WAVES IN PLANETARY RINGS: HYDRODYNAMIC MODELING OF RESONANTLY FORCED DENSITY WAVES AND VISCous OVERSTABILITY IN SATURN’S RINGS

MARIUS LEHMANN

University of Oulu Graduate School
University of Oulu
Faculty of Science
Astronomy Research Unit

Academic Dissertation to be presented with the assent of the Faculty of Science, University of Oulu, for public discussion in the Auditorium L6, on November 30th, 2018, at 12 o’clock noon.
Opponent
Dr. Henrik N. Latter, DAMTP, University of Cambridge, United Kingdom

Reviewers
Dr. Glen R. Stewart, LASP, University of Colorado, Boulder, USA
Dr. Pierre-Yves Longaretti, IPAG, CNRS & UGA, Université Grenoble, Alpes, France

Custos
Prof. Jürgen Schmidt, Astronomy Research Unit, University of Oulu, Finland
Lehmann, Marius
Waves in planetary rings: Hydrodynamic modeling of density waves and viscous overstability in Saturn’s rings

Astronomy Research Unit
P.O. Box 3000
FIN-90014 University of Oulu
FINLAND

Abstract

The present thesis investigates the dynamics of wave structures in dense planetary rings by employing hydrodynamic models, along with local N-body simulations of the particulate ring flow. The focus is on the large-scale satellite induced spiral density waves as well as on the free short-scale waves generated by the viscous overstability in Saturn’s A and B rings.

An analytic weakly nonlinear model is derived by using perturbation theory based on multi-scale methods to compute the damping behavior of nonlinear spiral density waves in a planetary ring subject to viscous overstability. In order to study the complex spatio-temporal evolution of these wave structures, numerical schemes are developed to integrate the hydrodynamical equations in time on large radial domains, taking into account collective self-gravity forces of the ring material, as well as the forcing by an external satellite. The required numerical stability and accuracy is achieved by applying Flux-Vector-Splitting methods aligned with advanced shock-capturing techniques. The free short-scale overstable waves are also investigated with local N-body simulations of the sheared ring flow. In particular, the influence of collective self-gravity between the ring particles as well as the periodic forcing due to a nearby Lindblad resonance on the overstable wave pattern is considered.

The linear stability criterion for spiral density waves in Saturn’s rings is found to be identical to the condition for the onset of spontaneous viscous overstability in the limit of long wavelengths and agrees with the stability criterion for density waves derived by Borderies et al. within the streamline formalism. The derived nonlinear damping behavior of density waves can be very different from what has previously been thought. The role of collective self-gravity on the nonlinear evolution of short-scale overstable waves is determined, reconciling the partly contradicting results of previous studies. It is shown that collective self-gravity plays an important role in setting the length-scale on which the nonlinear overstable waves saturate in a planetary ring. A co-existence of spiral density waves and short-scale overstable waves
is modeled in terms of one-dimensional large-scale hydrodynamical integrations. Due to the restriction to one space dimension, certain terms in the hydrodynamical equations that arise from the spiral shape of a density wave need to be approximated based on the weakly nonlinear model. These integrations reveal that density waves and spontaneous viscous overstability undergo complex interactions. In particular it is found that, depending on the relative magnitude of the two wave structures, the presence of short-scale overstable waves can lead to a damping of an overstable density wave and *vice versa*, density waves can suppress overstability. The effect of a density wave on the viscous overstability is also studied in terms of a simplified axisymmetric model of a ring perturbed by a nearby Lindblad resonance. A linear hydrodynamic stability analysis and local N-body simulations of this model system confirm the corresponding results of the large-scale hydrodynamical integrations.

*Key words*: planets and satellites: rings, hydrodynamics, waves, instabilities, collisional physics
Acknowledgments

The present PhD thesis has been carried out at the Astronomy Research Unit in the University of Oulu. I wish to express my gratitude to my supervisors Prof. Jürgen Schmidt and Prof. Heikki Salo for providing me the opportunity to perform this thesis work. Without your permanent support and guidance it would not have been possible.

I am grateful to my past and present fellow colleagues Simón Díaz-García, Sébastien Comerón, Jarkko Laine, Pertti Rautiainen, Martín Herrera-Endoqui, Xiaodong Liu, and Aku Venhola for help and company during the past years. I want to thank all other members of the Astronomy Research Unit for a friendly and supportive working environment.

I thank the pre-examiners Dr. Glen R. Stewart and Dr. Pierre-Yves Longaretti for their thorough reading of the thesis manuscript and for providing valuable scientific knowledge. I further wish to express my thanks to Dr. Päivi Pollari for providing a smoothrunning computer system and Dr. Reijo Rasinkangas for seamless technical support.

I acknowledge financial support from the University of Oulu Graduate School, the Astronomy Research Unit of University of Oulu, the Academy of Finland, and the University of Oulu Scholarship Foundation.

Finally, I am grateful to my family and relatives, providing me support throughout the years.

Oulu, November 2018        Marius Lehmann
LIST OF ORIGINAL PAPERS

The present thesis incorporates an introductory part and the following papers, referred in the text by their Roman numerals:


In Paper I the author provided relevant mathematical derivations and results. The author performed all the calculations and was the primary writer in Paper II. In Papers III and IV the author was the primary writer and performed all hydrodynamic computations, as well as a part of the N-body simulations.
vi
## Contents

Abstract i
Acknowledgments iii
List of Original Papers iv

1 Introduction 1
  1.1 Background .............................................. 1
  1.2 Research Goals and Methods of the Research Articles .... 2
  1.3 Fundamental Dynamics of Dense Planetary Rings ......... 7
    1.3.1 Kinetic Theory and Hydrodynamic Approximation .... 7
    1.3.2 Dense Ring Equilibrium ............................... 17
    1.3.3 Self-Gravity in the Rings ............................. 20
    1.3.4 Transport of Angular Momentum ....................... 22
  1.4 Viscous Overstability in Saturn’s Rings .................. 29
  1.5 Spiral Density Waves in Saturn’s Rings .................... 37
    1.5.1 Lindblad Resonances .................................. 37
    1.5.2 Linear Density Waves ................................. 38
    1.5.3 Nonlinear Density Waves ............................... 41

2 Summary of the Results and Discussion 47

Bibliography 51

Original papers 60
Chapter 1

Introduction

1.1 Background

In the early 1980’s the Voyager 1 and 2 missions revealed a surprising abundance of structure in Saturn’s ring system (Cuzzi et al. [1984]). Many physical mechanisms have been proposed to explain parts of this structure. Some of it can be explained by considering the ring constituents, which are typically centimeter to meter-sized icy boulders, as a fluid, taking into account their mutual gravitational attraction as well as the effects of numerous moons, either exterior to the rings or embedded within them. As such it was found that disk-satellite interactions are at work for instance in maintaining gaps in the ring system, constraining ring edges, as well as in launching spiral density waves and vertical corrugation waves. In particular, the theory of spiral density waves, originally proposed for the spiral structure of disk galaxies, has been applied to explain many features in the rings.

Data from the recent Cassini (NASA/ESA) mission show that Saturn’s rings are an amazingly rich dynamical system (Burns and Cuzzi [2006], Schmidt et al. [2009], Cuzzi et al. [2010]). A true wealth of phenomena is observed, partly relating to processes of accretion and disk-satellite interaction that may bear implications for processes in protoplanetary disks (Cuzzi et al. [2010], Latter et al. [2018]).

The work presented in this thesis explores the connection between the satellite induced spiral density waves in the rings (Section 1.5), for which detailed data are recorded by the Cassini instruments (Tiscareno et al. [2007],
Colwell et al. [2009]), and a new type of oscillatory instability, the so-called viscous overstability (Section 1.4). Overstability was predicted for perturbed rings (Borderies et al. [1985], Mosqueira [1996]), narrowly confined rings (Longaretti and Rappaport [1995]) and as a spontaneous instability of unperturbed rings (Schmit and Tscharnuter [1995, 1999]). The latter has been studied in terms of axisymmetric fluid models (Schmit and Tscharnuter [1995, 1999], Schmidt et al. [2001], Latter and Ogilvie [2009, 2010]) and N-body simulations (Salo et al. [2001], Rein and Latter [2013]) which predict that it leads to the formation of axisymmetric waves on a 100m scale. Cassini data of Saturn’s dense rings indeed provide evidence for the existence of quasi-axisymmetric, short-scale wavetrains (Thomson et al. [2007], Colwell et al. [2007], Hedman et al. [2014]). In this thesis it are these short-scale structures that are referred to as viscous overstability.

Preliminary analysis has shown that there is a deep connection between the damping behavior of density waves and the viscous overstability (Schmidt et al. [2011]). Traditionally, in the theory of density waves the approximation is made that the viscosity of the ring flow can be treated as a constant (Goldreich and Tremaine [1978]). If this approximation is lifted, it turns out that the damping behavior of density waves can be very different from what was previously thought.

Density waves are a classical topic of ring dynamics. They are excited by the gravitational perturbation due to external moons at isolated Lindblad resonance locations, with typical wavelengths of 1-10km (Goldreich and Tremaine [1978, 1979]). The study of these waves provides insight in the principles of disk-satellite interaction, and it is therefore of fundamental interest for astrophysical disks. Indeed, the dynamics of the moons that orbit interior and exterior to Saturn’s rings bears many similarities to the dynamics of planets forming in disks around young stars. For instance the detection of spiral structure in protoplanetary disks can be due to embedded planets and the application of density wave theory can provide estimates for their masses and locations (e.g. Muto et al. [2012]). Furthermore, by applying kinetic or hydrodynamical models, the analysis of density wave profiles in Saturn’s rings can provide important physical parameters which are otherwise hard to constrain, such as the mass surface density and the viscosity of the rings. A clear understanding of the damping behavior of spiral density waves is therefore of great importance.

1.2 Research Goals and Methods of the Research Articles

In this thesis the dynamical evolution of spiral density waves and short-scale waves arising from the spontaneous viscous overstability is studied in
the hydrodynamic approximation (Section 1.3.1) that is best applicable to a dense planetary ring, such as Saturn’s A and B rings.

In Paper I we show that the linear stability condition of spiral density waves in a dense planetary ring is identical to the criterion for spontaneous viscous overstability in the limit of long wavelengths (Schmit and Tscharnuter [1995]). In Paper II it is shown that this criterion is identical to the stability criterion for spiral density waves derived by Borderies et al. [1985] within the streamline formalism (cf. Section 1.5). If a density wave is unstable, the usually applied linear theory to describe its damping fails. Instead, one needs to consider the full nonlinear hydrodynamic equations. The complexity of these equations describing the current setup, i.e. a self-gravitating, viscous fluid in close vicinity to a Lindblad resonance with a perturbing satellite, prevents an exact analytical solution. To derive a comprehensive model for the damping behavior of such unstable density waves we apply the method of multiple scales (Cross and Hohenberg [1993], Kevorkian and Cole [1996]) to obtain an approximate solution. That is, for the nonlinear steady state of an unstable density wave we derive a Landau-type amplitude equation (Paper II). This result constitutes a novel nonlinear relation for the viscous damping of density waves that can be compared to observations and predictions of other theories.

Generally, such amplitude equations are essential tools in the study of patterns that form when a system is pushed outside of its equilibrium by an instability. As such they have been applied to a great variety of physical systems (Cross and Hohenberg [1993]). However, they are usually constructed in a heuristic manner to contain the essential physics and to respect certain symmetries of the problem at hand. In contrast, in Paper II we derive an amplitude equation from the underlying hydrodynamic equations in a formal, rigorous manner by applying a multi-scale expansion. Patterns typically emerge when a certain “control parameter” exceeds a critical value, marking the onset of an instability. The idea is that one first obtains a solution of the linearized equations, which can be computed exactly in the limit that the control parameter takes its critical value. Then one accounts for the nonlinearities in the equations that emerge for a non-vanishing amplitude of the pattern, i.e. when the control parameter exceeds the threshold value. In a multi-scale expansion one connects the effect of all terms in the underlying equations that now emerge to a modulation of the envelope of the basic pattern on a large time and/or length scale, which are/is defined in addition to the regular scales. Thus, the resulting amplitude equation describes slow modulations in space and/or time of the basic pattern that represents the solution of the linearized equations at marginal stability. Strictly, amplitude equations derived in this manner are only valid in the weakly nonlinear regime, i.e. close to the threshold for instability. However, in Paper II we perform a comparison with the streamline model for spiral density waves by
Borderies et al. [1986] (Section 1.5.3) for a variety of parameter sets and find that our derived amplitude equation provides quantitatively correct results even for waves well within the nonlinear regime.

In Paper III we focus on the small-scale waves generated by the viscous overstability mechanism, which have thus far mainly been studied in greatly simplified hydrodynamic models (but see Section 1.4 for more details). All published studies of the viscous overstability to date ignore potentially important physical elements. As pointed out by Latter and Ogilvie [2010] the perhaps most important of these is the collective self-gravity among the ring particles. In a fluid description of a planetary ring the self-gravity force acting on a fluid element is a collective force arising from the superposition of the gravitational forces of neighboring fluid elements. Each fluid element typically contains many ring particles so that in this description the gravitational forces between individual particles are not taken into account in a direct manner. The latter lead to a scattering of particles upon their mutual encounters and act to increase their velocity dispersion, so that the effect of particle-particle gravitational forces can be taken into account indirectly in a hydrodynamic model by adjusting the velocity dispersion (see Sections 1.3.1, 1.3.3).

In principle the collective self-gravity force acts in all spatial directions (vertical and planar). Due to the very small vertical extent of Saturn’s rings, the vertical component has a qualitatively different effect than the planar components, i.e. those acting within the ring plane. The vertical component leads to an increased frequency of collisions between ring particles and to a flattening of the ring (Araki and Tremaine [1986], Wisdom and Tremaine [1988]). In a fluid description (Section 1.3.1) these aspects can be incorporated in the transport coefficients (see Salo et al. [2001] and Section 1.3.1). These effects have important consequences for the stability properties of Saturn’s rings, as they affect the behavior of the ring’s viscosity in such a manner that the viscous overstability can actually occur (Section 1.4). The planar components of the collective self-gravity also play an important role as they can instigate various instabilities of the ring flow, such as the gravitational instability, leading to the formation of self-gravity wakes (Section 1.3.3), the viscous instability (Section 1.3.4) and also the here discussed viscous overstability, the latter resulting in the formation of short-scale density waves. The occurrence of self-gravitational wakes in Saturn’s main rings (Section 1.3.3) indicates that planar collective self-gravity forces are important on length scales similar to those on which overstable waves develop. Furthermore, these forces are crucial for the propagation of spiral density waves in the rings (Section 1.5). It is therefore necessary to understand their effect on the wavetrains generated by viscous overstability, in order to gain a deeper understanding of the apparent relation between density waves and viscous overstability. As the overstable waves discussed here are (quasi-)axisymmetric (see Section 1.4)
the planar component of the self-gravity force associated with these waves is purely radial. In contrast, for the description of self-gravity wakes, which are substantially non-axisymmetric, both components are required. In what follows, if not stated otherwise by collective self-gravity is always meant its radial component.

In Paper III we study the effect of collective self-gravity on the nonlinear evolution of short-scale viscous overstability within an axisymmetric fluid model. We develop a numerical scheme to solve the non-isothermal hydrodynamic equations, including the thermal balance equation (Section 1.3.1) as well as the (radial component of) collective self-gravity. This scheme utilizes several modern methods from the field of computational fluid dynamics, such as Flux-Vector-Splitting and shock capturing techniques (e.g. Hirsch [1991] and Toro [1999]) which are necessary to obtain the required accuracy and stability. Using this scheme we conduct large-scale hydrodynamical integrations, mimicking many-kilometer sized ring portions. Additionally, we employ local N-body simulations using the code by Salo [1995] and Salo et al. [2001], also restricting to the radial component of the collective self-gravity, thereby omitting the emergence of self-gravity wakes, which would make impossible a direct comparison with our fluid approach (see below). The collective radial self-gravity in the simulations is computed using the method of Salo and Schmidt [2010]. Our results show that the collective self-gravity is indeed a crucial element in the nonlinear saturation of short-scale viscous overstability, as it sets, together with pressure forces, a characteristic length scale on which the nonlinear wave pattern tends to settle (see Section 1.4 for more details).

