Strong Radiation-Matter Interaction in a Driven Superconducting Quantum System

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Abstract

In this thesis we study the interaction between radiation and matter using superconducting circuits that behave analogously with the conventional photon-atom interaction in quantum optics. The research is done with a system consisting of a waveguide resonator (radiation) strongly coupled to a transmon device (matter). We focus on the phenomena caused by strong coupling between the radiation and matter, and by driving the resonator to higher excited states with a strong monochromatic radiation. These have been studied little in the traditional radiation-matter systems. Increasing the strength of the monochromatic radiation drive, the dynamics of the system experiences a transition from the quantum to the classical regime. Also, the free-particle states of the transmon start being populated.

In the weak driving limit, the transmon can be regarded as a two-state system. As a consequence, the resonator-transmon system is conventionally discussed in terms of the linear Jaynes–Cummings model. However, for strong coupling the Bloch–Siegert shift, caused by the terms neglected in the Jaynes–Cummings model, is strong and the Jaynes–Cummings model is insufficient for describing the dynamics of the system.

We study the effects caused by strong coupling and the excitation of the higher transmon states instigated by the driving of the resonator. With reflection spectroscopy, we measure the absorption spectrum of the system and compare this with the spectrum calculated numerically using the Floquet–Born–Markov approach. We find that, in the region of the quantum-to-classical transition, the two-state approximation for the transmon is insufficient and the higher transmon states are necessary for accurate simulations. By calculating the average resonator occupation, we compare different numerical models: the Lindblad master equation, the Floquet–Born–Markov, and the semiclassical model.

Coupling a transmon to a resonator shifts the energy levels of the resonator. This shift in the energy levels prevents the higher resonator states from being populated if the system is weakly driven with a frequency that is near the resonance frequency of the resonator. We simulate this photon blockade numerically and show that the blockade is substantially different for the two-state and multistate transmon approximations.

Keywords: Superconducting qubits, Stark shift, Bloch–Siegert shift, Floquet method, Photon blockade
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Oulu, March 2019  Iivari Pietikäinen
List of original publications

The present thesis contains an introductory part and the following publications which will be referred to in the text by their Roman numbers.


The author had an important role in developing the theory and he contributed to the writing process of all articles. All numerical simulations in Publication II and the majority of numerical simulations in Publications I and III were conducted by the author.
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Chapter 1

Introduction

Cavity quantum electrodynamics (cavity QED) studies the interaction between atoms and electromagnetic waves confined inside a cavity. A simple system for studying the interaction between radiation and matter is a single atom in an electromagnetic field. This can be realized, e.g., with an atom in a Fabry–Pérot cavity [1, 2, 3] or with ion traps where a charged particle is trapped in a confined space using electromagnetic fields [4]. Applying laser light to the system, couples the internal electronic degrees of freedom and the motion of the particle.

This thesis will focus on the phenomena caused by strong coupling and strong driving in a radiation-matter system. To the best of my knowledge, this has not been studied in traditional cavity QED systems. Starting recently, it has become possible to manufacture superconducting circuits with similar properties as in cavity quantum electrodynamics [5, 6]. This field of study is called circuit quantum electrodynamics (circuit QED). Circuit QED systems can reach parameter ranges beyond the standard cavity QED systems, allowing the study of the strong-coupling and the strong-driving regime.

Superconductivity plays an important role in circuit QED. Superconductivity was discovered by Heike Kamerlingh Onnes in 1911 when he noticed that mercury lost its electrical resistance completely below a certain critical temperature [7]. Since then, many other materials exhibiting the same phenomenon have been discovered. The critical temperature, below which the material becomes superconducting, is typically very low. It took over 40 years from the discovery of superconductivity until a microscopic theory of superconductivity was created. In 1957, John Bardeen, Leon Cooper, and John Robert Schrieffer formulated the BCS-theory [8, 9]. The theory explains that at low temperatures, electrons can bind together as pairs. These pairs are called Cooper pairs [10], and they are the charge carriers in superconductors. In 1962, Brian Josephson predicted a flow of electric current between two superconducting metals separated by a thin insulating layer [11]. This
CHAPTER 1. INTRODUCTION

phenomenon, known as Josephson effect, is an essential part of circuit QED.

In circuit QED, the cavity and the atom are replaced by a transmission line resonator and an artificial atom, often a Cooper-pair box [12, 13, 14, 15, 16]. The Cooper-pair box is a superconducting island connected to the rest of the circuit by a capacitor and a Josephson junction [11, 17]. In 2007 a charge-insensitive qubit design, the transmon, was introduced [18]. This design improves the decoherence times of the qubit compared to the Cooper-pair box design.

If the transmon population is small, it can be approximated that only the two lowest states of the system become populated and the transmon can be truncated to a quantum two-state system, i.e., a qubit. This requires that the energy levels of the system are not equally spaced. When the transmon population increases, also the higher states become occupied and the qubit approximation is no longer valid [19, 20]. In this thesis we will focus on the regime where the qubit approximation breaks down. This is accomplished in a system consisting of a transmon coupled to a driven transmission-line resonator. A lot of the research on the subject has focused on the qubit approximation [21, 22, 23] but also the effects of the higher states have been studied [24, 25, 26, 27, 28].

In this thesis we study a system consisting of a waveguide resonator strongly coupled to a transmon device. The resonator is excited with monochromatic radiation and as the strength of this drive is increased the system experiences a transition from the quantum realm to the classical one. The system is modelled using Floquet theory which is shown to be efficient in the case of strongly-driven quantum systems. This thesis consists of three original publications. In Publication I we study the Bloch–Siegert shift in the resonator-qubit system as the system undergoes the driven quantum-to-classical transition. Typically, the resonator-qubit system is modelled using the Jaynes–Cummings model. The Bloch–Siegert shift is caused by the coupling terms left out in the Jaynes–Cummings model. In Publications II and III the numerical simulations with different numbers of transmon states are compared with each other and with the experimental absorption spectrum. The spacing between the energy levels in our system is non-equidistant. If the system is driven with a frequency close to the transition frequency from the ground state to one of its excited states, all the other transitions are suppressed and the system behaves effectively as a two-state system. This suppression of the resonator excitation is known as photon blockade and studied in Publication III. Also in Publication III we compare two different dissipation models of the system.

The thesis starts with an introduction to important concepts and theories in Chapter 2. The harmonic oscillator and pendulum are used as models for the radiation and matter components, respectively. Chapter 3 focuses on the components of the superconducting circuits. In Chapter 4 the numerical
methods that were used in Publications I-III are introduced alongside with some of the most important results. In Chapter 5, the work is summarized and a few potential topics for future studies are discussed.
Chapter 2

Radiation and matter

Interaction between radiation and matter is a fundamental process of nature. Radiation is often modelled by harmonic oscillators and matter by two-state systems. In this thesis, we focus on electric circuits. However, to get a better understanding of our system, we will begin by introducing the classical harmonic oscillator and pendulum and quantizing these. Later, in Chapter 3, we will demonstrate that the components of superconducting circuits behave analogous to these.

2.1 Harmonic oscillator

An example of a classical harmonic oscillator is a mechanical oscillator with a mass \( m_r \) attached to a spring with a spring constant \( k \). With no other forces or friction affecting the system, the position \( q_r \) of the mass obeys the equation of motion

\[
m_r \ddot{q}_r = -k q_r.
\]  

(2.1)

The total energy of the system, in this case, is described by the Hamiltonian

\[
H_r = \frac{p_r^2}{2m_r} + \frac{1}{2} m_r \omega_r^2 q_r^2,
\]  

(2.2)

where we have defined the momentum \( p_r = m_r \dot{q}_r \) and the oscillator frequency \( \omega_r = \sqrt{k/m_r} \).

In quantum mechanics, the variables \( p_r \) and \( q_r \) are replaced with the corresponding operators \( \hat{p}_r \) and \( \hat{q}_r \) that obey the commutation relation \([\hat{p}_r, \hat{q}_r] = \imath \hbar\) .

Introducing the creation and annihilation operators

\[
\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}} \left( \frac{\hat{p}_r}{\sqrt{m_r \omega_r}} - i \sqrt{m_r \omega_r} \hat{q}_r \right),
\]  

(2.3)

\[
\hat{a} = \frac{1}{\sqrt{2\hbar}} \left( \frac{\hat{p}_r}{\sqrt{m_r \omega_r}} + i \sqrt{m_r \omega_r} \hat{q}_r \right),
\]  

(2.4)
such that $[\hat{a}, \hat{a}^\dagger] = 1$, the Hamiltonian can be written as

$$\hat{H}_r = \hbar \omega_r \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right).$$

(2.5)

Unlike the classical harmonic oscillator whose energy is a continuous quantity, the energy of the quantum harmonic oscillator is quantized. The energy levels of the oscillator are $E_n = \hbar \omega_r (n + 1/2)$, where $n = 0, 1, 2, \ldots$. Often, the latter term in Eq. (2.5), related to the zero-point fluctuations, is neglected and one obtains

$$\hat{H}_r = \hbar \omega_r \hat{a}^\dagger \hat{a}.$$  

(2.6)

### 2.2 Pendulum

The classical pendulum is an anharmonic oscillator and it shall be used to model the matter in our radiation-matter system. Consider a mass $m_q$ attached to a fixed point with a massless rod of length $l$ in a gravitational field. The kinetic energy of the mass is

$$T = \frac{1}{2} m_q l^2 \dot{\theta}^2,$$

(2.7)

where $\theta$ is the angle of the rod defined such that $\theta = 0$ is at the lowest point. The potential energy is

$$V = -m_q g_a l \cos \theta,$$

(2.8)

where $g_a$ is the gravitational acceleration. Thus, the corresponding classical Hamiltonian can be written as

$$H_q = \frac{p_q^2}{2 m_q l^2} - m_q g_a l \cos \theta,$$

(2.9)

where the angular momentum is $p_\theta = m_q l^2 \dot{\theta}$. Defining the position $q_q = l \theta$ and momentum $p_q = p_\theta / l$, the Hamiltonian becomes

$$H_q = \frac{p_q^2}{2 m_q} - m_q g_a l \cos(q_q / l).$$

(2.10)

Additionally, let us assume that the mass $m_q$ also carries an electric charge $Q$ and the system is in a homogeneous magnetic field that is perpendicular to the plane of motion of the pendulum ($xy$-plane). The field can be represented by a vector potential $A = B_0 (-y, x, 0) / 2$, where $B_0$ is the strength of the magnetic field [18]. This leads to a Hamiltonian

$$H_q = \frac{(p_q - \frac{Q B_0 l}{2})^2}{2 m_q} - m_q g_a l \cos(q_q / l).$$

(2.11)
The canonical quantization leads to the Hamiltonian operator

\[ \hat{H}_q = \left( \hat{p}_q - \frac{QB_0}{2} \right)^2 - m_q g \hat{q} \cos(\hat{q}q/l). \]  

(2.12)

The energy-level structure of the pendulum is nonlinear, i.e., the difference between adjacent eigenenergies in the Hamiltonian (2.12) is not constant contrary to those of the harmonic oscillator. The energy states of the pendulum can be obtained from the eigenvalue problem \( \hat{H}_q |\Psi_i\rangle = E_i |\Psi_i\rangle \), where \( E_i \) are the eigenenergies and \( |\Psi_i\rangle \) are the corresponding eigenvectors. The Hamiltonian operator (2.12) can be written in its eigenbasis \( \{|\Psi_i\rangle, i \in \mathbb{N} \} \).

