficients introduced in (4.13) above. Comparing to the ballistic limit expressions (3.65), we see the connections $H_c \rightarrow A_i$, $Q_c \rightarrow B_i$, $B_c \rightarrow \frac{2}{5}P_F(B_i + C_i)$ and $H_w \rightarrow D_i$ in the case of diffuse wire. The real parts of coefficients $Q^{(h)}_m$ and $H^{(h)}_m$ are shown in Fig. 4.2, and those for $g^{(h)}_{w,m}$ and $g^{(h)}_{c,m}$ are shown in Fig. 4.3.

All these coefficients depend on the parameters $\ell/a$, $a\omega/v_F$, $b/a$, $S$, $F_0$ and $F_1$. Once the coefficients have been calculated, we can solve the distribution $\psi$ from the matrix equation

$$\Psi = D\Psi + B \Leftrightarrow \Psi = (I - D)^{-1}B,$$  \hspace{1cm} (4.21)

where $\Psi$ is a $3N + 2$ component vector $\Psi = (\varphi^{(1)}_1, \varphi^{(1)}_2, \ldots, \varphi^{(3)}_N, g_c, g_w)^T$, and $B = (B^{(1)}_1, B^{(1)}_2, \ldots, B^{(3)}_N, B_c, 0)^T$, and the $(3N + 2) \times (3N + 2)$ matrix $D$ contains the terms $d_{nm}^{(h)}$, $H_c$, $Q_c$, $H_w$, $g^{(h)}_{c,m}$, and $g^{(h)}_{w,m}$ in appropriate order:

$$D = \begin{pmatrix} d^{11}_{11} & d^{11}_{12} & \cdots & d^{11}_{1N} & d^{12}_{11} & \cdots & d^{12}_{1N} & d^{13}_{11} & \cdots & d^{13}_{1N} & H^{(1)}_1 & Q^{(1)}_1 \\
 d^{21}_{11} & d^{21}_{12} & \cdots & d^{21}_{1N} & d^{22}_{11} & \cdots & d^{22}_{1N} & d^{23}_{11} & \cdots & d^{23}_{1N} & H^{(2)}_1 & Q^{(2)}_1 \\
 \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
 d^{31}_{N1} & d^{31}_{N2} & \cdots & d^{31}_{NN} & d^{32}_{N1} & \cdots & d^{32}_{NN} & d^{33}_{N1} & \cdots & d^{33}_{NN} & H^{(3)}_N & Q^{(3)}_N \\
 g^{(1)}_{c,1} & g^{(1)}_{c,2} & \cdots & g^{(1)}_{c,N} & g^{(2)}_{c,1} & \cdots & g^{(2)}_{c,N} & g^{(3)}_{c,1} & \cdots & g^{(3)}_{c,N} & H_{c} & Q_{c} \\
 g^{(1)}_{w,1} & g^{(1)}_{w,2} & \cdots & g^{(1)}_{w,N} & g^{(2)}_{w,1} & \cdots & g^{(2)}_{w,N} & g^{(3)}_{w,1} & \cdots & g^{(3)}_{w,N} & H_{w} & 0 \end{pmatrix}.$$  \hspace{1cm} (4.22)

The solutions of $c$ and $b$ for diffuse and specular wire are shown in Fig. 4.4. In order to be more illustrative, we limit the consideration to the symmetric case, and plot the reduced quantities $c/(u_F \cos \theta)$, $b_c/(u_F \cos \theta)$ and $b_w/(u_F \sin \theta)$. We see in panels a and b that the density component $c$ starts to deviate from zero at larger $\ell$; in the hydrodynamic limit $\ell \ll a$ the fluid can be considered incompressible. In panels c and d we see the boundary conditions for the radial component: at $b_r(a)/(u_F \cos \theta) \equiv 1$ and $b_r(b) \equiv 0$. From the tangential velocity component in panels e and f we see how in the hydrodynamic limit the fluid follows the wire’s motion, especially in the case of diffuse wire. The magnitude of the tangential flow is reduced at lower temperatures. For the smallest mean free path $\ell = 0.01a$ at larger distances from the wire, the fluid approaches the ideal fluid behavior, more so in the case of specular wire. The values of $g_c$ and $g_w$ are shown in Fig. 4.5. The solution is illustrated also in Fig. 4.6 in terms of velocity fields around the wire. We see how the fluid flows around the wire to make way to it. In the hydrodynamic limit, in the case of diffusely reflecting wire, part of the fluid sticks to the wire and moves along with it.
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Figure 4.4: The real parts of local equilibrium components $c$, $b_r$, and $b_\theta$ for diffuse (left) and specular (right) wire in the cylindrically symmetric case. Four mean free paths are used, $\ell = 0.01a$ (orange), $\ell = 0.1a$ (green), $\ell = 1.0a$ (violet), and $\ell = 100a$ (red). For larger $\ell$ the profiles do not change significantly from the case $\ell = 100a$. The solid lines correspond to diffuse chamber of radius $b = 10a$, the dashed lines represent absorbing chamber of the same size, and the black dashed line shows the ideal fluid solution in unlimited fluid. We see that the results for the two chamber types differ only for large $\ell$; for $\ell = 0.01a$ they overlap.

**Force on the Wire**

To be able to calculate the force on the wire, we need to know the full distribution $\psi_\beta$ on the surface of the wire, not only the local equilibrium
components c and b. Once $\psi^l_p$ is known, it is straightforward to calculate $\psi^r_p$ on the wire surface using equation (4.6) [or (4.1) directly]. The force on the wire in the cylindrically symmetric case is (2.58)

$$\mathbf{F} = i \omega a^2 \pi \rho_s \mathbf{G} \mathbf{u} - 3n_3 a \pi (d''_{rr} - d''_{r\theta}) \hat{\mathbf{u}}.$$  (4.23)

We can calculate the forces due to the different factors separately, using

$$\psi_{ajk} = I_{ajk} + B_{ajk} + g_w Q_{ajk} + g_c H_{ajk},$$  (4.24)

where the index $a$ now emphasizes that we are on the wire surface. For the force on the wire we can now write

$$F/u = i \omega a^2 \pi \rho_s \mathbf{G} \mathbf{u} + F_{BC} + \sum_{m,h} f_{m,h} \psi_{m}^{(h)} + F_Q g_w + F_H g_c,$$  (4.25)

where the contribution of the boundary condition part $F_{BC}$ is calculated in (2.61) for specular and diffusive wire,

$$F_{\text{spec}} = -\frac{3\pi}{4} \pi an_3 p_F \mathbf{u}, \quad F_{\text{diff}} = -\frac{43\pi}{48} \pi an_3 p_F \mathbf{u},$$  (4.26)

and for the rest of the terms (we replace index $n$ by $a$ on the wire surface)

$$f_{m,h} = -3n_3 \pi a \left[ \langle (\hat{n} \cdot \hat{p})^2 a_{amjk}^{(h)} \rangle / \cos \theta - \langle (\hat{\theta} \cdot \hat{p}) (\hat{n} \cdot \hat{p}) a_{amjk}^{(h)} \rangle / \sin \theta \right],$$

$$F_Q = -3n_3 \pi a \left[ \langle (\hat{n} \cdot \hat{p})^2 Q_{ajk}^{(h)} \rangle / \cos \theta - \langle (\hat{\theta} \cdot \hat{p}) (\hat{n} \cdot \hat{p}) Q_{ajk}^{(h)} \rangle / \sin \theta \right],$$

$$F_H = -3n_3 \pi a \left[ \langle (\hat{n} \cdot \hat{p})^2 H_{ajk}^{(h)} \rangle / \cos \theta - \langle (\hat{\theta} \cdot \hat{p}) (\hat{n} \cdot \hat{p}) H_{ajk}^{(h)} \rangle / \sin \theta \right].$$

The division of the force into components is illustrated in Fig. 4.7.
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Figure 4.6: The numerically calculated velocity fields close to the wire for diffuse (red) and specular (blue) scattering (real parts). In the hydrodynamic region (panel a with $\ell = 0.01a$) we see how the fluid sticks to the diffuse wire, but for specular wire the fluid simply moves out of the way. When the ballistic limit is approached ($\ell = 10a$ in panel b), the tangential velocity components are much smaller near the wire. Far from the wire, difference between the two boundary conditions is small.

Absorbing Walls

Instead of diffuse container walls we can use an alternative wall type, absorbing walls. The diffuse boundary condition can be used to model scattering from a rough but solid wall. In the experiments, however, the sample volume is often surrounded by sintered silver for cryotechnical reasons. Some of the quasiparticles are then more likely to be absorbed rather than scattered. In diffuse scattering the outgoing quasiparticle distribution has knowledge of the incoming distribution only on the average. In the case of fully absorbing walls, on the other hand, the outgoing distribution has no dependence on the incoming quasiparticles.

The absorbing condition means that there can be flow across the container wall. Depending on the complex phase of $\langle \hat{p}\psi_p \rangle_p$, the flow can be in or out of the container. Recalling the harmonic time dependence $\exp(-i\omega t)$, we see that on the average there is no net flow across the wall, and no helium flows in or out the chamber.

In terms of the distribution $\psi_p$, we have $\psi_{p_{\text{out}}} = 0$, where $\hat{n}_c \cdot \hat{p}_{\text{out}} > 0$. In the numerical method described above this means that $g_{c}^{(i)} \equiv 0$, and it is not necessary to calculate the coefficients $H_n^{(i)}$ or $g_{c,m}^{(h)}$. The rows and columns corresponding to these terms can be removed from the matrix equations as well.
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Figure 4.7: The components of the force on the wire as functions of the mean free path \( \ell \). In panel a we show the real parts: the dashed line gives the constant contribution from the boundary condition (2.61), the blue line gives the contribution from \( \psi \), i.e. from the integral term \( I_{ajk} \) in Eq. (4.24), the green line gives \( F_w = F_Q g_w \), and the red line gives the contribution of the quasiparticles reflected from the container, \( F_c = F_h g_c \). From panel a we see that in the limit \( \ell \to 0 \) the other contributions exactly cancel the constant part \( F_{BC} \). In the ballistic limit the direct contribution from \( \psi \) nearly vanishes, i.e. the ballistic quasiparticle gas does not directly contribute to the dissipation, but the contributions from \( g_c \) and \( g_w \) increase the total dissipation above the constant \( F_{BC} \). For panel b the constant boundary condition part has no effect, and the effect of the container wall \( F_c \) is seen only for \( \ell \) well above \( a \). We also see that the direct contribution from \( \psi \) has more structure than is seen in the total force: the largest increase of frequency due to the \( \psi \)-part is at \( \ell \sim 10a \), rather than in the ballistic limit. The Landau force is seen as \( \text{Im}(F_\psi) > 0 \) in limit \( \ell \to \infty \).

4.1.2 General Case

The cylindrically symmetric case is useful for studying the theory numerically without unnecessary complications. However, the experiments are rarely performed in an idealized geometry, or in infinite medium. In the experiments of Martikainen et al. [29] the wire oscillated between two parallel plates, in a slab of thickness \( 2h = 16a \), the other dimension being large enough to be consider infinite. In this section we will briefly present the equations of the numerical method for a general geometry, where the cylindrical symmetry is lost.