In local N-body simulations one solves directly the Newtonian equations of motion for a large ensemble of individual particles moving in the central planet’s force field (Wisdom and Tremaine [1988], Salo [1991, 1992]; Richardson [1993, 1994], Salo [1995], Mosqueira [1996], Lewis and Stewart [2000], Daisaka et al. [2001]). One takes into account collisions between the particles and potentially their mutual gravitational attraction. The Newtonian equations of motions are usually solved in the Hill-approximation, involving a linearization of the planet’s force field with respect to small excursions in planar as well as vertical direction from a given reference circular orbit. Consequently, in the absence of any additional forces the particle motions are small amplitude oscillatory motions (epicycles) about their individual reference circular orbits. As one considers only a small rectangular portion of the planetary ring, boundary conditions must be imposed to particles that leave the computational domain at one of its boundaries. Usually one works with periodic boundaries so that a particle which leaves at one boundary is replaced by a “new” particle entering at the opposite boundary, taking into account that the two particles are subjected to different orbital velocities due to the Keplerian shear. One of the main profits of performing N-body
Simulations of a dense planetary ring is to step away from the hydrodynamic approximation which usually implies a Newtonian stress-strain relation (Section 1.3.1), thereby ignoring potentially relevant dynamical aspects. As such, in N-body simulations one obtains a more realistic description of the (angular) momentum transport resulting from inter-particle collisions, which eventually governs the entire dynamical evolution of an unperturbed ring, as long as inter-particle gravitational forces are sub-dominant (Section 1.3.2). In many simulation studies the particles are assumed spherical and their mutual impacts are considered instantaneous, corresponding to “hard spheres”. More realistic models that take into account the forces that act on the particles during a collision of finite duration have been developed (Dilley [1993], Salo [1995]) and are necessary when the particles’ mutual gravitational forces are taken into account (Section 1.3.3).

In Paper III we employ local N-body simulations to study the nonlinear saturation of short-scale viscous overstability. This means that the simulation region must have an appreciable radial extent, containing many wave cycles of the studied waves, amounting to several kilometers. On the other hand, since the waves are axisymmetric, the azimuthal size of the simulation region can be quite small, but should at be least several times the particles’ mean free path.

One can also apply such simulations to explore the different possible mechanisms that give rise to a viscosity in a planetary ring (Section 1.3.1). As such one can obtain values for these viscosities, or other transport coefficients (Salo et al. [2001], Daisaka et al. [2001]), which can subsequently be used in hydrodynamic models as is done in the present thesis work. It is also possible to investigate the effects due to the presence of different particle sizes, surface irregularities, or frictional forces at collision. An extensive overview of possible applications can be found in Salo et al. [2018].

Due to the high CPU demand, N-body simulations of Saturn’s dense rings incorporating inter-particle self-gravity and plausible values for the particle size and optical depth are to date restricted to radial sizes of significantly less than one kilometer. From N-body simulations restricted to relatively small spatial domains it became clear that self-gravity wakes and viscous overstability interact and compete in a non-trivial manner (Salo et al. [2001], Ballouz et al. [2017]). At present, however, it is computationally not feasible to study the large-scale behavior and the evolution on long time scales of such systems. Therefore, the investigation of the effect of non-axisymmetric self-gravity on the nonlinear evolution of a viscously overstable wave-system must be deferred to future study. Problematic would also be the simulation of a spiral density wave (Section 1.5) as this would in principle require to cover a substantial radial (and azimuthal) domain. Generally this means that for

\footnote{It also implies an isotropic hydrostatic pressure, as well as a specific shape of the term describing heat transport in the balance equations.}
phenomena that occur on large radial (and possibly azimuthal) length scales, including eccentric ring modes (Borderies et al. [1985], Longaretti and Rapaport [1995]) or structures resulting from the ballistic transport instability (Latter et al. [2012, 2014a,b]) one still needs to resort to hydrodynamic models.

In Paper IV we are concerned with a modeling of the co-existence of a spiral density wave and the short-scale modes generated by viscous overstability to arrive at a qualitative understanding of the possible interactions between the two wave types. That such a co-existence occurs in Saturn’s rings has been found by evaluating Cassini data (Hedman et al. [2014]). This modeling of structures that possess very different length scales (tens of kilometers compared to hundreds of meters) requires a robust numerical scheme. We apply a Weighted Essentially Non-Oscillatory [WENO (Jiang and Shu [1996], Borges et al. [2008])] method to compute the hydrodynamic flow vector. Furthermore, in Paper IV we describe the effect of a density wave on viscous overstability in terms of a simplified axisymmetric model of a perturbed ring near an isolated satellite resonance, introduced by Mosqueira [1996]. This model admits a hydrodynamic linear stability analysis, as well as the application of local N-body simulations. In the hydrodynamic treatment of this setup the density wave is taken into account by adopting a modified (time-dependent) equilibrium solution of the hydrodynamic equations (Section 1.3.1), describing the periodically changing collective motion of the ring flow within a small rectangular region that is penetrated by the density wave. In this model the wavelength of the density wave is assumed to be infinite. In the N-body simulations the presence of the density wave is accounted for through an oscillating radial size of the simulation region, so that for instance the ground state surface mass density within the simulation region oscillates about a certain mean value, as is the case in the hydrodynamic treatment. Furthermore, modified boundary conditions are applied for particles leaving the simulation region which take into account that the collective motion of the ring flow now departs from simple Keplerian shear.

1.3 Fundamental Dynamics of Dense Planetary Rings

1.3.1 Kinetic Theory and Hydrodynamic Approximation

The most suitable theoretical framework to study the dynamics of a dense planetary ring is provided by a kinetic description based on a Boltzmann or Enskog-Equation (Goldreich and Tremaine [1978], Hämeen-Anttila [1978], Shukhman [1984], Stewart et al. [1984], Borderies et al. [1983], Borderies
et al. [1985], Shu and Stewart [1985], Araki and Tremaine [1986], Latter and Ogilvie [2006, 2008], Araki [1988, 1991]). As starting point we write the Boltzmann equation

$$\frac{\partial}{\partial t} f({\bf v}, {\bf x}, t) + v_i \frac{\partial}{\partial x_i} f({\bf v}, {\bf x}, t) = \partial_t [f({\bf v}, {\bf x}, t)]_{\text{coll}}$$

(1.1)

with the one-particle distribution function $f$ defined so that $f({\bf v}, {\bf x}, t) d^3x d^3v$ denotes the number of particles located within the volume element $d^3x$ centered at $x$ with velocities within $d^3v$ around $v$. The potential $U$ is due to the central planet, any satellites (either embedded in or external to the rings), as well as the collective self-gravity force generated by the ring material. The term on the right hand side describes the change of $f$ due to collisions between ring particles and will be discussed below.

In general, $f$ also depends on the particle spins, sizes and surface irregularities. While in an unperturbed dense ring in equilibrium these additional dependencies have minor effects, they may impact structures evolving due to dynamical instabilities. In this section we will ignore these dependencies. Furthermore, the gravitational encounters between particles are omitted. Also the possibility of destructive collisions and aggregate formation, which would lead to a evolving particle size distribution is not considered.

First we make a change of variables

$$w = v - u({\bf x}, t)$$

(1.2)

where $w$ is the peculiar velocity and $u({\bf x}, t)$ is the local mean velocity field of the ring flow which will be defined below. In order to write (1.1) in terms of the variables $(w, x, t)$ we need to take into account that $f = f[w(u({\bf x}, t), v), x, t]$ and make the following replacements

$$\frac{\partial_t f}{\partial_t f} \rightarrow \frac{\partial_t f}{\partial_t f} + (\partial_{w_i} f) \frac{\partial_t u_i}{\partial_t u_i}$$

(1.3)

$$v_i \frac{\partial_x f}{\partial_x f} \rightarrow (w_i + u_i) [\frac{\partial_x f}{\partial_x f} + (\partial_{w_k} f) \frac{\partial_x u_k}{\partial_x u_k}].$$

(1.4)

Saturn’s main rings are located at relatively large radial distance from the central planet. That is, their radial distance from the planet is much greater than the radial length scale of any fine structures that form within these rings, so that the local curvature of the ring shape can usually be neglected. In order to describe the local dynamics, it is most convenient to adopt a rectangular coordinate system $(x, y, z)$ which rotates on a circular orbit at radius $r_0$ in the ring plane, with the local velocity $u(r_0)$. We assume that the disk is in a state of Keplerian differential rotation, which is a sufficiently accurate assumption to describe most dynamical phenomena in Saturn’s dense rings. Small deviations from Keplerian motion exist due to the non-spherical shape of the central planet or local pressure gradients and self-gravity, but are ignored in what follows. Hence we have

$$u = \Omega r \mathbf{e}_\theta$$

(1.5)
expressed in polar coordinates \((r, \theta)\), with the Kepler frequency

\[ \Omega = \sqrt{\frac{GM_p}{r^3}} \tag{1.6} \]

where \(G\) is the gravitational constant and \(M_p\) the central planet’s mass. We represent the ring as a small co-moving rectangular patch at radius \(r_0\) with planar dimensions \(L_x \ll r\) and \(L_y \ll r\). The \(x\)-coordinate points radially outward and the \(y\)-coordinate is taken to coincide with the local direction of collective orbital motion. Then we can approximate

\[ u = (\Omega - \Omega_0) r e_y \approx -\frac{3}{2} \Omega_0 x e_y \tag{1.7} \]

where \(\Omega_0 = \Omega(r_0)\) and \(x = r - r_0\).

In the rotating frame, inertial forces appear in Equation (1.1) so that

\[ \partial_x U \to \partial_x U + [\Omega_0 \times (\Omega_0 \times r) + 2 \Omega_0 \times v]_i \tag{1.8} \]

where \(\Omega_0 = \Omega_0 \, e_z\). In what follows we will absorb the centrifugal term in the central planet’s gravity term which is contained in \(\partial_x U\). The Boltzmann equation now reads

\[
\begin{align*}
\partial_t f + (u_k + w_k)\partial_{x_k} f & - [\partial_t u_k + (u_l + w_l)\partial_{x_l} u_k + \partial_{x_k} U - 2\epsilon_{kzm} \Omega_0 (w_m + u_m)] \partial_{w_k} f \\
& = \partial_t [f(v, x, t)]_{\text{coll}}
\end{align*}
\tag{1.9}
\]

where \(\epsilon\) denotes the Levi-Civita symbol. Solving this equation is difficult in general, since the collision term is a nonlinear functional of \(f\).

In the context of planetary rings it is common practice (Goldreich and Tremaine [1979], Shukhman [1984], Shu et al. [1985a], Shu et al. [1985b], Shu and Stewart [1985]) to replace the Boltzmann equation by its first three moment equations and to solve these in a self-consistent manner. The moment equations are obtained by multiplying (1.9) subsequently by \(m_p\), \(m_p w_i\) and \(m_p w_i w_j\) (with \(i, j = x, y, z\) and particle mass \(m_p\)) and integrating over all velocities \(w\). The result is

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u_i) & = \partial_t [\rho]_{\text{coll}}, \\
\partial_t (\rho u_i) + \partial_x (\rho u_i u_k) + \rho \partial_x U + \partial_x \dot{P}_{ij} & + 2\rho \epsilon_{izm} \Omega_0 u_m = \partial_t [\rho u_i]_{\text{coll}}, \tag{1.10}
\end{align*}
\]

\[
\begin{align*}
\partial_t \dot{P}_{ij} + \partial_x (u_k \dot{P}_{ij}) + \dot{P}_{jk} \partial_{x_k} u_i + \dot{P}_{ik} \partial_{x_k} u_j & + \partial_{x_k} \dot{P}_{ijk} \\
+ 2\Omega_0 (\epsilon_{izm} \dot{P}_{mj} + \epsilon_{jzm} \dot{P}_{im}) & = \partial_t [\dot{P}_{ij}]_{\text{coll}}. \tag{1.11}
\end{align*}
\]
In these equations we define

\begin{align*}
\rho &= m_p \int f d^3w, \quad (1.13) \\
\rho u_i &= m_p \int f v_i d^3w, \quad (1.14) \\
\hat{P}_{ij} &= m_p \int f w_i w_j d^3w, \quad (1.15) \\
\hat{P}_{ijk} &= m_p \int f w_i w_j w_k d^3w, \quad (1.16)
\end{align*}

as well as

\begin{align*}
\partial_t [\rho]_{\text{coll}} &= \int [\partial_t f(\mathbf{w}, \mathbf{x}, t)]_{\text{coll}} m_p d^3w, \quad (1.17) \\
\partial_t [\rho u_i]_{\text{coll}} &= \int [\partial_t f(\mathbf{w}, \mathbf{x}, t)]_{\text{coll}} m_p w_i d^3w, \quad (1.18) \\
\partial_t \left[ \hat{P}_{ij} \right]_{\text{coll}} &= \int [\partial_t f(\mathbf{w}, \mathbf{x}, t)]_{\text{coll}} m_p w_i w_j d^3w. \quad (1.19)
\end{align*}

The quantities $\rho$, $u_i$ and $\hat{P}_{ij}$ are the mass volume density, the mean velocity field and the pressure tensor, respectively. These quantities have straightforward physical interpretations, unlike the moments of the velocity $w_i$ of order higher than 3.

It is furthermore useful to decompose the pressure tensor into its isotropic and anisotropic parts. To this end one writes

\begin{equation}
\hat{P}_{ij} = \left( \frac{1}{3} \hat{P}_{kl} \delta_{kl} \right) \delta_{ij} - \left[ \left( \frac{1}{3} \hat{P}_{kl} \delta_{kl} \right) \delta_{ij} - \hat{P}_{ij} \right] \quad (1.20)
\end{equation}

and defines the isotropic (or scalar) pressure

\begin{equation}
p = \frac{1}{3} \hat{P}_{kl} \delta_{kl} \quad (1.21)
\end{equation}

along with the viscous stress tensor

\begin{equation}
\hat{\Pi}_{ij} = p \delta_{ij} - \hat{P}_{ij}. \quad (1.22)
\end{equation}

In above expressions $\delta$ is the Kronecker-symbol. In studies of planetary ring dynamics one usually works with the velocity dispersion of the ring particles instead of the pressure $p$. The velocity dispersion $c$ is defined through

\begin{equation}
p = \frac{1}{3} m_p \int w^2 f d^3w \equiv \rho c^2. \quad (1.23)
\end{equation}
With the definitions (1.21), (1.22) and (1.23) one can split (1.12) into the equations
\[
\frac{3}{2} \left[ \partial_t (\rho c^2) + \partial_{x_k} (\rho c^2 u_k) \right] + p \partial_{x_k} u_k - \hat{\Pi}_{ik} \hat{D}_{ik} + \partial_{x_k} F_k = \frac{3}{2} \partial_t [\rho c^2]_{\text{coll}} \tag{1.24}
\]
and
\[
\partial_t \hat{\Pi}_{ij} + \partial_{x_k} (u_k \hat{\Pi}_{ij}) - \left( \frac{2}{3} \hat{\Pi}_{ik} \hat{D}_{ik} \delta_{ij} - \hat{\Pi}_{jk} \partial_{x_k} u_i - \hat{\Pi}_{ik} \partial_{x_k} u_j \right) - \partial_{x_k} \left( \hat{P}_{ijk} - \frac{2}{3} F_k \delta_{ij} \right) = -2\Omega_0 (\epsilon_{izm} \hat{\Pi}_{mj} + \epsilon_{jzm} \hat{\Pi}_{im}) + 2p \hat{S}_{ij} + \partial_t [\hat{\Pi}_{ij}]_{\text{coll}} \tag{1.25}
\]
where we define the energy flux vector as the contracted third order moment
\[ F_k = \frac{1}{2} \hat{P}_{ijk} \delta_{ij}, \tag{1.26} \]
describing the transport of kinetic energy of the particles’ random motion. Furthermore, we define the deformation rate tensor
\[ \hat{S}_{ij} = \hat{D}_{ij} - \frac{1}{3} \partial_{x_k} u_k \delta_{ij}, \tag{1.27} \]
along with the rate of strain tensor
\[ \hat{D}_{ij} = \frac{1}{2} (\partial_{x_i} u_j + \partial_{x_j} u_i), \tag{1.28} \]
both of which are symmetric tensors. The collisional changes in (1.24) and (1.25) are given by
\[
\partial_t [\rho c^2]_{\text{coll}} = \int [\partial_t f(\mathbf{v}, \mathbf{x}, t)]_{\text{coll}} m_p \delta_{ij} w_i w_j \, d^3 w, \tag{1.29}
\]
\[
\partial_t [\hat{\Pi}_{ij}]_{\text{coll}} = \int [\partial_t f(\mathbf{v}, \mathbf{x}, t)]_{\text{coll}} m_p [\delta_{kl} w_k w_l \delta_{ij} - w_i w_j] \, d^3 w. \tag{1.30}
\]
Equations (1.10), (1.11), (1.24) and (1.25) describe the local balance of mass, momentum, random kinetic energy and viscous stress, respectively.