Due to the nonlinearity of the energy levels, the excitation of the higher states is suppressed and the system can often be restricted to include only the two lowest energy states. This means that the pendulum is approximated as a quantum two-state system, a qubit. In this case, the Hamiltonian operator can be written as

\[ \hat{H}_q = \frac{\hbar \omega_q}{2} \hat{\sigma}_z, \]  

(2.13)

where \( \hbar \omega_q = E_1 - E_0 \) is the energy difference between the two states and \( \hat{\sigma}_z \) is one of the Pauli matrices

\[ \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(2.14)

### 2.3 Quantum Rabi model

The interaction between radiation and matter can be obtained by considering a particle with a dipole moment in a harmonic electromagnetic field. This forms the Rabi model which is a commonly used model for radiation-matter interaction because of its simplicity. In the quantum Rabi model the particle is modelled as a qubit.

The Rabi Hamiltonian is

\[ \hat{H} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_q}{2} \hat{\sigma}_z + \hbar g_0 (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x, \]  

(2.15)

where \( \hat{a} \) and \( \hat{a}^\dagger \) are the harmonic oscillator annihilation and creation operators, and \( \hat{\sigma}_i \) are the Pauli matrices for the two-state system. The first two terms in the Hamiltonian describe the harmonic oscillator and the two-state system, respectively. The third term is an interaction term where \( g_0 \) is a coupling constant describing the strength of the coupling.

### 2.3.1 Jaynes–Cummings model

The Jaynes–Cummings Hamiltonian is obtained from the Rabi Hamiltonian by using the rotating-wave approximation (RWA). Typically the radi-
ation-matter interaction in cavity QED systems is so small that the Jaynes–
Cummings model is accurate. The eigenstates of the Jaynes–Cummings
Hamiltonian can be solved analytically and that is why it is a popular model
when studying interaction between a two-state system and a harmonic oscil-
lator.

Introducing the two-state system creation and annihilation operators
\[ \hat{\sigma}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\sigma}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \] (2.16)
one can write the interaction term from Hamiltonian (2.15) as
\[ \hat{H}_{\text{int}} = \hbar g_0 (\hat{a}^\dagger \hat{\sigma}^- + \hat{a} \hat{\sigma}^+ + \hat{a}^\dagger \hat{\sigma}^+ + \hat{a} \hat{\sigma}^-). \] (2.17)

The first two terms move one excitation state from the two-state system to
the harmonic oscillator or vice versa. These terms conserve the total excita-
tion number. The last two terms either create or annihilate one excitation
in both the harmonic oscillator and the two-state system.

In order to remove the terms that do not conserve the total excitation
number, we will perform the RWA. First, the Hamiltonian (2.15) is trans-
fomed into the interaction picture. This is done by separating the Hamil-
tonian in two parts  \( \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \), where \( \hat{H}_{\text{int}} \) is Hamiltonian (2.17) and
\[ \hat{H}_0 = \hbar \omega_r \hat{a}^\dagger \hat{a} + \hbar \omega_q \hat{\sigma}^+ \hat{\sigma}^- . \] (2.18)

Using
\[ e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{a} \hat{a}^\dagger e^{-\frac{i}{\hbar} \hat{H}_0 t} = \hat{a} e^{-i \omega_r t}, \quad e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{a}^\dagger \hat{a} e^{-\frac{i}{\hbar} \hat{H}_0 t} = \hat{a}^\dagger e^{i \omega_r t}, \] (2.19)
\[ e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{\sigma}^- \hat{\sigma}^+ e^{-\frac{i}{\hbar} \hat{H}_0 t} = \hat{\sigma}^- e^{-i \omega_q t}, \quad e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{\sigma}^+ \hat{\sigma}^- e^{-\frac{i}{\hbar} \hat{H}_0 t} = \hat{\sigma}^+ e^{i \omega_q t}, \] (2.20)
the interaction picture Hamiltonian is
\[ \hat{H}^{(\text{int})} = e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{H} e^{-\frac{i}{\hbar} \hat{H}_0 t} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_q}{2} \hat{\sigma} z + \hbar g_0 (\hat{a}^\dagger \hat{\sigma}^- e^{i (\omega_r - \omega_q) t} + \hat{a} \hat{\sigma}^+ e^{i (\omega_r + \omega_q) t} + \hat{a} \hat{\sigma}^- e^{-i (\omega_r - \omega_q) t} + \hat{a}^\dagger \hat{\sigma}^+ e^{-i (\omega_r + \omega_q) t}). \] (2.21)

Assuming that the frequencies \( \omega_r \) and \( \omega_q \) are close to each other, the
terms with the exponent \( \omega_r + \omega_q \) are oscillating in time much faster than
the other terms. We make an assumption that such terms oscillate so fast
that they can be thought to average to zero and are therefore ignored. The
Jaynes–Cummings model is accurate only if the coupling is weak (\( g_0 \ll \omega_q \))
and the frequencies \( \omega_r \) and \( \omega_q \) are near each other.

The Hamiltonian in the interaction picture is
\[ \hat{H}^{(\text{int})} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_q}{2} \hat{\sigma} z + \hbar g_0 (\hat{a}^\dagger \hat{\sigma}^- e^{i (\omega_r - \omega_q) t} + \hat{a} \hat{\sigma}^+ e^{-i (\omega_r - \omega_q) t}). \] (2.22)
2.3. QUANTUM RABI MODEL

Transforming this back to the Schrödinger picture results in
\[ \hat{H}_{JC} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_q}{2} \hat{\sigma}_z + \hbar g_0 (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+) . \] (2.23)

This is the Jaynes–Cummings Hamiltonian. The interaction term now conserves the total excitation number.

The ground state of Hamiltonian (2.23) is \(|E_0\rangle = |0, 0\rangle\), with energy \(E_0 = -\frac{1}{2} \hbar \omega_q\), and the other states are
\[ |E_{n,+}\rangle = \sin\left(\frac{\theta_n}{2}\right) |n, 0\rangle + \cos\left(\frac{\theta_n}{2}\right) |n - 1, 1\rangle , \] (2.24)
\[ |E_{n,-}\rangle = \cos\left(\frac{\theta_n}{2}\right) |n, 0\rangle - \sin\left(\frac{\theta_n}{2}\right) |n - 1, 1\rangle , \] (2.25)
where \(\theta_n = \arctan[2g_0\sqrt{n}/(\omega_q - \omega_r)]\), with energies
\[ E_{n,\pm} = \left(n - \frac{1}{2}\right) \hbar \omega_i \pm \frac{\hbar}{2} \sqrt{4g_0^2n + (\omega_q - \omega_r)^2} . \] (2.26)

Here \(n = 1, 2, ...\) and \(|m, l\rangle = |m\rangle \otimes |l\rangle\), where \(|m\rangle\) and \(|l\rangle\) are the number states of the harmonic oscillator and the two-state system, respectively.

2.3.2 Displacement transformation

The system is excited by coupling an external monochromatic radiation source to the harmonic oscillator part of the system. The drive Hamiltonian is
\[ \hat{H}_d = \hbar A \cos(\omega_d t)(\hat{a}^\dagger + \hat{a}) , \] (2.27)
where \(A\) and \(\omega_d\) are the drive amplitude and angular frequency, respectively.

The Hamiltonian of the harmonic oscillator (2.6) has an infinite amount of energy states. For numerical simulations, the eigenbasis of this Hamiltonian has to be truncated to a finite number of states. The more strongly the system is excited, the more basis states are needed in the truncation. In order to lower the number of states that are needed, one can perform a displacement transformation. With the displacement transformation, the harmonic oscillator can be moved to a frame where fewer states get occupied allowing an efficient numerical implementation with small truncation [30].

The displacement transformation is defined by the unitary operator
\[ \hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} , \] (2.28)
where the parameter \(\alpha\) will be defined later. This operator transforms the annihilation operator according to
\[ \hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha . \] (2.29)
Let us consider the Rabi Hamiltonian \((2.15)\) with the drive \((2.27)\). Applying the displacement transformation, we obtain \((\hat{H} \to \hat{D}^\dagger \hat{H} \hat{D} + i\hbar \hat{D}^\dagger \hat{D})\)

\[
\hat{H} = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_a}{2} \hat{\sigma}_z + \hbar g_0 (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x + \hbar g_0 (\alpha^* + \alpha) \hat{\sigma}_x
\]

\[
+ \hbar \omega_t (\alpha^* \hat{a} + \alpha \hat{a}^\dagger) + \frac{\hbar A}{2} (\hat{a}^\dagger e^{-i\omega_d t} + \hat{a} e^{i\omega_d t}) + i\hbar (\hat{\alpha}^* \hat{a} - \hat{\alpha} \hat{a}^\dagger),
\]

where we have also performed the RWA for the drive. The displacement transformation can also be performed for the master equation of the density operator, resulting in

\[
\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] + \kappa \mathcal{L}[\hat{a}] \hat{\rho} + \gamma \mathcal{L}[\hat{\sigma}_-] \hat{\rho}
\]

\[
- i[\hat{a}, \hat{\rho}] \left( i\alpha^* + \omega_t \alpha + \frac{A}{2} e^{i\omega_d t} + \frac{\kappa}{2} \alpha^* \right)
\]

\[
- i[\hat{a}^\dagger, \hat{\rho}] \left( -i\alpha + \omega_t \alpha + \frac{A}{2} e^{-i\omega_d t} - \frac{\kappa}{2} \alpha \right),
\]

where \(\mathcal{L}[\hat{A}] \hat{\rho} = \frac{1}{2} (2\hat{A} \hat{\rho} \hat{A}^\dagger - \hat{A}^\dagger \hat{A} \hat{\rho} - \hat{\rho} \hat{A}^\dagger \hat{A})\) is the Lindblad superoperator and

\[
\hat{H}_{\text{eff}} = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_a}{2} \hat{\sigma}_z + \hbar g_0 (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x + \hbar g_0 (\alpha^* + \alpha) \hat{\sigma}_x.
\]

For simplicity, we assume zero temperature in the master equation. Defining \(\alpha\) to obey the equation of motion of the driven and damped harmonic oscillator

\[
\dot{\alpha} = -i \omega_t \alpha - i \frac{A}{2} e^{-i\omega_d t} - \frac{\kappa}{2} \alpha,
\]

Eq. \((2.31)\) can be written in the form

\[
\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] + \kappa \mathcal{L}[\hat{a}] \hat{\rho} + \gamma \mathcal{L}[\hat{\sigma}_-] \hat{\rho}.
\]

The steady-state solution for \(\alpha\) is

\[
\alpha_{ss} = \frac{-A e^{-i\omega_d t}}{2(\omega_t - \omega_d - i\frac{\kappa}{2})} = \frac{A e^{-i\omega_d t}}{2\sqrt{(\omega_t - \omega_d)^2 + \frac{\kappa^2}{4}}},
\]

where in the last equality we have included the constant phase shift \(\theta = \arctan[\kappa/(2(\omega_t - \omega_d))]\) in the drive. The effective Hamiltonian \((2.32)\) can thus be written in the form

\[
\hat{H}_{\text{eff}} = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_a}{2} \hat{\sigma}_z + \hbar g_0 (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x + 2h g_0 |\alpha_{ss}| \cos(\omega_d t) \hat{\sigma}_x.
\]