The generalization of the numerical method to the asymmetric case is straightforward, but the notation is somewhat heavier. The lattice points are now in two dimensions, not only in the radial direction as before. This is not visible in the equations: the number of lattice points \( N \) is just larger. Moreover, we have no a priori known \( \theta \) dependencies, so we do not normalize the averages by \( \cos \theta \) and \( \sin \theta \) as in equations (4.14-4.17). The most explicit change in the equations is that the diffuse boundary condition terms \( g_c \) and \( g_w \) have now different values at the \( N_c \) or \( N_w \) lattice points at the surfaces,
respectively. Now, the equations for \( \psi_{njk} \) reads as before,

\[
\psi_{njk} = I_{njk} + B_{njk} + Q_{njk} + H_{njk},
\]

(4.27)

but the definitions of the last two terms are changed. We have \( Q_{njk} = \sum_t g_{w,t} Q_{ntjk} \) and \( H_{njk} = \sum_t g_{c,t} H_{ntjk} \), where the sums go through all the lattice points \( t \) on the boundaries. The effect of \( g_{c,t} \) to \( \psi_{njk} \) is found from

\[
H_{ntjk} = w_t e^{ks_c},
\]

where \( s_c \) is the distance to the wall along the trajectory, and \( w_t \) is a weight coming from linear interpolation on the container wall and (\( w_t \) is non-zero for the two lattice points closest to the point where the container is hit). We then define the averaged values by

\[
H_{nt}^{(1)} = \langle H_{ntjk} \rangle \hat{p}, \quad H_{nt}^{(2)} = 3 \hat{r} \cdot \langle \hat{p} H_{ntjk} \rangle \hat{p}, \quad H_{nt}^{(3)} = 3 \hat{\theta} \cdot \langle \hat{p} H_{ntjk} \rangle \hat{p}.
\]

(4.28)

Similarly,

\[
Q_{ntjk} = w_t e^{ks_w}, \quad Q_{nt}^{(i)} = \langle Q_{ntjk} \rangle \hat{p}, \quad Q_{nt}^{(2)} = 3 \hat{r} \cdot \langle \hat{p} Q_{ntjk} \rangle \hat{p}, \quad Q_{nt}^{(3)} = 3 \hat{\theta} \cdot \langle \hat{p} Q_{ntjk} \rangle \hat{p}.
\]

(4.29)

For the local equilibrium distribution we find

\[
\psi_n^{(i)} = \sum_{m,h} d_{nm}^{(h)} \phi_m^{(h)} + B_n^{(i)} + \sum_t Q_{nt}^{(i)} g_{w,t} + \sum_t H_{nt}^{(i)} g_{c,t}.
\]

(4.30)

The calculation of coefficients \( d_{nm}^{(h)} \) is the same as before, apart from that we have to use interpolation with respect to \( \theta \) as well. The boundary condition parameters \( B_n^{(i)} \) remain effectively the same, but the coefficients \( g_{c,t}, g_{w,t} \) get new expressions. We have

\[
g_{c,t} = -2 \hat{n}_{c,t} \cdot \langle \hat{p} \phi \rangle_{in} = \sum_{h,m} g_{c,t,m}^{(h)} \phi_m^{(h)} + B_{c,t} + \sum_s g_{c,s} H_{c,ts} + \sum_s g_{w,s} Q_{c,ts},
\]

(4.31)

and

\[
g_{w,t} = -2 \hat{n}_{w,t} \cdot \langle \hat{p} \phi_{njk} \rangle_{p} = \sum_{m,h} g_{w,t,m}^{(h)} \phi_m^{(h)} + \sum_s H_{w,ts} g_{c,s}.
\]

(4.32)

Equations (4.30), (4.31), and (4.32) can be presented as a matrix equation similar to before, \( \Psi = D\Psi + B \), yet now the matrices are larger. In calculating the force on the wire one cannot use symmetries, but needs to integrate over \( \theta \) explicitly.

In the slab geometry we have used two lattices, a cylindrical one near the wire, and a rectangular one close to the container walls, see Fig. 4.1. The lattices overlap partly for easier interpolation methods. In the \( y \)-direction the rectangular lattice is extended so far that the numerical results converge, usually at distance \( 24a \) or so. Formally, absorbing walls are assumed in this direction, as they are the simplest to model and have the least effect on the distribution. Effectively this corresponds to infinite length of the slab in this direction. The effect of the slab geometry to the results is discussed in Fig. 4.8.
4.2. Asymmetric Geometries

In Paper V we compared our numerical calculations to experiments [29, 31] made with two different wires at different concentrations. For some cases considered, the quantitative agreement between experiment and theory was worse than for others. In particular, there seems to be a systematic difference between the two wires, although they are assumed identical, apart from the
obvious difference in resonant frequency. The helium containers for the two wires are also supposed to have identical geometries. With the aim to explain these differences, we consider some deviations from the usual geometry. In the basic geometries, the cylindrical and the slab containers, the wire is assumed to be in the center of the container. We will now consider containers with wire displaced from the center, in both cylindrical and slab geometry. In the slab geometry, we will also consider case where the wire’s motion is not perpendicular to the walls.

Wire off the Center

It is possible that the wire is not exactly at the center of the slab in the direction of oscillation. We have shifted the wire by a small amount $\Delta x$ while keeping the distance between the walls constant at $2h = 16a$. In figure 4.9 (b) and (c) we show the results for $\Delta x = 0$, $\Delta x = 0.8a$ and $\Delta x = 1.6a$, and notice that the effect is negligible, and in the wrong direction for wire 2.

Figure 4.9: If the wire oscillates with an angle $\alpha$ to the $x$-axis, which is perpendicular to the walls, the minimum frequency becomes larger (panel a). Similarly, if the wire is not centered between the walls, but is shifted by $\Delta x$, the minimum frequency increases slightly (panel b). Therefore these asymmetries do not help in fitting the results to wire 2.

A numerical issue arises when shifting the wire’s position: It is numerically most efficient to use a cylindrical lattice as large as possible around the wire, i.e. one with radius $b = h$ so that the cylindrical lattice touches the container walls. If a smaller radius is used, the results differ significantly, unless the rectangular lattice is made very dense, with enormous cost in processor time. For $\Delta x > 0$ we must have $b \leq h - \Delta x$. Instead of increasing the number of lattice points to obtain realistic simulation, we give a ball park estimate for the effect of $\Delta x$. 
4.2. ASYMMETRIC GEOMETRIES

Motion not Perpendicular to Walls

Another possible asymmetry is the direction of the wire oscillation. If the fixed ends of the semicircular loop of wire are not exactly in the same level (as measured from the container walls), the direction of oscillation is not necessarily perpendicular to the (nearest) straight walls of the container. Apparently, if \( \mathbf{x} \cdot \mathbf{u} = u \cos \alpha \) with some nonzero, but small \( \alpha \), the only difference in the calculations comes from the analytic part of the boundary condition (4.8), where now \( \mathbf{n} \cdot \mathbf{u} = u \cos(\theta - \alpha) \), instead of the usual \( \mathbf{n} \cdot \mathbf{u} = u \cos \theta \). The results are shown in Fig. 4.9.

Wire in Non-Coaxial Cylindrical Container

![Figure 4.10: The discrete lattice for slightly non-coaxial container. Lattice points are at intersections of the lines, blue lines correspond to constant \( \eta \), and red lines to constant \( \xi \). The asymmetry parameter is \( c = 100 \) on left and \( c = 10 \) in the middle. On right, preliminary results for the asymmetric lattice, compared to symmetric case. The curves should overlap in the hydrodynamic region, but due to problems with interpolation, which is crucial in the limit \( \ell \to 0 \), this does not happen. For larger \( \ell \), however, the curve with large \( c \) agrees with the symmetric case. The interpolation problems for small \( \ell \) can be lifted for large \( c \), resulting in much better agreement with the symmetric case, but for small \( c \) these corrections fail. Assuming that the model is correct for large \( \ell \), the main conclusion from the preliminary calculations is that the slope of the ballistic branch of the \( (Z'',Z') \) curve is slightly affected by displacement of the wire from the center of the cylinder. Since the cylindrical container model is not very realistic to start with, we have not investigated this model further.

In this section we consider the asymmetry caused by a displacement \( d \) of the centers of the wire and the cylindrical container. For that purpose we use the so-called dipolar coordinates, and following Basset [42] we define new
coordinates $\eta$ and $\xi$ by

$$x + iy = c \tan \frac{\xi + i\eta}{2}, \quad (4.33)$$

where $c$ is a parameter that describes the asymmetry: we find for the displacement of the centers $d = \sqrt{b^2 + c^2} - \sqrt{a^2 + c^2}$, i.e. large $c$ corresponds to more symmetric case. Solving the real and imaginary parts separately gives

$$x = c \frac{2e^{\eta}\sin \xi}{e^{2\eta} + 2e^{\eta}\cos \xi + 1}, \quad y = c \frac{e^{2\eta} - 1}{e^{2\eta} + 2e^{\eta}\cos \xi + 1}, \quad (4.34)$$

or for the inverse transform

$$\eta = \tanh^{-1}\left(\frac{2cy}{c^2 + x^2 + y^2}\right), \quad \xi = \arctan \frac{2cx}{c^2 - x^2 - y^2}. \quad (4.35)$$

A constant $\eta$ corresponds to a circle of radius $r = c/\sinh \eta$ whose center is at $y = c \coth \eta = \sqrt{r^2 + c^2}$. The center of the wire, radius $a$, is then at $y_0 = \sqrt{a^2 + c^2}$. So far we have followed the notation on Basset, but now change to a (primed) coordinate system where the origin is at the center of the wire. We also rotate the coordinate system by $\pi/2$ counterclockwise, so that the displacement of the centers is in the direction of wire’s motion, the $x$-axis. We have $x' = y_0 - y$, and $y' = x$, with $x$ and $y$ from (4.34).

We discretize $\xi$ to $T$ equal intervals of length $\Delta \xi = 2\pi/T$ in the range $(0, 2\pi)$, and $\eta$ to $R$ equal intervals of length

$$\Delta \eta = \frac{\eta_a - \eta_b}{R}, \quad \text{where} \quad \eta_a = \ln(\frac{c}{a} + \sqrt{\frac{c^2}{a^2} + 1}), \quad \text{and} \quad \eta_b = \ln(\frac{c}{b} + \sqrt{\frac{c^2}{b^2} + 1}). \quad (4.36)$$

Note that the lattice points are not equidistant. There are difficulties in the interpolation with dipolar coordinates. We could naively interpolate linearly with respect of the coordinates $\eta$ and $\xi$, but they do not correspond to lengths in the $xy$-plane. A linear change in $\eta$ corresponds to exponential change in the distance from the origin, $\rho$. We use interpolation with respect to $\rho$ instead of $\eta$. Now the nearest lattice points used in interpolation may all have different $\rho$, which complicates the interpolation with respect to both $\rho$ and $\xi$.

### 4.3 Second Relaxation Time

In this section, we generalize the collision term to include a second relaxation time. We can expand $\psi_{\hat{p}}$ in real valued spherical harmonics:

$$\psi(\hat{p}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_{lm} Y_{lm}(\hat{p}), \quad \text{where} \quad \psi_{lm} = 4\pi \langle Y_{lm}(\hat{p}') \psi(\hat{p}') \rangle_{\hat{p}'} \quad (4.37)$$
4.4. Iterative Method

We use locally rotated real valued spherical harmonics, i.e. use angles defined by \( \hat{p} \cdot \hat{r} = \sin \zeta \cos \beta, \hat{p} \cdot \hat{\theta} = \sin \zeta \sin \beta, \) and \( \hat{p} \cdot \hat{z} = \cos \zeta. \) We write

\[
D_l(\hat{p}) = \sum_{m=-l}^{l} Y_{lm}(\hat{p}) 4\pi \langle Y_{lm}(\hat{p}') \psi(\hat{p}') \rangle_{\hat{p}'},
\]

so that \( \psi(\hat{p}) = \sum_{l} D_l(\hat{p}). \) A straightforward calculation gives

\[
D_0 = c, \quad D_1 = \hat{p} \cdot b,
\]

and for the new term, using cylindrical symmetry,

\[
D_2 = (\hat{p} \cdot \hat{r})(\hat{p} \cdot \hat{\theta})d_1 \sin \theta + [3(\hat{p} \cdot \hat{z})^2 - 1]d_2 \cos \theta + (\hat{p} \cdot \hat{r})^2 (\hat{p} \cdot \hat{\theta})^2 d_3 \cos \theta,
\]

where we have defined three new, \( \theta \)-independent variables

\[
d_1 = 15d_{r\theta}^{'}, \quad d_2 = \frac{5}{4}(3d_{zz}^{' - c'}), \quad d_3 = \frac{15}{4}(d_{rr}^{' - d_{r\theta}'}). \]

The integral in the Landau-Boltzmann equation (4.1) now takes the form

\[
I_{\hat{p}} = \int_{s_0}^{0} ds \left[ \frac{i\psi^{(\ell)}(s)}{\ell} - i\frac{\omega}{v_F} \delta \epsilon_{\hat{p}}(s) + \frac{1}{\ell}(1 - \frac{\tau}{\tau_2})D_2(\hat{p}, s) \right] e^{ks}.
\]

We notice that if \( \tau_2 = \tau, \) we return to the previous, simpler case. We denote \( X = 1 - \tau/\tau_2, \) which takes the value 0.65 according to Flowers and Richardson [43]. The role of parameter \( \ell \) in previous calculations is now taken by \( \ell_2 = v_F \tau_2 = v_F \tau/(1 - X), \) which should be used, for example, in the viscosity expression (3.7).

The method described in section 4.1 can now be generalized to take into account the three new fields. We have

\[
I_{njk} = \sum_{h=1}^{6} \sum_{m} a_{nmjk}^{(h)} \varphi_m^{(h)},
\]

where \( \varphi_m^{(4)} = d_1(r_m), \varphi_m^{(5)} = d_2(r_m), \) and \( \varphi_m^{(6)} = d_3(r_m). \) Similar generalization is made for all other quantities, \( B_n^{(i)}, Q_n^{(i)}, H_n^{(i)}, g_{c,n}^{(i)}, \) and \( g_{w,n}^{(i)}. \) The averaged quantities can then be solved from a matrix equation, and the force on the wire can be calculated. The preliminary results are shown in Fig. 4.11. We see that using a second relaxation time does not change the results significantly, except in the ballistic limit.