Various studies have shown that for phenomena varying slowly in time and space, one can arrive at a self-consistent dynamical description in terms of Equations (1.10), (1.11), (1.24) with constitutive relations for the viscous stress tensor $\Pi_{ij}$ and the energy flux $F_k$ (Chapman and Cowling [1970], Résibois and De Leener [1977], Jenkins and Richman [1985]) which are then linear functions of the spatial gradients of the quantities $\rho$, $\mathbf{u}$ and $c^2$ and read
\[
\hat{\Pi}_{ij} = 2\eta \left[ \hat{D}_{ij} - \left( \frac{1}{3} \hat{D}_{kl} \delta_{kl} \delta_{ij} \right) \right], \tag{1.31}
\]
\[
F_k = -\kappa_D \partial_{x_k} c^2, \tag{1.32}
\]
where \( \eta = \eta(\rho, c^2) \) and \( \kappa_D = \kappa_D(\rho, c^2) \) are the dynamic shear viscosity and dynamic heat conductivity, respectively. These functions are called transport coefficients since the terms in which they appear in Equations (1.11), (1.24) and (1.25) induce a transport of momentum, energy and viscous stress, respectively. The subscript \( D \) of the heat conductivity is used to distinguish this quantity from the epicyclic frequency \( \kappa \) which will be defined in Section 1.3.3. Note that in the following we will not write the dependence on \( \rho \) and \( c^2 \). At this level of approximation the distribution function takes the form

\[
f(\mathbf{w}, \mathbf{x}, t) = f^{(0)}(\mathbf{w}, \mathbf{x}, t) \left[ 1 + \Phi(\mathbf{w}, \mathbf{x}, t) \right] \tag{1.33}
\]

with the Maxwellian

\[
f^{(0)}(\mathbf{w}, \mathbf{x}, t) = \frac{\rho/m_p}{(2\pi c^2)^{3/2}} \exp \left[ -\frac{(\mathbf{w} - \mathbf{u})^2}{2c^2} \right], \tag{1.34}
\]

which is the distribution function of the molecules of a gas in equilibrium and where \( \Phi(\mathbf{w}, \mathbf{x}, t) \) is a linear function of the first spatial derivatives of \( \rho \), \( c^2 \) and \( \mathbf{u} \). An implicit \( \mathbf{x} \)-dependence in (1.34) can exist through \( \rho \), \( \mathbf{u} \) and \( c^2 \). This notation is chosen to emphasize that the deviations from a Maxwellian distribution function are small. The resulting set of equations is called the hydrodynamic approximation and it correctly describes phenomena involving small spatial and temporal variations of the gas.

In order to obtain a qualitatively correct hydrodynamic description of a dense ring, the collisional change of \( f \), and hence the collisional changes of \( \rho \), \( \mathbf{u} \), \( c^2 \), and \( \Pi_{ij} \) have to be modeled in a way that these take into account the circumstance that the mean free path of particles between successive collisions need not be large compared to the particle size. This means that the transfer of energy and momentum at collisions over one particle diameter cannot be neglected, or can even dominate the transfer between the collisions due to particle random motion. In order to account for the former (collisional, or nonlocal) transfer, the positions of the two colliding particles must be distinguished at collision. Furthermore, it should be taken into account that the frequency of collisions of ring particles is affected by the presence of neighboring ring particles. The most simple generalization of the Boltzmann equation which takes these effects into account is due to Enskog. Enskog’s equation differs from the Boltzmann equation only in that the collisional change of \( f \) is modified.

Let us first consider the probability of collision of two particles (diameter \( D \)) in a dilute gas of hard spheres in the Boltzmann theory. The two particles possess velocities \( \mathbf{v} \) and \( \mathbf{v}_1 \) and collide at location \( \mathbf{x} \) (the location of the particle with velocity \( \mathbf{v} \)) such that the unit vector \( \mathbf{k} \) connects the particle centers at collision. We define the relative velocities before and after collision as \( \mathbf{g} = \mathbf{v}_1 - \mathbf{v} \) and \( \mathbf{g}' = \mathbf{v}_1' - \mathbf{v}' \), respectively. The collision is inelastic so that

\[
(\mathbf{g}' \cdot \mathbf{k}) = -\epsilon(\mathbf{g} \cdot \mathbf{k}) \tag{1.35}
\]
with the normal coefficient of restitution \(0 \leq \epsilon \leq 1\). In the current approximation, the particle’s relative tangential velocity is not changed by the collision.

At collision, the center of the particle with velocity \(v_1\) is located within the surface element \(D^2d\Omega_k\) of the sphere with radius \(D\), centered at \(x\) and where \(d\Omega_k\) denotes the incremental solid angle increment centered at \(k\). In order for such collision to take place within a time interval \(dt\), the center of the particle with velocity \(v_1\) must be within a cylinder of volume \(D^2(g \cdot k)d\Omega_k dt\) with \(g \cdot k > 0\). Therefore, the number of such collisions per unit time where the first particle lies in the volume \(d^3x\) is given by

\[
f(v, x, t)f(v_1, x, t)D^2(g \cdot k)d\Omega_k d^3v_1 d^3x.
\] (1.36)

Note that the particle locations are not distinguished. In the Enskog-theory this quantity is modified and reads

\[
\chi \left( x + \frac{1}{2}Dk \right) f(v, x, t)f(v_1, x + Dk, t)D^2(g \cdot k)d\Omega_k d^3v_1 d^3x.
\] (1.37)

The difference is that the particle locations are now distinguished and that there is an additional factor \(\chi\) (the Enskog-factor) which takes into account two effects which arise from the finite size of the particles. One aspect is due to the obvious circumstance that the volume in which the particles can freely move becomes smaller with increasing particle volume number density. The available volume is reduced by a factor

\[
1 - \frac{4\pi}{3}D^3\frac{\rho}{m} \equiv 1 - \text{ff}
\] (1.38)

with \(\text{ff}\) denoting the volume filling factor. Thus, a larger number density implies a smaller mean free path and an increased frequency of collisions. The second effect is a decrease of the collision frequency due to partial shielding by neighboring particles. That is, if two particles are sufficiently close to each other, the range of solid angles describing the direction of a potentially impacting (third) particle is reduced. The factor \(\chi\) approaches unity for a rarefied gas and becomes infinitely large as the particles approach the state of densest possible packing which prevents any free motion. Significant departures from unity occur whenever the mean free path becomes comparable to the particle dimensions. Using the above expressions, the collisional change of an arbitrary quantity \(\Psi\) per unit volume at location \(x\) is given by

\[
\partial_t [\Psi]_{\text{coll}} = \int \int \int \chi \left( r + \frac{1}{2}Dk \right)
\times \left( \Psi' - \Psi \right) f(v, x, t)f(v_1, x + Dk, t)D^2(g \cdot k)d\Omega_k d^3v_1 d^3x.
\] (1.39)
Here Ψ and Ψ′ denote the particle property before and after a collision. It can be shown [e.g. Jenkins and Richman [1985]] that the collisional changes of ρui and ρc^2 can be expressed as

\[ \partial_t [\rho u_i]_{\text{coll}} = \partial_{x_j} \hat{P}_{ij} \]  

(1.40)

\[ \partial_t [\rho c^2]_{\text{coll}} = \partial_{x_i} \hat{F}_{i} - \Gamma \]  

(1.41)

where one refers to \( \hat{P}_{ij} \) and \( \hat{F}_{i} \) as the collisional pressure tensor and the collisional heat flux, respectively. These terms describe the transport of momentum and random kinetic energy at collisions and arise because the colliding particle’s positions are distinguished. The quantity Γ is the rate of dissipation of the particle’s random kinetic energy due to the inelasticity of mutual collisions [Equation (1.35)] and is also present if the gas is dilute such that (1.36) is used instead of (1.37).

The collisional change (1.39) is a functional of the two particle locations. In order to obtain a spatially local description one needs to perform Taylor series expansions of the collisional integrals in the quantity \( D_k \), which is justified if \( f \) and \( \chi \) do not vary appreciably on length scales of the order of a particle diameter. If these expansions are truncated at the next leading order, it can be shown that the collisional pressure tensor and the collisional heat flux can be written in the form

\[ \hat{P}_{ij} = \left( p_{\text{coll}} - \xi \hat{D}_{kl} \delta_{kl} \right) \delta_{ij} - 2 \eta_{\text{coll}} \left[ \hat{D}_{ij} - \left( \frac{1}{3} \hat{D}_{kl} \delta_{kl} \delta_{ij} \right) \right], \]  

(1.42)

\[ F_{k} = -\kappa_{D} \partial_{x_k} c^2. \]  

(1.43)

Comparison of these expressions with (1.21), (1.22) (1.31) and (1.32) shows that the collisional heat flux has the same form as its transport counterpart. The collisional pressure tensor involves a new quantity \( \xi \), the (dynamic) bulk viscosity, which measures the irreversible transfer of collective motion into random motions of particles when material is compressed (Chapman and Cowling [1970]). The transport coefficients \( \eta_{\text{coll}} \), \( \xi \) and \( \kappa_{D} \) depend on \( \rho \) and \( c^2 \) similar as (1.31) and (1.32).

It should be noted that the establishment of a local stress-strain relation (1.31, 1.42) in a planetary ring generally requires, apart from small spatial and temporal variations, large collision frequencies \( \omega_c \), much larger than any velocity gradient in the system. If the latter condition is not fulfilled, non-locality in time of the viscous stress is expected to occur (Latter and Ogilvie [2006]), which means that the latter cannot be expressed as a simple function of time but rather in terms of a functional because the immediate history of the viscous stress needs to be taken into account. For an adequate description of the dynamics it is then necessary to solve the moment equations for the stress tensor (1.25). The existence of a local stress-strain relation in an
unperturbed dilute ring even if the collision frequencies are small turns out to be a special circumstance tied to the azimuthal symmetry of the ground state, combined with small superimposed particle random motions [see Shu and Stewart [1985] for more details].

The denser regions in Saturn’s rings have a small vertical extent of considerably less than \( \sim 100\) m. This is a consequence of the frequent dissipative mutual collisions of ring particles, resulting in small values of the velocity dispersion \( c \). Therefore one is usually not interested in the detailed vertical structure of the rings. Most theoretical studies on Saturn’s ring dynamics assume from the outset symmetry about the mid-plane \((z = 0)\) and that the disk is vertically isothermal (Goldreich and Tremaine [1978], Borderies et al. [1983], Araki and Tremaine [1986], Latter and Ogilvie [2008]). With these assumptions a vertical averaging of the moment equations can be performed (Shu and Stewart [1985]) where the third order moments can be neglected altogether, as long as the investigation is restricted to structures in the ring plane with characteristic length scales that are much greater than the vertical disk thickness. This procedure then leads to a self-consistent description of the ring dynamics within which the pressure tensor can be rigorously computed through Equation (1.12), provided the collisional change can be computed. The aforementioned studies assume that the velocity distribution function is a tri-axial Gaussian

\[
f(\mathbf{w}, \mathbf{x}, t) = \frac{\rho}{m_p} \frac{1}{\sqrt{(2\pi)^3 \det[\hat{C}_{ij}]}} \exp \left( -\frac{1}{2} \mathbf{w}^T \hat{C}^{-1} \mathbf{w} \right)
\]

(1.44)

in order to compute the collisional changes, with the velocity dispersion tensor

\[
\hat{C}_{ij} = \frac{1}{\rho} \hat{P}_{ij}.
\]

(1.45)

Since \( \hat{P}_{ij} \) and \( \hat{C}_{ij} \) are per definition symmetric tensors, it is always possible to change to the (orthogonal) principal axis system in which \( \hat{C}_{ij} \) is a diagonal tensor. In this coordinate system the distribution function takes the form

\[
f(\mathbf{w}, \mathbf{x}, t) = \frac{\rho}{m_p} \frac{1}{(2\pi)^{3/2} c_1 c_2 c_3} \exp \left[ -\sum_{i=1}^{3} \frac{w_i^2}{2c_i^2} \right]
\]

(1.46)

where \( c_1, c_2, c_3 \) are the principal values of the velocity dispersion tensor \( \hat{C}_{ij} \). The main advantage of this formalism is that it can generally provide a more realistic description of the pressure tensor than provided by (1.31), including anisotropy of the velocity dispersion and non-Newtonian behavior of the viscous stress. For instance, from (1.7) and (1.12) we obtain the equilibrium relation \( \hat{P}_{xx} = 4\hat{P}_{yy} \) together with \( \hat{P}_{xy} = \hat{P}_{yx} = 0 \) by assuming reflection symmetry of the disk with respect to the vertical plane.
\( f(w_x, w_y, -w_z, x, t) = f(w_x, w_y, w_z, x, t) \) so that \( \hat{P}_{xz} = \hat{P}_{yz} = 0 \) and neglecting the collisional changes. Although particle collisions tend to eliminate velocity anisotropies, the pressure tensor retains anisotropy even at large optical depths. The disadvantage of this formalism which involves a rigorous solution of the pressure tensor equation (1.12) is a highly increased mathematical complexity.

In summary, when we apply the hydrodynamic approximation, the dynamics of a dense ring is described in terms of the balance equations for mass, momentum, and energy which read

\[
\begin{align*}
\partial_t \rho &= -\nabla (\rho u), \quad (1.47) \\
\partial_t \rho u &= -\nabla \cdot (\rho uu) - \rho \nabla (\phi_p + \phi_{sg} + \phi_s) - \nabla \cdot (\hat{P} + \hat{P}_{\text{coll}}) \\
&\quad - \Omega_0 \times (\Omega_0 \times r) - 2\Omega_0 \times u, \quad (1.48) \\
\partial_t \left( \frac{3}{2} \rho T \right) &= -\nabla \left( \frac{3}{2} \rho uT \right) - (\hat{P} + \hat{P}_{\text{coll}}) : \nabla u - \nabla \cdot (F + F_{\text{coll}}) - \Gamma, \quad (1.49)
\end{align*}
\]

where we use the definition of the granular temperature \( T \equiv c^2 \) and where the relations (1.21), (1.22), (1.31), (1.32), (1.42) and (1.43) are to be applied. Furthermore, we have written out the potential \( U \). The symbols \( \phi_p, \phi_{sg}, \phi_s \) denote the planet’s potential, the self-gravity potential and the potential of external or embedded satellites. So far we have denoted by \( \hat{P} \) and \( F \) the transport pressure tensor and energy flux, and by \( \hat{P}_{\text{coll}} \) and \( F_{\text{coll}} \) their collisional counterparts, resulting from the mutual physical contact of particles during their collisions. In the context of planetary rings one usually refers to the transport contribution as the local contribution and to the collisional contribution as the nonlocal contribution.

