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Marko Leinonen

ON VARIOUS IRRATIONALITY MEASURES

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FACULTY OF SCIENCE*

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A Scientiae Rerum Naturalium 699

MARKO LEINONEN

ON VARIOUS IRRATIONALITY MEASURES

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Abstract

This dissertation consists of four articles on irrationality measures. In the first paper we derive explicit irrationality measures by using the simple continued fraction expansions in a completely new way. In the second and third articles we use Padé approximations to construct irrationality measures. In the second paper we obtain an explicit irrationality measure for the values of q-exponential series, for which the earlier corresponding results are not as explicit. Furthermore, we construct a restricted irrationality measure for the values of q-exponential series, which is an improvement on the earlier results in the restricted case. In the third article we derive the best possible asymptotic restricted irrationality exponent for the values of Jacobi's triple product. In the last paper we consider Cantor series. We generalize the earlier results by deriving Sondow's irrationality measure for some Cantor series.

Keywords: Cantor series, continued fraction, irrationality measure, Jacobi's triple product, Padé approximation, q-exponential series

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Tiivistelmä

Tämä väitöskirja koostuu neljästä artikkelista, jotka kaikki käsittelevät irrationaalisuusmittoja. Ensimmäisessä artikkelissa irrationaalisuusmittoja johdetaan uudella tavalla irrationaalilukujen yksinkertaisista ketjumurtolukuesityksistä. Toisessa ja kolmannessa artikkelissa irrationaalisuusmitat konstruoidaan Padé-approksimaatioiden avulla. Toisessa artikkelissa saadaan eksplisiittinen irrationaalisuusmitta q-eksponenttisarjan arvoille, joiden vastaavat aikaisemmat irrationaalisuusmitat eivät ole näin eksplisiittisiä. Lisäksi samassa artikkelissa konstruoidaan q-eksponenttisarjan arvoille rajoitettu eksplisiittinen irrationaalisuusmitta, mikä parantaa aikaisempia tuloksia rajoitetussa tapauksessa. Kolmannessa artikkelissa johdetaan paras mahdollinen asymptoottinen irrationaalisuuseksponentti Jacobin kolmitulon arvoille. Viimeisessä artikkelissa käsitellään Cantorin sarjoja. Siinä yleistetään aikaisempia tuloksia johtamalla Sondowin irrationaalisuusmitta tietylle joukolle Cantorin sarjoja.

Asiasanat: Cantorin sarja, irrationaalisuusmitta, Jacobin kolmitulo, ketjumurtoluku, Padé-approksimaatio, q-eksponenttisarja

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Oulu, September 2017 Marko Leinonen

List of original publications

I Hančl J, Leinonen M, Leppälä K & Matala-aho T (2012) Explicit irrationality measures for continued fractions. J. Number Theory 132: 1758–1769.

II Leinonen L, Leinonen M & Matala-aho T (2016) On approximation measures of q-exponential function. Int. J. Number Theory 12: 287–303.

III Leinonen L & Leinonen M On restricted approximation measures of Jacobi's triple product. Manuscript.

IV Leinonen M On Sondow's irrationality measure for some Cantor series. Manuscript.

Article IV is an independent work of the author. Author's contribution to article III is one half, to article II it is one third and to article I it is one quarter.

Contents

1 Introduction

In this thesis we consider rational approximations of irrational numbers. A real number α is irrational if and only if

$$\left|\alpha - \frac{M}{N}\right| > 0$$

for all $M, N \in \mathbb{Z}$, $N \geq 1$. Irrationality measures of an irrational number α are lower bounds on the distances between the irrational number α and all rational numbers. Typically an irrationality measure of an irrational number α is any function of N bounding

$$\left|\alpha - \frac{M}{N}\right|, \ M, N \in \mathbb{Z} \text{ with } N \geq 1,$$

from below for $N \geq N_0 \in \mathbb{Z}_+$. We say that μ is *an irrationality exponent* of the irrational number α if for any rational M/N there exist positive constants C and N_0 such that

$$\left|\alpha - \frac{M}{N}\right| > \frac{C}{N^{\mu}},$$

when $N \geq N_0$. If C and N_0 can be determined effectively, the irrationality measure is called *effective*. Furthermore, the irrationality measure is *explicit*, if C and N_0 are explicit.