4.4 Iterative Method

We have used an iterative method as an alternative numerical tool. There, we start from an initial guess \( c_0 \) and \( b_0, \) use these in the integrand, and calculate
new values \( c_1 \) and \( b_1 \). The process is repeated until the solution converges. Calculation of one iteration is much faster than calculating the numerous coefficients needed in the "matrix method", but the number of iterations required may become large, especially so in the short mean free path limit. Close to the ballistic limit, however, the convergence is fast, and iterative method may easily overtake the matrix method in terms of efficiency. It has other advantages as well, one of them being that we can start from a "good guess" for \( c_0 \) and \( b_0 \), e.g. if we change \( \ell \) only little, the solution for the previous \( \ell \) is a good estimate for the new solution.

**Figure 4.11**: a): Preliminary results for the two relaxation times approximation for diffuse wire. We use parameter \( X = 1 - \tau/\tau_2 \), so that \( X = 0 \) corresponds to the previously calculated, single relaxation time approximation. The case \( X = 0.01 \) is indistinguishable from the single relaxation time case. There is a peculiar bend in the curve in the ballistic region, in the case \( X = 0.65 \); the curve reaches higher frequencies, but returns to the same end point with the other curves in the extreme limit \( \ell \to \infty \). In panels b and c we show the components of \( \tilde{Z} \) as functions of \( \ell = v_F \tau \) (short-dashed lines) and as functions of \( \ell_2 = v_F \tau_2 = v_F \tau/(1 - X) \) (solid lines). The latter parameter is now the "correct" one, and gives better agreement between the curves with different \( X \). (These calculations were made with a scarce lattice, and numerical inaccuracy is visible deep in the hydrodynamic region.)
4.5 Parameters

In the numerical calculations we use dimensionless quantities, such as $m^*/m_3$, $a\omega/v_F$, and $\psi/(u_F)$, and the results are obtained in form $Z/(an_3p_F)$. In order to have some connection to reality, we need to know the numerical values of these parameters. For that end, we study some of the properties of helium mixtures at different concentrations and pressures. The molar volume of a mixture of molar $^3$He concentration \( x_3 = N_3/(N_3 + N_4) \) at temperature \( T \) and pressure \( p \) is [44]

\[
V_m(x_3, p, T) = V_4(p, T)[1 + \alpha(x_3, p, T)x_3],
\]

where \( V_4(p, T) \) is the molar volume of pure \(^4\)He and \( \alpha \) is the BBP parameter [19]. The \(^3\)He atom has smaller mass than \(^4\)He, and takes more space due to larger zero point motion; the parameter \( \alpha \) describes how much more space is needed when a \(^4\)He atom is replaced by \(^3\)He. Edwards et al. [45] give (for small \( x_3 \))

\[
\alpha(x_3, 0, T) = 0.284 - 0.032 T,
\]

so \( \alpha = 0.284 \) at zero pressure and temperature. For other values of \( \alpha(p, 0) \) we use the results of Watson et al. [44]; for the molar volume of pure \(^4\)He, we use the data of Tanaka et al. [46], see Table 4.1 below. (Here \( \alpha \) is assumed independent of \( x_3 \).)

<table>
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<th>( p ) (atm)</th>
<th>( \alpha )</th>
<th>( V_4 )</th>
</tr>
</thead>
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<td>2.75793</td>
</tr>
<tr>
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<td>0.24</td>
<td>2.62019</td>
</tr>
<tr>
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<td>0.21</td>
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</tr>
<tr>
<td>20</td>
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<td>2.37761</td>
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Table 4.1: Experimental values of \( \alpha \) at \( T = 0 \) from Watson et al. [44], and \(^4\)He molar volume \( V_4 \) (\( \times 10^{-5} \) m\(^3\) at \( T = 50 \) mK) from Tanaka et al. [46].

The total density of the mixture is

\[
\rho = \frac{M}{V} = \frac{N_A(x_3m_3 + (1 - x_3)m_4)}{V_m} = \frac{N_A(x_3m_3 + (1 - x_3)m_4)}{V_4(1 + \alpha x_3)},
\]

(4.45)

where \( N_A = 6.022 \times 10^{23} \) is the Avogadro’s number, and the densities of the components are

\[
\rho_4 = \frac{N_A(1 - x_3)m_4}{V_4(1 + \alpha x_3)}, \quad \rho_3 = \frac{N_A x_3m_3}{V_4(1 + \alpha x_3)},
\]

(4.46)

The densities of the normal (n) and superfluid (s) parts of the fluid are (2.27)

\[
\rho_s = \rho_4 - \frac{Dm^*n_3}{1 + F_1/3}, \quad \rho_n = \frac{m^*n_3}{1 + F_1/3}.
\]
Once the $^3\text{He}$ density $\rho_3$ is known, we can calculate the $^3\text{He}$ number density $n_3 = \rho_3/m_3$ and the Fermi momentum $p_F$:

$$p_F = \sqrt[3]{3\pi^2\hbar^3 n_3}. \quad (4.47)$$

The values of $\rho_3$, $\rho_4$, $n_3$ and $p_F$ for some concentrations used in experiments [31] are given in table 4.2.

If we know the $^3\text{He}$ quasiparticle effective mass $m^*$, we can also calculate the Fermi velocity $v_F = p_F/m^*$, and various other parameters. The effective mass depends on the $^3\text{He}$ concentration and pressure, but the dependence is not accurately known. The first values for $m^*$ are given by the specific-heat experiments of Anderson et al. [47], namely $m^*/m_3 = 2.38$ for a 1.3% mixture, and $m^*/m_3 = 2.46$ for a 5% mixture. After that, numerous experiments have been conducted for various concentrations and at various pressure, using heat capacity, second sound velocity, normal density, and other methods. To mention a few, at saturated vapor pressure (SVP) only, Polturak and Rosenbaum [48] give specific heat effective mass $m^*/m_3 = 2.45$ for $x_3 = 0.0502$, and $m^* = 2.44$ for $x_3 = 0.0299$, Chocolacs et al. [49] find $m^*/m_3 = 2.37$ for a 1.3% mixture, and $m^*/m_3 = 2.46$ for a 5.1% mixture, and more recently Simons and Müller [50] report the somewhat smaller values $m^*/m_3 = 2.25$ for a 1.3% mixture, $m^*/m_3 = 2.29$ for a 3.7% mixture, and $m^*/m_3 = 2.31$ for a 5.5% mixture.

In order to obtain values at arbitrary concentration, a theoretical model is needed. One of the first was given by Bardeen, Baym and Pines (BBP) [19], followed by Disatnik and Brucker [51], and Hsu and Pines [52]. More recent models are presented by Yorozu et al. [53], and by Krotscheck et al. [15].

The theoretical models are largely based on zero concentration limit effective mass $m^*_0$, that is, the effective mass of a single $^3\text{He}$ atom immersed in

<table>
<thead>
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<th>$x_3$</th>
<th>$\rho_3$</th>
<th>$\rho_4$</th>
<th>$\rho$</th>
<th>$n_3 \times 10^{26}$</th>
<th>$p_F \times 10^{-25}$</th>
<th>$an_{3PF}$</th>
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<td>143.7</td>
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<td>152.2</td>
<td>22.28</td>
<td>4.26</td>
<td>0.0589</td>
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</table>

Table 4.2: Various density-related quantities are well known, and are tabulated here for the concentrations used in experiments [29]. In the last two lines pressure is 10 MPa, and the rest are at saturated vapor pressure. The units are kg/m$^3$ for the densities, $10^{26}$/m$^3$ for $n_3$, kg m/s $\times 10^{-25}$ for $p_F$, and kg/(ms) for $an_{3PF}$. The values of $\rho_n$ and $\rho_s$ are not shown here, since they depend on $m^*$ and $F_1$. 

<table>
<thead>
<tr>
<th>$x_3$</th>
<th>$\rho_3$</th>
<th>$\rho_4$</th>
<th>$\rho$</th>
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<th>$p_F \times 10^{-25}$</th>
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Figure 4.12: Parameters $m^*$ and $F_1$ used in calculations, and their relation to experimental results, at saturated vapor pressure $p \approx 0$. The solid curves on the left are calculated from Eq. (4.49), with $m_H$ given in the legend (in units of $m_3$), and the dashed curves on the right for $F_1$ are obtained from these using (4.48). The experimental results are from Anderson et al. [47], Polturak et al. [48], Chocolacs et al. [49], and Simons et al. [50]. The values of $F_1$ for experimental data are calculated from $m^*$ using Eq. (4.48), with $m_H = 2.34 m_3$ for all data except $m_H = 2.18 m_3$ for Simons et al. (if smaller $m_H$ were used, the values would be larger). The black dashed line on the left is calculated using the fit given by Hsu and Pines [52].

$^4$He. This mass is often obtained from experiments, see Ref. [48] for a table of values, but theoretical values exist as well. Many names and notations are used for $m_0^*$, not all of which necessarily mean the same. We will use the Landau-Pomeranchuck assumption that in the zero concentration limit the specific heat effective mass, $m_0^*$, equals the inertial effective mass $m_i = \rho_n/n_3$. In this thesis, we follow Krotscheck et al. using term hydrodynamic mass, and we denote it by $m_H$. From the assumption above it follows that

$$m_H = \frac{\rho_n}{n_3} = \frac{m^*}{1 + F_1/3}. \quad (4.48)$$

This is a consequence of Galilean invariance, but it should be noted that the Galilean invariance applies to the whole liquid, not to $^3$He part alone[6, 15], so Eq. (4.48) is approximative, but should be adequate for our purposes. The values of $F_1$ at various concentrations and pressures are calculated from this formula.

Krotscheck et al. [15] have calculated the dependence of effective mass $m^*$ from the concentration and pressure using microscopic theory. Based on their calculations, they give a simple fitting formula

$$m^*(x_3) = m_H + ax_3^{2/3} + bx_3 + cx_3^{5/3} + dx_3^{7/3} + \ldots, \quad (4.49)$$

with pressure dependent coefficients $a$, $b$, $c$, and $d$ given in Ref. [54]. (Note that the formula given in Ref. [15] is erroneous; instead of $cx_3^{5/2}$ there should
CHAPTER 4. NUMERICAL CALCULATIONS

Table 4.3: Various parameters that depend on the choice of $m^*$ and $m_H$. Here we use $m_H = 2.15$ at SVP, $m_H = 2.39$ at $p = 10$ atm (for 7% and 9.5%), and Eq. (4.49) for $m^*$. The second sound velocity $u_2$ is calculated from (5.3), using $F_0$ from the second column in table 4.4. The frequency parameters $\Omega = \omega/v_F$, correspond to the vacuum frequencies $f^{(1)}_{\text{vac}} = 1202.85$ Hz and $f^{(2)}_{\text{vac}} = 1857.7$ Hz [29, 31]. The velocities $v_F$ and $u_2$ are given in units m/s, the density is in units kg/m$^3$, and $m^*$ is given in units of m$^3$. Table adapted from Paper V.

<table>
<thead>
<tr>
<th>$x_3$</th>
<th>$m^*$</th>
<th>$v_F$</th>
<th>$F_1$</th>
<th>$\rho_n$</th>
<th>$u_2$</th>
<th>$\Omega_1$</th>
<th>$\Omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8%</td>
<td>2.26</td>
<td>21.10</td>
<td>0.151</td>
<td>4.21</td>
<td>11.10</td>
<td>0.022</td>
<td>-</td>
</tr>
<tr>
<td>3.6%</td>
<td>2.31</td>
<td>25.97</td>
<td>0.219</td>
<td>8.38</td>
<td>12.71</td>
<td>0.018</td>
<td>-</td>
</tr>
<tr>
<td>5.6%</td>
<td>2.34</td>
<td>29.60</td>
<td>0.266</td>
<td>12.96</td>
<td>13.38</td>
<td>0.016</td>
<td>-</td>
</tr>
<tr>
<td>6.6%</td>
<td>2.35</td>
<td>31.07</td>
<td>0.284</td>
<td>15.23</td>
<td>13.50</td>
<td>0.015</td>
<td>0.023</td>
</tr>
<tr>
<td>7.0%</td>
<td>2.60</td>
<td>29.56</td>
<td>0.269</td>
<td>19.75</td>
<td>12.56</td>
<td>0.016</td>
<td>0.024</td>
</tr>
<tr>
<td>9.5%</td>
<td>2.63</td>
<td>32.36</td>
<td>0.301</td>
<td>26.67</td>
<td>12.25</td>
<td>0.014</td>
<td>0.022</td>
</tr>
</tbody>
</table>

be $cx_3^{5/3}$. The correct formula is given in Ref. [54].) These results show that the effective mass is not linear in concentration, especially for small $x_3$. In our numerical calculations we use Eq. (4.49), with two choices of the zero concentration effective mass: $m_H = 2.15$ and $m_H = 2.34$. The former value gives a decent fit to $m^*$ measured by Simons et al. [50], while the latter corresponds to earlier experiments [47] and theory [19]. We have intentionally chosen the values far apart to give limits to the effect of $m^*$ in the calculations. The corresponding values at $p = 10$ atm are $m_H = 2.39m_3$ and $m_H = 2.64m_3$. Various parameters using the former values are given in table 4.3, for the latter values see Paper V. Comparison of calculated and measured values of $m^*$ and $F_1$ is shown in Fig. 4.12. In the calculations we use also $m_3 = 3.01603$ u, and $m_4 = 4.00260$ u, where $u = 1.660538 \times 10^{-27}$ kg.