We have already assumed that the ring is symmetric with respect to the mid-plane. If the ring is vertically isothermal, and if we neglect vertical motions of the disk \( (u_z = 0) \), then, in the thin-disk approximation (i.e. by assuming \( \rho \equiv \sigma(x, y)\delta(z) \), with the Dirac delta-function) a vertical integration yields the set of equations

\[
\begin{align*}
\partial_t \sigma &= -\nabla (\sigma u), \quad (1.50) \\
\partial_t (\sigma u) &= -\nabla \cdot (\sigma uu) - \sigma \nabla (\phi_p + \phi_{sg} + \phi_s) - \nabla \cdot (\hat{P} + \hat{P}_{\text{coll}}) \\
&\quad - \Omega_0 \times (\Omega_0 \times r) - 2\Omega_0 \times u, \quad (1.51) \\
\partial_t \left( \frac{3}{2} \sigma T \right) &= -\nabla \left( \frac{3}{2} \sigma uT \right) - (\hat{P} + \hat{P}_{\text{coll}}) : \nabla u - \nabla \cdot (F + F_{\text{coll}}) - \Gamma, \quad (1.52)
\end{align*}
\]

with the surface mass density

\[
\sigma(x, y) = \int_{-\infty}^{\infty} \rho \, dz \quad (1.53)
\]
and where \( \mathbf{u} = (u_x, u_y) \). The quantities \( \hat{P}, \hat{P}_{\text{coll}}, \mathbf{F}, \mathbf{F}_{\text{coll}} \) and \( \nabla \) are now understood to be two-dimensional tensors and vectors, respectively. Furthermore, the quantities \( (p, p_{\text{coll}}, \eta, \eta_{\text{coll}}, \xi, \kappa_D, \kappa_{D_{\text{coll}}}, \Gamma) \) are now vertically integrated. The remaining quantities are to be evaluated in the plane \( z = 0 \).

The ring is of course not infinitesimally thin, but possesses a certain (small) scale height \( H \). In the present approximation the constant scale height follows from the vertical component of the simplified momentum equation (1.48)

\[
0 = -\rho \Omega^2 z - \rho \partial_z \phi_{sg} - \partial_z \left( \rho c^2 + \hat{P}_{zz_{\text{coll}}} \right).
\]

which describes a vertical hydrostatic equilibrium and where the vertical force exerted by the central planet has been obtained by Taylor-expanding its potential. If the ring is dilute, so that the vertical component of self-gravity and the collisional stress can be neglected, vertical integration yields

\[
\rho(z) = \rho_0 \exp \left( -\frac{\Omega^2}{c^2} z^2 \right),
\]

i.e. a Gaussian distribution with the scale height

\[
H = \frac{c}{\Omega},
\]

In a dense ring the collisional stress, and also the vertical self-gravity can in general not be neglected as they can even dominate the balance (Araki and Tremaine [1986]).

### 1.3.2 Dense Ring Equilibrium

The ground state of a dense unperturbed ring is governed by the thermal energy equation (1.52) which describes the balance of the heating by viscous dissipation and the cooling by inelastic mutual particle collisions [cf. (1.35)]. Rather than presenting explicit solutions of the collisional integrals for a given distribution function, we give a heuristic description of this balance which qualitatively explains well the basic features of the ground states calculated in kinetic studies of dense rings (Araki and Tremaine [1986], Latter and Ogilvie [2008]) as well as non-selfgravitating local N-body simulations [Wisdom and Tremaine [1988], Salo [1991] (Note that we have thus far omitted gravitational forces between individual particles)].

The loss of random kinetic energy is estimated by

\[
[\partial_t c^2]_{\text{loss}} = -\omega_c c^2 (1 - \epsilon^2),
\]

where \( \omega_c \) is the frequency of binary particle collisions [cf. (1.36), (1.37)] and the velocity dispersion \( c \) is used as a proxy for the typical value of the impact
velocity. For a dilute, vertically isothermal ring in equilibrium, modeled by the Maxwellian (1.34) one can derive the estimate (Shu and Stewart [1985])

\[
\omega_c \approx 3\Omega \tau \tag{1.58}
\]

with the geometric optical depth

\[
\tau = \pi (D/2)^2 \frac{\sigma}{m_p} \tag{1.59}
\]

which describes the surface fraction covered by particles. In a dense ring the collision frequency is significantly larger than the orbital frequency. In the description of the ring dynamics based on the moment equations (1.10)-(1.12), (1.24), (1.25), the collisional cooling is contained in \( \Gamma \).

The gain of random kinetic energy arises from an effective viscous coupling of the ring particle flow at different radii and reads

\[
\left[ \partial_t c^2 \right]_{\text{gain}} = \frac{9}{4} \Omega^2 (\nu_{\text{local}} + \nu_{nl}). \tag{1.60}
\]

This expression follows from the second term on the right hand side of Equation (1.52). Here we define the kinematic shear viscosity \( \nu \) by \( \eta = \rho \nu \) and denote by the subscripts \( \text{local} \) and \( \text{nl} \) the contributions arising from the transport (1.20) and collisional (1.42) pressure tensor. Furthermore, we inserted the Keplerian shear velocity (1.7) to evaluate the rate of strain tensor (1.28). An expression for the local shear viscosity \( \nu_{\text{local}} \) in a dilute ring was first derived by Goldreich and Tremaine [1978] (see also Hämén-Anttila [1978] and Shu and Stewart [1985]) and reads

\[
\nu_{\text{local}} = \frac{c^2}{\Omega} \frac{\tau}{1 + \tau^2}. \tag{1.61}
\]

This expression can be motivated by considering that microscopically, the basic relation for the viscosity is

\[
\nu = \omega_c l^2 \tag{1.62}
\]

where \( l \) denotes the particles’ mean free path (in radial direction in the present context). At larger optical depths \( \tau \gtrsim 1 \) particles undergo many collisions during one orbital period and \( l \sim c/\omega_c \). On the other hand, at low optical depths, collisions are infrequent and the radial excursions are limited by the particles’ epicyclic motions so that \( l \sim c/\Omega \). By using (1.58) for \( \omega_c \) the expressions (1.61) and (1.62) agree with each other in both limits for \( \tau \).

By the nonlocal viscosity mechanism momentum is transported in a collision over the particle diameter, so that \( l \sim D \) and we have

\[
\nu_{nl} \sim \omega_c D^2. \tag{1.63}
\]
All together we can write

$$\partial_t c^2 = -k_1 \omega_c c^2 (1 - \epsilon^2) + \frac{9}{4} \Omega^2 \left( k_2 \frac{c^2}{\Omega} \frac{\tau}{1 + \tau^2} + k_3 \omega_c D^2 \right)$$

(1.64)

where $k_1, k_2, k_3$ are dimensionless constants of order unity and $\omega_c$ is taken from Equation (1.58).

If the ring is hot and dilute (i.e. $c \gg$) the nonlocal viscosity is negligible and one obtains the $\epsilon$-$\tau$ relation (Goldreich and Tremaine [1978])

$$(1 - \epsilon_r^2) (1 + \tau^2) = \frac{3k_2}{4k_1} \approx 0.61,$$

(1.65)

where we used (1.58) and where the numerical value follows by setting in the left hand side of the equation $\tau = 0$ and $\epsilon_r = 0.627$, which was computed by Goldreich and Tremaine [1978] for this limit. If $\epsilon$ were a constant (independent of $c$) this balance could for a specific optical depth $\tau$ only be fulfilled by a unique elasticity value $\epsilon_r$. If $\epsilon > \epsilon_r$, the heating exceeds the cooling and no equilibrium is possible. If on the other hand $\epsilon < \epsilon_r$, cooling dominates and $c$ reduces until the nonlocal viscous heating balances the cooling. In the latter case the velocity dispersion yields values $c \sim \Omega D$.

However, the restitution coefficient describing collisions in Saturn’s dense rings depends on the particles’ mutual impact velocity. Laboratory measurements of collisions between water-ice (the main constituent of Saturn’s rings) particles have been carried out (Bridges et al. [1984], Hatzes et al. [1988]) at conditions mimicking those in Saturn’s rings (low pressure and temperature). The most widely used elasticity law in studies of Saturn’s rings is by Bridges et al. [1984], which describes fairly inelastic collisions

$$\epsilon(v) = \begin{cases} \left( \frac{v_n}{v_c} \right)^{-0.234} & \text{if } v_n > v_c, \\ 1 & \text{if } v_n \leq v_c, \end{cases}$$

(1.66)

with the scale parameter $v_c = 0.077 m/s$. This relation was obtained from collisions between frost-covered ice spheres. Assuming a nominal particle radius of $R = 1 m$, this law results for optical depths corresponding to Saturn’s main rings in a flattened ring ($c \sim \Omega D$), dominated by nonlocal momentum transport (at least if mutual self-gravity forces are omitted). By comparing photometric modeling of Saturn’s dense rings (based on N-body simulations) and observations it has been concluded that collisions between ring particles are indeed very dissipative (Karjalainen and Salo [2004], French et al. [2007], Porco et al. [2008]), perhaps even more dissipative then described by (1.66).

Nevertheless, there exist also measurements (Hatzes et al. [1988]) using ice particles covered by compacted frost, that lead to an overall larger $\epsilon$ and
a weaker dependence on impact velocity. Such setups describe fairly elastic collisions and would result, again assuming \( R = 1 \text{ m} \), in a hotter, more dilute ring at low and intermediate optical depths \( \tau \lesssim 1 \), so that nonlocal effects are unimportant at these optical depths. Note that also relation (1.66) could result in a hot, dilute ring, if the particle size were substantially smaller than \( R = 1 \text{ m} \).

The velocity dependence of \( \epsilon \) implies that in a dilute ring \( (c \gg \Omega D) \) the effective value \( \epsilon_{\text{eff}} \), which represents the weighted average of \( \epsilon \) over different impact velocities, will take the value \( \epsilon_\tau \). This value specifies the equilibrium velocity dispersion through (1.66). If the ring is sufficiently dense, so that \( \nu_{\text{nl}} \) cannot be neglected, the equilibrium value of \( \epsilon \) is smaller than what is implied by (1.65). The time scale to achieve this equilibrium corresponds to the collisional time scale \( 1/\omega_c \).

Furthermore, from the momentum equation (1.51) in the ground state follows

\[
\begin{align*}
\partial_x \phi_p - \Omega_0^2 (r_0 + x) - 2\Omega_0 u_y = 0, \\
\partial_y \phi_p - \Omega_0^2 y + 2\Omega_0 u_x = 0,
\end{align*}
\]

(1.67)

(1.68)

as the collective self-gravity and satellite potentials are negligible compared to the planetary potential. If we write the latter as double Taylor series

\[
\phi_p = \frac{-GM}{\sqrt{(r_0 + x)^2 + y^2}}
\]

\[
= -GM \left[ \frac{r_0^2 - r_0 x + x^2 - y^2/2}{r_0^3} \right] + \mathcal{O}(x^3, y^3, xy^2),
\]

then Equations (1.67,1.68) are identically fulfilled with the velocity field (1.7). In summary, the ground state of a (unperturbed) dense ring is described by an equilibrium velocity dispersion \( \epsilon_0 \) resulting from the balance (1.64), together with a constant equilibrium surface mass density \( \sigma_0 = \text{const} \) and the velocity field \( u_0 = -\frac{3}{2} \Omega_0 xe_y \).

### 1.3.3 Self-Gravity in the Rings

As mentioned above, all considerations thus far have neglected the effect of gravitational forces between individual particles. From a linear stability analysis of the collisionless Boltzmann equation for a thin axisymmetric disk Toomre [1964] found that local axisymmetric perturbations can grow without bound if

\[
Q \equiv \frac{\kappa c}{3.36 G \sigma} \lesssim 1
\]

(1.70)

with the epicyclic frequency \( \kappa \) [cf. Equation (1.92)]. The first collisional N-body simulations of an unperturbed dense planetary ring incorporating a
realistic description of self-gravity forces (Salo [1992]) revealed a rather com-
plicated, inhomogeneous ground state when adopting parameters appropriate
to model Saturn’s dense rings. It was found that if $Q \lesssim 2 - 3$ a planetary
ring is susceptible to the emergence of non-axisymmetric trailing density en-
hancements, known as self-gravity wakes (cf. Figure 1.4). In the context of
spiral disk galaxies the occurrence of such trailing density enhancements was
already predicted by Julian and Toomre [1966].

To quantify the effects of self-gravity (in N-body simulations) an ad-
ditional parameter must be introduced which quantifies the importance of
gravitational two-particle forces. A convenient choice is This is the parame-
ter (Daisaka et al. [2001])

$$r_H = \frac{R_H}{R_1 + R_2} = \left( \frac{\rho}{\rho_P} \right)^{\frac{1}{3}} \left( \frac{a}{R_P} \right) \frac{(1 + \mu)^{\frac{1}{3}}}{1 + \mu^{\frac{1}{3}}},$$

(1.71)

which describes the ratio of the mutual Hill-radius of a particle pair (with
masses $M_1, M_2$) and the sum of their physical radii. In this relation
$\mu = M_1/M_2$ and $R_H = ((M_1 + M_2)/3M_P)^{1/3} a$ is the particles’ Hill radius,
within which the mutual gravitational attraction exceeds the disruptive tidal
force due to the planets gravitational field at distance $a$ from the planet.
Furthermore, $M_P, R_P, \rho_P$ are the planet’s mass, radius and mean internal
mass density. Adopting values typical for Saturn’s dense rings we have

$$r_H = 0.82 \left( \frac{M_P}{5.69 \cdot 10^{26} \text{ kg}} \right)^{-\frac{1}{2}} \left( \frac{\rho}{900 \text{ kg m}^{-3}} \right)^{\frac{1}{3}} \left( \frac{a}{100,000 \text{ km}} \right).$$

(1.72)

where we assume $\mu = 1$.

As noted above, the non-selfgravitating “frosty ice” model (1.66) with
meter-sized particles will result in a velocity dispersion $c \sim \Omega D$. The criterion
for the emergence of self-gravity wakes can also be written as $\tau r_H^2 \gtrsim 0.1$ (Salo
et al. [2018]). In the presence of self-gravity wakes the velocity dispersion
will take values so that $Q \sim 2 - 3$ due to the heating by the wakes. For larger
values of the velocity dispersion the wakes are dissolved and the ring cools
down until wakes can be formed again, and so on. Even if the criterion for
wakes is not fulfilled, self-gravitational forces contribute to the local viscosity
mechanism as particles are scattered by mutual encounters. The associated
heating will keep the velocity dispersion above the two-body escape speed
$v_{esc} = \sqrt{2GM/R} \approx 4.9 \Omega D r_H^{3/2}$ (Salo et al. [2018]).