The average steady-state photon number in the harmonic oscillator is \(N_{\text{osc}} = |\alpha_{ss}|^2 + \langle \hat{a}^\dagger \hat{a} \rangle\). In the displaced coordinates the oscillator is transformed to the vacuum state and the drive is affecting the qubit.
2.3. QUANTUM RABI MODEL

2.3.3 Stark shift

If we perform the RWA for the displaced Hamiltonian in Eq. (2.36), we obtain

\[ \hat{H} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_q}{2} \hat{\sigma}_z + \hbar g_0 (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+) + \hbar g_0 |\alpha_{ss}| \left( e^{i \omega_d t} \hat{\sigma}_- + e^{-i \omega_d t} \hat{\sigma}_+ \right). \]  

(2.37)

Transforming this with

\[ \hat{U} = \exp \left[ -\frac{g_0}{\omega_q - \omega_r} (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+) \right], \]  

(2.38)

using the Baker-Hausdorff lemma \[31\]

\[ e^{\hat{S}} \hat{H} e^{-\hat{S}} = \hat{H} + [\hat{S}, \hat{H}] + \frac{1}{2!} [\hat{S}, [\hat{S}, \hat{H}]] + \ldots, \]

(2.39)

and taking into account only terms up to second order in \( g_0 \), one obtains

\[ \hat{U}^\dagger \hat{H} \hat{U} \approx \hbar \left( \omega_r + \frac{g_0^2}{\omega_q - \omega_r} \right) \hat{a}^\dagger \hat{a} + \frac{\hbar}{2} \left( \omega_q + \frac{g_0^2}{\omega_q - \omega_r} \right) \hat{\sigma}_z \]

\[ + \hbar g_0 |\alpha_{ss}| \left( e^{i \omega_d t} \hat{\sigma}_- + e^{-i \omega_d t} \hat{\sigma}_+ \right) + \frac{\hbar g_0^2 |\alpha_{ss}|}{\omega_q - \omega_r} (e^{i \omega_d t} \hat{a} + e^{-i \omega_d t} \hat{a}^\dagger) \hat{\sigma}_z. \]

(2.40)

Here, we assume that we are working in the dispersive regime \((\omega_q - \omega_r) \gg g_0\) so we can ignore the higher order terms and the weak drive to the harmonic oscillator. The Hamiltonian can be written as

\[ \hat{H} = \hbar \left( \omega_r + \chi_0 \hat{\sigma}_z \right) \hat{a}^\dagger \hat{a} + \frac{\hbar}{2} \left( \omega_q + \chi_0 \right) \hat{\sigma}_z + \hbar g_0 |\alpha_{ss}| \left( e^{i \omega_d t} \hat{\sigma}_- + e^{-i \omega_d t} \hat{\sigma}_+ \right), \]

(2.41)

where \( \chi_0 = \frac{g_0^2}{\omega_q - \omega_r} \). The shift \( \chi_0 \) is called the vacuum AC Stark shift due to its analogy to the Stark effect where an external electric field shifts and splits the energy levels of an atom. In Eq. (2.41) the harmonic oscillator frequency depends on the qubit state but other than that there is no interaction between the oscillator and the qubit. We have effectively two uncoupled systems. The oscillator frequency is shifted by \( \omega_r \pm \chi_0 \) for the qubit in the ground state (-) and in the excited state (+). When the system is driven at high power, the qubit saturates and the harmonic oscillator frequency averages over both qubit states \[32\], resulting in just the bare oscillator frequency \( \omega_r \) \[33, 34, 35\].

2.3.4 Multistate Rabi model

In the case of weak driving and weak coupling, the Hilbert space of the pendulum can be truncated to include only the two lowest states. In Publications \[I\] we study systems with strong driving and strong coupling. In
this regime, also the higher states of the pendulum become excited. Thus, we need to include also eigenstates outside the low-energy subspace of the pendulum. This can be done by generalizing the quantum Rabi model (2.15) to the form of the multistate Rabi model
\[
\hat{H} = \hbar \omega_r \hat{a} \hat{a}^\dagger + \sum_{i=0}^{K-1} \hbar \omega_i |i\rangle \langle i| + \sum_{i,j=0}^{K-1} \hbar g_{ij} (\hat{a}^\dagger + \hat{a}) |i\rangle \langle j| + \hbar A \cos(\omega_d t) (\hat{a}^\dagger + \hat{a}) ,
\]
(2.42)
where \( K \) is the number of pendulum states included in the system and \(|i\rangle\) is the eigenstate of the pendulum corresponding to the eigenenergy \( \hbar \omega_i \). Here, the coupling strengths \( g_{ij} \) depend on the pendulum states and we have included the driving term in the Hamiltonian. We also introduce the total excitation number operator
\[
\hat{N} = \hat{a}^\dagger \hat{a} + \sum_{i=0}^{K-1} i |i\rangle \langle i|.
\]
(2.43)

Similarly, the displaced two-state Hamiltonian (2.36) can be generalized to the multistate form written as
\[
\hat{H}_{\text{eff}} = \hbar \omega_r \hat{a} \hat{a}^\dagger + \sum_{i=0}^{K-1} \hbar \omega_i |i\rangle \langle i| + (\hat{a}^\dagger + \hat{a}) \sum_{i,j=0}^{K-1} \hbar g_{ij} |i\rangle \langle j| + 2 |\alpha_{ss}| \cos(\omega_d t) \sum_{i,j=0}^{K-1} \hbar g_{ij} |i\rangle \langle j|.
\]
(2.44)

\section{2.4 Floquet theory}

Floquet theory can be used if the Hamiltonian is time-periodic. It transforms the time-dependent Schrödinger equation into a time-independent form. Due to the driving, we are dealing with a time-periodic system with the period of \( \tau = 2\pi / \omega_d \). As a consequence, the eigenstates and eigenenergies of the multistate Rabi Hamiltonian are dressed by the drive. These drive-dressed states, i.e., quasienergy states, can be found using the Floquet formalism. The Floquet theorem states that for a \( \tau \)-periodic Hamiltonian \( \hat{H}(t) = \hat{H}(t + \tau) \), the solution to the time-dependent Schrödinger equation
\[
\frac{\text{i} \hbar}{\text{d}t} |\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle ,
\]
(2.45)
can be written as
\[
|\Psi(t)\rangle = e^{-i\epsilon t/\hbar} |\Phi(t)\rangle ,
\]
(2.46)
2.4. FLOQUET THEORY

where $|\Phi(t)\rangle = |\Phi(t+\tau)\rangle$ is $\tau$-periodic. The variable $\varepsilon$ is called the quasienergy and $|\Phi(t)\rangle$ is the corresponding quasienergy state. By plugging Eq. (2.46) into Eq. (2.45), one obtains

$$\left( \hat{H}(t) - i\hbar \frac{d}{dt} \right) |\Phi(t)\rangle = \varepsilon |\Phi(t)\rangle.$$  \hfill (2.47)

Expanding the quasienergy state and the Hamiltonian as a Fourier series

$$|\Phi(t)\rangle = \sum_{n=-\infty}^{\infty} e^{im\omega_d t} |\Phi^{(n)}\rangle,$$  \hfill (2.48)

$$\hat{H}(t) = \sum_{n=-\infty}^{\infty} e^{im\omega_d t} \hat{H}^{(n)},$$  \hfill (2.49)

where

$$|\Phi^{(n)}\rangle = \frac{\omega_d}{2\pi} \int_{-\pi/\omega_d}^{\pi/\omega_d} e^{-in\omega_d t} |\Phi(t)\rangle dt,$$  \hfill (2.50)

$$\hat{H}^{(n)} = \frac{\omega_d}{2\pi} \int_{-\pi/\omega_d}^{\pi/\omega_d} e^{-in\omega_d t} \hat{H}(t) dt,$$  \hfill (2.51)

the Schrödinger equation becomes

$$\sum_{m} e^{im\omega_d t} \left[ \sum_{n} \hat{H}^{(m-n)} |\Phi^{(n)}\rangle + m\hbar \omega_d |\Phi^{(m)}\rangle \right] = \varepsilon \sum_{m} e^{im\omega_d t} |\Phi^{(m)}\rangle.$$  \hfill (2.52)

We can write the equation for the $m$th Fourier component as

$$\sum_{n} \left( \hat{H}^{(m-n)} + n\hbar \omega_d \delta_{nm} \right) |\Phi^{(n)}\rangle = \varepsilon |\Phi^{(m)}\rangle.$$  \hfill (2.53)

Here, the Schrödinger equation (2.45) is written in the form of a time-independent eigenvalue problem

$$\hat{H}_F |\Phi\rangle = \varepsilon |\Phi\rangle,$$  \hfill (2.54)

where

$$\hat{H}_F = \begin{pmatrix} \ddots & \hat{H}^{(0)} + (m-1)\hbar \omega_d & \hat{H}^{(-1)} & \hat{H}^{(-2)} & \vdots \\ \cdots & \hat{H}^{(1)} & \hat{H}^{(0)} + m\hbar \omega_d & \hat{H}^{(-1)} & \cdots \\ \hat{H}^{(2)} & \hat{H}^{(1)} & \hat{H}^{(0)} + (m+1)\hbar \omega_d & \hat{H}^{(-1)} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$  \hfill (2.55)
and

\[
|\Phi\rangle = \begin{pmatrix}
|\Phi^{(m-1)}\rangle \\
|\Phi^{(m)}\rangle \\
|\Phi^{(m+1)}\rangle \\
\vdots
\end{pmatrix}.
\] (2.56)

The Hamiltonian \(\hat{H}_F\) operates in the Hilbert space \(\mathcal{H} \otimes \mathcal{T}\) where \(\mathcal{H}: \{\langle \psi_k|, k \in \mathbb{N}\}\) is the Hilbert space of the Hamiltonian in Eq. (2.45) and \(\mathcal{T}: \{\langle m|, m \in \mathbb{Z}, \langle m|\Phi(t)\rangle = \langle \Phi^{(m)}|\}\}^{[36, 37]}\).

The quasienergy solutions for Eq. (2.54) have such a property that if a quasienergy \(\varepsilon_\alpha\) is a solution to the eigenvalue problem, quasienergies \(\varepsilon_\alpha, l = \varepsilon_\alpha + l\hbar\omega_d\), where \(l \in \mathbb{Z}\), are also solutions. The corresponding quasienergy states \(|\alpha, l\rangle\) can also be easily obtained from each other. Namely, if

\[
|\alpha, l\rangle = \sum_{k,n} C_{k,n,\alpha,l} |\psi_k, n\rangle,
\] (2.57)

then

\[
|\alpha, l + m\rangle = \sum_{k,n} C_{k,n,\alpha,l} |\psi_k, n + m\rangle.
\] (2.58)

This means that it is sufficient to solve for the quasienergy states in one energy interval of width \(\hbar\omega_d\), for example \(|\alpha, 0\rangle\) and the intervals with \(l \neq 0\) are then obtained using the relations mentioned above. The intervals labelled by the integer \(l\) are referred to as Brillouin zones \([38]\).

A numerical method of finding the eigenergies and eigenvectors by solving the eigenproblem (2.54) was used in Publications I and II. The implementation unavoidably requires truncation of the basis of \(\mathcal{H} \otimes \mathcal{T}\). For strong driving one has to include many basis states in both of the spaces \(\mathcal{H}\) and \(\mathcal{T}\) in order to obtain converged results. This makes the method computationally demanding at strong drive powers.