For the Fermi liquid parameter $F_0$ there seems to be some variation in the existing data as well. Based on Anderson’s experiments [47], BBP give $F_0 = -0.202$ for the 1.3% mixture, and $F_0 = -0.423$ for the 5% mixture. Khalatnikov [17] gives the approximation $F_0 = -1.15x_3^{1/3}$, apparently based on the BBP values. Corruccini [20] has analyzed second sound and osmotic pressure data, and shows a collection of values for $F_0$ at varying concentration and pressure; at SVP he gives $F_0 = -0.10$ for the 1.3% mixture, and $F_0 = -0.26$ for the 5% mixture, using data of Murdock et al. [55].

Other values given by Corruccini’s analysis are based on the data from J. Landau et al. [56, 57] (osmotic pressure), Greywall and Paalanen [58, 59], and Brubaker et al. [60]. With the exception of the data by Landau et al., the data is obtained from measurements of second sound velocity $u_2$. As described by Corruccini, the data is extrapolated to $T = 0$ using the observed linear dependence of $u_2^2$ on the internal energy $U$ of an ideal Fermi gas having
Table 4.4: The fitting parameters $A$ and $B$ and the resulting values of $F_0$ from fits of the form $-F_0 = Ax^{1/2} + Bx^{1/3}$ to experimental data. The values in the second column from the right (†) are corrected by using effective masses from Krotscheck et al. [15] in the analysis.

<table>
<thead>
<tr>
<th>$x_3$</th>
<th>Fit to Ref. [55]</th>
<th>Fit to Ref. [56]</th>
<th>Fit to Ref. [60]</th>
<th>Fit to Ref. [19]</th>
<th>Fit to Ref. [60]†</th>
<th>Fit to Ref. [17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ (SVP)</td>
<td>2.511</td>
<td>2.113</td>
<td>1.640</td>
<td>1.5</td>
<td>0.942</td>
<td>0.0</td>
</tr>
<tr>
<td>$B$ (SVP)</td>
<td>-0.811</td>
<td>-0.446</td>
<td>-0.292</td>
<td>0.0</td>
<td>0.378</td>
<td>1.150</td>
</tr>
<tr>
<td>$A$ 10atm</td>
<td>1.287</td>
<td>2.860</td>
<td>-1.371</td>
<td>-0.0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$B$ 10atm</td>
<td>-0.236</td>
<td>-0.913</td>
<td>-0.0</td>
<td>-1.15</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1.8% (SVP)</td>
<td>-0.12</td>
<td>-0.17</td>
<td>-0.14</td>
<td>-0.20</td>
<td>-0.23</td>
<td>-0.30</td>
</tr>
<tr>
<td>3.6% (SVP)</td>
<td>-0.21</td>
<td>-0.25</td>
<td>-0.21</td>
<td>-0.28</td>
<td>-0.30</td>
<td>-0.38</td>
</tr>
<tr>
<td>5.6% (SVP)</td>
<td>-0.28</td>
<td>-0.33</td>
<td>-0.28</td>
<td>-0.35</td>
<td>-0.37</td>
<td>-0.44</td>
</tr>
<tr>
<td>6.6% (SVP)</td>
<td>-0.32</td>
<td>-0.36</td>
<td>-0.30</td>
<td>-0.39</td>
<td>-0.39</td>
<td>-0.46</td>
</tr>
<tr>
<td>7.0% 10atm</td>
<td>-0.24</td>
<td>-0.38</td>
<td>-0.36</td>
<td>-</td>
<td>-0.47</td>
<td>-</td>
</tr>
<tr>
<td>9.5% 10atm</td>
<td>-0.29</td>
<td>-0.47</td>
<td>-0.42</td>
<td>-</td>
<td>-0.52</td>
<td>-</td>
</tr>
</tbody>
</table>

The experimental results are typically given in terms of the resonant frequency of the wire $f_0$ and the line width $\Delta f$, associated with dissipation. The results are presented as $(f_0, \Delta f)$ pairs, where no precise knowledge of the temperature is required. We prefer to present our results in terms of $\tilde{Z}$ (2.64), where the strong dependence on $n_3$ is lifted. In order to convert the experimental results to this form we use

$$\tilde{Z}' = \frac{2\pi^2 a^2 \rho_w}{an_3 p_F} \Delta f, \quad \tilde{Z}'' = \frac{4\pi^2 a^2 \rho_w}{an_3 p_F} [(1 + \frac{1}{2} G \frac{\rho}{\rho_w}) f_0 - f_{vac}].$$

In Fig. 4.14 we show the experimental results of Martikainen et al. [29] and Pentti et al. [31] in terms of $\tilde{Z}$. We see that all the curves have roughly the
Figure 4.13: **Left:** The fits used for the Landau parameter $F_0$, and comparison to experiments. The fits using table 4.4 are shown by the dashed lines, except for the BBP fit, which is given by the solid red line. The experimental data is from Ref. [20], based on data from Refs. [55, 56, 57, 58, 59, 60]. Two sets of data points and one fit are corrected by using $m_H = 2.15$ and $m^*$ from (4.49), with the specification "(alt.)" in the legend. The correction results in smaller values of $F_0$, and removes the problem concerning one data point with suspicious value $F_0 > 0$.

**Right:** The extrapolation of $u^2$ to zero temperature for one specific case is shown with the two choices for $m^*$. Agreement with the earlier analysis is shown by the circles, corresponding to data from Ref. [20].

same size, unlike the $(f, \Delta f)$ curves. We see that there are two basic shapes for the curves, one corresponding to wire 1 and another corresponding to wire 2 with a steeper bend in the curve. The curve corresponding to $x_3 = 1.8\%$ (wire 1) is between these, with $\Omega$ close to those appropriate for wire 2, but smaller $|F_0|$ (Table 4.3). Now the results of numerical calculations presented in terms of $\tilde{Z}$ are not very sensitive to concentration, but the conversion (4.51) is, as seen in Fig. 4.14 b).
Figure 4.14: The experimental results [29, 31] presented in terms of the reduced impedance $\tilde{Z}(2.64)$. The results must be shifted in $Z''$ so that they extrapolate to zero for $\ell \to 0$. The shifts are different for each case, and correspond to shifts of order 1 Hz to the vacuum frequency. In panel b) we show that the conversion (4.51) is sensitive to concentration, unlike the numerical calculations. We also show a curve calculated with 5% smaller wire density, for which no shift in $Z''$ is needed (this overlaps with the 1.9% curve).
Chapter 5

Results

In this chapter we show some new aspects of the results presented in Paper V. We also present some new, unpublished results: in section 5.2 we make comparison to earlier experiments by Guénault et al. [27] and König et al. [28]. In Paper V we made comparison to the experiments of Martikainen et al. [29, 31] in nine different cases: six concentrations for wire 1 and three concentrations for wire 2, see table 4.3 for the parameters. Here we review these results from a different point of view. In Paper V the results were presented in terms of the reduced impedance $\tilde{Z}$. Here we show the same results also in terms of the resonant frequency $f$ and line width $\Delta f$. We show the ballistic limit effects 1-4 described in Paper III in this form as well, to give a more concrete picture of the magnitude of these Fermi liquid effects. In Paper V we preferred to present all results in terms of the quasiparticle mean free path $\ell$, and made no analysis of the corresponding temperatures. This temperature analysis is presented here, in section 5.1.1. In section 5.1.2 we revisit the second sound resonances, and show results for the slab geometry at higher frequencies.

In addition to the parameter values manifested by the available experimental data, we studied the dependence of $Z$ on parameters $b$ (or $h$), $S$, $\Omega$ and $F_0$ in a wider range in Paper V. We continue this line of study here. In Fig. 4.8 we already studied the effects of $b$ and $h$ in the case of absorbing walls. In Fig. 5.5 we study the effect of $\Omega$ in slab geometry, in connection to the second sound resonances. We speculate with a larger value of $|F_0|$ in Fig. 5.2. The effect of $S$ is studied in Fig. 5.6, where it is used as a fitting parameter.

5.1 Review of Paper V

In Paper V the comparison to experiments was made for each concentration separately. Here, in Fig. 5.1, we show a combination of the numerical results corresponding to different concentrations, and different wires. The
parameters are obtained from Table 4.3 and second column of Table 4.4, and the solid lines here correspond to the solid red lines in Fig. 10 of paper V. The governing parameters $\Omega$ and $F_0$ change so modestly between the different cases, that the reduced impedance $\tilde{Z}$ for different concentrations and even for different wires changes only little, especially so for the dissipative part. For the reactive component $\tilde{Z}''$ there is more deviation: in the ballistic limit $\tilde{Z}''$ is larger for wire 2 than for wire 1. We also see that for cylindrical container with diffusive walls $\tilde{Z}''$ is larger than for slab container, while the container with absorbing walls gives the smallest $\tilde{Z}''$ in the ballistic limit.

When the same data is presented as pairs $(f, \Delta f)$, using conversion (4.51) backwards, the curves for different concentrations have different scales, as seen in Fig. 5.2. We see that for the two curves at higher pressure, $x_3 = 7.0\%$ and $x_3 = 9.5\%$ at $p = 10$ atm for wire 1, the agreement between the experiments and calculations is poor, but for the four other concentrations at saturated vapor pressure the fit is somewhat better.

In Fig. 5.2 the vertical dashed lines show the four effects in the ballistic limit, described in Paper III, for concentrations 6.6\% and 9.5\%. The dotted vertical line corresponds to the high temperature frequency, or $Z''_{\text{ideal}} = -\omega a^2 \pi \rho G$. The first deviation from this (1) is due to decoupling of the $^3\text{He}$ part from the motion of the wire in the ballistic limit, as explained in connection to Fig. 3.6. This corresponds to density $\rho_3$ subtracted from the total density $\rho$ in the expression for $Z''_{\text{ideal}}$. The second contribution (2) is due to the fraction of $^4\text{He}$ that moves with the $^3\text{He}$ quasiparticles and is therefore also decoupled. This corresponds to density $Dm^* n_3/(1 + F_1/3)$ subtracted from $\rho$, see (2.27). Together the effects (1) and (2) result in decoupling of
5.1. REVIEW OF PAPER V

Figure 5.2: The numerical results for wire 1 and wire 2 for all concentrations, as resonant frequency $f_0$ vs. frequency width $\Delta f$. The vertical lines correspond to the four effects on the impedance, described in text, and to the (shifted) high temperature frequency used in conversion (4.51). In panels a) and c) the vertical lines correspond to $x_3 = 9.5\%$, and in panel b) to $x_3 = 6.6\%$. The numerical calculations are done using slab geometry with $h = 8a$, for diffusive wire and diffusive walls. The red dashed lines correspond to curves with larger $|F_0|$ with absorbing container walls.

the normal fluid component from the motion of the wire, and thus to $\rho_n$ subtracted from the total density. The third effect (3) is caused by the Fermi liquid interactions, mainly $F_0$, and needs to be calculated numerically. This effect was named ”Landau force” in Paper V. Finally, the fourth effect (4) is due to confinement, and for the small containers considered here, it has the largest influence. The third effect is obtained by comparing the numerical results using $F_i = 0$ to the usual calculations using $F_i$ from table 4.3, with
both calculations in a large container. The fourth effect is obtained in a similar way, by comparing the calculation made in the large container to those in the restricted geometry, in this case a slab of thickness $2h = 16a$.