The infimum of all irrationality exponents of the irrational number α is called *an asymptotic irrationality exponent* of α and is denoted by $\mu_I(\alpha)$. Dirichlet's approximation theorem [8] implies that the asymptotic irrationality exponent $\mu_I(\alpha) \geq 2$ for all irrational numbers α. Moreover, it is known that the asymptotic irrationality measure of almost all real numbers is 2. Roth [14] has proved that the asymptotic irrationality exponent $\mu_I(\alpha) = 2$ for all algebraic irrational numbers α (see also [9]). This result is not effective. However, it ultimately remains to be shown how to even prove that the explicit irrationality measure for the given algebraic number is less than $2 + \varepsilon$ for a given ε. Only partial solutions are known, for example the best known upper bound for an explicit irrationality exponent of $\sqrt[3]{2}$ is 2.47, which has been derived by Bennett [3]. The situation for transcendental numbers is more challenging compared to algebraic numbers. Namely, there is not such a general asymptotic result as Roth's result. On the whole there is not a hint of how to derive an upper bound for the asymptotic irrationality exponent of a particular transcendental number α. However, for example Marcovecchio [12] has proved $\mu_I(\log 2) < 3.57455391$ and Salikhov [15] has derived

$\mu_I(\pi) < 7.606309$, which are the best known upper bounds for asymptotic irrationality exponents of $\log 2$ and π.

When the irrational number α is approximated by an infinite proper subset of the rational numbers, we say that the irrationality measure of α is *restricted*. In the restricted case the irrationality exponent is at most equal to the usual irrationality exponent. The restricted asymptotic irrationality exponent for the irrational number α is at least 1. In the most studied restricted case the denominators of rational numbers are d^s, $d \geq 2$. Amou and Bugeaud [2] have noted that in this case the restricted asymptotic irrationality measure is 1 for almost all irrational numbers. However, if α is a classical mathematical constant like $\sqrt{2}$, e or π, we do not know whether the restricted asymptotic irrationality measure is 1 for any d.

Normally, it is not easy to construct the usual irrationality measure from the restricted irrationality measure. Sondow [16] proved an irrationality measure for e in the restricted case $N = n!$ from which he derived a new explicit irrational measure for e,

$$\left| e - \frac{M}{N} \right| > \frac{1}{(S(N)+1)!},$$

where $S(N)$ is the smallest positive integer such that $S(N)!$ is a multiple of N. This type of irrationality measure is called Sondow's irrationality measure.

2 Summary of the original articles

This thesis consists of four articles, which all concern irrationality measures. In article I we establish a very useful formulation to estimate the irrationality measure of an irrationality number given by the continued fraction expansion with the infinite nested logarithm function. In articles II and III we use Padé approximations to construct irrationality measures. In paper II we give an explicit measure for the values of q-exponential series, which is an important function both in analytical and in number theoretical sense. While this measure is a well-known bound of an asymptotic irrationality exponent obtained by Bundschuh [4] almost half a century ago, it gives more precise information than the mere upper bound. Moreover, we improve Bundschuh's bound of an irrationality exponent of these values under a certain restriction on rational numbers in the approximation. In paper III we consider rational approximations for values of Jacobi's triple product, where the denominators of rational numbers are powers of a fixed integer, whose absolute value is at least 2. In that case we determine new restricted irrationality measures with more precise information. In article IV we consider Cantor series. We treat a wide class of Cantor series and generalize Sondow's result for them.

2.1 Article I: Explicit irrationality measures for continued fractions

Every irrational number α has an infinite simple continued fraction expansion

$$\alpha = c_0 + \frac{1}{c_1}{}_{+\dots}\frac{1}{c_n}{}_{+\dots} = [c_0; c_1, c_2, \dots, c_n, \dots],$$

where c_0 is an integer and c_i is a positive integer for all $i \in \mathbb{Z}_+$. This expansion is unique. If M/N is closer than any other fraction with a smaller or equal denominator than N to α, the fraction M/N is the best approximation of the irrational number α. The convergents of the simple continued fraction expansion

$$\frac{p_n}{q_n} = [c_0; c_1, c_2, \dots, c_n],$$

where $\gcd(p_n, q_n) = 1$, are the best rational approximations for the irrational number α. Because the convergents are the best approximations, it is reasonable to utilize them to obtain an irrationality measure.