### 2.4.1 Time-evolution operator

There exists a less computationally demanding way to solve the Schrödinger equation (2.45). One can solve for the Floquet quasienergies \(\varepsilon\) and quasienergy states \(|\Phi(t)\rangle\) by introducing a unitary time-evolution operator \(\hat{U}(t_2, t_1)\), defined as \([38]\)

\[
\hat{U}(t_2, t_1)|\Psi(t_1)\rangle = |\Psi(t_2)\rangle.
\] (2.59)

Plugging this to the Schrödinger equation (2.45), we acquire

\[
i\hbar \frac{d}{dt} \hat{U}(t, 0) = \hat{H}(t)\hat{U}(t, 0),
\] (2.60)
from which we can numerically solve for $\hat{U}(t,0)$. Here, using Eqs. (2.46) and (2.59) we obtain

$$\hat{U}(\tau,0)|\Phi(0)\rangle = e^{-i\varepsilon\tau/\hbar}|\Phi(\tau)\rangle = e^{-i\varepsilon\tau/\hbar}|\Phi(0)\rangle,$$

(2.61)

where in the last equality we have used the periodicity of $|\Phi(t)\rangle$. From this, the quasienergies $\varepsilon$ and quasienergy states $|\Phi(0)\rangle$ can be obtained straightforwardly. The quasienergy states for other times $|\Phi(t)\rangle$ can be obtained from

$$\hat{U}(t,0)|\Phi(0)\rangle = e^{-i\varepsilon t/\hbar}|\Phi(t)\rangle.$$

(2.62)

In the above time-evolution operator method, the eigenproblem (2.61) is presented in the Hilbert space $\mathcal{H}$: $\{|\psi_k\rangle, k \in \mathbb{N}\}$. Therefore, the dimensions of the matrices in the eigenvalue problem (2.61) are considerably smaller than in Eq. (2.54), meaning that this method requires less memory in the numerical implementation. This method was used in Publication III.
Chapter 3

Circuit quantum electrodynamics

The interaction between radiation and matter has been typically studied using an atom in a Fabry–Pérot cavity \[1\ 2\ 3\]. With the help of nonlinear circuit components, such as Josephson junctions, one can build circuits that behave analogous to the Fabry–Pérot cavity. Electric circuits have the benefit that the parameters of the system are easier to manipulate, allowing for a better control in experiments.

3.1 Quantum network theory

For the analysis of the electric circuits in the Hamiltonian formalism, it is helpful to introduce the quantum network theory, developed by Yurke and Denker \[39\], and Devoret \[40\]. Here we shall follow the approach of Ref. \[40\]. In the quantum network theory, a circuit consists of branches. In Fig. 3.1, we have depicted an arbitrary branch \(b\) with the voltage difference \(V_b(t) = V_1(t) - V_0(t)\) and the current \(I_b(t)\) across the branch. The points that connect branches are called nodes. For each branch, one defines a branch flux and a branch charge from the voltage difference and the current between the

![Figure 3.1](image)

Figure 3.1: Branch \(b\) shown with the current \(I_b(t)\) and the voltage \(V_b(t) = V_1(t) - V_0(t)\).
CHAPTER 3. CIRCUIT QUANTUM ELECTRODYNAMICS

ends of the branch as

\[ \Phi_b(t) = \int_{-\infty}^{t} V_b(t') \, dt', \quad (3.1) \]

\[ Q_b(t) = \int_{-\infty}^{t} I_b(t') \, dt', \quad (3.2) \]

respectively. Here \( t = -\infty \) is considered such a time that the system is at rest with no voltages or currents. The energy in a branch can be calculated from

\[ E_b(t) = \int_{-\infty}^{t} V_b(t') I_b(t') \, dt'. \quad (3.3) \]

For example, for a capacitor \( V_b = Q_b/C = \dot{\Phi}_b \) and \( I_b = \dot{Q}_b \), leading into

\[ E_b = \frac{Q_b^2}{2C} = \frac{C\dot{\Phi}_b^2}{2}. \quad (3.4) \]

Similarly, for an inductor with \( I_b = \Phi_b/L \), we obtain

\[ E_b = \frac{\Phi_b^2}{2L}. \quad (3.5) \]

Kirchhoff’s laws can be expressed in terms of branch fluxes and charges as

\[ \sum_{\text{all } b \text{ around } l} \Phi_b = \tilde{\Phi}_l, \quad (3.6) \]

\[ \sum_{\text{all } b \text{ at } n} Q_b = \tilde{Q}_n, \quad (3.7) \]

where the equations mean, respectively, that the total flux \( \tilde{\Phi}_l \) through a loop \( l \) and the total charge \( \tilde{Q}_n \) at node \( n \) are constants. Thus,

\[ \sum_{\text{all } b \text{ around } l} \dot{\Phi}_b = 0, \quad (3.8) \]

\[ \sum_{\text{all } b \text{ at } n} \dot{Q}_b = 0. \quad (3.9) \]

To construct the Hamiltonian of a circuit, all the nodes \( n \) of the system are assigned a flux \( \Phi_n \) with one node chosen as ground, \( \Phi_{\text{ground}} = 0 \). With Eq. (3.6), all branch fluxes can be expressed in terms of node fluxes. Using the Lagrangian formalism and choosing the fluxes \( \Phi_n \) as the generalized coordinates, the Lagrangian \( L \) of an electric circuit can be obtained by substracting the energy terms depending on the flux from the energy terms depending on the derivative of the flux. The Hamiltonian is then

\[ H(\Phi_1, Q_1, \ldots) = \sum_i \dot{\Phi}_i Q_i - L(\Phi_1, \dot{\Phi}_1, \ldots), \quad (3.10) \]

where \( Q_i = \frac{\partial L}{\partial \dot{\Phi}_i} \) and the summation goes over all the nodes.
3.2 Josephson junction

In standard circuit elements such as capacitors and inductors, the charge and voltage or the flux and current have linear dependence on each other. With these components one can, however, only model linear systems. In order to model nonlinear systems, such as the pendulum, one needs a nonlinear circuit component. The Josephson junction is such a component.

A Josephson junction consists of two superconducting leads separated by an insulating layer. The electrons in the superconducting state have formed Cooper pairs. There is a possibility for the Cooper pairs to tunnel through the insulating layer. This causes a current across the layer that can be written as

\[ I = I_c \sin \phi, \]  

where \( I_c \) is called the critical current and \( \phi \) is the phase difference across the insulating layer. With constant voltage across the junction, the phase \( \phi \) evolves as

\[ \frac{d\phi}{dt} = \frac{2eV}{\hbar}. \]  

(3.12)

With \( V = 0 \) there is a constant current across the layer. This is called the DC Josephson effect. For non-zero constant voltage there is an alternating current across the layer. This is the AC Josephson effect. Using Eqs. (3.1) and (3.12), we can write the flux in terms of the phase difference \( \phi \) as

\[ \Phi = \frac{\hbar}{2e} \phi, \]  

(3.13)

and Eq. (3.3) implies that the potential energy of the Josephson junction is given by

\[ E = -E_J \cos \left( \frac{2\pi \Phi}{\Phi_0} \right), \]  

(3.14)

where \( E_J = I_c \Phi_0/(2\pi) \) and \( \Phi_0 = \hbar/(2e) \) is the flux quantum.

3.3 Transmon

In Publications I-III we are dealing with a specific circuit design called the transmon [18]. The transmon is a superconducting circuit consisting of two islands coupled through Josephson junctions. We use tunable transmon setup that has two Josephson junctions and a capacitor of capacitance \( C_B \) coupled in parallel. The capacitance \( C_B \) is added to increase the ratio between the potential and kinetic energies of the transmon. We will show later why this is important. Adding a capacitively coupled gate electrode that controls
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the effective offset charge \( n_g \), we obtain a circuit shown in Fig. 3.2. Using standard circuit analysis \[18\], we obtain the Hamiltonian

\[
\hat{H} = \frac{(\hat{Q}_J + C_g V_g)^2}{2(C_B + C_g)} - E_J \cos \left( \frac{2\pi}{\Phi_0} \hat{\Phi} \right) .
\] (3.15)

Here, the quantization has been done in analogy to the case of the harmonic oscillator in Section 2.1, \( Q_J \rightarrow \hat{Q}_J \) and \( \Phi \rightarrow \hat{\Phi} \), where the operators follow the commutation relation \([\hat{\Phi}, \hat{Q}_J] = i\hbar\). Due to its periodicity, the flux has to be limited to a range \( \Phi \in [0, \Phi_0] \) for the quantization. Also, we have assumed that the Josephson junctions are identical with the total Josephson energy \( E_J \) and that the junction capacitances are included in \( C_B (C_B + C_{J1} + C_{J2} \rightarrow C_B) \). This Hamiltonian is of the same form as that of the pendulum in Eq. (2.12). By defining the dimensionless charge and flux operators \( \hat{n}_J = -\hat{Q}_J / (2e) \) and \( \hat{\phi} = -(2\pi/\Phi_0)\hat{\Phi} \), respectively, we can write Hamiltonian (3.15) in the form

\[
\hat{H} = 4E_C(\hat{n}_J - n_g)^2 - E_J \cos \hat{\phi},
\] (3.16)

where \( E_C = e^2/(2(C_B + C_g)) \), \( n_g = C_g V_g / (2e) \). The operators \( \hat{n}_J \) and \( \hat{\phi} \) can be interpreted as the number of Cooper pairs that are transferred to and from the island and the phase difference across the Josephson junction, respectively.

In reality, the Josephson junctions are not identical but there exists some asymmetry \( d = E_{J2} - E_{J1} / E_{J2} + E_{J1} \). With asymmetric Josephson junctions, the potential energy in Hamiltonian (3.16) is written as

\[
\hat{H}_J = -E_{J1} \cos \hat{\phi}_1 - E_{J2} \cos \hat{\phi}_2 ,
\] (3.17)

where \( E_{J1,2} \) are the Josephson energies and \( \varphi_{1,2} \) are the phase differences across the junctions. The difference between the phases can be controlled with a magnetic flux \( \Phi_B \) through the circuit loop formed by the junctions. The phase difference is

\[
\varphi_1 - \varphi_2 = 2\pi n + 2\pi \Phi_B / \Phi_0 ,
\] (3.18)
where \( n \) is an integer. This leads to

\[
\hat{H}_J = -E_{J\Sigma} \cos \left( \frac{\pi \Phi_B}{\Phi_0} \right) \sqrt{1 + d^2 \tan^2 \left( \frac{\pi \Phi_B}{\Phi_0} \right) \cos(\hat{\phi} - \phi_0)},
\]

where \( \hat{\phi} = (\hat{\phi}_1 + \hat{\phi}_2)/2 \), \( E_{J\Sigma} = E_{J1} + E_{J2} \), and \( \tan \phi_0 = d \tan(\pi \Phi_B/\Phi_0) \). With constant flux, \( \phi_0 \) is just a constant phase shift and this Hamiltonian is the same as the one in Eq. (3.16) with only a different Josephson energy

\[
E_J = E_{J\Sigma} \cos \left( \frac{\pi \Phi_B}{\Phi_0} \right) \sqrt{1 + d^2 \tan^2 \left( \frac{\pi \Phi_B}{\Phi_0} \right)}.
\]

Therefore, it is sufficient to consider Hamiltonian (3.16) as long as we remember that the Josephson energy \( E_J \) can be controlled with an external magnetic flux.