### 5.1.1 Temperature Dependence

<table>
<thead>
<tr>
<th>$x_3$ (%)</th>
<th>$c_2$</th>
<th>$c_1$</th>
<th>$c_0$</th>
<th>$\bar{c}_2$</th>
<th>$\bar{c}_1$</th>
<th>$\bar{c}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>4.2</td>
<td>0</td>
<td>0.008</td>
<td>0.75</td>
<td>0</td>
<td>0.0014</td>
</tr>
<tr>
<td>3.6</td>
<td>13</td>
<td>0.2</td>
<td>0.007</td>
<td>0.89</td>
<td>0.014</td>
<td>0.0005</td>
</tr>
<tr>
<td>5.6</td>
<td>22</td>
<td>0.45</td>
<td>0.0025</td>
<td>0.85</td>
<td>0.017</td>
<td>0.0001</td>
</tr>
<tr>
<td>6.6</td>
<td>28</td>
<td>0.5</td>
<td>0.0035</td>
<td>0.87</td>
<td>0.016</td>
<td>0.0001</td>
</tr>
<tr>
<td>average</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.87</td>
<td>0.016</td>
<td>0.00023</td>
</tr>
</tbody>
</table>

Table 5.1: The left side of the table gives the values from Pentti et al.\([31]\), the right side gives the corresponding reduced values (see text). The units are Pa s K\(^{-9}\), Pa s K\(^{-6}\), and Pa s 10\(^{-3}\) for $c_2$, $c_1$, and $c_0$, respectively. The last line gives averaged values (over the three largest concentrations) of the reduced quantities: these values are used in calculating $T(\ell)$ in the figures in this chapter.

In order to fix the temperature, we need to know how it is related to the mean free path $\ell$. Previously we have assumed that for Fermi liquid $\eta = \frac{1}{2} n_3 p_F \ell$ (3.7), and that $\eta T^2$ is (nearly) constant (for fixed $x_3$). For higher temperatures, there are small corrections to this. Following Pentti et al.\([31]\), we write

$$\eta = c_2 T^{-2} + c_1 T^{-1} + c_0,$$  \hspace{1cm} (5.1)

with the values $c_i$ given in table 5.1.

We use reduced, dimensionless, coefficients $c'_0 = c_0/(an_3 p_F)$, $c'_1 = c_1/(an_3 p_F T_0)$, $c'_2 = c_2/(an_3 p_F T_0^2)$, $\eta' = \eta/(an_3 p_F) = \ell/(5a)$, and $T = T'/T_0$, where $T_0 = 1$ mK. Based on the simple expression (3.7) for $\eta$, we expect the reduced coefficients to be the same for all concentrations. The reduced coefficients calculated from the experimental parameters are shown on the right hand side of table 5.1, and we see some variation between the concentrations. For comparison, we find from the data of König and Pobell \([28]\) $c'_2 = 0.70$ for $x_3 = 0.98\%$ and $c'_2 = 0.93$ for $x_3 = 6.12\%$.

Based on (3.7) and (5.1) the temperature can now be expressed in terms of $\ell$ (in mK) as

$$T' = \frac{2c'_2}{-c'_1 + \sqrt{c'_1^2 - 4c'_2[c'_0 - \ell/(5a)]}}.$$  \hspace{1cm} (5.2)
Figure 5.3: **Left:** Here we show the numerical results with various boundary conditions, and compare to experimental results for $x_3 = 3.6\%$ [29]. The solid lines correspond to diffusive containers: a slab of thickness $16a$ (blue) and a cylinder of radius $8a$ (red). The dashed lines correspond to similar containers with absorbing walls, and to a larger cylinder with radius $80a$ (green). The dotted and dashed black lines give hydrodynamic solution for specular and no-slip boundary conditions in unlimited fluid. The thick dashed line corresponds to a wire in a cylinder of radius $8a$ with no-slip condition. The yellow diamonds show the experimental data. We have calculated some temperatures using (5.2) for the slab geometry, for diffusive and absorbing (values in cursive) walls. **Right:** The dissipation as a function of the mean free path $\ell$ (lower horizontal axis) or the temperature (upper axis). The experimental data is from Pentti et al. [31] in the high-temperature region, where independent measurement of the temperature was made. We see that above 3 mK the experimental data fits rather well to most of the calculations, which are nearly the same for all boundary conditions at these temperatures.

This expression diverges for small $\ell$ if $c'_0 > c'_2/(4c'_2)$. We note that at the low temperatures that we concentrate on, $c'_2$ is the dominating term, and the other two are less important.

It should be noted that viscosity is often directly confused with dissipation, using some pre-determined hydrodynamic formula. We see from Fig. 5.3 and 5.4 that this may be a fair approximation at higher temperatures, but below 4 mK the boundary conditions including confinement have a crucial role and need to be taken into account. Interpreting experiments in terms of viscosity alone may easily lead to apparent concentration and temperature dependence more complicated than expected from the simple Fermi liquid form (3.7).

In Fig. 5.3 a) we show the calculated temperatures for two cases in the slab geometry with $h = 8a$, one with diffusive walls and the other with absorbing walls. The curves overlap for small $\ell$, and follow the same tem-
Figure 5.4: The temperature dependence of the dissipative ($Z'$) and reactive ($Z''$) components. Apart from the reactive part of large cylindrical container, the values saturate below approximately 0.4 mK. The minimum frequency takes place at around 3 mK, although the exact value depends on the boundary conditions. The temperature is calculated from (5.2) using the averaged values in table 5.1.

5.1.2 Second Sound Resonances

Khalatnikov gives for the second sound velocity in mixtures [17]

$$u_2^2 = \frac{1}{3} \left( 1 + \frac{F_1}{3} \right) \left( 1 + F_0 \right) \left[ 1 - \frac{\rho_n}{\rho_4} \left( \alpha_1 (1 + \frac{F_1}{3}) + D \right) \right]^2 v_F^2, \quad (5.3)$$

where $\alpha_1 m^*/m_4 = 1 + \alpha$. The associated wavelength is $\lambda = u_2/f = 2\pi u_2/\omega$. The values for $u_2$ at different concentrations are given in table 4.3. In the numerical calculations with diffuse container we observe peculiar behavior in both reactive and dissipative parts of the impedance, when $b\omega/v_F$ approaches unity. We associate this behavior with second sound resonances.

The resonances for a cylinder oscillating inside a stationary coaxial cylinder in the case of ideal fluid were discussed in section 3.2.2. From (3.46) we find the first resonance at $kb = 1.80$ for $b = 10a$ ($kb = 1.78$ for $b = 8a$), where $k = \omega/c$. For the velocity of sound $c$ we can now use the reduced second
5.2. ADDITIONAL RESULTS

In this section we show previously unpublished results, illustrating generalizations of our method. In figure 5.6 we compare our calculations to the experimental results of Guénauld et al. [27], and König et al. [28]. We do not know the details of the experimental geometries, but for the former case the authors give a minimum distance 24a to the walls, and at least 30a for the latter case. At least in the latter case the geometry of the container is much more complicated than our model, but since the distance to the walls is so large, this should not be important. There exist other data on vibrating wire experiments in helium mixtures [26, 40, 61, 30], but these are not considered here.

In figure 5.6 we see that using \( S = 0 \), following the analysis in Fig. 5.1, results in poor agreement between the calculations and experiments. We have therefore used the fraction of specular scattering \( S \) as a fitting parameter to obtain better agreement. We see that values of \( S \) as high as 0.6 are
required for decent fits, keeping the other parameters fixed. All experiments considered here use nominally identical wires made of Tantalum, with diameter \( d = 125 \, \mu \text{m} \) and density \( \rho_w = 16700 \, \text{kg/m}^3 \). It seems improbable that the nature of scattering in terms of \( S \) would change from \( S = 0 \) used for the experiments of Martikainen et al. [29] to \( S = 0.6 \) suggested by the fits in Fig. 5.6.

Figure 5.6: a) : Comparison to results of Guénault et al. [27] for \( x_3 = 6.5\% \). The vacuum frequency \( f_{\text{vac}} = 897 \, \text{Hz} \) is shifted so that \( \tilde{Z} \) starts from zero at \( \ell = 0 \). We have no knowledge of the geometry of the container, other than that the distance to the closest wall is \( 24a \). The fit is not very good using either \( h = 24a \) for slab geometry or \( b = 24a \) for cylinder using diffusive wire, but for \( S = 0.6 \) a decent fit is obtained. b): Comparison to results of König et al. [28] for a 6.12% mixture at three different pressures. When presented in terms of \( \tilde{Z} \), the measured points follow almost the same curve. The vacuum frequency \( f_{\text{vac}} = 1883 \, \text{Hz} \) is slightly adjusted so that \( \tilde{Z} \) extrapolates to zero for \( \ell \to 0 \). The numerical calculations are done at zero pressure, in cylinder of radius \( b = 30a \) with absorbing walls, and a decent fit is obtained for a wire with specular scattering fraction \( S = 0.6 \). We note that the fit is rather arbitrary, since the experimental geometry is complicated, and the measurements apparently did not reach long mean free paths.
Chapter 6
Conclusions

We have expressed the Fermi liquid theory for $^3$He-$^4$He mixtures in a form where the effects of the superfluid and normal components on the mechanical properties of the fluid are explicit. We have utilized a transformed distribution which is particularly suitable for numerical calculations. We have presented a numerical method for solving the quasiparticle distribution for full range of mean free path for fluid disturbed by an oscillating wire. We studied the reliability of this numerical solution, and found that it is in excellent agreement with semi-analytic results in the limits of vanishing and infinite mean free path.

We have shown that due to the Fermi liquid interactions, the $^3$He-$^4$He mixture at low temperatures acts like an elastic medium. Related to this, we have introduced a new phenomenon, the Landau force, which explains the observed increase of resonant frequency of the wire in the ballistic limit. We have studied the effect of Fermi liquid parameters $F_l$ and $m^*$ in detail, as well as the effects of frequency and concentration of the helium mixture. We have shown that the geometry of the experimental setup has crucial role in interpreting the measurements, along with boundary conditions on the wire and container surfaces. We have found that the quasiparticles reflected from the container walls back to the wire cause second sound resonances in the hydrodynamic region, and other type of quasiparticles interference in the ballistic limit. Finally, we have made comparison to experiments, and found fair quantitative agreement between measurements and theory, with only few open questions remaining.

The theory and numerical method presented in this thesis could be directly applied to other types of oscillating viscometers. Experimental data exists for measurements made with torsional oscillators, quartz tuning forks, oscillating grids, ions, etc. In addition to different geometries, there are boundary conditions that were not studied here, such as (partially) specular container walls, or angle dependent scattering. Another natural generalization would be to study superfluid $^3$He.
Perhaps the most obvious improvement of the numerical method would be to take the coupling to the superfluid component into account. This would allow to study larger frequencies, in particular for examining the second sound resonances in more detail. A more realistic collision term could also be used. Finally, a full three dimensional calculating taking the exact experimental geometry into account might give some additional information on the validity of the theory.

To summarize, we note that the Fermi liquid theory is a fundamental theory of interacting many body systems, a standard against which other theories can be compared. We have studied interacting fermions in a system where the mean free path can be varied through the full range from the hydrodynamic regime to the ballistic limit. We have shown that the Fermi liquid theory beautifully explains the dynamics of $^3$He quasiparticles in superfluid $^4$He background, as measured in vibrating wire experiments. Using numerical methods, the theory explains the transition from the hydrodynamic region to the ballistic limit, which has been poorly understood so far. In the ballistic limit, the theory explains how the Fermi interactions lead to a completely new feature, elasticity of the quasiparticle fluid.
Appendix A

Hydrodynamic Limit

The Navier-Stokes equation can be derived from the kinetic equation. The quasiparticle distribution function $\psi_\hat{p}$ can be expanded in spherical harmonics. Due to rapid scattering in the hydrodynamic limit, the higher terms are assumed small, and we consider the three first terms in the form

$$\psi_\hat{p} = c + \hat{p} \cdot b + \hat{p} \cdot \hat{\leftrightarrow} Y \cdot \hat{p},$$  \hspace{1cm} (A.1)

where we have already anticipated that the first two terms are nothing but the familiar $c$ and $b$. The definitions of the terms are found by taking the (weighted) angular averages of (A.1) over the momentum directions $\hat{p}$

$$\langle \psi_\hat{p} \rangle_\hat{p} = c + \frac{1}{3} \text{Tr } \hat{\leftrightarrow} Y, \quad \text{and } \langle \hat{p} \psi_\hat{p} \rangle_\hat{p} = \frac{1}{3} b.$$  \hspace{1cm} (A.2)

The terms in series expansion need to be orthogonal, so we must set $\text{Tr } \hat{\leftrightarrow} Y = \delta_{ij} Y_{ij} = 0$ (summation over repeated indices). The definitions of $c$ and $b$ assume their familiar form. Using weight $\hat{pp}$ in the average gives

$$\langle \hat{pp} \psi_\hat{p} \rangle_\hat{p} = \frac{1}{3} c \hat{1} + \langle \hat{pp} \hat{p} \rangle_\hat{p} \hat{Y}.$$  \hspace{1cm} (A.3)