Let $\alpha = [c_0; c_1, c_2, \ldots]$ be a simple continued fraction expansion for an irrational number α determined by an infinite integer sequence (c_n). All theorems of this article are based on the next result.

Lemma 1. *Let p_n/q_n be a convergent of α. Then*

$$\left|\alpha - \frac{p_n}{q_n}\right| > \frac{1}{(c_{n+1}+3)q_n^2}.$$

If the rational number M/N is not a convergent of the irrational number α, we can prove that

$$\left|\alpha - \frac{M}{N}\right| > \frac{1}{4N^2}.$$

Hence we can concentrate to evaluate the lower bound of Lemma 1. First we consider such continued fractions that the terms c_n are restricted from above by a power function. We approximate the term c_{n+1} by using the inverse function $z(y)$ of the function

$$y(z) = z\log z,\ z \geq \frac{1}{e}.$$

Theorem 1 (I, Theorem 2.6)**.** *Let real numbers $a, B, c, l > 0$ be given and suppose*

$$c_n \leq an^l, \quad (Bn)^{cn} \leq q_n$$

for all $n \in \mathbb{Z}_+$. Then

$$\left|\alpha - \frac{M}{N}\right| > \frac{1}{N^2\left(a\left(\frac{1}{B}z\left(\frac{B}{c}\log N\right)+1\right)^l+3\right)}$$

for all $M, N \in \mathbb{Z}$ with $N > e^{\frac{c}{Be}}$.

If we replace $z(y)$ by $z_2(y) = \frac{y}{\log\frac{y}{\log y}}$, we can derive explicit irrationality measures for certain numbers. For example, we obtain explicit irrationality measures for e, which generalizes and improves earlier results of Alzer [1], Bundschuh [5] and Davis [7]. Furthermore, if in Theorem 1 we replace the condition $(Bn)^{cn} \leq q_n$ with the condition $bn^h \leq c_n$ $(b, h > 0)$, we obtain a weaker result but in a more useful form.

By using Lemma 1 we can handle continued fractions in which the terms c_n are restricted from above by an exponential function. Hence we obtain results without using any other tools.

Theorem 2 (I, Theorem 2.8)**.** *Let real numbers $a, B > 1$, $h, l > 0$ be given. Suppose*

$$c_n \leq a^{n^l}, \quad B^{n^{h+1}} \leq q_n$$

for all $n \in \mathbb{Z}_+$. *Then*

$$\left|\alpha - \frac{M}{N}\right| > \frac{1}{N^2\left(N^{\frac{\log a}{\log N}\left(\sqrt[h+1]{\frac{\log N}{\log B}}+1\right)^l}+3\right)}$$

for all $M, N \in \mathbb{Z}$ *with* $N \geq 2$.

The measure in this theorem is explicit. If in Theorem 2 we replace the condition $B^{n^{h+1}} \leq q_n$ with the condition $b^{n^h} \leq c_n$ ($b > 1, h > 0$), we obtain a weaker result but in a more useful form.

2.2 Article II: On approximation measures of q-exponential function

Let $f(t) = \sum_0^\infty a_k t^k \in \mathbb{Q}[[t]]$ be a formal power series. A triple $(Q_m(t), P_n(t), R_{m,n}(t))$, where

$$Q_m(t) \in \mathbb{Q}[t],\ \deg Q_m(t) \leq m,$$
$$P_n(t) \in \mathbb{Q}[t],\ \deg P_n(t) \leq n$$

and

$$R_{m,n}(t) \in \mathbb{Q}[[t]],\ \operatorname{ord} R_{m,n}(t) = m+n+1,$$

where $\operatorname{ord} R_{m,n}(t)$ is an order of the series $R_{m,n}(t)$, is called a Padé approximation of the series $f(t)$ if

$$Q_m(t)f(t) - P_n(t) = R_{m,n}(t).$$

The Padé approximation always exists but it is not unique. The Padé approximations yield good irrationality results. Next we show how we obtain an irrationality measure by using Padé approximations. If we have a Padé approximation for the series $f(t)$ such that $Q_m(t), P_n(t) \in \mathbb{Q}[t]$, we can construct a Padé approximation that $Q_m(t), P_n(t) \in \mathbb{Z}[t]$. We now study series $f(t)$ at $t \in \mathbb{Q} \setminus \{0\}$. For a given Padé approximation, we obtain

$$\begin{aligned} |Q_m(t)|N\left|f(t) - \frac{M}{N}\right| &= |Q_m(t)Nf(t) - Q_m(t)M| = |NP_n(t) - Q_m(t)M + R_{m,n}(t)N| \\ &\geq |NP_n(t) + Q_m(t)M| - |R_{m,n}(t)|N. \end{aligned} \tag{1}$$