### 3.3.1 Eigenvalues

Operators \( \hat{\phi} \) and \( \hat{n}_J \) are canonically-conjugate operators following the commutation relation \([\hat{\phi}, \hat{n}_J] = i\). Analogous to the position \( \hat{x} \) and momentum \( \hat{p} \) operators, the operator \( \hat{n}_J \) can be written as

\[
\hat{n}_J = -i \frac{\partial}{\partial \varphi},
\]

and the Schrödinger equation associated with the Hamiltonian (3.16) is

\[
4E_C \left( -i \frac{\partial}{\partial \varphi} - n_g \right)^2 \Psi(\varphi) - E_J \cos(\varphi) \Psi(\varphi) = E \Psi(\varphi),
\]

with the boundary condition \( \Psi(\varphi) = \Psi(\varphi + 2\pi) \). Looking for a solution of the form \( \Psi(\varphi) = e^{in_g \varphi} u(\varphi/2) \), the Schrödinger equation (3.22) can be written in the form of the Mathieu equation

\[
\frac{d^2 u(x)}{dx^2} + (a - 2q \cos(2x)) u(x) = 0,
\]

where \( x = \varphi/2 \), \( a = E/E_C \) and \( q = -E_I/(2E_C) \).

The eigenenergies and eigenfunctions of Eq. (3.22) are

\[
E_m = E_C \mathcal{M}_A \left( 2(n_g + k(m)), -\frac{E_I}{2E_C} \right),
\]

\[
\Psi_m(\varphi) = \frac{e^{i n_g \varphi}}{\sqrt{2\pi}} \mathcal{M}_C \left( \frac{E_m}{E_C}, \frac{E_I}{2E_C}, \varphi/2 \right) + i (-1)^{m+1} \mathcal{M}_S \left( \frac{E_m}{E_C}, \frac{E_I}{2E_C}, \varphi/2 \right),
\]

where

\[
\mathcal{M}_A(2(n_g + k(m)), -\frac{E_I}{2E_C}) = \mathcal{M}_A(2(n_g + k(m))) + \mathcal{M}_A(2(n_g + k(m)))
\]

and

\[
\mathcal{M}_C \left( \frac{E_m}{E_C}, \frac{E_I}{2E_C}, \varphi/2 \right) = \mathcal{M}_C \left( \frac{E_m}{E_C}, \frac{E_I}{2E_C}, \varphi/2 + \pi \right).
\]

\[
\mathcal{M}_S \left( \frac{E_m}{E_C}, \frac{E_I}{2E_C}, \varphi/2 \right) = \mathcal{M}_S \left( \frac{E_m}{E_C}, \frac{E_I}{2E_C}, \varphi/2 + \pi \right).
\]
where $M_A(r,q)$ is the Mathieu characteristic value, $M_C(a,q,x)$ and $M_S(a,q,x)$ are the Mathieu cosine and sine functions, and \[18\]

$$k(m) = \sum_{l=\pm 1} [\text{int}(2n_g + l/2) \mod 2][-\text{int}(n_g) + l(-1)^m((m+1) \text{div} 2)], \quad (3.26)$$

where $\text{int}(x)$ rounds to the integer closest to $x$, $a \mod b$ is the modulo operator, and $a \text{ div } b$ denotes the integer quotient of $a$ and $b$.

The energy state structure of the transmon is visible in Fig. 3.3. The offset charge $n_g$ has a negligible effect on the lowest energies provided that the ratio between the potential and kinetic energies of the transmon $E_J/E_C$ is large. When $E_J/E_C \gtrsim 20$, the two lowest energies can be regarded as independent of the offset charge making the transmon qubit $n_g$-independent.

The advantage of the transmon compared to the Cooper-pair box is its longer coherence time. In quantum systems, the coherence time tells us how long a system stays quantum mechanical. Longer coherence times are important as they allow us to perform more quantum operations in the system [43]. One source of loss of coherence is the dephasing which is caused by the fluctuations of the energies of the system due to its coupling to environment. By increasing $E_J/E_C$ of the transmon qubit, one can make the system practically charge insensitive, eliminating the effects of the charge fluctuations and by this way decreasing the dephasing. Additionally, this also increases the relaxation time of the system [18].
3.3. TRANSMON

3.3.2 Coupling to harmonic oscillator

Let us return to the discussion of the radiation-matter interaction. As shown above, the transmon is analogous to the pendulum, i.e., the matter. The role of the radiation will be played by a transmission-line resonator. The circuit schematic is shown in Fig. 3.4, where the transmission-line resonator is depicted as a lumped-element LC-circuit. Also, we have included a drive $V_d$ on the LC-circuit. The corresponding Hamiltonian operator can be written as \[ H = \frac{1}{2C_{\Sigma}^2} \left[ (C_r + C_g + C_c)\dot{Q}_J^2 + (C_B + C_g)\dot{Q}_r^2 + 2C_g\dot{Q}_r\dot{Q}_J \right. \\
+ 2C_cC_g\dot{Q}_JV_d + 2C_c(C_B + C_g)\dot{Q}_rV_d \left. + \frac{1}{2L_r}\dot{\Phi}_r^2 - E_J\cos\left(\frac{2\pi}{\Phi_0}\dot{\Phi}_J\right) \right], \]

where \[ C_{\Sigma}^2 = C_cC_g + C_cC_g + C_B(C_r + C_g + C_c). \] (3.28)

Assuming that $C_r \gg C_B, C_c, C_g$, we have

\[ H = \frac{\dot{Q}_r^2}{2C_r} + \frac{\dot{\Phi}_r^2}{2L_r} + \frac{\left(\dot{Q}_J + \frac{C_cC_g}{C_r}V_d\right)^2}{2C_{\Sigma}} - E_J\cos\left(\frac{2\pi}{\Phi_0}\dot{\Phi}_J\right) \]

\[ + \frac{C_g}{C_rC_{\Sigma}}\dot{Q}_r\dot{Q}_J + \frac{C_c}{C_r}\dot{Q}_rV_d, \]

where $C_{\Sigma} = C_B + C_g$. The two first terms of the Hamiltonian (3.29) can be identified as the kinetic and potential energy of a harmonic oscillator, the third and fourth terms are the same transmon terms as in Eq. (3.15), the fifth term is a coupling term between the harmonic oscillator and the transmon, and the last term is a driving term.
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The charge and flux operators for the harmonic oscillator can be written using the annihilation and creation operators

\[ \hat{Q}_r = \sqrt{\frac{\hbar}{2C_r}} \omega_r (\hat{a}^\dagger + \hat{a}) , \quad (3.30) \]

\[ \hat{\Phi}_r = i \sqrt{\frac{\hbar}{2L_r}} (\hat{a}^\dagger - \hat{a}) , \quad (3.31) \]

where \( \omega_r = 1/\sqrt{L_r C_r} \). Thus, the Hamiltonian (3.29) becomes

\[ \hat{H} = \hbar \omega_r \hat{a}^\dagger \hat{a} + 4EC (\hat{n}_J - n_g)^2 - E_J \cos \hat{\varphi} + h g \hat{n}_J (\hat{a}^\dagger + \hat{a}) + \hbar A \cos (\omega_d t) (\hat{a}^\dagger + \hat{a}) , \quad (3.32) \]

where \( n_g = \frac{C_s C_g}{2e C_g} V_d \), \( g = \frac{2e C_g}{C_s C_C} \sqrt{\frac{C_s \omega_r}{2\hbar}} \) and \( A = \frac{C_C}{C_s} \sqrt{\frac{C_s \omega_r}{2\hbar}} \).

We can further write the Hamiltonian (3.32) in the eigenbasis of the transmon as

\[ \hat{H} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \sum_{n=0}^{\infty} \hbar \omega_n |n\rangle \langle n| + h g \sum_{n,m=0}^{\infty} \hat{\Pi}_{nm} (\hat{a}^\dagger + \hat{a}) + \hbar A \cos (\omega_d t) (\hat{a}^\dagger + \hat{a}) , \quad (3.33) \]

where \( \hbar \omega_n = E_n \) from Eq. (3.24), \( \hat{\Pi}_{nm} = \langle n| \hat{\varphi} |m\rangle |n\rangle \langle m| \), and \( |n\rangle \) is the transmon eigenstate corresponding to energy \( E_n \). The matrix elements \( \langle n| \hat{\varphi} |m\rangle \) can be obtained using the eigenfunctions (3.25) according to

\[ \langle n| \hat{\varphi} |m\rangle = \int_0^{2\pi} \Psi^*_n(\varphi) \left( -i \frac{d}{d\varphi} \right) \Psi_m(\varphi) \ d\varphi . \quad (3.34) \]

We can see that Hamiltonian (3.33) is the multistate Rabi Hamiltonian (2.42) that we introduced earlier (\( g_{ij} = g \langle i| \hat{\varphi} |j\rangle \)).

Truncating Hamiltonian (3.33) to two states results in

\[ \hat{H} = \hbar \omega Q \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_0 Q}{2} \hat{\sigma}_z + h g_0 (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x + \hbar A \cos (\omega_d t) (\hat{a}^\dagger + \hat{a}) , \quad (3.35) \]

where \( \omega_0 = \omega_1 - \omega_0 \) and \( g_0 = g \langle 0| \hat{\varphi} |1\rangle \).
Chapter 4
Numerics and results

The simulations in Publications [I][II][III] involved several different numerical methods. Here, we will explain the methods and show some of the important results obtained with them.

The calculations were performed using multiple computing environments and languages. The eigenvalues and the matrix elements \( \langle n | \hat{n}_J | m \rangle \) for the transmon were calculated with Mathematica using the built-in Mathieu functions. This method was quite time consuming and could likely be done more efficiently but since the calculation had to be performed only once for a fixed value of \( E_J / E_C \), this method was sufficient for our purposes. The reflection coefficient was calculated using MATLAB as were the Floquet energies and states. The time-evolution in the Lindblad and the semiclassical models was implemented with Fortran using a fourth-order Runge-Kutta method.

4.1 Reflection measurement

In Publications [I] and [II] we compare our simulations with experiments done by the Kvantti group in the Low Temperature Laboratory in Aalto University using a \( \lambda/4 \)-waveguide-resonator cavity capacitively coupled to a transmon. The experimental data are obtained through reflection spectroscopy which gives us the reflection coefficient \( \Gamma \) of the system. We will now show how to acquire the reflection coefficient from the steady-state eigenstates and eigenenergies of the system (3.33).

Let us assume that the system is probed with a weak probe of the form

\[
\hat{H}_p(t) = \hbar A_p \cos(\omega_p t)(\hat{a}^\dagger + \hat{a}),
\]

(4.1)

where \( A_p \) and \( \omega_p \) are the probe amplitude and frequency, respectively. Using linear-response theory, the reflection coefficient can be written as

\[
\Gamma(\omega_p) = \frac{Z(\omega_p) - Z_0}{Z(\omega_p) + Z_0},\]

(4.2)
where $Z_0$ is the impedance of the transmission line and

$$Z(\omega_p) = \frac{1}{i\omega_p C_c} + Z_R(\omega_p),$$  \hspace{1cm} (4.3)$$

where $Z_R(\omega_p)$ is the impedance of the driven Rabi system. For our calculations the transmission-line impedance is $Z_0 = 50 \Omega$.