Daisaka et al. [2001] have shown that in the presence of strong self-gravity
wakes, the angular momentum transport in the ring is dominated by the grav-
itational torques exerted by the non-axisymmetric wakes on the surrounding
ring material, as well as the bulk motion of the wakes themselves. By re-sorting to the relation for the viscous angular momentum luminosity of an unperturbed ring (which will be discussed in the following section)

\[ L_{\text{visc}}^H(r) = 3\pi \nu \sigma \Omega r^2 \]

with the kinematic shear viscosity \( \nu \), Daisaka et al. [2001] defined an associated self-gravitational viscosity due to the wake-torques (the bulk motion of the wakes contributes to the local viscosity). It turns out that under these circumstances the nonlocal viscosity is sub-dominant and the total viscosity can be approximated by

\[ \nu \approx \nu_{\text{grav}} + \nu_{\text{local}} \propto \frac{r^5 H G^2 \sigma^2}{\Omega^3} . \] (1.73)

It can be expected that the effective viscous stress induced by the wakes exhibits non-Newtonian behavior, as the wakes are being created and dissolved on the orbital time scale and on length scales which are roughly given by the Toomre critical wavelength (Toomre [1964])

\[ \lambda_T = \frac{4\pi^2 G \sigma}{k^2} , \] (1.74)

with \( \lambda_T \lesssim 100 \) m in Saturn’s dense rings. For phenomena that occur on much larger length scales, such as spiral density waves (Section 1.5) this might be less of a problem. However, the concept of gravitational viscosity should be used with caution as the effective viscous stress associated with the wakes is in general not known, apart from its contribution to the angular momentum transport in an unperturbed ring (see Tajeddine et al. [2017] and Longaretti [2018] for more details).

### 1.3.4 Transport of Angular Momentum

Large parts of the structure formation in Saturn’s rings are directly related to the redistribution of angular momentum within the rings. The main agents for angular momentum transport in Saturn’s rings are the viscous stress (Section 1.3.2) and self-gravity. A self-gravitational transport of angular momentum occurs only through the formation of non-axisymmetric density structures (see below). The gravitational transport of angular momentum due to spontaneous gravitational instability in form of (non-axisymmetric) self-gravity wakes (Salo [1992]; Richardson [1994]; Daisaka et al. [2001]) can be interpreted as a component of the viscous stress (Section 1.3.3). The effective viscosity due to self-gravity wakes is likely to be the dominant component in Saturn’s A ring.
Furthermore, non-axisymmetric density structures are generated in response to gravitational perturbations by nearby satellites. Angular momentum exchange between a satellite and ring material occurs mainly at the former's Lindblad resonances (Section 1.5.1) and to a lesser degree at its corotation resonances if the satellite evolves on an eccentric orbit. At these resonance locations the satellite exerts a torque on the ring material (Goldreich and Tremaine [1979]). In the case of an isolated Lindblad resonance the induced perturbations of the ring material survive many synodic periods with the satellite before they damp, forming a spiral density wave (Section 1.5).

Closer to the satellite, its Lindblad resonances become increasingly closely spaced and eventually start to overlap. The latter means that particle orbits with semi-major axes corresponding to adjacent resonances will acquire forced eccentricities large enough to result in a crossing of their orbits. Perturbations of ring material in this region which are excited at the encounter with the satellite significantly damp within one synodic period. The collective motion of the particles organizes in form of density wakes (Showalter et al. [1986], Borderies et al. [1989], Spahn et al. [1994]). This phenomenon occurs at the edges of the circumferential Encke and Keeler gaps in the outer A-ring, generated by the moonlets Pan and Daphnis around their orbits, respectively.

Periodic perturbations acting on the rings can also be due to anomalies of Saturn’s internal mass distribution, or by normal oscillations within Saturn (Marley and Porco [1993], Hedman and Nicholson [2013, 2014]). Hedman and Nicholson [2013] showed that the latter are responsible for the excitation of several density waves in Saturn’s C ring which were first discovered by Rosen et al. [1991a] in radio occultations carried out with the Voyager spacecraft.

Figure 1.2 shows a stellar and a radio occultation of Saturn’s A-ring, obtained with the Cassini Ultraviolet Imaging Spectrograph (UVIS) and the Cassini Radio Science Subsystem (RSS), respectively. It also shows the Encke and the Keeler gaps, generated by the torques of the aforementioned moonlets Pan and Daphnis, respectively. Visible as well are optical depth variations caused by spiral density waves. Those with the moons Janus and Mimas are marked by their perturbing moon and the frequency ratio of the moon and the ring material at resonance (Section 1.5). Many density waves induced by the moons Prometheus and Pandora are also visible in the profile (not marked).

An equation for the evolution of angular momentum can be derived from the hydrodynamical momentum equation (1.51), written in cylindrical coordinates \((r, \theta, z)\). In the following we assume that perturbations to the equilibrium values of surface mass density, radial and azimuthal velocities \((\sigma - \sigma_0), (u - u_0), (v - v_0)\), vary on typical radial length scales \(L_r\) and azimuthal length scales \(L_\theta\) such that \(L_r/r \ll 1\), as well as \(L_r \ll L_\theta\). These conditions
apply to all structures investigated in this thesis. With these assumptions the azimuthal component of the momentum equation reads

\[
\partial_t (\sigma v) = - (\Omega - \Omega_0) \partial_\theta (\sigma v) - \frac{1}{2} \Omega \sigma u - \partial_r \left( \sigma u v + \dot{P}_{r\theta} \right) - \frac{\sigma}{r} \partial_\theta (\phi_d + \phi_s). \tag{1.75}
\]

We are interested in the change of angular momentum contained in a cylinder annulus \([r_1, r_2]\), centered at radius \(r\) and with \(r_2 - r_1 \ll r\):

\[
L_{12} = \int_0^{2\pi} \int_{r_1}^{r_2} r dr d\theta (r \sigma v). \tag{1.76}
\]

For its time derivative we obtain

\[
\partial_t L_{12} = \int_0^{2\pi} \int_{r_1}^{r_2} r^2 dr d\theta \left[ - \frac{1}{2} \Omega \sigma u \right] - \partial_r \left( \sigma u v + \dot{P}_{r\theta} \right) - \frac{\sigma}{r} \partial_\theta \left( \phi_s + \phi_d \right). \tag{1.77}
\]

Generally, the time derivative is given by the sum of all external torques acting on the region, and the flux of angular momentum through the boundaries. The azimuthal derivative (1) vanishes upon integration over azimuth. The inertial term (2) arises from the epicyclic motion of the ring fluid and does not directly lead to a (secular) transport of angular momentum. The term labeled (3) describes the transport of angular momentum by advection
(Reynold’s stress):

\[
(3) \approx -r^2 \int_{0}^{r_2} \int_{0}^{2\pi} d\theta \int_{r_1}^{r_2} dr \partial_r (\sigma uv)
\]

\[
= -\left[ r^2 \int_{0}^{2\pi} d\theta \sigma uv \right]_{r_2}^{r_2} + \left[ r^2 \int_{0}^{2\pi} d\theta \sigma uv \right]_{r_1}^{r_1}.
\]

The terms in parentheses describe the rates of angular momentum which pass the radii \(r_1\) and \(r_2\), respectively. Generally, one can define the advective angular momentum luminosity at an arbitrary radius \(r\) as

\[
L_{H}^{adv}(r) = r^2 \int_{0}^{2\pi} d\theta \sigma uv.
\]

The term labeled (4) denotes the viscous transport of angular momentum and we have

\[
(4) \approx -\left[ r^2 \int_{0}^{2\pi} d\theta \hat{P}_{r\theta} \right]_{r_2}^{r_2} + \left[ r^2 \int_{0}^{2\pi} d\theta \hat{P}_{r\theta} \right]_{r_1}^{r_1}.
\]

In the same fashion we define the viscous angular momentum luminosity at radius \(r\) by

\[
L_{H}^{visc}(r) \approx r^2 \int_{0}^{2\pi} d\theta \hat{P}_{r\theta}.
\]

The non-diagonal component of the pressure tensor is given by

\[
\hat{P}_{r\theta} = -\eta \left[ \frac{\partial(v + (\Omega - \Omega_0)r)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} - \frac{v + (\Omega - \Omega_0)r}{r} \right]
\]

\[
= -\nu \sigma \left[ -\frac{3}{2} \Omega + \frac{\partial v}{\partial r} \right]
\]

where the term \(v/r\) is neglected as explained above. For an unperturbed disk we obtain the result of Lynden-Bell and Pringle [1974]

\[
L_{H}^{visc}(r) = 3\pi \nu \sigma \Omega r^2.
\]

Furthermore, the term labeled (5) in Equation (1.77) is the accumulated satellite torque within the cylindrical annulus. In this study we assume
that this torque is due to the gravitational potential of an external satellite. Goldreich and Tremaine [1979] have shown that an external potential exerts a torque only at Lindblad and corotation resonances. They calculated this torque in the linear limit, which is valid if the perturbations to the field quantities $\sigma, u$ and $v$ are small. An expression for the torque which also holds in the nonlinear case was derived by Shu et al. [1985a] by using the streamline formalism (Section 1.5).

In order to compute the contribution of self-gravity (6) we follow Lynden-Bell and Kalnajs [1972] and rewrite this term using the Poisson equation

$$f_i^d = -\sigma \partial_i \phi_d = - \int_{-\infty}^{+\infty} dz \left( \frac{1}{4\pi G} \partial_j \partial_j \phi_d \right) \partial_i \phi_d$$

(1.84)

where we adopt an index notation with $\partial_i, \partial_j$ and $\partial_l$ respectively denoting ($\partial_r, \frac{1}{r} \partial_\theta$ or $\partial_z$) and we can write

$$f_i^d = - \int_{-\infty}^{+\infty} dz \frac{1}{4\pi G} \left[ \partial_j \left( \partial_j \phi_d \partial_i \phi_d \right) - \left( \partial_l \phi_d \partial_l \phi_d \right) \partial_i \phi_d \right]$$

$$= - \int_{-\infty}^{+\infty} dz \frac{1}{4\pi G} \partial_j \left[ \partial_j \phi_d \partial_i \phi_d - \frac{1}{2} \delta_{ij} (\partial_l \phi_d \partial_l \phi_d) \right]$$

$$= - \int_{-\infty}^{+\infty} dz \partial_j \left[ \frac{\partial_i \phi_d \partial_j \phi_d}{4\pi G} - \frac{\partial_l \phi_d \partial_l \phi_d}{8\pi G} \delta_{ij} \right].$$

(1.85)

This result implies that the gravitational force can be expressed as

$$-\partial_i \phi_d = - \frac{1}{\sigma} \partial_j \hat{T}_{ij}$$

(1.86)

with the so defined (vertically integrated) self-gravitational stress tensor

$$\hat{T}_{ij} = \int_{-\infty}^{+\infty} dz \left[ \frac{\partial_i \phi_d \partial_j \phi_d}{4\pi G} - \frac{\partial_l \phi_d \partial_l \phi_d}{8\pi G} \delta_{ij} \right].$$

(1.87)

Specializing on the case $i = \theta$ in (1.85) leads to a vanishing of all terms after
integration by $\theta$ except the first term with $j = r$. Hence we have

$$
(6) = - \int_{-\infty}^{+\infty} dz \int_{0}^{r_1} r^2 dr d\theta \left( -\frac{\sigma}{r} \partial_\theta \phi_d \right)
$$

$$
= - \int_{-\infty}^{+\infty} dz \int_{0}^{r_1} r^2 dr d\theta \left( \frac{1}{r} \partial_r \left[ \partial_\theta \phi_d \partial_r \phi_d \right] \right)
$$

$$
= - \int_{-\infty}^{+\infty} dz \left[ \frac{r}{4\pi G} \int_{0}^{2\pi} d\theta \left( \partial_\theta \phi_d \partial_r \phi_d \right) \right]_{r_1}^{r_2} + \int_{-\infty}^{+\infty} dz \left[ \frac{r}{4\pi G} \int_{0}^{2\pi} d\theta \left( \partial_\theta \phi_d \partial_r \phi_d \right) \right].
$$

In analogy to the considerations of the advective contribution (3) and viscous contribution (4) we define the angular momentum luminosity due to self-gravity at radius $r$ by

$$
L_{H}^{sg}(r) = \frac{r}{4\pi G} \int_{-\infty}^{+\infty} dz \int_{0}^{2\pi} d\theta \left( \partial_\theta \phi_d \partial_r \phi_d \right).
$$

This result shows that self-gravitational forces can transport angular momentum only through non-axisymmetric potential (and density) variations.

In an unperturbed ring region, far away from any strong resonance site, angular momentum flows outward due to viscous stresses [Equation (1.83)]. This outward flow of angular momentum goes along with a net inward flow of mass (Lynden-Bell and Pringle [1974]). Also, it causes initially narrow ringlets to spread to a width $W$ on the viscous time scale $W^2/\nu_0$, assuming a constant viscosity and the absence of perturbing satellites. Salmon et al. [2010] showed that the viscous spreading of a ring is more complicated when applying a more realistic viscosity model. They investigated the global viscous spreading of a dense planetary ring in a one-dimensional hydrodynamical model which utilizes a viscosity that includes the different contributions (1.61), (1.63) and (1.73) and takes into account their local dependence on the evolving surface mass density of the ring. They found that in a disk that initially forms self-gravity wakes the viscous spreading undergoes two evolutionary stages. Initially there is a transient rapid spreading due to the viscosity generated by the self-gravity wakes (which the authors assume to be present as long as the Toomre-parameter fulfills $Q \leq 2$, c.f. Section 1.3.3), followed by an asymptotic regime where the spreading rate continuously drops since larger parts of the disk attain $Q > 2$ so that the viscosity due to the wakes (1.73) vanishes, as a consequence of the decreasing surface mass density. Eventually the whole disk lacks self-gravity wakes and the evolution
slows down dramatically. This could have important consequences for the lifetime of Saturn’s ring system.

Furthermore, resonant interactions of Saturn’s rings with external satellites play an important role in the rings’ global spreading. Charnoz et al. [2010] developed a hybrid model which calculates hydrodynamically the viscous spreading of the rings as well as the orbital evolution of aggregates forming beyond the Roche limit (inside this limit the tidal forces by Saturn prevent the gravitational accretion of ring material). They showed that the viscous spreading of Saturn’s rings beyond the Roche limit can explain well the mass-distribution and orbital arrangement of the small ring moons Atlas, Prometheus, Pandora, Epimetheus and Janus. They also showed that the current confinement of Saturn’s A ring arises from the interplay of the viscous evolution of the rings and its interaction with the forming ring moons (see also Tajeddine et al. [2017]).

In the presence of strong perturbations by a satellite, i.e. close to a Lindblad resonance with a massive satellite, the flow of angular momentum can be very different from its unperturbed behavior. For instance, if the azimuthal velocity perturbation $v$ increases outwards rapidly enough so that $\partial_r v > 3/2 \Omega$ within a given azimuth, the viscous luminosity of angular momentum, i.e. the rate of angular momentum passing this radius due to viscous stress, can vanish or even reverse (Borderies et al. [1982]). A vanishing of the viscous angular momentum luminosity is assumed to occur at the outer edges of Saturn’s A- and B-rings, which coincide with the strong Janus 7:6 and Mimas 2:1 inner Lindblad resonances (cf. Section 1.5), respectively (but see Tajeddine et al. [2017] and references therein for a more accurate picture of the situation). Furthermore, viscous angular momentum flux reversal occurs in the vicinity of certain isolated Lindblad resonances in Saturn’s C ring, leading to the formation of narrow ringlets within narrow gaps (Hänninen and Salo [1995], Goldreich et al. [1995]). Also the gap edges around embedded moonlets such as Pan and Daphnis (Borderies et al. [1989], Lewis et al. [2011]) are subject to viscous flux cancellation. The consequence of the vanishing of the viscous luminosity is that all the affected edges are very sharp. In particular they are much sharper than expected from the radial range throughout which angular momentum is exchanged between ring particles and the involved satellite. The reader is referred to Longaretti [2018] for a detailed discussion of the above phenomena.