We define $\hat{d} = \langle \hat{pp} \psi_\hat{p} \rangle_\hat{p}$, and using the tensor identity $(\langle \hat{pp} \hat{p} \rangle_\hat{p})_{ijkl} = (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})/15$, and the condition $\delta_{ij} Y_{ij} = 0$ we can write $d_{ij} - \delta_{ij}/3 = (Y_{ij} + Y_{ji})/15$. The left side is clearly symmetric, so $\hat{Y}$ must be symmetric as well, and we get

$$\hat{Y} = \frac{15}{2} \hat{d} - \frac{5}{2} c \hat{1}.$$  \hspace{1cm} (A.4)

For later use, we need the divergence of the third order tensor average

$$\nabla \cdot \langle \hat{pp} \psi_\hat{p} \rangle_\hat{p} = \nabla \cdot \langle \hat{pp} \hat{p} \rangle_\hat{p} \cdot b = (\nabla b + b \hat{\nabla} \nabla \cdot b) \hat{1},$$  \hspace{1cm} (A.5)

where $\hat{\nabla}$ operates to the left.
Using $k = 1/\ell - i\omega/v_F$, we can rewrite the kinetic equation as

$$
k\psi + \nabla \cdot \hat{p}\psi = \frac{\psi^{le}}{\ell} - \frac{i\omega}{v_F} \delta\epsilon. \quad \text{(A.6)}$$

We notice that $\langle (\hat{p}\hat{p} - \frac{\hat{1}}{3})\delta\epsilon\rangle \frac{\hat{p}}{\rho} = 0$, so we multiply equation (A.6) by $\hat{p}\hat{p} - \frac{\hat{1}}{3}$ and take the angular average to get

$$
k\left(\frac{\hat{1}}{3} - \frac{1}{c}\right) + \nabla \cdot \langle \hat{p}\hat{p}\psi\rangle \frac{\hat{p}}{\rho} - \frac{1}{9} \nabla \cdot b = 0, \quad \text{(A.7)}$$

where the averages over $\psi^{le}$ vanish. Using relation $b = p_F v_n$ and the equation (A.5) we then get

$$
\frac{\hat{1}}{3} = \frac{1}{3} + \frac{p_F}{15k} (\nabla v_n + v_n \nabla - \frac{2}{3} \hat{1} \nabla \cdot v_n). \quad \text{(A.8)}
$$

Inserting this in the momentum flux expression

$$\
\hat{\Pi} = p_0 \hat{1} + \rho_s \frac{\delta \mu_4}{m_4} \hat{1} + 3n_3 \langle \hat{p}\hat{p}\psi\rangle \frac{\hat{p}}{\rho},
$$

(A.9)

gives

$$\
\hat{\Pi} = \left( p_0 + \rho_s \frac{\delta \mu_4}{m_4} + \frac{1}{3} \right) \hat{1} - \frac{p_F}{15k} (\nabla v_n + v_n \nabla - \frac{2}{3} \hat{1} \nabla \cdot v_n), \quad \text{(A.10)}
$$

Comparing to the hydrodynamic equation

$$\Pi_{ij} = p\delta_{ij} - \eta \left( \frac{\partial v_i}{\partial v_j} + \frac{\partial v_j}{\partial v_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial v_k} \right) - \zeta \delta_{ij} \frac{\partial v_k}{\partial v_k}, \quad \text{(A.11)}$$

we can identify the viscosity

$$\eta = \frac{p_F n_3}{5k} \approx \frac{1}{5} p_F n_3 \ell, \quad \text{(A.12)}$$

(in the limit $\ell \ll v_F/\omega$) and the pressure

$$p = p_0 + c n_3 + \rho_s \frac{\delta \mu_4}{m_4}, \quad \text{(A.13)}$$

where we can write separately the component of pressure corresponding to the normal and superfluid components $p_n = n_3 c = n_3 \langle \psi_\rho \rangle \rho$ and $\rho_s = \rho_\mu \delta \mu_4 / m_4$. Using the definition of pressure above we can rewrite the momentum equation (A.9) as

$$\
\hat{\Pi} = n_3 \langle (3\hat{p}\hat{p} - \frac{\hat{1}}{3})\psi_\rho \rangle \frac{\rho}{\rho} + p \hat{1}. \quad \text{(A.14)}
$$
Comparison with the hydrodynamic equation $\Pi = p \mathbb{I} + \rho \mathbf{v} \mathbf{v} - \mathbf{\sigma}'$ gives (neglecting the second order term in velocity) the viscous stress tensor

$$\mathbf{\sigma}' = n_3 (\mathbb{1} - 3 \hat{p} \hat{p}) \psi \hat{p}.$$  \hspace{1cm} (A.15)

We can identify $Y = -\frac{5}{2} \mathbf{\sigma}'$. The quasiparticle distribution $\psi_\mathbf{p}$ of kinetic theory can then be expressed in term of the hydrodynamic variables as

$$\psi(\mathbf{p}) = c + \mathbf{p} \cdot \mathbf{b} + \mathbf{p} \cdot Y \cdot \mathbf{p} = \frac{p_n}{n_3} + p_F \hat{p} \cdot \mathbf{v}_n - \frac{5}{2} \mathbf{p} \cdot \mathbf{\sigma}' \cdot \mathbf{p}.$$  \hspace{1cm} (A.16)

Here, we have expressed $c$ in terms of the normal pressure $p_n$. In terms of total pressure $p$ we have $c = (p - p_0 - \rho_s \delta \mu_s/m_4)/n_3$. 

Appendix B

Ballistic Limit

In this appendix we describe how the diffuse boundary condition terms $g_c$ and $g_w$ can be expressed in terms of relatively simple integrals in the ballistic limit, when the Fermi liquid interactions are neglected. Since the boundary condition terms determine $\hat{\psi}_p$ in this approximation, the force on the wire can also be expressed in a simple integral form. We concentrate on the case of a diffuse wire inside a diffuse coaxial cylindrical container, but give expressions for the specular wire as well. Some of the results are given in Papers I and II, but here we give more general expressions.

The diffuse boundary condition on the wire surface is (2.44)

$$\psi_{p_{\text{out}}} = g_w + p_F (\hat{p}_{\text{out}} + \frac{2}{3} \hat{n}_w) \cdot \mathbf{u}, \quad \text{(B.1)}$$

where we have used the definition (2.45)

$$g_w = -2 \langle \hat{n}_w \cdot \hat{p}_{\text{in}} | \psi_{\hat{p}_{\text{in}}}, \hat{p}_{\text{in}} \rangle \cdot \hat{p}_{\text{in}}. \quad \text{(B.2)}$$

The average is over incoming trajectories, $\hat{n}_w \cdot \hat{p}_{\text{in}} > 0$. On the container wall we have (2.47)

$$\psi_{\hat{p}_{\text{out}}} = g_c = -2 \hat{n}_c \cdot (\hat{p}_{\text{in}} | \psi_{\hat{p}_{\text{in}}}, \hat{p}_{\text{in}}) \hat{p}_{\text{in}}. \quad \text{(B.3)}$$

In the cylindrical symmetric case we define (2.53)

$$g_w = g'_w \hat{n}_w \cdot \mathbf{u}, \quad g_c = g'_c \hat{n}_c \cdot \mathbf{u}, \quad \text{(B.4)}$$

where $\hat{n}_w$ and $\hat{n}_c$ are the surface normal vectors on the wire and container surfaces pointing to the fluid, respectively.

Since $\psi_p$ is now constant along a trajectory, we notice that the distribution $\psi_{\hat{p}_{\text{in}}} (r_w)$ for an incoming trajectory on the wire surface (at $r_w$) is the same as the distribution $\psi_{\hat{p}_{\text{out}}} (r_c)$ on the container wall (at $r_c$) for the same trajectory $\hat{p}$, which is outgoing on the container wall. In other words,

$$\psi_{\hat{p}_{\text{in}}} (r_w) = \psi_{\hat{p}_{\text{out}}} (r_c). \quad \text{(B.5)}$$
To calculate $\psi_p$ for the incoming trajectories $\hat{p}_m$ on the wire surface, we need to track the point $r_c$ on the container wall, where the trajectory comes from. This is obtained from

$$r_c = a\hat{n}_w - s_1\hat{p}', \quad \text{where} \quad s_1 = a\hat{n}_w \cdot \hat{p}' + \sqrt{b^2 - a^2(1 - (\hat{n}_w \cdot \hat{p}')^2)}. \quad (B.6)$$

The corresponding surface normal vector on the container wall, pointing to the fluid, is $\hat{n}_c = -\hat{r}_c$. Here we have introduced notation where $\hat{p}'$ is in the $xy$-plane, $\hat{p} = \cos \zeta \hat{z} + \sin \zeta \hat{p}'$. Now we can express $g_w$ in terms of $g_c$ as

$$g_w = (\hat{n}_w \cdot \mathbf{u})g_w' = -2(\hat{n}_w \cdot \hat{p}_m \psi_p)\hat{p}_m = 2g_c'\langle \hat{n}_w \cdot \hat{p}_m \hat{n}_c \cdot \mathbf{u} \rangle. \quad (B.7)$$

Taking the average over $\zeta$ separately, inserting (B.6), denoting $\hat{n} \cdot \hat{p}' = \cos \beta$, and dividing by $\hat{n}_w \cdot \mathbf{u}$ gives, after some manipulation,

$$g_w' = g_c' \int_0^{\pi/2} \cos \beta \left[ \frac{a}{b} \sin \beta + \cos \beta \sqrt{1 - \frac{a^2}{b^2} \sin^2 \beta} \right] d\beta = D_4 g_c'. \quad (B.8)$$

where we have used the symmetry of the integrand. Here we have defined the integral $D_4$, which can be expressed also as

$$D_4 = \int_0^1 \left( \frac{a}{b} x^2 + \sqrt{1-x^2} \sqrt{1-\frac{a^2}{b^2} x^2} \right) dx. \quad (B.9)$$

The function $D_4$ can not be expressed in terms of elementary functions, but must be evaluated numerically, or using expansion in the limit $a \ll b$. The latter approach gives

$$D_4 = \frac{\pi}{4} + \frac{1}{3} \frac{a}{b} - \frac{\pi}{32} \frac{a^2}{b^2} - \frac{\pi}{256} \frac{a^4}{b^4} - \frac{5\pi}{4096} \frac{a^6}{b^6} + O(a/b)^7. \quad (B.10)$$

The term $g_c$ can be calculated in two parts, $g_{cc}$ for the contribution of the other points on the container wall, and $g_{cw}$ for the contribution of the boundary condition on the surface of the wire. We write

$$g_c' = g_{cc} + g_{cw}. \quad (B.11)$$

To calculate the $g_{cc}$ part, we note that, similar to (B.5),

$$\psi_p^{\text{in}}(r_{c1}) = \psi_p^{\text{out}}(r_{c2}), \quad (B.12)$$

where the incoming trajectory $\hat{p}$ on the outer boundary at $r_{c1} = -b\hat{n}_{c1}$ comes from other point on the container wall $r_{c2} = b[\hat{r}_{c1} - 2\hat{p}'(\hat{r}_{c1} \cdot \hat{p}')]$. The corresponding surface normal vector is $\hat{n}_{c2} = \hat{n}_{c1} - 2\hat{p}'(\hat{n}_{c1} \cdot \hat{p}')$. The distance
between points \( \mathbf{r}_{c1} \) and \( \mathbf{r}_{c2} \) is \( s_2 = 2b\mathbf{p}' \cdot \mathbf{r}_{c1} \). It is now straightforward to calculate

\[
g_{cc}(\mathbf{n}_{c1} \cdot \mathbf{u}) = 2\mathbf{n}_{c1} \cdot \langle \mathbf{p}_{in} g_c(\mathbf{n}_{c2} \cdot \mathbf{u}) \rangle_{\mathbf{p}_{in}}^{\text{walls}} = 2g_c\mathbf{n}_{c1} \cdot \langle \mathbf{p}_{in} [\mathbf{n}_{c1} - 2\mathbf{p}'(\mathbf{n}_{c1} \cdot \mathbf{p}')] \cdot \mathbf{u} \rangle_{\mathbf{p}_{in}}^{\text{walls}} = g'_c A_i(\mathbf{n}_{c1} \cdot \mathbf{u}).
\]

The superscript indicates that the average is over the trajectories coming from other points on the container wall (those coming from the wire are excluded). The wire is seen in angle \( 2\phi = 2\arcsin(a/b) \) from the container wall, which is employed in calculating the integral above. Here we have defined an integral \( A_i \), which can be expressed as

\[
A_i = \int_{\arcsin(a/b)}^{\pi/2} \cos \beta (1 - 2 \cos^2 \beta) \, d\beta = -\frac{1}{3} + \frac{a}{b} - \frac{2a^3}{3b^3}.
\]

The first term on the right hand side is due to other points on the container wall, and the last two terms are corrections due to the shadow of the wire.