In order to obtain the impedance $Z_R(\omega_p)$, we need to consider the probe as a charge probe and the Hamiltonian (4.1) can be written as

$$\hat{H}_p(t) = Q(t)\hat{V},$$  \hspace{1cm} (4.4)$$

where $\hat{V} = V_{zp}(\hat{a}^\dagger + \hat{a})$, $V_{zp} = \sqrt{\hbar\omega_r/(2C_r)}$, $Q(t) = Q_0 \cos(\omega_p t)$, and $Q_0 = \hbar A_p/V_{zp}$. Here, we have assumed that the probe is coupled to the system in the same way as the drive in Section 3.3.2 and therefore the probe can be written in the same way as the drive in Eq. (3.29).

For the electrical impedance $Z(\omega)$, we have equation [44]

$$V(\omega) = Z(\omega)I(\omega).$$  \hspace{1cm} (4.5)$$

Using the Fourier transformation

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\omega)e^{-i\omega t}d\omega,$$  \hspace{1cm} (4.6)$$

we can write

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega)I(\omega)e^{-i\omega t}d\omega.$$  \hspace{1cm} (4.7)$$

Knowing that $I(t) = \dot{Q}(t) = -\omega_p Q_0 \sin(\omega_p t)$, we obtain

$$I(\omega) = \int_{-\infty}^{\infty} I(t)e^{-i\omega t}dt = -i\pi \omega_p Q_0 (\delta(\omega - \omega_p) - \delta(\omega + \omega_p)), $$  \hspace{1cm} (4.8)$$

and thus,

$$V(t) = \frac{1}{2}i\omega_p Q_0 \left[ Z(-\omega_p)e^{i\omega_p t} - Z(\omega_p)e^{-i\omega_p t} \right].$$  \hspace{1cm} (4.9)$$

The average dissipated power in the system is

$$P(\omega_p) = \frac{\omega_p}{2\pi} \int_{0}^{2\pi/\omega_p} V(t)I(t)dt = \frac{\omega_p^2 Q_0^2}{2} Z'(\omega_p),$$  \hspace{1cm} (4.10)$$

where we have assumed $Z(-\omega_p) = Z^*(\omega_p)$ and defined $Z_1(\omega_p) = Z'(\omega_p) + iZ''(\omega_p)$. The average is taken over one probe period.

The average dissipated power in the system can also be defined with Fermi’s golden rule by using the eigenenergies and eigenstates of the system. In the Floquet basis (Section 2.4), the absorptive transition rate of the
transition from initial quasienergy state $|i, 0\rangle$ to final quasienergy state $|f, l\rangle$ caused by the probe can be written using the Fermi’s golden rule as

$$P_{ifl}(\omega_p) = \frac{A_p^2}{4} p_i \frac{\gamma_{fi}}{\omega_{fi} + l\omega_d - \omega_p} |\langle f, l|(\hat{a} + \hat{a}^\dagger)|i, 0\rangle|^2$$

where $p_i$ is the occupation probability of the initial state $|\Psi_i(t)\rangle$, $\gamma_{fi}$ is the relaxation rate of the transition $|\Psi_f(t)\rangle \rightarrow |\Psi_i(t)\rangle$, and $\hbar\omega_{fi} = \varepsilon_f - \varepsilon_i$, where the quasienergies are calculated in the Brillouin zone $l = 0$. The different Brillouin zones correspond to the same initial state $|\Psi_i(t)\rangle$, so we need to only consider the transitions from one Brillouin zone ($l = 0$).

The average dissipated power can be expressed in the form of the transition rate (4.11) as

$$P(\omega_p) = \sum_{i,f,l} \hbar (\omega_{fi} + l\omega_d) P_{ifl}(\omega_p) + \hbar (\omega_{fi} + l\omega_d) P_{ifl}(-\omega_p),$$

where the first term describes absorption and the second emission. Plugging this into Eq. (4.11) we obtain

$$P(\omega_p) = \frac{\hbar A_p^2}{4} \sum_{i,f,l} (\omega_{fi} + l\omega_d) p_i \gamma_{fi} |\langle f, l|(\hat{a} + \hat{a}^\dagger)|i, 0\rangle|^2 \times \left[ \frac{1}{(\omega_{fi} + l\omega_d - \omega_p)^2 + \frac{1}{4} \gamma_{fi}^2} + \frac{1}{(\omega_{fi} + l\omega_d + \omega_p)^2 + \frac{1}{4} \gamma_{fi}^2} \right],$$

where $\omega_p \geq 0$. Here, the summation goes over all $i \neq f$. Thus,

$$Z'(\omega_p) = \frac{\omega_t}{4\omega_p^2 C_t} \sum_{i,f,l} (\omega_{fi} + l\omega_d) p_i \gamma_{fi} |\langle f, l|(\hat{a} + \hat{a}^\dagger)|i, 0\rangle|^2 \times \left[ \frac{1}{(\omega_{fi} + l\omega_d - \omega_p)^2 + \frac{1}{4} \gamma_{fi}^2} + \frac{1}{(\omega_{fi} + l\omega_d + \omega_p)^2 + \frac{1}{4} \gamma_{fi}^2} \right],$$

where $\omega_t = \sqrt{1/L_tC_t}$. When the probe frequency is close to the resonator frequency $\omega_p \approx \omega_t$, the factor is $\omega_t/(\omega_p^2 C_t) \approx \sqrt{L_r/C_r} = Z_t$.

The imaginary part is obtained using the Kramers–Kronig relation

$$Z''(\omega_p) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{Z'(\xi)}{\xi - \omega_p} d\xi = \frac{\omega_t}{4\omega_p^2 C_t} \sum_{i,f,l} p_i |\langle f, l|(\hat{a} + \hat{a}^\dagger)|i, 0\rangle|^2 \times \frac{4\omega_p (\omega_{fi} + l\omega_d)}{[(\omega_{fi} + l\omega_d - \omega_p)^2 + \frac{1}{4} \gamma_{fi}^2] [(\omega_{fi} + l\omega_d + \omega_p)^2 + \frac{1}{4} \gamma_{fi}^2]}$$

$$\times \left[ (\omega_{fi} + l\omega_d - \omega_p) (\omega_{fi} + l\omega_d + \omega_p) - \frac{1}{4} \gamma_{fi}^2 \right],$$

(4.15)
where P stands for principal value. Notice that the sign convention is different than in the standard Kramers–Kronig relation since the poles of the function are in the upper half of the complex plane. The impedance of the system is $Z_R(\omega_p) = Z'(\omega_p) + iZ''(\omega_p)$. We still need to get values for the relaxation rates $\gamma_i$ and the probabilities $p_i$, in order to calculate the impedance from Eqs. (4.14) and (4.15).

### 4.1.1 Floquet–Born–Markov model

In order to get the relaxation rates $\gamma_i$ and the probabilities $p_i$, we will use the Floquet–Markov–Born master equation. The equation was first derived by Blümel et al. [45, 46]. Using the approach of Grifoni and Hänggi [38], we assume that the system is connected to a bath of harmonic oscillators and modelled with the total Hamiltonian

$$\hat{H}(t) = \hat{H}_S(t) + \hat{H}_B + \hat{H}_{SB},$$

where $\hat{H}_S(t)$ is the Hamiltonian of the driven system,

$$\hat{H}_B = \sum_i \hbar \omega_i \hat{a}_i^{\dagger} \hat{a}_i$$

is the Hamiltonian of the bath and

$$\hat{H}_{SB} = -X \sum_i c_i \frac{i\sqrt{\hbar}}{\sqrt{2C_i \omega_i}} (\hat{a}_i^{\dagger} - \hat{a}_i) + \hat{X}^2 \sum_i \frac{c_i^2}{2C_i \omega_i^2}$$

is the Hamiltonian of the coupling between the system and the bath. Here $\hat{a}_i$, $C_i$, $\omega_i$, and $c_i$ are the annihilation operator, capacitance, angular frequency, and the coupling coefficient of the $i$th bath oscillator, respectively. The bath is coupled to the system via the operator $\hat{X}$ which in this case is the resonator flux operator

$$\hat{X} = i\sqrt{\frac{\hbar}{2C_r \omega_r}} (\hat{a}_r^{\dagger} - \hat{a}_r).$$

In the derivation of the typical Lindblad master equation, the Hamiltonian $\hat{H}_S$ is thought of as undriven. Therefore, the dissipative transitions occur between the eigenstates of the undriven system. In the Floquet–Born–Markov master equation, the drive is included already in the derivation of the dissipators.

The Floquet–Born–Markov equations for the reduced density matrix $\rho$, obtained by tracing over the bath degrees of freedom, is given as [45, 46]

$$\dot{\rho}_{\alpha\alpha}(t) = -\frac{1}{2} \sum_{\nu} [\Gamma_{\alpha\nu} + \Gamma_{\beta\nu}] \rho_{\alpha\beta}(t), \quad \alpha \neq \beta,$$

$$\dot{\rho}_{\alpha\beta}(t) = \sum_{\nu} [\Gamma_{\nu\alpha} \rho_{\nu\nu}(t) - \Gamma_{\alpha\nu} \rho_{\alpha\alpha}(t)],$$

where $\Gamma_{\alpha\nu}$ is the transition rate from state $\nu$ to state $\alpha$.

(4.20)
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where the density matrix is represented in the Floquet eigenbasis \{ |\alpha, 0 \rangle, \alpha \in \mathbb{N} \} of a single Brillouin zone \( l = 0 \) and

\[
\Gamma_{\alpha\beta} = \sum_{l=-\infty}^{\infty} \left[ \gamma_{\alpha\beta l} + n_{\text{th}}(\Delta_{\alpha\beta l})(\gamma_{\alpha\beta l} + \gamma_{\beta\alpha -l}) \right], \tag{4.21}
\]

\[
\gamma_{\alpha\beta l} = \frac{2\pi}{\hbar} \theta(\Delta_{\alpha\beta l}) J(\Delta_{\alpha\beta l}) |X_{\alpha\beta l}|^2. \tag{4.22}
\]

In the above, \( \Gamma_{\alpha\beta} \) is the transition rate of the transition \( |\Psi_\alpha(t)\rangle \rightarrow |\Psi_\beta(t)\rangle \), \( n_{\text{th}}(\omega) = 1/(\exp[\hbar\omega/(k_B T)] - 1) \) is the thermal occupation of the environment, \( \theta(\omega) \) is the Heaviside step-function, \( J(\omega) \) is the spectral function of the environmental coupling, and \( \hbar\Delta_{\alpha\beta l} = \varepsilon_\alpha - \varepsilon_\beta + l\hbar\omega_d \). Also, we have

\[
X_{\alpha\beta l} = i\sqrt{\frac{\hbar}{2C_1\omega_r}} \langle \alpha, l |(\hat{a}^\dagger - \hat{a}) |\beta, 0 \rangle. \tag{4.23}
\]

Assuming an Ohmic spectral density \[47\]

\[
J(\omega) = \pi \sum_i \frac{c_i}{2C_i\omega_i} \delta(\omega - \omega_i) \approx C_1\kappa \omega, \tag{4.24}
\]

one can write

\[
\gamma_{\alpha\beta l} = \pi\kappa \theta(\Delta_{\alpha\beta l}) \frac{\Delta_{\alpha\beta l}}{\omega_r} |\langle \alpha, l |(\hat{a}^\dagger - \hat{a}) |\beta, 0 \rangle|^2. \tag{4.25}
\]