Furthermore, if the ring viscosity $\nu$ depends, in a certain radial region, on the surface mass density $\sigma$ in such a way that $d(\sigma \nu)/d\sigma < 0$, small perturbations in $\sigma$ will result in the formation of ringlets, separated by gaps. This diffusion-type instability of the ring flow (also known as viscous instability) was proposed as a structure forming mechanism for Saturn’s B-ring in the early 1980s (Lukkari [1981], Lin and Bodenheimer [1981], Ward [1981]). Later studies considered the nonlinear saturation of this instability in terms
of a fluid model (Schmit and Tscharnuter [1995]) and N-body simulations (Salo and Schmidt [2010]). From kinetic theory it follows that dilute rings with a viscosity (1.61) could support this instability for a range of optical depths. However, in dense rings the nonlocal viscosity dominates already at quite low optical depths so that the required decrease of $\nu$ with increasing $\sigma$ does not occur and the instability is prevented. The instability might in principle still occur in form of a “size-selective instability“ so that it mainly affects the smallest particles in a ring with distribution of particle sizes (Salo and Schmidt [2010]).

1.4 Viscous Overstability in Saturn’s Rings

In the previous section we argued that much of the structure observed in Saturn’s dense rings is related to perturbations with external or embedded satellites. However, there exist fine-scale structures in Saturn’s A and B-rings on scales down to the resolution limit ($\sim 100$ m) that cannot be connected to resonances (Colwell et al. [2007], Thomson et al. [2007], Hedman et al. [2014]) and are likely generated by intrinsic instabilities.

Note that until recently self-gravitational wakes (Section 1.3.2) were not detectable by direct means, since they do not possess a continuous azimuthal coherence. However, since the orientation of the (elongated) wakes with respect to the orbital direction depends essentially only on the radial location, their presence gives rise to observable azimuthal brightness variations of the rings at a certain ring radius, so that the observed ring brightness varies with ring longitude and path optical depth (e.g. Salo et al. [2003], Nicholson et al. [2005], French et al. [2007], Hedman et al. [2007]). Recently, Rehnberg et al. [2017] were able to directly confirm the existence of the nearly void gaps between the self-gravity wakes in the A-ring by using high resolution Cassini UVIS HSP (High Speed Photometer) stellar occultations. The radial number density of such detected gaps correlates with the magnitude of observed azimuthal brightness variations measured by Dones et al. [1993] and the ring transparency model for the A-ring by Colwell et al. [2006].

It is generally accepted that some of the finest resolvable structures are most likely due to the short-scale viscous overstability (Schmit and Tscharnuter [1995, 1999], Salo et al. [2001], Schmidt et al. [2001]). The high resolution required to detect these structures has been realized in two different types of observational methods. On the one hand, Cassini RSS observations (Thomson et al. [2007]) were carried out such that the spacecraft emits a radio signal through the rings, which is then received by an Earth based antenna. The pattern of the ring opacity generated by viscous overstability can act as a diffraction grating in the transmission process. The method relies on the fact that the antenna beam emitted by Cassini covers a certain radial width.
on the ring plane. Part of the signal is transmitted through the ring with the original frequency. However, the ring material covered by the beam is subject to Keplerian shear such that different radial locations will cause a different Doppler-shift of the transmitted signal. The geometry of the transmission through the rings is such that there are two radial regions, symmetrically located with respect to the beam center, where first order diffraction lobes originate that are diffracted under the right angle to be received by the Earth based antenna. These signals are detected as (Doppler-shifted) side bands of the original signal. Thomson et al. [2007] state that the detected structures are approximately axisymmetric in that they are best fit with grating models that are tilted by less then 3°. The second observational method to detect the viscous overstability are stellar occultations that are passing a turnaround point in the ring plane. This means that the apparent track of the star behind the rings is tangential to the orbital direction at a certain radius. This setup allows for a particularly high radial resolution and has led to the detection of short-scale wavetrains in the B-ring (Colwell et al. [2009]) and the A-ring (Hedman et al. [2014]). In the latter study it was possible to confirm an azimuthal coherence of the wave pattern of about 3,000 kilometers. This was done by comparing the detailed shape of the ingress and egress profiles of two stellar occultations with a wavelet-technique. Figure 1.3 shows the wavelet power of the optical depth profiles inferred from these occultations. The wavelet power can be thought of as a Fourier-power spectrum obtained as a function of radial location in the rings.

Viscous overstability is also believed to operate on global scales in Saturn’s rings in maintaining (non-axisymmetric) eccentric ring modes as well as \( m \)-lobed normal modes (Borderies et al. [1985], Longaretti and Rappaport [1995], Longaretti [2018]). Such modes have been detected near the sharp edges of Saturn’s main rings as well as near the sharp edges of the many narrow gaps and ringlets which have been discovered in the A- and C-rings and in the Cassini Division. Also in many of the Uranian narrow rings normal modes were found (see Nicholson [2018] for an overview). In some cases these modes are evidently forced by an external satellite resonance, such as in the case of the outer edges of the A- and B-rings, which are strongly influenced by the Mimas 2:1 and Janus-Epimetheus 7:6 inner Lindblad resonances [Section 1.5.1], respectively (Spitale and Porco [2010], Nicholson et al. [2014], Porco et al. [1984], Spitale and Porco [2009]).

The viscous overstability studied here describes an intrinsic oscillatory instability of the ring flow which amplifies the collective epicyclic motion of the ring particles (on the orbital time scale). This occurs through a coupling of the viscous stress-oscillations to the background orbital shear. This coupling can induce a transfer of energy from the orbital motions to the epicyclic oscillations. In the hydrodynamic description of the ring flow such an amplification of the epicyclic motions by the viscous stress requires that the latter
Figure 1.3: Wavelet transform of two stellar occultations of a region in Saturn’s A ring (top and bottom panel) recorded with the Cassini Visual and Infrared Mapping Spectrometer (VIMS). The diagram shows the signal strength as a function of radial distance in the ring and the signal’s wavelength. Darker gray shading corresponds to a stronger signal. The prominent features with wavelengths $\lambda \sim 200\,\text{m}$ are wavetrains, most likely resulting from viscous overstability. The data also reveal three density waves and their wavelength dispersion is indicated with dashed curves. The two waves corresponding to the Atlas 7:6 and the Pan 10:9 inner Lindblad resonances [cf. Section 1.5.1] co-exist with the short-scale overstable waves. The wavelength of viscous overstability in the region of the Atlas 7:6 density wave shows complex variations with radial distance and it can be expected that this is at least partially due to the presence of the density wave. Notable as well is that all of the three density waves possess relatively short damping lengths of less than 150 km (see Section 1.5.2) (Figure from Hedman et al. [2014]).

is a steeply increasing quantity with increasing surface mass density of the ring. Schmit and Tscharnuter [1995] studied the conditions for the onset of viscous overstability in Saturn’s B-ring in terms of an axisymmetric, isothermal hydrodynamic model [i.e. neglecting variations of the temperature from its equilibrium value which are described by Equation (1.24)], assuming a
powerlaw parameterization for the shear viscosity

\[ \eta = \nu_0 \sigma_0 \left( \frac{\sigma}{\sigma_0} \right)^{\beta + 1}. \]  

(1.90)

They adopted a value \( \beta = 1.26 \) which was based on results from N-body simulations by Wisdom and Tremaine [1988]. Schmit and Tscharnuter [1995] found that the condition for viscous overstability can be formulated as \( \beta > \beta_c \) with

\[ \beta_c = \frac{1}{3} \left( \gamma - \frac{2}{3} \right). \]  

(1.91)

Their model assumed a bulk viscosity \( \xi = \gamma \eta \) with \( \gamma = 1 \) so that \( \beta_c = 1/9 \). On the other hand, non-gravitating N-body simulations, as well as N-body simulations including the ring’s vertical self-gravity (Salo et al. [2001]) revealed a substantially larger critical value \( \beta_c \approx 1 \) for the emergence of overstable oscillations. In terms of hydrodynamic modeling, this can be explained by including thermal effects (Schmidt et al. [2001]), as well as larger values of the bulk viscosity \( \gamma \sim 2 - 4 \) (Salo et al. [2001]).

In non-selfgravitating N-body simulations adopting the "frosty“ ice elasticity model (1.66) for meter-sized particles it was found that \( \beta \approx 1 \) for optical depths \( \tau \gtrsim 4 \). On the other hand, in simulations where vertical self-gravity was taken into account in terms of an enhanced vertical frequency of the particles \( \Omega_z/\Omega = 3.6 \) (the same value was originally used by Wisdom and Tremaine [1988]) overstability occurred for \( \tau \gtrsim 1 \). The reason for this difference is that the onset of viscous overstability requires a critical filling factor [cf. Eq. (1.38)] of the particle configuration to be exceeded. Recall that a larger filling factor leads to a larger collision frequency, which in turn results in an increased nonlocal viscosity (1.63) and an increased value of \( \beta \equiv d \ln \nu / d \ln \sigma \), with the kinematic shear viscosity \( \nu = \eta/\sigma \). In simulations of uni-sized particles which neglect vertical self-gravity, the particles tend to spread in vertical direction, thereby reducing the filling factor. This vertical extension of the ring is significantly reduced if vertical self-gravity is included. Also the kinetic study of viscous overstability in Saturn’s rings by Latter and Ogilvie [2008], based on an Enskog-equation, lends support for the criterion for the onset of viscous overstability in terms of a critical filling factor. They computed the growth rates of axisymmetric overstable modes from linearized kinetic moment equations in the second-order approximation [using Equations (1.10)-(1.12)] for different elasticity laws. While it was found that the critical optical depths vary significantly between different elasticity models, the critical filling factors do not.

The N-body simulations of Salo et al. [2001] including particle-particle self-gravity revealed that axisymmetric overstable oscillations can co-exist with non-axisymmetric self-gravity wakes (Figure 1.4). But if the self-gravitational perturbations are too strong (resulting in too large density
contrasts), overstability is suppressed. Note that the viscosity (1.73) induced by the wakes would imply $\beta = 2$, so that one could naively expect overstability to occur, since, at least in the absence of wakes $\beta_c \approx 1$. While the interaction between the two types of structures is far from understood it can be expected that the non-Newtonian nature of the stress induced by the wakes afflicts overstability, as it varies on length and time scales comparable to those which characterize overstable oscillations. At this point it should be noted again that the contribution of self-gravitational wakes to the stress acting on a 100 m scale in the presence of nonlinear overstable waves is less clear than their contribution to the angular momentum flux in an unperturbed ring. Furthermore, Latter and Ogilvie [2006] have shown that the viscous overstability mechanism is very sensitive to any non-locality of the viscous stress in time. In other words, the instability mechanism requires that the stress oscillations are (sufficiently) in phase with the oscillations of the (epicyclic) velocity perturbations. This requires that the relaxation time of the stress, which is set by the collisional time scale, is appreciably shorter than the orbital time scale, i.e. $\omega_c \gg \Omega$. In Paper IV we found that also the perturbation due to a nearby Lindblad resonance can destroy the phase relation of the viscous stress perturbation and the epicyclic oscillation in an overstable wave, offering an explanation for the mitigation of overstable waves by density waves seen in our hydrodynamic integrations presented in Paper IV. Recently, Ballouz et al. [2017] studied the effect of surface irregularities of ring particles on the co-existence of axisymmetric overstability and self-gravity wakes in terms of N-body simulations where particles in contact are subject to static and rolling friction. They found that with increasing inter-particle friction axisymmetric viscous overstability oscillations become more pronounced, whereas non-axisymmetric self-gravity wakes tend to be suppressed. They also showed that with increasing friction the nonlocal viscosity increases, which is likely to be crucial in promoting overstability.

The nonlinear saturation of short-scale viscous overstability in Saturn’s rings has, thus far, mainly been investigated in terms of hydrodynamic models (Schmit and Tscharnuter [1999], Schmidt and Salo [2003], Latter and Ogilvie [2009, 2010]). An exception are the non-gravitating N-body simulations by Rein and Latter [2013]. The result of these studies is that the viscous overstability saturates in form of axisymmetric traveling waves with typical wavelengths in the range $100 \text{ m} - 500 \text{ m}$. The papers by Latter and Ogilvie [2009], Latter and Ogilvie [2010] and Rein and Latter [2013] showed that the saturated state most likely consists of counter-propagating wavetrains, separated by defect structures (so called sink and source structures) which are reminiscent of solutions of the coupled complex Ginzburg-Landau equations describing (weakly) nonlinearly interacting, counter-propagating waves (e.g. van Hecke et al. [1999]). The only existing nonlinear study, which takes into account collective self-gravity forces generated by the wave pattern, is
by Schmit and Tscharnuter [1999]. They argued, based on hydrodynamical integrations, that the collective self-gravity limits the saturation wavelength of the overstable pattern to a finite value, as their integrations without self-gravity did not show a halt in the increase of the wavelength within the time scale accessible by their integrations. This finding contradicts in some sense with results by Latter and Ogilvie [2009, 2010], who showed that there is a well-defined wavelength which the nonlinear saturated waves can attain in the absence of self-gravity. Note that the studies by Salo et al. [2001], Ballouz et al. [2017] and Latter and Ogilvie [2008] investigated under which conditions viscous overstability is expected to occur in a given setup, rather than how it eventually saturates in the nonlinear regime. The first two of these studies applied local N-body simulations, whereas the latter used a kinetic formalism based on an Enskog-equation (Section 1.3.1).

Even though the detailed saturated state of viscous overstability in Saturn’s rings is certainly affected by the presence of self-gravitational wakes (Salo et al. [2001], Ballouz et al. [2017]), it can be expected that the (radial) collective self-gravity forces generated by the wavetrains themselves play an important, distinctive role in the saturation mechanism. In Paper III we study this in detail. On the one hand we perform hydrodynamic model calculations in the isothermal and non-isothermal approximation. The latter includes the dynamical evolution of the ring temperature [Equation (1.24)]. We compare these calculations with results from local N-body simulations of a dense ring that incorporate (collective) radial self-gravity as well as important aspects of the vertical component of self-gravity. The key result of this study is that the overstable pattern subject to axisymmetric self-gravity forces tends to evolve into a state of minimum oscillation frequency, or equivalently, vanishing group velocity of the wavetrains. Hence, in the saturated state, the pattern’s wavelength is governed by the frequency minimum of the nonlinear dispersion relation of overstable waves. The group velocity determines the propagation speed of (small) perturbations to the wavetrains and is thus expected to set the time scale of nonlinear evolution of the waves which practically diverges at the nonlinear frequency minimum.

In accordance with Latter and Ogilvie [2010] we find in Paper III that the boundary conditions applied in the hydrodynamical integrations (and also in the local N-body simulations) have an appreciable influence on the saturation behavior of viscous overstability. As such, we find that the usage of periodic boundary conditions in a homogeneous model ring yield a saturation wavelength which is notably larger than what is predicted based on the minimum of the nonlinear dispersion relation. In such a setup the overstable pattern eventually settles on a single uni-directional wavetrain, subsequent to a more complicated stage consisting of multiple counter-propagating wave patches, separated by sink and source structures. However, Latter and Ogilvie [2010] argued that it is most likely this prior, more complicated state which the vis-
cous overstability in Saturn’s rings adopts, due to the inevitable presence of perturbations and local changes in the ring parameters. The formation of a single unidirectional wavetrain that fills out the entire computational domain can thus be considered an artifact of the periodic boundary conditions.