For \( g_{cw} \) we use, similar to (B.5) and (B.12) above,

\[
\psi^\text{in}_p(\mathbf{r}_c) = \psi^\text{out}_p(\mathbf{r}_w),
\]

where

\[
\mathbf{n}_w = \frac{\mathbf{r}_c - s_3\mathbf{p}'}{a}, \quad \text{and} \quad s_3 = \mathbf{p}' \cdot \mathbf{r}_c - \sqrt{a^2 - b^2(1 - (\mathbf{p}' \cdot \mathbf{r}_c)^2)}.
\]

For \( \psi^\text{out}_p(\mathbf{r}_w) \) we use the boundary condition (2.44), and integrate over only those trajectories that come from the wire

\[
g_{cw}(\mathbf{r}_c \cdot \mathbf{u}) = 2(\mathbf{r}_c \cdot \mathbf{p}_{gw} \mathbf{n}_w \cdot \mathbf{u} + p_F(\mathbf{p} + \frac{2}{3}\mathbf{n}_w) \cdot \mathbf{u})_{\mathbf{p}_{in}}^{\text{wire}},
\]

We then get, denoting \( \mathbf{r}_c \cdot \mathbf{p}' = \cos \beta \) and using the symmetry of the integrand,

\[
g_{cw}(\mathbf{r}_c \cdot \mathbf{u}) = \frac{8}{3\pi} p_F \int_0^\phi \mathbf{r}_c \cdot \mathbf{p}' \mathbf{p}' \cdot \mathbf{u} \, d\beta + (g_w + \frac{2}{3}p_F) \int_0^\phi \mathbf{r}_c \cdot \mathbf{p}' \mathbf{n}_w \cdot \mathbf{u} \, d\beta
\]

\[
= \frac{2}{3} C_i p_F + B_i (g_w + \frac{2}{3}p_F).
\]

The integrals \( B_i \) and \( C_i \) can be written as

\[
B_i = \frac{a}{b} \int_0^1 \left( \frac{a}{b} x^2 + \sqrt{1 - \frac{a^2}{b^2} x^2} \sqrt{1 - x^2} \right) \, dx = \frac{a}{b} D_i,
\]

and

\[
C_i = \frac{4}{\pi} \int_0^\gamma \cos^2 \beta \, d\beta = \frac{4}{3\pi} \left( \frac{a}{b} \sqrt{1 - \frac{a^2}{b^2}} + \phi \right)
\]

\[
= \frac{4a}{\pi b} - \frac{2}{3\pi} \frac{a^3}{b^3} - \frac{1}{10\pi} \frac{a^5}{b^5} + O(a/b)^7.
\]
Collecting the results gives

\[ g_{cc} = A_i g'_c, \quad g_{cw} = \frac{2}{3} C_i p_F + B_i (g'_w + \frac{2}{3} p_F), \quad g'_w = D_i g'_c, \quad (B.21) \]

and

\[ g'_c = \frac{2}{3} (B_i + C_i) \frac{1}{1 - A_i - B_i D_i p_F}. \quad (B.22) \]

In the limit \( a \ll b \) we have, to third order in \( a/b \),

\[ g'_c = p_F \left[ \frac{2}{\pi} + \frac{1}{b} \left( \frac{3}{2} + \frac{3}{2\pi} + \frac{9\pi^3}{16} \right) \frac{a^2}{b^2} + \left( \frac{3}{8} + \frac{19}{24\pi} + \frac{3\pi^2}{128} + \frac{27\pi^3}{2048} + \frac{9\pi^5}{32768} \right) \frac{a^3}{b^3} \right]. \quad (B.23) \]

Next, we calculate the force on the wire due to \( g_w \) and \( g_c \); the remaining part due to boundary condition terms is calculated in \( (3.62) \). The force on a surface element of the wire \((2.29)\) can be expressed as

\[ \frac{dF_g}{da} \cdot \hat{u} = -3n_3 \hat{n}_w \cdot \langle \hat{p} \hat{p} \psi_g(\hat{p}) \rangle \hat{p} \cdot \hat{u} = -n_3 u \left[ \frac{1}{2} g'_w (\hat{n}_w \cdot \hat{u})^2 + \frac{3}{2} \hat{n} \cdot \langle \hat{p} \hat{p} \hat{p} \psi'_c \hat{r}_c \cdot \hat{u} \rangle \hat{p}_m \cdot \hat{u} \right]. \quad (B.24) \]

Here, \( \psi_g \) emphasizes that the simple boundary condition terms are neglected. The total force on the wire due the chamber walls is obtained by integrating over the wire surface. Using \((B.6)\) for \( r_c \) and some algebra, we get

\[ F_g = -an_3 g'_c \left( \frac{\pi}{2} D_i + \frac{\pi b}{2 a} C_i \right) u = -an_3 \pi a \left( D_i + \frac{b}{a} C_i \right)^2 \left( \frac{1}{b} \right)^2 u. \quad (B.25) \]

This is the force caused by quasiparticles scattered from the container walls back to the wire, in the limit of small oscillations. In the limit \( a \ll b \) we have, to third order in \( a/b \),

\[ F_g \approx -an_3 p_F \left[ \left( \frac{4}{\pi} + \frac{3}{2} + \frac{9\pi^3}{64} \right) \frac{a^2}{b^2} + \left( \frac{3}{2} + \frac{3}{\pi} + \frac{9\pi}{16} + \frac{\pi^2}{24} + \frac{9\pi^3}{256} + \frac{9\pi^5}{4096} \right) \right] u. \quad (B.26) \]

**Specular Wire**

For specular wire the treatment is similar, but now we have to consider the trajectories reflected from the wire. Three new integrals need to be defined. The specular boundary condition \((2.43)\) on the surface of the wire is

\[ \psi_{\hat{p}_{\text{out}}} = \psi_{\hat{p}_{\text{out}}} - 2\hat{n}_w (\hat{n}_w \cdot \hat{p}_{\text{out}}) + 2p_F (\hat{n}_w \cdot \hat{p}_{\text{out}}) (\hat{n}_w \cdot \hat{u}), \quad (B.27) \]
We write
\[ g'_c = g_{cw} + g_{cc} = 2\hat{r}_c \cdot \langle \hat{p}_w \psi_{p_{out}}(r_w) \rangle_{p_{in}}^{wire} + 2\hat{r}_c \cdot \langle \hat{p}_w \gamma_c \cdot u \rangle_{p_{in}}^{walls}, \tag{B.28} \]
where the averages are calculated on the surface of the container over trajectories coming from the wire, and other points on the container wall, respectively. The \( g_{cc} \) part is the same as above (B.13). The \( g_{cw} \) part is separated further into two parts by \( g_{cw} = g_{cw}^{BC} + g_{cw}^{refl} \), where the former is due to the second term in (B.27), and the latter is due to the reflected trajectory. For the first term we get
\[ g_{cw}^{BC} = 2\hat{r}_c \cdot \langle \hat{p}_w 2p_F(\hat{n}_w \cdot \hat{p})(\hat{n}_w \cdot u) \rangle_{p_{in}}^{wire} / (\hat{r}_c \cdot u) = p_F E_i \tag{B.29} \]
where we have defined the integral \( E_i \) using (B.16) and (B.20)
\[ E_i = \frac{16}{3\pi} \int_0^1 dx \left[ \frac{a^2}{b^2} \sqrt{1-x^2} + \frac{a}{b} (1-x^2) \right]. \tag{B.30} \]
In order to calculate \( g_{cw}^{refl} \) at point \( \hat{r}_{c1} \), we need to track the trajectory reflected from the wire to its origin \( \hat{r}_{c3} \) on the container wall. Due to the nature of specular scattering, the distances from the reflection point on wire surface \( a\hat{n}_w \) to \( \hat{r}_{c1} \) and \( \hat{r}_{c3} \) are the same, \( s_3 \) from (B.16). The point \( \hat{r}_{c3} \) can then be expressed as
\[ \hat{r}_{c3} = a\hat{n}_w - s_3[\hat{p}' - 2\hat{n}_w(\hat{n}_w \cdot \hat{p}')]. \tag{B.31} \]
We then get \( g_{cw}^{refl} = F_i g'_c \), where we have defined the integral
\[ F_i = \int_0^1 dx \left[ (1 - 2x^2)(1 - 2x^2) + 4a^2 \sqrt{1-x^2} \right]. \tag{B.32} \]
Combining these,
\[ g'_c = g_{cw} + g_{cc} = p_F E_i + a \gamma_c (F_i + A_i) \Rightarrow g'_c = p_F \frac{E_i}{1 - F_i - A_i}. \tag{B.33} \]
Next, we calculate the force on the wire due to \( g_c \). For incoming trajectories on the wire surface we use (B.6), and for the reflected, outgoing trajectories we have \( \hat{r}_c = a\hat{n}_w - s_4[\hat{p}' - 2\hat{n}_w(\hat{n}_w \cdot \hat{p}')] \), where \( s_4 = s_1 = -a\hat{n}_w \cdot \hat{p}' + \sqrt{b^2 - a^2(1 - (\hat{n}_w \cdot \hat{p}')^2)} \). We get
\[ \frac{dF_g}{da} = -3n_3\hat{n}_w \cdot \langle \hat{p} \hat{p} g'_c(\hat{r}_c \cdot u) \rangle \hat{p} = -n_3\hat{n}_w(\hat{n}_w \cdot u)g'_c G_i, \tag{B.34} \]
where
\[ G_i = \frac{4}{\pi} \int_0^1 dx \left[ \frac{a}{b} \frac{x^2 \sqrt{1-x^2} + (1-x^2) \sqrt{1-x^2}}{x^2 \sqrt{1-x^2} + (1-x^2) \sqrt{1-x^2}} \right]. \tag{B.35} \]
Then, total force is \( F_g = -n_3 a \pi g'_c G_i u \), or, to third order in \( a/b \),
\[ F_g \approx -n_3 a \pi G_i \left[ \frac{64}{9a} + \frac{1}{3} + \frac{64}{9a^2} \right] \frac{256}{45} \frac{a^3}{b^3} u. \tag{B.36} \]
Frequency Dependence

The previous treatment can be generalized to arbitrary $\omega$, i.e. to solve equation (3.70). We do this by generalizing the integrals $A_i$, $B_i$, $C_i$ and $D_i$, while the expressions for $g_c$, $g_w$ and $F$ remain implicitly the same. Instead of constant $\psi_p$ along trajectories, we must include a complex phase term $\exp(-i\omega s/v_F)$. Equations (B.5), (B.12), and (B.15) take the form

$$
\psi_p^{\text{in}}(r_w) = \psi_p^{\text{out}}(r_c)e^{-i\omega s'/v_F}, \quad \psi_p^{\text{in}}(r_{c1}) = \psi_p^{\text{out}}(r_{c2})e^{-i\omega s'/v_F},
$$

$$
\psi_p^{\text{in}}(r_c) = \psi_p^{\text{out}}(r_w)e^{-i\omega s'/v_F}.
$$

(B.37)

The distances $s_1$, $s_2$, and $s_3$ in $xy$-plane were given in the previous section; now we must use the actual distances along the trajectories, $s'_i = s_i / \sin \zeta$. For this reason the angle $\zeta$ will appear in a non-trivial form in the integrals. We get

$$
A_2 = \frac{2}{\pi} \int_0^\pi d\zeta \sin^2 \zeta \int_{\phi}^{\pi/2} d\gamma \cos \beta (1 - 2 \cos^2 \beta)e^{\varepsilon_1},
$$

(B.38)

$$
B_2 = \frac{a}{b} \frac{2}{\pi} \int_0^\pi d\zeta \sin^2 \zeta \int_0^1 dx \left( \frac{a}{b} x^2 + \sqrt{1 - \frac{a^2}{b^2} x^2} \sqrt{1 - x^2} \right) e^{\varepsilon_2} = \frac{a}{b} D_2, \quad (B.39)
$$

$$
C_2 = \frac{a}{b} \frac{3}{\pi} \int_0^\pi d\zeta \sin^3 \zeta \int_0^1 dx \sqrt{1 - \frac{a^2}{b^2} x^2} e^{\varepsilon_2},
$$