If a Padé approximation $(Q_m(t), P_n(t), R_{m,n}(t))$ exists for the series $f(t)$ such that the number $NP_n(t) + Q_m(t)M$ is a non-zero integer and $|R_{m,n}(t)N| < \frac{1}{2}$, then by equation (1)

$$\left|f(t) - \frac{M}{N}\right| > \frac{1}{2Q_m(t)N}.$$

If we can optimize $Q_m(t)$ and $R_{m,n}(t)$, we will obtain an explicit irrationality measure for $f(t)$ at $t \in \mathbb{Q} \setminus \{0\}$.

In this article we study q-exponential series

$$E_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\prod_{n=1}^{k}(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{1-q^k t}, \; 0 < |q| < 1, \; |t| < 1, \; q,t \in \mathbb{C}.$$

In 1947 Lotosky [10] has proved that $E_q(t)$ with $q = 1/d$, $|d| \in \mathbb{Z}_{\geq 2}$, is irrational. We obtain an explicit irrationality measure for $E_q(t)$ by using Padé approximations.

Theorem 3 (II, Theorem 1.1)**.** *Let $q = 1/d$, where $|d| \in \mathbb{Z}_{\geq 2}$, and $t = u/v \in \mathbb{Q}$, where $u \in \mathbb{Z}$, $v \in \mathbb{Z}_+$, $\gcd(u,v) = 1$ and $0 < |t| < 1$. Then*

$$\left| E_q(t) - \frac{M}{N} \right| \geq C_1 (2|N|)^{-\left(\frac{7}{3}+\varepsilon_1\right)}$$

for all $M,N \in \mathbb{Z}$, $|N| \geq 1$ with an explicit constant

$$\begin{aligned} C_1 &= C_1(d,v) = e^{-c_4}, \\ c_4 &:= \frac{14 \log v}{3 \log|d|} \left(\frac{4}{3} \log v + \sqrt{\log 4 \log|d|} \right) \\ &\qquad + 8\sqrt{\log 4 \log|d|} + \log(2^7 v^{\frac{44}{3}} |d|^8 \prod_{k=0}^{\infty}(1+\frac{1}{2^k})) \end{aligned}$$

and

$$\varepsilon_1 = \varepsilon_1(v,d,N) = \frac{\frac{22}{3}\log v + 8\log|d| + 4\sqrt{(\log 4)\log|d|}}{\sqrt{\frac{3}{2}(\log|d|)\log(2|N|)}}.$$

This result gives that $\mu_I(E_q(t)) \leq 7/3$. It is the same irrationality measure as Bundschuh presented in [4] but in a more explicit form. When we approximate q-exponential series by rational numbers d^s/N, we can choose Padé approximations in different ways. Hence we obtain a better measure than in Theorem 3, but only in a restricted case.

Theorem 4 (II, Theorem 1.2)**.** *Let $|d| \in \mathbb{Z}_{\geq 2}$ and $t = u/v \in \mathbb{Q}$, where $u \in \mathbb{Z}$, $v \in \mathbb{Z}_+$, $\gcd(u,v) = 1$ and $0 < |t| < 1$. Then there exists an effective positive constant $C_2 = C_2(d,v)$ such that*

$$\left| E_q(t) - \frac{d^s}{N} \right| \geq C_2 (2|N|)^{-\left(2+\frac{1}{3+2\sqrt{3}}+\varepsilon_2\right)}$$

for all $s \in \mathbb{Z}_+, N \in \mathbb{Z}$, $|N| \geq 82836$, with $\varepsilon_2 = \varepsilon_2(d,v,N) \in \mathbb{R}_+$ satisfying $\varepsilon_2(d,v,N) \to 0$, when $|N|$ increases.