In a hydrodynamic model this unwanted influence of periodic boundaries can be eliminated by adding (small) radial regions in the computational domain where the condition for viscous overstability is not fulfilled, i.e. where \( \beta < \beta_c \) [Equation (1.90)]. The boundaries of these domains will then serve as sink and source structures and in such setups we find that the saturation wavelengths do approach the nonlinear frequency minimum. In N-body simulations a similar result is achieved when the simulation starts in a periodic domain which is not entirely filled out with particles, so that a central region occupied by particles only slowly spreads under the action of viscosity. The void regions constitute barriers for overstable waves so that also in this situation defect structures will form at the boundaries.
Figure 1.4: Snapshots of local N-body simulations incorporating particle-particle gravitational forces. Seen in each top view are the particle positions after about 50 orbital periods. The $x$-axis points into the radial direction, away from the planet. The $y$-axis coincides with the direction of orbital motion. From top to bottom panel the particles’ internal density $\rho$ (in units $\text{kg m}^{-3}$) is reduced while the particle radius $R$ (in meters) is increased. This is done so as to maintain the same mean optical depth $\tau = 1.4$ and mean surface mass density $\sigma = 840 \text{ kg m}^{-2}$ in all runs, which are plausible values for Saturn’s B ring. In all cases the simulation region is located at saturnocentric distance $r = 100,000 \text{ km}$. It has radial and tangential sizes of 583 m and 233 m, respectively. This amounts to $10 \times 4$ critical wavelengths (1.74). The number of particles used varies between 15,000 and 60,000. The plots reveal self-gravitational wakes, which are canted by about 20 degree with respect to the azimuthal direction, as well as axisymmetric waves due to viscous overstability. With increasing particle internal density $\rho$ the self-gravity wakes become more pronounced, thereby mitigating the overstable waves (Figure from Salo et al. [2001]).
1.5 Spiral Density Waves in Saturn’s Rings

1.5.1 Lindblad Resonances

As noted in Section 1.3.4 spiral density waves are generated by resonant periodic perturbations acting on the ring material. For simplicity, we consider here the case of a satellite, evolving on an un-inclined, nearly-circular orbit, with semi-major axis \( r_s \), eccentricity \( \epsilon_s \) and mean motion \( \Omega_s \). We assume the same cylindrical coordinates as in Section 1.3.4. The satellite exerts a periodic gravitational force on a ring particle which evolves on a nearly circular orbit at radius \( r \), with orbital frequency \( \Omega(r) \). In response to the gravitational perturbation the particle performs an epicyclic oscillation about its reference orbit with the epicyclic frequency

\[
\kappa(r) = \sqrt{r^{-3}\partial_r(r^2\Omega)^2}. \tag{1.92}
\]

If \( \Omega \) is the Keplerian frequency (1.6), we have \( \kappa = \Omega \). Small deviations from a Keplerian motion occur due to Saturn’s oblateness, as well as due to the ring’s local self-gravity forces and pressure gradients. For our purposes we can neglect these deviations.

Following the argumentation in Shu [1984], the satellite potential can be decomposed in Fourier modes of the form

\[
\phi^{mp}_s(r, \theta, t) = \text{Re} \left[ \hat{\phi}^{mp}_s(r) \exp \{ i(m\theta - \omega_{mp} t) \} \right] \tag{1.93}
\]

where \( \text{Re} \) stands for the real part. The perturbing frequency reads

\[
\omega_{mp} = m\Omega_s \pm p\kappa_s, \tag{1.94}
\]

and mode amplitudes \( \hat{\phi}^{mp}_s(r) \propto \epsilon_s^{[p]} \), where \( m \) and \( p \) are positive integers. The strongest modes are those independent of the satellite’s orbital eccentricity (with \( p = 0 \)).

The perturbation by the satellite is resonant (so that it drives oscillations with the epicyclic frequency \( \kappa \)) at radii \( r_L \) that fulfill

\[
\omega_{mp} - m\Omega(r_L) = \pm \kappa(r_L). \tag{1.95}
\]

These are the inner (−) and outer (+) Lindblad resonances (ILR and OLR, respectively), located interior and exterior to the satellite’s orbit. For example, if \( m = 2, p = 0 \) and we assume \( \Omega \approx \kappa \), then from Equations (1.94) and (1.95) follows for the inner resonance (−) that \( \Omega(r_L) = 2\Omega_s \). Hence this resonance is labeled the 2:1 ILR.

Provided that these resonance locations lie within the ring system, angular momentum is exchanged between the forcing satellite and the ring material at the resonant sites. That is, in response to the resonant perturbation
the disk launches a (non-axisymmetric) spiral density wave, propagating outward (inward) from the ILR (OLR) [Goldreich and Tremaine [1979]]. With increasing distance from resonance, these spiral waves rapidly become tightly wound (cf. Figure 2.1). The satellite potential (1.93) varies slowly with $r$, so that a torque coupling to the disk [described by the term labeled (5) in Equation (1.77)] occurs only within a narrow radial region away from the resonance. The propagating wave carries away the deposited angular momentum. As it is subsequently damped by viscosity, the angular momentum is transferred to the ring material. In the case of an ILR (OLR), the deposited torque is negative (positive), so that attenuation of the density wave will cause material to drift inward (outward). Goldreich and Tremaine [1978] proposed that such disk-satellite interactions at strong resonances could lead to the creation of gaps in the ring system, and suggested that the 2:1 ILR with Mimas has cleared the Cassini division in Saturn’s rings. This process requires that the exerted satellite torque exceeds the viscous angular momentum luminosity of the unperturbed ring (1.83). Numerous examples of spiral density waves\(^2\) were observed by the Voyager and Cassini spacecrafts, launched at specific Lindblad resonances with various moons.

### 1.5.2 Linear Density Waves

From the linear (hydrodynamic) theory of density waves (Goldreich and Tremaine [1979], Shu [1984]) it is known that density waves in Saturn’s rings are tightly wound $m$-armed spiral patterns, rotating uniformly with pattern speed

$$\Omega_p = \frac{\omega}{m}. \quad (1.96)$$

The wave can be described in terms of the perturbed surface mass density

$$\sigma(r, \theta, t) = \sigma_0 + \text{Re}[A(r) \cdot \exp \left\{ i \int^r k(s) \, ds \right\} \cdot \exp \{i (m \theta - \omega t)\}] \quad (1.97)$$

with radial wavenumber

$$k = \frac{Dx}{2\pi G\sigma_0}, \quad (1.98)$$

where $x = (r - r_L)/r_L$ and $D = 3(m - 1)\omega_L^2$ (Lissauer and Cuzzi [1982]). The amplitude $A$ is slowly varying such that $(\partial_r A/A) \ll k$. This inequality holds for all $x > 0$ except for a small radial region close to resonance, typically a few tens of kilometers in Saturn’s dense rings. This is precisely the region within which the satellite torque couples to the disk.

The characteristic dependence of the wavelength of spiral density waves on $\sigma_0$, combined with its relatively large magnitude (typically tens of kilometers near a resonance) provides a robust means to estimate the surface

\(^2\)mostly in Saturn’s A ring due to its closer proximity to the perturbing moons
mass density of the ring material through which the wave propagates (Esposito et al. [1983]; Lissauer et al. [1985]; Nicholson et al. [1990]; Rosen et al. [1991a]; Rosen et al. [1991b]; Spilker et al. [2004]; Tiscareno et al. [2007]; Colwell et al. [2009]; Baillie et al. [2011]; Hedman and Nicholson [2016]). More recent analyses using Cassini Image Science Subsystem (ISS) optical images (Tiscareno et al. [2007]) and stellar occultation profiles recorded with the Cassini UVIS (Colwell et al. [2009]; Baillie et al. [2011]) and Cassini VIMS (Hedman and Nicholson [2016]) instruments, applied wavelet techniques (Torrence and Compo [1998]) which are particularly suitable to analyze the profiles of density waves due to the waves’ continuously varying wavenumber with radial location (see Figure 1.5).

Figure 1.5: Example of a wavelet-analysis of a density wave in Saturn’s A ring, associated with the Pan 19:18 ILR. The top panel shows the reflectance profile of the considered ring region obtained from an optical image (recorded with the Cassini ISS instrument, cf. Figure 2.1). In the second panel a high pass filter is applied so as to remove variations that arise from changes in the background properties of the ring region and are not related to the density wave. The green curve is a fit to the density wave using Equations (1.97) and (1.98). Since the moon Pan has a fairly small mass, its associated density waves are generally weak and linear theory can be applied. The third panel shows the wavelet power of the filtered signal. In the same panel the blue line indicates the filter boundary and the green dashed line represents the wavenumber dispersion of the fitted density wave [Equation (1.98)] (Figure from Tiscareno et al. [2007]).
From the characteristic damping length of the waves one can infer an estimate of the ring viscosity. A theoretical relation for the damping of a density wave by a constant shear viscosity $\nu_0$, as the wave propagates radially away from resonance, has been derived from a linear hydrodynamic model (Goldreich and Tremaine [1978]; Shu [1984]). It can be formulated in terms of an imaginary wavenumber, complementing the real-valued wavenumber (1.98) such that

$$k = \frac{D_D}{2\pi G\sigma_0} + i \frac{\Omega_L D^2 x^2}{(2\pi G\sigma_0)^3} \frac{7}{3} \nu_0 .$$

(1.99)

By comparing this relation with the actual damping of observed wave profiles, values for the shear viscosity in the A-, B- and C-rings, as well as the Cassini division have been determined (see Esposito [1983]; Tiscareno et al. [2007]; Colwell et al. [2009]; Baillié et al. [2011]).

In many studies, the value of $\nu_0$ obtained in this manner is used to estimate the vertical scale height of the corresponding ring region. To this end one uses measurements of the optical depth $\tau$ (cf. Figure 1.2) and the viscosity $\nu_0$ to compute the velocity dispersion, by assuming relation (1.61). Then, Equation (1.56) is used to obtain the scale height. Such estimates involve many approximations which are most likely irrelevant to the A and B rings. On the one hand, the expressions (1.56) and (1.61) have been derived for a dilute ring. In a dense ring the viscosity is not given by (1.61), as the nonlocal contribution (1.63) is likely to dominate (Shukhman [1984]; Araki and Tremaine [1986]). This is expected to be the case in Saturn’s C-ring and inner B-ring. On the other hand, if self-gravity wakes are present, as in the outer B-ring (Colwell et al. [2007]) and throughout the A-ring (Hedman et al. [2007]), the viscosity is expected to follow Equation (1.73). Therefore, the obtained values for $H$ can at best be interpreted as upper limits. Remarkably, in Saturn’s A-ring it was found that the viscosity obtained from the damping of a large number of linear density waves according to (1.99) follows quite well relation (1.73) [Tiscareno et al. [2007]].

However, the application of the damping relation (1.99) is most likely inappropriate for many waves in Saturn’s rings. First of all, it neglects the contribution of the bulk viscosity, as well as any dependence of the shear (and bulk) viscosity on the surface mass density. Values for these quantities appropriate for Saturn’s main rings have been measured from (small-scale) steady state and mildly perturbed N-body simulations (Salo et al. [2001]). In Paper I we have shown that these contributions can drastically alter the damping behavior of density waves in Saturn’s dense rings. It follows that the damping length of density waves is closely related to the proximity of the ring to the threshold for axisymmetric viscous overstability, which can be expressed through a modified damping relation

$$k = \frac{D_D}{2\pi G\sigma_0} + i \frac{\Omega_L D^2 x^2}{(2\pi G\sigma_0)^3} \frac{7}{3} \nu_0 F,$$

(1.100)
where

\[ F = 1 - \frac{9}{7} (\beta + 1) + \frac{3}{7} \gamma. \]  

(1.101)

The parameters \( \beta \) and \( \gamma \) are defined in Section 1.4. Note that this relation predicts instability of a density wave if the condition for viscous overstability in the long wavelength limit [Equation (1.91)] is fulfilled, so that its amplitude would grow with increasing distance from resonance. In Paper II we show that the instability condition \( F < 0 \) is identical to the condition \( T_1 > 0 \) where \( T_1 \) is a fundamental (viscous) dynamical quantity defined within the streamline formalism of a planetary ring [see for instance Equation (22) of Borderies et al. [1985]]. Borderies et al. [1986] have shown that the condition \( T_1 > 0 \) results in a linear instability of a spiral density wave, consistent with our findings. Note that this condition is not sufficient to describe correctly the stability boundary of the short-scale overstable modes (Section 1.4) which are studied in Papers II and IV. The reason is that the effects of self-gravity and pressure, neither contained in (1.101), nor in the viscous coefficient \( T_1 \), cannot be neglected for the short-scale modes.

Nevertheless, if \( \beta \lesssim \beta_c \) (so that \( F \gtrsim 0 \)) density waves might propagate over very long distances before they eventually damp. In this case the application of the classical relation (1.99) will underestimate the ring’s shear viscosity. On the other hand, if \( F \approx 1 \), the damping behavior could be very similar to what is predicted by (1.99), particularly for linear density waves. If \( \beta \geq \beta_c \), a density wave will eventually (sufficiently far away from resonance) attain a large amplitude so that nonlinear effects control its damping. In Paper I we present estimates of the quantity \( F \) using values of \( \beta \) and \( \gamma \) obtained by Salo et al. [2001] for a range of optical depths. As such it is found that in Saturn’s inner B ring \( F \) could take positive as well as negative values, depending on the precise value of the optical depth. This might be an explanation for the remarkable length of the Janus 2:1 wave (Figure 1.6).

### 1.5.3 Nonlinear Density Waves

If a density wave is nonlinear such that \(|(\sigma - \sigma_0)/\sigma_0| \sim 1\), the linear hydrodynamic description cannot be applied. Existing models for nonlinear density waves (Shu et al. [1985a], Shu et al. [1985b], Borderies et al. [1986]) in Saturn’s rings describe the ring flow in terms of streamlines, which basically represent the shape of the orbits of ”fluid test particles” which are subject to collective forces, generated by neighboring streamlines. In Saturn’s rings these streamlines are represented accurately by ellipses with very small eccentricities \( e \) (typically \( e \sim 10^{-5} \)). In this ”streamline approach” (Longaretti and Borderies [1991]) the relevant kinematic and dynamical equations are expanded to the lowest nontrivial order in the eccentricity parameter \( e \), which describes the deviation from circular motion (see Longaretti [2018] for a de-
tailed explanation).

The studies by Borderies et al. and Shu et al. describe a streamline at unperturbed radius $a$ near a Lindblad resonance as an $m$-lobed pattern (with positive integer $m$) with the kinematic equation

$$r = a \left[1 - e(a) \cos m \left(\phi + \Delta(a)\right)\right], \quad \text{(1.102)}$$

where $(r, \phi)$ are polar coordinates in a frame rotating with frequency $\Omega_p$ [Equation (1.96)]. One then uses $(a, \phi)$ as Lagrangian labels of the streamlines. Furthermore, $\Delta(a)$ is a phase angle which is related to the periapse angle $\bar{\omega}$ of a precessing streamline in an inertial frame. Thus, the streamlines are stationary in the rotating frame, as is the density wave pattern they are expected to describe [cf. (1.97)] with appropriate functions $e(a)$ and $\Delta(a)$. The degree of compression of the ring material is quantified by

$$J(a, \phi) = \frac{\partial r}{\partial a} = 1 - q \cos \left(m [\phi + \Delta] + \gamma\right), \quad \text{(1.103)}$$

where

$$q \cos \gamma = \frac{d(ae)}{da}, \quad \text{(1.104)}$$

$$q \sin \gamma = mae \frac{d\Delta}{da},$$

with the nonlinearity parameter $q \geq 0$ and $0 \leq \gamma \leq 2\pi$. From this follows for the perturbed surface mass density due to the density wave

$$\sigma(a, \phi) = \sigma_0 \frac{J}{J}. \quad \text{(1.105)}$$

The degree of nonlinearity of the wave is given by $q$. Linear waves are described by the limit $q \to 0$, such that Equation (1.97) can be recovered from (1.103) and (1.105), taking into account that the latter Equations are defined in a rotating frame. If $q \lesssim 1$ the surface mass density takes values $|\sigma - \sigma_0|/\sigma_0 \sim 1$ and the wave is considered nonlinear (see Figure 1.6). Collective effects generally prevent $q$ from becoming anywhere near 1 (Shu et al. [1985a], Borderies et al. [1985]).