(B.40)

where

$$
\varepsilon_1 = \frac{2i\omega b \cos \beta}{v_F \sin \zeta}, \quad \text{and} \quad \varepsilon_2 = \frac{i\omega (\sqrt{b^2 - a^2 x^2} - a \sqrt{1 - x^2})}{v_F \sin \zeta}.
$$

(B.41)

For the specular scattering we have integrals $E_i$, $F_i$ and $G_i$. For $E_i$, we have $c = (b \cos \beta - \sqrt{a^2 - b^2 \sin^2 \beta})/\sin \zeta$, where $\cos \beta = \hat{p} \cdot \hat{r}$, and

$$
E_2 = \frac{4}{\pi} \int_0^\pi d\zeta \sin^3 \zeta \int_0^1 dx \left[ \frac{a^2}{b^2} \sqrt{1 - x^2} + \frac{a}{b} (1 - x^2) \sqrt{1 - \frac{a^2}{b^2} x^2} \right] e^{\varepsilon_2}.
$$

For $F_i$, the distance is $c = 2(b \cos \beta - \sqrt{a^2 - b^2 \sin^2 \beta})/\sin \zeta$

$$
F_2 = \frac{2}{\pi} \int_0^\pi d\zeta \sin^2 \zeta \int_0^1 dx \left[ (1 - 2x^2)(1 - 2 \frac{a^2}{b^2} x^2) + 4 \frac{a^2}{b^2} \sqrt{1 - \frac{a^2}{b^2} x^2} \sqrt{1 - x^2} \right] e^{2\varepsilon_2}
$$

For $G_i$ the distance is $c = (-a \cos \beta + \sqrt{b^2 - a^2 \sin^2 \beta})/\sin \zeta$, and

$$
G_2 = \frac{3}{\pi} \int_0^\pi d\zeta \sin^3 \zeta \int_0^1 dx \left[ \frac{a}{b} x^2 \sqrt{1 - x^2} + (1 - x^2) \sqrt{1 - \frac{a^2}{b^2} x^2} \right] e^{\varepsilon_2}.
$$
Appendix C

Numerical Integration

Here we describe the method of integration used in the numerical calculations. We present the general case, where cylindrical symmetry is not assumed. For brevity, we will only consider the basic integral of the same form as in (2.50), and not the case including boundary conditions (2.49). The generalization is straightforward. The integral to be solved is

\[
I(r_n, \hat{p}_{j\kappa}) = \int_{s_0}^{s} ds \, e^{ks} \left[ c \left( \frac{1}{\ell} - i \frac{\omega}{v_F} \frac{F_0}{1 + F_0} \right) + \left( \frac{1}{\ell} - i \frac{\omega}{v_F} \frac{F_1}{3 + F_1} \right) \hat{p}_{j\kappa} \cdot \mathbf{b} \right], \quad (C.1)
\]

where \( r \) is a functions of \( s \) (\( r = r_0 + s\hat{p} \), unless the trajectory reflects from the wire). Instead of calculating the value of the integral as such, we calculate \( 3 \times N \) integrals for the weights with which \( c'(r_m), b'_r(r_m), \) and \( b'_\theta(r_m) \) contribute to the integral (C.1), for all lattice points \( r_m \). These weights are then used to form a matrix equation, from which the averaged values \( c \) and \( b \) in all lattice points can be solved for.

In other words, we calculate the factors \( a_{nmjk}^{(h)} \), where the index \( n \) designates the point \( r_n \) in question, \( m \) gives the other lattice points, and \( j \) and \( k \) designate the trajectory direction. We write the integral (C.1) as

\[
I_{njk} = \sum_{m=a}^{R} \left[ a_{nmjk}^{(1)} c + a_{nmjk}^{(2)} b_{r,m} + a_{nmjk}^{(3)} b_{\theta,m} \right] = \sum_{h=1}^{3} \sum_{m=a}^{R} a_{nmjk}^{(h)} \psi_m^{(h)}. \quad (C.2)
\]

Next, we write the explicit formulae for the factors \( a_{nmjk}^{(h)} \). For the \( c \) component we find

\[
\left( \frac{1}{\ell} - i \frac{\omega}{v_F} \frac{F_0}{1 + F_0} \right) \int_{s_0}^{s} ds \, e^{ks} = \left( \frac{1}{\ell} - i \frac{\omega}{v_F} \frac{F_0}{1 + F_0} \right) \sum_{m=a}^{R} c_m \int_{s_0}^{s} ds W_m e^{ks} ds = \sum_{m=a}^{R} a_{nmjk}^{(1)} c_m. \quad (C.3)
\]
so that

\[ a_{nmjk}^{(1)} = \left( \frac{1}{\ell} - i \frac{\omega}{v_F} \frac{F_0}{1 + F_0} \right) \int_{s_0}^{0} ds \, e^{ks} W_m(s), \quad (C.4) \]

where \( W_m \) is the weight factor for lattice point \( m \). The weight factor for a particular lattice point \( m \) at point \( (r(s), \theta(s)) \) along the trajectory is zero for all except the nearest four lattice points. For the nearest points the weight is calculated using interpolation twice, first with respect to \( \theta \) and then with respect to \( r \). Let \( r_1 < r < r_2 \) and \( \theta_1 < \theta < \theta_2 \) define the nearest four lattice points. We then have

\[
W_1 = \frac{r_2 - r}{\Delta r} \sin(\theta_2 - \theta) - \frac{\sin \Delta \theta}{\Delta r}, \quad W_2 = \frac{r_2 - r}{\Delta r} \sin(\theta_1 - \theta), \quad W_3 = \frac{r - r_1}{\Delta r} \sin(\theta_2 - \theta) - \frac{\sin \Delta \theta}{\Delta r}, \quad W_4 = \frac{r - r_1}{\Delta r} \sin(\theta_1 - \theta), \quad (C.5)
\]

where \( W_1 \) is the weight for lattice point \( (r_1, \theta_1) \); \( W_2 \) for \( (r_1, \theta_2) \); \( W_3 \) for \( (r_2, \theta_1) \); and \( W_4 \) for \( (r_2, \theta_2) \), and \( W_m = 0 \) for any other lattice point. Here we have used \( \Delta r = r_2 - r_1 \) and \( \Delta \theta = \theta_2 - \theta_1 \), and assumed \( \theta \)-dependence of the form \( f(\theta) = A \cos \theta + B \sin \theta \) (between each pair \( \theta_1, \theta_2 \)). In rectangular lattice the weights are calculated similarly with respect to \( x \) and \( y \).

The integration path in (C.9) is divided into \( N_i \) intervals, and at each node \( i \) along the integral the weight factor \( W_m \) for each lattice points \( m \) is calculated, see Fig. C.1. We end up with a table \( W_{m,i} \) for the weights of each lattice point \( m \) at each node \( i \). The integral is then approximated as a sum over the intervals \( i \), using a modified Simpson’s rule which will be described shortly. This is done for each \( m \) separately, resulting in the factors \( a_{nmjk}^{(1)} \).

For the components of \( b \) we find

\[
\int_{s_0}^{0} ds \, e^{ks} \hat{p} \cdot b(s) = \sin \zeta_j \int_{s_0}^{0} ds \, e^{ks} \left[ \cos(\beta_k - \theta) b_r(s) + \sin(\beta_k - \theta) b_\theta(s) \right] = \left( \frac{1}{\ell} - i \frac{\omega}{v_F} \frac{F_1}{3 + F_1} \right)^{-1} \sum_{m=a}^{R} a_{nmjk}^{(2)} b_{r,m} + \sum_{m=a}^{R} a_{nmjk}^{(3)} b_{\theta,m}. \quad (C.6)
\]

In the above formulas \( \theta \) depends on the position on the trajectory, as well as \( \beta \), if there is a reflection from the wire, but \( \zeta \) is constant along a trajectory. Now we can identify

\[
a_{nmjk}^{(2)} = \left( \frac{1}{\ell} - i \frac{\omega}{v_F} \frac{F_1}{3 + F_1} \right) \sin \zeta_j \int_{s_0}^{0} ds \, e^{ks} W_m(s) \cos(\beta_k - \theta), \quad (C.7)
\]
\[
a_{nmjk}^{(3)} = \left( \frac{1}{\ell} - i \frac{\omega}{v_F} \frac{F_1}{3 + F_1} \right) \sin \zeta_j \int_{s_0}^{0} ds \, e^{ks} W_m(s) \sin(\beta_k - \theta), \quad (C.8)
\]

where the weight function \( W_m(s) \) is defined as in (C.5).
Figure C.1: Each trajectory (pink line) is divided into separate modes (green crosses). At each trajectory point the weights of four nearest lattice points are calculated and tabulated (inset A). The lattice points that affect this particular trajectory are marked as blue squares. In inset B), we have picked a lattice point \( n \), which has nonzero weight at six consecutive trajectory points. For the chosen trajectory \( \vec{p}_{jk} \), starting from point \( m \), the effect of lattice point \( n \) (i.e. the coefficient \( a_{mnjk}^{(h)} \)) is calculated by integrating over these tabulated weights, taking into account the exponential decay factor and the angle dependent terms. This is illustrated in the right hand figure, where the weight is nonzero for only the six trajectory points mentioned above. (The actual exponential term is complex; here we show only the real (decay) part.)

**Modified Simpson’s Rule**

Here we describe the modified Simpson’s rule, which takes the exponential term into account exactly. The integrals that need to be calculated in the previous section are of the form

\[
\int_0^s ds \ e^{ks} W_m(s), \quad \text{or} \quad \int_0^s ds \ e^{ks} W_m(s) \sin(\beta_k - \theta),
\]

but the method can be applied to any integral of the form

\[
I = \int_0^{s_0} f(x) e^{-kx} dx.
\]

This can be also used for calculating the force on the wire, when \( c \) and \( b \) are already solved. The part \( f(x) \) has known values \( y_i = f(x_i) \) at discrete points \( x_i \), with a constant distance \( h \) between the points. The function is approximated by a second degree polynomial between the points:

\[
f(x) = f(\bar{x} + ph) = \frac{1}{2} \left[ p(p-1)y_1 - 2(p^2-1)y_2 + p(p+1)y_3 \right],
\]

where \( p = (x - x_2)/h \). Now \( dp/dx = h^{-1} \), so

\[
f'(x) = f'(p)h^{-1} = \frac{1}{2h} \left[ (2p-1)y_1 - 4py_2 + (2p+1)y_3 \right],
\]

(C.12)
APPENDIX C. NUMERICAL INTEGRATION

and \( f'' \) is constant \( f''(x) = (y_1 - 2y_2 + y_3)h^{-2} \). Using partial integration we get

\[
I = \int f(x)e^{-kx}dx = -\frac{1}{k}f(x)e^{-kx} + \frac{1}{k} \int f'(x)e^{-kx}dx \quad \text{(C.13)}
\]

and for definite integral

\[
I = \int_{x_1}^{x_3} f(x)e^{-kx}dx = \left[ -\frac{1}{k}f(x_3) - \frac{1}{k^2}f'(x_3) - \frac{1}{k^3}f''(x_3) \right] e^{-kx_3} \quad \text{(C.14)}
\]

Inserting \( f'(x_1) = \frac{1}{2h}(-3y_1 + 4y_2 - y_3) \) and \( f'(x_3) = \frac{1}{2h}(y_1 - 4y_2 + 3y_3) \), we can express the integral as a sum of \( y_i \) with appropriate factors, including the exponential terms \( e^{-kx_1} \) and \( e^{-kx_3} \). Using \( e^{-kx_3} = e^{-2kh}e^{-kx_1} = qe^{-kx_1} \), where \( q = e^{-2kh} \), the integration can be presented as

\[
I = \int_{x_0}^{x_2} f(x)e^{-kx}dx = (ay_0 + by_1 + cy_2)e^{-kx_0}, \quad \text{(C.15)}
\]

where

\[
a = \frac{1}{k} - \frac{q + 3}{2k^2h} + \frac{1 - q}{k^3h^2}, \quad b = 2\left( \frac{1 + q}{k^2h} - \frac{1 - q}{k^3h^2} \right), \quad c = -\frac{q}{k} - \frac{1 + 3q}{2k^2h} + \frac{1 - q}{k^3h^2}.
\]

Now the whole integral can be calculated piecewise

\[
I = \int_{a}^{b} f(x)e^{-kx}dx = \sum_{i=0,2,4,...,n-2}^{x_{i+2}} \int_{x_i}^{x_{i+2}} f(x)e^{-ikx}dx. \quad \text{(C.16)}
\]

The method can be generalized for higher degree polynomials. The results do not depend on the degree of polynomial, provided the distance \( h \) between trajectory points is small enough, and the lattice is dense enough.

When considering a trajectory reflected from the wire, the integration can be done piecewise. It must be taken into account that same lattice points may have non-zero weights on both parts of the trajectory.
Bibliography