From Eq. (4.20), we can identify

\[
p_\alpha = \frac{\sum_{\nu \neq \alpha} \Gamma_{\nu\alpha} p_\nu}{\sum_{\nu \neq \alpha} \Gamma_{\alpha\nu}}, \tag{4.26}
\]

\[
\gamma_{\alpha\beta} = \frac{1}{2} \sum_\nu [\Gamma_{\alpha\nu} + \Gamma_{\beta\nu}], \tag{4.27}
\]

where we have denoted \( p_\alpha = \rho_{\alpha\alpha} \). The probabilities can be solved for with the normalization condition \( \sum_\nu p_\nu = 1 \). These are the relaxation rates and probabilities needed for Eqs. (4.14) and (4.15). We can also calculate the average steady-state resonator occupation via

\[
N_{\text{osc}} = \langle \hat{a}^\dagger \hat{a} \rangle = \sum_\alpha p_\alpha \langle \alpha, 0 |\hat{a}^\dagger \hat{a} |\alpha, 0 \rangle. \tag{4.28}
\]

4.2 Bloch–Siegert shift in driven resonator-qubit system

In Publications I and II we studied the dispersive regime \(|\omega_q - \omega_i| \gg g_0\), where the Jaynes–Cummings model \( (2.23) \) predicts a shift in the energy levels of both the resonator and the qubit. This is the AC Stark shift.
Additionally, the counter-rotating terms, left out in the RWA used for the Jaynes–Cummings model, produce an additional shift of the energy levels. This is the Bloch–Siegert shift \([48]\). The size of the shift depends on the coupling strength and is typically very small. The vacuum Bloch–Siegert shift can be derived in a way similar to the Stark shift by transforming the Rabi Hamiltonian (2.15) by the unitary matrix

\[
\hat{U} = \exp \left[ -\frac{g_0}{\omega_q + \omega_r} (\hat{a}^\dagger \hat{\sigma}_+ - \hat{a} \hat{\sigma}_-) \right]. \tag{4.29}
\]

Using the Baker–Hausdorff lemma and ignoring two-photon processes and terms that are higher than second order in \(g_0\), we obtain

\[
\hat{U}^\dagger \hat{H} \hat{U} \approx \hbar (\omega_r + \chi_{BS} \hat{\sigma}_z) \hat{a}^\dagger \hat{a} + \hbar (\omega_q + \chi_{BS}) \frac{\hat{\sigma}_z}{2} + \hbar g_0 (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+) , \tag{4.30}
\]

where \(\chi_{BS} = g_0^2 / (\omega_q + \omega_r)\) is the vacuum Bloch–Siegert shift. The coupling is of the same form as in the Jaynes–Cummings Hamiltonian (2.23) but the resonator frequency has a shift that depends on the two-state system. As the experiments are entering the regime of strong coupling, the Bloch–Siegert shift is becoming an important part in understanding the behavior of these systems.

### 4.2.1 Counter-rotating hybridized rotating-wave approximation

The rotating-wave approximation depends on the basis. Therefore, one way to look at the effects ignored by the approximation is to transform the system to a different basis before performing the RWA. We will use the counter-rotating hybridized rotating-wave (CHRW) model developed by Lü and Zheng \([49, 50]\). This method takes into account the counter-rotating terms neglected in the Jaynes–Cummings model.

Starting from the displaced Hamiltonian (2.36) we make a transformation into a non-uniformly rotating frame with

\[
\hat{U} = \exp \left( -i \frac{2\hbar g_0 |\alpha_{ss}| \xi \sin(\omega_d t)}{\omega_d} \hat{\sigma}_x \right) , \tag{4.31}
\]

where the constant \(\xi\) will be defined later. The transformed Hamiltonian can be written as

\[
\hat{H} = \hbar \omega_x \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_d}{2} \left[ \cos \left( \frac{4\hbar g_0 |\alpha_{ss}| \xi \sin(\omega_d t)}{\omega_d} \right) \hat{\sigma}_z \right. \\
+ \sin \left( \frac{4\hbar g_0 |\alpha_{ss}| \xi \sin(\omega_d t)}{\omega_d} \right) \hat{\sigma}_y \left. + \hbar g_0 (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x + 2\hbar g_0 |\alpha_{ss}| \cos(\omega_d t) (1 - \xi) \hat{\sigma}_x \right]. \tag{4.32}
\]
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In the following, we apply the Jacobi-Anger relations

\[
\cos(x \sin \theta) = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos(2n \theta), \\
\sin(x \sin \theta) = 2 \sum_{n=1}^{\infty} J_{2n-1}(x) \sin((2n-1) \theta),
\]

where \( J_n(x) \) are Bessel functions. Ignoring the second and higher harmonics, the Hamiltonian in Eq. (4.32) becomes

\[
\hat{H} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_q}{2} J_0 \left( \frac{4 \hbar g_0 |\alpha_{ss}|}{\omega_d} \xi \right) \hat{\sigma}_z + \hbar \omega_q J_1 \left( \frac{4 \hbar g_0 |\alpha_{ss}|}{\omega_d} \xi \right) \sin(\omega_d t) \hat{\sigma}_y \\
+ \hbar g_0 (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x + 2 \hbar g_0 |\alpha_{ss}| \cos(\omega_d t) (1 - \xi) \hat{\sigma}_x.
\]

We fix the parameter \( \xi \) as

\[
2 \hbar g_0 |\alpha_{ss}| (1 - \xi) = \hbar \omega_q J_1 \left( \frac{4 \hbar g_0 |\alpha_{ss}|}{\omega_d} \xi \right),
\]

and by defining variables

\[
\tilde{\omega}_q = \omega_q J_0 \left( \frac{4 \hbar g_0 |\alpha_{ss}|}{\omega_d} \xi \right), \\
\tilde{\alpha}_{ss} = 2 |\alpha_{ss}| (1 - \xi),
\]

the Hamiltonian can be written as

\[
\hat{H} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar \tilde{\omega}_q}{2} \tilde{\sigma}_z + \hbar g_0 (\hat{a}^\dagger + \hat{a}) \tilde{\sigma}_x \\
+ \hbar g_0 \tilde{\alpha}_{ss} (\cos(\omega_d t) \tilde{\sigma}_x + \sin(\omega_d t) \tilde{\sigma}_y) \\
= \hbar \omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar \tilde{\omega}_q}{2} \tilde{\sigma}_z + \hbar g_0 (\hat{a}^\dagger + \hat{a}) \tilde{\sigma}_x \\
+ \hbar g_0 \tilde{\alpha}_{ss} (e^{i \omega_d t} \sigma_- + e^{-i \omega_d t} \sigma_+).
\]

This Hamiltonian is similar to the Hamiltonian in Eq. (2.36) if one replaces \( \omega_q \rightarrow \tilde{\omega}_q \) and \( \alpha_{ss} \rightarrow \tilde{\alpha}_{ss} \), and makes a RWA for the last term. Following the Section 2.3.3 we can write Eq. (4.39) as

\[
\hat{H} = \hbar (\omega_r + \chi_0 \tilde{\sigma}_z) \hat{a}^\dagger \hat{a} + \frac{\hbar}{2} (\tilde{\omega}_q + \tilde{\omega}_0) \tilde{\sigma}_z + \hbar g_0 |\tilde{\alpha}_{ss}| (e^{i \omega_d t} \sigma_- + e^{-i \omega_d t} \sigma_+),
\]

where \( \tilde{\chi}_0 = g_0^2 / (2 \tilde{\omega}_q - \omega_r) \).

In Fig. 4.1 the simulated reflection coefficient for the two-transmon-state Hamiltonian (2.36) is shown with the resonator resonance frequencies obtained with the RWA and CHRW approximations. At low drive power (small
Figure 4.1: Simulated reflection coefficient $|\Gamma|$ as a function of the probe frequency $\omega_p$ and the average resonator occupation $N_{osc}$. The simulation is done with the two-transmon-state Hamiltonian (2.36). The resonance frequencies for the RWA (blue dotted), obtained from Hamiltonian (2.23) with driving (2.27), and CHRW approximation (magenta), obtained from Hamiltonian (4.40), are shown with lines. The used parameter values are $\omega_q/\omega_r = 1.18$, $\omega_d/\omega_r = 0.99$, $\kappa/\omega_r = 9.1 \times 10^{-4}$, $g_0/\omega_r = 0.018$, $E_1/E_C = 28$, $n_g = 0$, and $k_B T/(\hbar \omega_r) = 0.1$. The figure is taken from Publication I.
resonator occupation), both approximative solutions differ from the numerical solution. This is the vacuum Bloch–Siegert shift \( \chi_{\text{BS}} = \frac{g_0^2}{\left(\omega_0 + \omega_r\right)} \). As the drive power is increased, the vacuum Bloch–Siegert shift averages out in a way similar to the vacuum AC Stark shift. At high power, the Bloch–Siegert effect appears as an oscillatory behavior as a function of the drive power. This behavior is captured qualitatively by the CHRW model, although the location of the average resonance is bit off due to the neglect of the higher harmonics.

4.3 Multistate Transmon

In Publications I and II, we studied how the number of transmon states affects the system in the case when the harmonic oscillator and the qubit are not in resonance and \( \omega_r < \omega_q \). In Fig. 4.2, the reflection coefficient minima from the simulations of Eq. (4.2) with different transmon state numbers \( (K = 2, 5, 7) \) are plotted on top of a contour of the measured reflection coefficient. The simulations are done using the displaced multistate Rabi Hamiltonian (2.44). We have calculated the reflection coefficient as explained in Section 4.1 using the Floquet quasienergies and the corresponding quasienergy states from Section 2.4.

At low drive amplitude, the resonator frequency is shifted from the bare resonator frequency \( \omega_r \) by the vacuum Stark and Bloch–Siegert shifts \( \omega_r - \chi_0 - \chi_{\text{BS}} \). As the drive power is increased, the transmon becomes saturated and the resonance frequency is shifted towards the bare resonator frequency \( \omega_r \). This was observed already with the two-state model but as can be seen in Fig. 4.2 a similar behavior occurs also if more transmon states are included into the truncation. We have confirmed that the seven-state simulation is converged up to \( N_{\text{osc}} \approx 100 \). For stronger drive powers, also the transmon states not included in the simulation start to become populated \cite{27, 28}.

The seven-state simulation follows the experimental resonance frequency quite well. Also, the two-state approximation for the transmon gives the locations of the resonance frequencies for low and high resonator occupations but in the intermediate region the additional transmon states are needed to reproduce the experimental data.

Here, the system experiences a quantum-to-classical transition. For low driving powers, the resonator occupation is small and the system behaves as a quantum system. At high power, a large number of states are involved in the dynamics of the system and it starts to behave as a classical system.
Figure 4.2: Measured reflection coefficient $|\Gamma|$ as a function of the probe frequency $\omega_p$ and the average resonator occupation $N_{osc}$. Numerically calculated resonance frequencies are shown with colored dots for $K = 2, 5, 7$ transmon states. The used parameter values are $\omega_q/\omega_r = 1.18$, $\omega_d/\omega_r = 0.99$, $\kappa/\omega_r = 9.1 \times 10^{-4}$, $g_0/\omega_r = 0.018$, $E_J/E_C = 28$, $n_g = 0$, and $k_B T/(\hbar \omega_r) = 0.1$. This figure is taken from Publication [I].

4.4 Photon blockade

In a harmonic oscillator that is coupled to another system, the coupling can shift the harmonic energy levels of the system in such a way that the system experiences a photon blockade. Typically this is studied with the harmonic oscillator coupled to a two-state system [51, 23]. In Publication [III], we investigated the photon blockade using the multistate transmon. In Ref. [24] the higher transmon states are included. In this research the value of $E_J/E_C$ is much higher and the effects of the higher transmon states are not as dramatic. Also, the differences might be partly due to the difference in the measured variable. We calculate the photon number ($|\langle \hat{a}^\dagger \hat{a} \rangle|$) and in Ref. [24] a heterodyne detection is used (corresponds to $|\langle \hat{a} \rangle|$).