The approaches by Shu et al. and Borderies et al. differ in that the former attack directly the Lagrangian equations of motion of a particle, subject to forces due to self-gravity, pressure, viscosity and an external satellite. Upon performing an expansion in powers of $e$, the lowest order equations $[O(e)]$ yield the ”trivial“ relation (1.102). From the second order equation $[O(e^2)]$ they obtain a solvability condition in form of a complex nonlinear integral equation. Solving this equation can in principle provide the unknowns $e(a)$ and $\Delta(a)$ in (1.102). However, the equation cannot be solved analytically due
Figure 1.6: The strongly nonlinear Janus 2:1 density wave, as seen in an optical depth profile of Saturn’s inner B-ring, inferred from a stellar occultation which was recorded with the Cassini UVIS instrument. In comparison with the Pan 19:18 density wave (Figure 1.5) the Janus 2:1 wave possesses a non-sinusoidal shape featuring sharp peaks and broad troughs [Equation (1.105)]. This wave is particularly remarkable as it propagates over a very long distance of at least 500 km before it eventually damps. It is possible that this is a consequence of the corresponding ring region being close to the threshold for viscous overstability (Figure from Paper I).

to the (nonlinear) self-gravity integral which sums up the gravitational forces exerted by neighboring fluid streamlines. Even a numerical solution is challenging so that the authors find it necessary to replace the integral equation by a first order differential equation (for some complex generalization of the deviatoric quantity \( [r(a) - a] \)) in the radial distance \( a \) from resonance. This differential equation is then justified on the grounds that it fulfills important asymptotic relations that can be derived from the original integral equation, also it yields the correct description in the linear limit. On the other hand, Borderies et al. describe the effects of all forces (self-gravity, pressure, viscosity,...) which the streamlines exert on one another by their contributions to the rates of change of the epicyclic elements \( de/dt, da/dt, d\tilde{\omega}/dt \). In this manner they can compute the dispersion relation \( k(a) = m\Delta a \) of a nonlinear density wave as well as its angular momentum luminosity \( L_{H}^{sg}(a) \) [cf. (1.89)]. Within both of the above approaches the damping of a density wave can be described by a time-independent first order differential equation for \( L_{H}^{sg}(a) \) in the radial distance from the resonance.

In a fluid description of the ring dynamics the damping of a density wave depends on different components of the pressure tensor. The nonlinear model for the damping of density waves in Saturn’s dense rings by Shu et al. [1985b] computes the pressure tensor from the kinetic second order moment equations [cf. (1.12)], using a Krook-collision term. It predicts reasonable damping lengths of a density wave as long as the assumed ground state optical depth (or surface mass density) of the ring does not exceed a certain critical value (which depends on the details of the collision term). Under these circumstances the density wave dispersion relation is found to follow the
linear relationship \( \lambda \propto 1/x \), but with a larger coefficient so that the nonlinear wavelength is larger by typically 10 – 20\%. For optical depths larger than this critical value, the wave damping becomes very weak so that the resulting wavetrains propagate with ever increasing amplitude and nonlinearity. In this situation the dispersion relation is practically identical to the nonlinear dispersion relation for inviscid waves (Shu et al. [1985a]), i.e. \( \lambda \propto x^{-1/3} \). This behavior has never been observed for a density wave in Saturn’s rings. The paper by Shu et al. [1985b] aims to describe with this model the observed wave profiles (by Voyager) of the Mimas 5:3 and the Janus 2:1 density waves. The Mimas 5:3 wave lies within the A-ring and the assumed optical depth is low enough so that a reasonable damping length is obtained. The adopted ground state optical depth for the Janus 2:1 wave (B-ring) however, is large and the resulting model wave behaves practically undamped. The major issue of this model is the application of a Krook-collision term for the description of a dense ring, which both of Saturn’s A- and B-rings are. The model neglects effects resulting from a finite particle size (Section 1.3.1).

On the other hand, Borderies et al. [1986] assume a Newtonian pressure tensor [cf. (1.31)] using transport coefficients derived with a granular flow model by Haff [1983] which yields the correct qualitative behavior for a dense ring. The density wave dispersion relation resulting from this model resembles the linear version \( \lambda \propto x^{-1} \), although (expected) deviations still occur in regions of large wave amplitude.

In Paper II we approach the problem of a nonlinear density wave in a completely different manner. We consider the density wave as a non-equilibrium pattern which forms in response to an instability of an isothermal hydrodynamic flow (Cross and Hohenberg [1993]). This flow is fully described by the nonlinear hydrodynamic equations [cf. (1.47), (1.48)]. The nonlinear pattern is computed by performing a weakly nonlinear stability analysis of the marginally unstable ring flow [\( \beta \gtrsim \beta_c \), cf. (1.100)], employing the method of multiple scales. The two scales involved are the length scale of phase oscillations (i.e. the wavelength of the density wave) and the radial scale on which the density wave amplitude varies. The main result of this analysis is a complex, ordinary nonlinear differential equation which describes the evolution of the wave’s amplitude and phase with increasing distance from the resonance. Solving this equation yields the wave’s nonlinear dispersion relation and evolution of its angular momentum luminosity. We compare our weakly nonlinear model with the model by Borderies et al. [1986] in the approximation that the viscous stress follows a powerlaw parameterization (1.90) and adopt values for the transport coefficients and pressure measured by Salo et al. [2001]. We find an overall good agreement of the two models. Significant departures occur only where the nonlinearity parameter takes fairly large values \( q \gtrsim 0.5 \), which is remarkable, considering that our model is, strictly, valid only in the weakly nonlinear regime.
The damping relation for nonlinear density waves is generally more complicated than Equation (1.99), regardless of whether or not the considered ring region is subjected to viscous overstability. Consequently, obtaining the ring’s viscosity from data of nonlinear density waves is in general more involved than in the case of a linear wave, as in the former case many additional physical quantities need to be considered (see for instance Borderies et al. [1986]). As mentioned before, all values of the viscosity acquired from observed density waves in Saturn’s rings so far are based on Equation (1.99), even though in some cases the probed waves show clear signatures of nonlinearity (cf. Figure 1.6). Up to date only the two studies by Longaretti and Borderies [1986] and Rappaport et al. [2009] have applied the streamline formalism for nonlinear density waves to actual profiles of density waves inferred from data. Both studies were able to reconstruct all the different kinematic quantities as a function of ring radius that culminate in Equation (1.105) to describe the Mimas 5:3 density wave in Saturn’s A ring. Longaretti and Borderies [1986] used data collected with the Voyager 2 photo-polarimeter experiment during a single flyby, while Rappaport et al. [2009] used multiple Cassini RSS occultations. In both cases the reconstructed surface density profiles describe the data quite well. However, in order to infer the ring’s viscosity, or more generally, the behavior of the rings’ stress tensor within this framework, one would need to include a nonlinear damping relation for density waves, such as the one derived in Borderies et al. [1985], to the reconstruction method, as well as the nonlinear wavenumber dispersion relation of a density wave, a step which has not yet been undertaken.

However, there exist density waves in Saturn’s rings whose damping behavior is not correctly described by any of the models discussed so far. Evaluation of Cassini data has revealed (Hedman et al. [2014]) that both spiral density waves and short-scale waves induced by viscous overstability co-exist in parts Saturn’s rings (see Figure 1.3). In Paper IV we study the damping behavior of a spiral density wave in such a situation within a one-dimensional hydrodynamic model. Due to the complexity of the hydrodynamical equations in this situation we numerically integrate these in time using a developed PDE-solver. In order to account for the spiral shape of a density wave in this one-dimensional model, certain terms in the hydrodynamical equations that describe advection due to the orbital motion of the ring fluid need to be approximated by using results from Paper II. Results of corresponding large-scale hydrodynamical integrations show that the emergence of nonlinear short-scale viscous overstability can lead to a damping of an overstable density wave [in the sense that \( F < 0 \) in Equation (1.101)]. This is in contrast to the predictions of the aforementioned nonlinear models, including our model developed in Paper II. Apparently the nonlinear viscous overstability can change the ring state (pressure tensor) in such a way that the density wave past a certain radial distance is not overstable anymore and damps out.
On the other hand, our results also show that a sufficiently strong density wave (excited by a sufficiently strong forcing potential) mitigates the growth of viscous overstability. In this case the damping of the density wave occurs according to the model derived in Paper II. In terms of an axisymmetric linear model of viscous overstability subjected to the perturbation by a density wave (cf. Section 1.2) we show in Paper IV that the latter brings out of phase the coupling of the viscous stress associated with viscous overstability to the background Keplerian shear and the epicyclic ring motion. A synchronization of both processes is essential for the viscous overstability mechanism (Latter and Ogilvie [2006]).
Chapter 2

Summary of the Results and Discussion

The detailed structure of the Saturnian ring system encompasses a vast diversity of length scales, ranging from thousands of kilometers down to tens of meters, the latter corresponding to the typical ring’s vertical thickness. This work has concentrated on two types of wave features in Saturn’s dense rings that originate from very different processes. The spiral density waves, on the one hand, which possess typical wavelengths of tens of kilometers, are the result of an angular momentum exchange between the ring system and Saturnian moons. On the other hand, the viscous overstability, an intrinsic instability of the ring flow, instigates waves on a 100 m-scale by extracting energy from the Keplerian shear and injecting it into the epicyclic motions of ring particles. In this thesis the nonlinear evolution of these wave structures has been studied by means of hydrodynamic models as well as local N-body simulations.

A weakly nonlinear model for the damping of spiral density waves has been derived from a fluid model (Paper II), thereby establishing a direct connection between density waves and the viscous overstability (Schmit and Tscharnuter [1995, 1999]). Indeed, compared to the classical linear damping relation for density waves which has been applied in most analyses of observed density waves, the model derived here predicts a rather wide range of possible damping lengths of a density wave, depending on the ring’s distance to the threshold for the onset of viscous overstability. Compared to the nonlinear model for the damping of density waves by Borderies et al. [1986] which is superior for the description of (strongly) nonlinear density waves, the newly derived model here is a comparably easy and transparent tool to gain insight in the evolution of the density wave amplitude with distance from resonance. It is likely that the large diversity in damping lengths observed for density waves in Saturn’s rings, such as the prominent first order waves driven by the satellite Janus and propagating in Saturn’s A- and B-rings, are directly
related to the rings’ affinity to viscous overstability and can now be better understood, at least qualitatively (Papers I and II).

In Paper III it has been shown that collective self-gravity forces have a direct bearing on the preferred saturation length scale of short-scale viscous overstability in Saturn’s rings. Large domain hydrodynamical integrations with radial scales of $5 - 30$ km together with N-body simulations have been carried out and the two approaches agree reasonably well for surface densities that are relevant for Saturn’s dense rings where short-scale overstability has been detected. It is found that the inclusion of the thermal balance equation in the hydrodynamic model is necessary to achieve this agreement. The length scale of saturation is found to be closely related to the wavelength of minimum oscillation frequency, or equivalently, vanishing group velocity of nonlinear short-scale overstable ring modes. This minimum exists only if the radial self-gravity force is taken into account and it shifts to smaller wavelengths with increasing strength of this force. The group velocity describes the propagation speed of small perturbations to the wavetrains and is thus expected to set the time scale of the waves’ nonlinear evolution. At the frequency minimum of the nonlinear dispersion relation this time scale diverges. As such, the derived saturation wavelengths in Paper III agree well with those measured for the periodic micro-structure in Saturn’s rings, i.e. $\lambda \sim 150 - 250$ m. However, notable discrepancies between the hydrodynamic description and the N-body simulations remain, manifesting for instance in the linear growth rates of short-wavelength overstable modes. Much of this can likely be attributed to the complicated equation of state of densely packed ring material, not fully captured by hydrodynamics. This can be improved on in future work by adopting more suitable constitutive relations for pressure and viscosity, combined with a self-consistent description of the disk’s vertical thickness.

By developing a one-dimensional numerical scheme to solve the hydrodynamical equations describing a density wave in a viscously overstable ring, it has been found (Paper IV) that both spiral density waves and short-scale viscous overstability can co-exist, and generally interact in a complex manner. If a density wave is sufficiently strong and the ring is overstable, it is shown in Paper IV that the density wave mitigates the axisymmetric waves generated by viscous overstability and retains a finite saturation amplitude at large distances from resonance. The latter is in agreement with the predictions of the derived weakly nonlinear model for the damping of a density wave (Paper II). In Paper IV the mitigation of viscous overstability by a density wave has additionally been demonstrated by employing a simplified axisymmetric model of a small ring region perturbed by a satellite resonance, introduced by Mosqueira [1996]. This model permits a hydrodynamic linear stability analysis, as well as the application of local N-body simulations, both of which show agreement with the large-scale hydrodynamical integrations.
On the other hand, in Paper IV it is also found that if a density wave is sufficiently weak and overstable, the presence of short-scale wavetrains due to spontaneous viscous overstability results in a damping of a density wave, a circumstance which is not captured by existing models for the damping of density waves.

It should be noted that in spite of the progress achieved with this work a detailed quantitative reproduction of the observed nonlinear density waves (Papers II and IV), or the precise wavelength of saturated viscous overstability in Saturn’s rings (Paper III) cannot yet be achieved with the here applied approaches. One reason is the omission of direct particle-particle self-gravity, leading to the formation of self-gravity wakes which are present throughout Saturn’s A ring and in parts of the B-ring. Studies have shown that viscous overstability and self-gravitational wakes interact in a complex manner (Salo et al. [2001], Ballouz et al. [2017]). It can be expected that the heating by the wakes, as well as their associated Non-Newtonian stress have a mitigating effect on overstable waves. But also other effects, such as particle friction or adhesive forces, will affect the nonlinear saturation of viscous overstability directly, and indirectly by altering the wakes. Furthermore, the presence of self-gravity wakes will also affect the co-existence of a density wave and viscous overstableness. While the direct effect of self-gravity wakes on a density wave may in principle be described through a self-gravitational viscosity (Daisaka et al. [2001]), the wakes can alter the damping of a density wave indirectly. That is, as the Toomre-parameter changes periodically within a density wave, the magnitude of the wakes will vary from peak to trough, thereby affecting the saturation of viscous overstability.

Moreover, recent studies (Stewart [2017]) suggest that the viscosity which is effectively generated by self-gravity wakes might be significantly underestimated in regions of strongly nonlinear density waves. The so called ”straw“ structures, seen in optical Cassini images of such density waves (Figure 2.1), are believed to represent the emergence of non-axisymmetric local gravitational instabilities on kilometer length scales. These length scales are much greater than the typical length scale of self-gravitational wakes that emerge in an unperturbed self-gravitating ring (Salo [1992]). As noted in Section 1.4, it is known that these wake structures contribute to, and under typical conditions even dominate the ring’s angular momentum flux (Daisaka et al. [2001]). The characteristic length scale for this ”gravitational viscosity“ is the Toomre-wavelength which scales with the surface mass density ($\lambda_T \propto \sigma$). Stewart [2017] shows that this length scale is greatly enhanced in the troughs of strongly nonlinear density waves (see also Salo and Schmidt [2014]), which suggests a similarly enhanced shear viscosity and as such perhaps a stronger damping of the wave.

Apart from self-gravitational wakes there are other processes affecting the damping of density waves that have not been considered thus far, such as
the dense packing of particles in wave crests and the related variation of the
disk’s scale height.

Figure 2.1: Left: Density wavetrain launched at the Prometheus 9:8 inner
Lindblad resonance. The typical $1/x$-decrease of the wavelength [cf. (1.98)]
is clearly visible (Image from Tiscareno et al. [2007]). Right: Cassini ISS
optical image of the density wavetrain associated with the Janus/Epimetheus
6:5 inner Lindblad resonance (upper portion of the image) located about
134,500 km from Saturn. The ropy structures known as “straw” are clearly
visible (Image: NASA/JPL).
Bibliography


Original papers

The original papers have been reprinted with the permission of

The American Astronomical Society
(aas.org/publications/publishing)

The Astrophysical Journal

Original publications are not included in the electronic version of the dissertation.