2.3 Article III: On restricted approximation measures of Jacobi's triple product

It is very hard to construct an irrationality measure for an infinite product. Hence we transform it into a series, when possible. In this article we focus on Jacobi's triple product

$$\Pi_q(t) := \prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m-1}t)(1+q^{2m-1}t^{-1}).$$

According to Jacobi's triple product identity

$$\Pi_q(t) = \sum_{n=-\infty}^{\infty} t^n q^{n^2},\ |q|<1. \tag{2}$$

Hence the irrationality of Jacobi's triple product at arbitrary rational $t \neq 0$ with $q = 1/d$, $d \in \mathbb{Z}\setminus\{0,\pm 1\}$, follows from the linear independence result of Bundschuh and Shiokawa [6] for Jacobi's Theta function

$$\Theta(q,t) := \sum_{n=0}^{\infty} t^n q^{n^2},\ |q|<1.$$

From formula (2) we can derive the restricted explicit irrationality measure for $\Pi_q(t)$ by using the simplest possible Padé approximation: $P_n(t)$ is the partial sum of order n of the series $f(t)$, $Q_0(t) = 1$, and $R_{0,n}(t)$ is the remainder term of $f(t)$ with $\operatorname{ord} R_{0,n} = n+1$.

Theorem 5 (III, Theorem 1)**.** *Let* $t = a/b \in \mathbb{Q}\setminus\{0\}$, $\gcd(a,b) = 1$, $d \in \mathbb{Z}\setminus\{0,\pm 1\}$ *and* $\max\{|a|,|b|\} < |d|$. *Then for all* $s, M \in \mathbb{Z}$ *with* $s \geq C_3$ *we have*

$$\left|\Pi_{\frac{1}{d}}(t) - \frac{M}{d^s}\right| > \frac{1}{2|d|^{s(1+\varepsilon_1(s))}}, \quad \varepsilon_3(s) = \frac{6}{\sqrt{s}} + \frac{8}{s},$$

where $C_3 = (3\max\{|a|,|b|\}-1)^2/4$.

In this result the restricted irrationality exponent is the best possible asymptotic.

2.4 Article IV: On Sondow's irrationality measure for some Cantor series

In this article we study Cantor series

$$\sum_{k=0}^{\infty} \frac{a_k}{b_0 \cdots b_k},$$

where $a_k, b_k \in \mathbb{Z}$, $b_0 = 1$ and $b_k \geq 2$ for all $k \geq 1$. We denote

$$\Theta := \sum_{k=0}^{\infty} \frac{a_k}{b_0 \cdots b_k},$$

where $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ are sequences of integers which satisfy conditions $b_0 = 1$, $b_k \geq 2$ for all $k \geq 1$, $0 \leq |a_k| \leq b_k - 1$ for all $k \geq 1$ and both conditions $|a_k| \geq 1$ and $|a_k| \leq b_k - 2$ hold infinitely often. Oppenheim [13] has proved that Θ is irrational if each prime divides infinitely many terms b_k. We generalize Sondow's irrationality measure obtained by Marques [11] for a larger set of Cantor series.

Theorem 6 (IV, Theorem 2). *Let M and N be integers with $N \geq 1$. Suppose that each prime divides infinitely many terms b_k. Let $s \geq 0$ be the smallest integer such that N divides $b_0 \cdots b_s$ and j be the smallest positive integer such that $j > s$ and $a_j \neq 0$. If*

(i) $j = s + 1$, *then*

$$\left| \Theta - \frac{M}{N} \right| > \frac{\min \{b_{s+1} - |a_{s+1}| - 1, |a_{s+1}| - 1\}}{b_0 \cdots b_{s+1}};$$

(ii) $j > s + 1$, *then*

$$\left| \Theta - \frac{M}{N} \right| > \frac{|a_j| - 1}{b_0 \cdots b_j}.$$

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Original publications

I Hančl J, Leinonen M, Leppälä K & Matala-aho T (2012) Explicit irrationality measures for continued fractions. J. Number Theory 132: 1758–1769.
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684. Karppinen, Pasi (2016) Studying user experience of health behavior change support systems : a qualitative approach to individuals' perceptions of web-based interventions

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