To explain the photon blockade phenomenon, we shall use the Jaynes–Cummings model, see Section 2.3.1. In the case where the resonator and the qubit are in resonance $\omega_r - \omega_q = 0$, the eigenenergies of the Jaynes–Cummings Hamiltonian (2.23) can be written as

\begin{align}
E_0 & = -\frac{1}{2} \hbar \omega_r, \\
E_{n,\pm} & = \left(n - \frac{1}{2}\right) \hbar \omega_r \pm \hbar g_0 \sqrt{n}.
\end{align}
Figure 4.3: Steady-state photon number $N_{\text{osc}}$ as a function of the drive detuning $\delta_d = \omega_d - \omega_r$ and amplitude $A$. The transmon is truncated to two states and the resonator and transmon are in resonance, i.e., $\omega_q = \omega_r$. (a) The eigenenergies of the resonator-transmon system. The states are labelled by the excitation number $N$ and the blue rectangles correspond to the $1, 2, 3, 4$, and $5$-photon transitions occurring in the range $\delta_d \in \{-0.06 \omega_r, 0.06 \omega_r\}$. These transitions are also indicated by the vertical arrows that are aligned with the locations of the corresponding multi-photon transition frequencies indicated by dashed lines in (b). The used parameter values are $\omega_q/\omega_r = 1$, $\kappa/\omega_r = 0.002$, $g_0/\omega_r = 0.04$, $E_J/E_C = 30$, $n_g = 0$, $k_B T/(\hbar \omega_r) = 0.1$. This figure is taken from Publication III.
Figure 4.4: Steady-state photon number $N_{osc}$ as a function of the drive detuning $\delta_d = \omega_d - \omega_r$ and amplitude $A$. The transmon is truncated to seven states and the resonator is in resonance with the lowest transition of the transmon, i.e., $\omega_q = \omega_r$. (a) The eigenenergies of the resonator-transmon system. The states are labelled by the excitation number $N$ and the blue rectangles correspond to the 1, 2, 3, 4, and 5 photon transitions occurring in the range $\delta_d \in \{-0.06\omega_r, 0.06\omega_r\}$. These transitions are also indicated by the vertical arrows that are aligned with the locations of the corresponding multi-photon transition frequencies indicated by dashed lines in (b). The red arrows indicate transitions that are not visible in (b). The used parameter values are $\omega_q/\omega_r = 1$, $\kappa/\omega_r = 0.002$, $g_0/\omega_r = 0.04$, $E_J/E_C = 30$, $n_g = 0$, $k_B T/(\hbar \omega_r) = 0.1$. This figure is taken from Publication [III].
Thus, the transition energy from the ground state to the first pair of excited states is $E_{1,+} - E_0 = \hbar \omega_r \pm \hbar g_0$. The higher transitions are $E_{n+1,\pm} - E_{n,\pm} = \hbar \omega_r \pm \hbar g_0 (\sqrt{n+1} - \sqrt{n})$. The transition energies between adjacent energy states are not constant contrary to the case of the bare harmonic oscillator. Assuming that we are driving the system with a frequency close to the transition frequency between the ground and the first excited state, $\omega_d = \omega_r - g_0$, if the coupling $g_0$ is strong enough, the next transition frequency is detuned far enough such that the system is not excited beyond the first excited state. The system thus behaves effectively as a two-state system \[51\]. This is referred to as photon blockade. The coupling to the harmonic oscillator changes the oscillator energies in such a way that it blocks population of the higher oscillator states. If the driving is strong enough, the photon blockade is broken and the higher states of the system start to get populated \[23\].

Multi-photon blockade happens when a multiple of the drive frequency is close to a transition from the ground state to an excited state, $n\omega_d = n\omega_r \pm g_0 \sqrt{n}$. Just as in the one-photon blockade case, the non-equidistant energy structure causes the system to behave as a two-state system spanned by a subspace $\{ |E_0\rangle, |E_{n,\pm}\rangle \}$, resulting in an $n$-photon blockade. The couplings to other states and dissipation change this picture slightly, causing transitions outside the subspace.

In Fig. 4.3(a), we show the 11 lowest eigenenergies of the two-state Rabi Hamiltonian (2.15). The energies are classified by the excitation number $N = \langle \hat{N} \rangle = \left\langle \left( \hat{a}^\dagger \hat{a} + \sum_{i=0}^{K-1} \sum_{|i\rangle} \langle i| \right) \right\rangle$. (4.43)

The coupling in the Rabi Hamiltonian does not conserve the excitation number, therefore the classification of the states in terms of $N$ is not exact. But since we have relatively small coupling and consider only a few of the lowest energystates, this is a useful classification. The excitations from the ground state are indicated by arrows in Fig. 4.3(a) and the dashed lines in Fig. 4.3(b) denote the detunings where the drive frequency is in resonance with the multi-photon transition frequencies. In Fig. 4.3(b), the steady-state photon number $N_{\text{osc}}$, obtained numerically with the Floquet–Born–Markov method, is shown as a function of the drive frequency $\omega_d$ and amplitude $A$. The excitation peaks agree well with the multi-photon transitions. We also observe that the photon number spectrum is symmetric with respect to the drive detuning $\delta_d = \omega_d - \omega_r$. Similar results were also obtained in Ref. \[23\].

The eigenstates of the system with a seven-state truncation for the transmon are shown in Fig. 4.4(a). Compared to the two-transmon-state case, the energy-state density is higher with the seven-state truncation. Again, the states are labelled by the excitation number $N$. The number of states corresponding to excitation number $N$ is $N + 1$, up to the point that $N + 1 > K$. 

4.4. PHOTON BLOCKADE
after which the number stays at $K$, where $K$ is the number of transmon states. With more than two transmon states, the steady-state photon number $N_{osc}$ is no longer symmetric with respect to the drive detuning $\delta_d$, as can be seen in Fig. 4.4(b).

The photon blockade was studied in Publication III. The eigenstates of the system were calculated using the time-evolution operator introduced in Section 2.4.1. The average steady-state photon number $N_{osc}$ is calculated from

$$N_{osc} = \sum_\alpha p_\alpha \langle \hat{a}^\dagger \hat{a} \rangle_\alpha,$$

(4.44)

where

$$\langle \hat{a}^\dagger \hat{a} \rangle_\alpha = \frac{1}{\tau} \int_0^\tau \langle \Phi_\alpha(t)|\hat{a}^\dagger \hat{a}|\Phi_\alpha(t) \rangle dt,$$

(4.45)

is the average photon number of a single quasienergy state and $p_\alpha$ is the average steady-state occupation probability of the state $|\Psi_\alpha(t)\rangle$. The probability $p_\alpha$ is calculated as shown in Section 4.1.1 with the exception that the matrix element in Eqs. (4.23) and (4.25) is replaced with

$$\langle \alpha, l|(\hat{a}^\dagger - \hat{a})|\beta, 0 \rangle \rightarrow \frac{1}{\tau} \int_0^\tau e^{-il\omega_\alpha t} \langle \Phi_\alpha(t)|(\hat{a}^\dagger - \hat{a})|\Phi_\beta(t) \rangle dt.$$  (4.46)

### 4.5 Lindblad master equation

Above, the dissipative time-evolution of the system has been handled using the Floquet–Born–Markov formalism. Alternatively, a common method is to use the Lindblad master equation. The derivation of the method proceeds similar to that of the Floquet–Born–Markov method, except that the effects of the drive are only taken into account in the coherent von Neumann part of the master equation.

The Lindblad master equation for the coupled resonator-transmon system can be written as [52, 53, 54]

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_S, \hat{\rho}] + \kappa(n_{th}(\omega_t) + 1)\mathcal{L}[\hat{a}]\hat{\rho} + \kappa n_{th}(\omega_t)\mathcal{L}[\hat{a}^\dagger]\hat{\rho} + \sum_{\omega_{nm} > 0} \Gamma_{nm} \mathcal{L}[|n\rangle \langle m|]\hat{\rho} + \sum_{\omega_{nm} < 0} \Gamma_{nm} \mathcal{L}[|n\rangle \langle m|]\hat{\rho} + \sum_{\omega_{nm} = 0} \Gamma_{nm} \mathcal{L}[|n\rangle \langle m|]\hat{\rho},$$

(4.47)

where $\hat{H}_S$ is the Hamiltonian (3.33), $\mathcal{L}[\hat{A}]\hat{\rho} = \frac{1}{2}(2\hat{A}\hat{\rho}\hat{A}^\dagger - \hat{A}^\dagger\hat{A}\hat{\rho} - \hat{\rho}\hat{A}^\dagger\hat{A})$ is the Lindblad superoperator, $\kappa$ is the resonator dissipation rate, $n_{th}(\omega_t) = 1/(\exp[\hbar\omega_t/(k_B T)] - 1)$ is the thermal occupation of the bath, $T$ is the temperature of the bath, $k_B$ is the Boltzmann constant, $\Gamma_{nm} = \langle n|\hat{n}_J|m\rangle S(\omega_{nm})$ is the transition rate from state $|m\rangle$ to $|n\rangle$, $S(\omega_{nm}) = \gamma_0 \omega_{nm}/(1 - e^{-\hbar\omega_{nm}/(k_B T)})$.
\[ \omega_{nm} = \omega_m - \omega_n, \quad \text{and} \quad \gamma_0 \text{ describes the strength of the coupling between the transmon and the bath} \ [55]. \] Here, we have assumed independent dissipation channels for the resonator and the transmon. However, the baths are assumed to have the same temperature. The first term on the right side of Eq. (4.47) describes the coherent time-evolution of the system. The second and third terms model the relaxation and excitation of the resonator caused by the environment, respectively. The last three terms are the relaxation, excitation and dephasing of the transmon caused by the environment, respectively.

If the reduced-system density operator has been obtained from Eq. (4.47), one can calculate the expectation value of operator \( \hat{A} \) as

\[ \langle \hat{A}(t) \rangle = \text{Tr} (\hat{A}\dot{\rho}(t)) , \] (4.48)

where \( \text{Tr}(x) \) is the trace. Eventually, the expectation values reaches a solution, independent of the initial conditions, that is oscillating with a constant frequency, i.e., a steady-state solution. The average resonator occupation number can be calculated by averaging the steady-state solution of the resonator occupation over \( m \) drive periods

\[ N_{\text{osc}} = \langle \hat{a}^\dagger \hat{a} \rangle = \frac{\omega_d}{m2\pi} \int_{t_0}^{t_0+m2\pi/\omega_d} \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle \, dt , \] (4.49)

where \( t_0 \) is a time after the system has reached the steady-state and \( \omega_d \) is the drive frequency. The steady-state solution is oscillating close to the drive frequency. Therefore, it is good to average over a multiple of the drive period. Typically, \( m \sim 10 \) is enough to obtain converged results. Similarly for a transmon state \( |i\rangle \), the occupation probability is obtained as

\[ \langle |i\rangle\langle i| \rangle = \frac{\omega_d}{m2\pi} \int_{t_0}^{t_0+m2\pi/\omega_d} \text{Tr} (|i\rangle\langle i|\dot{\rho}(t)) \, dt . \] (4.50)

There are two dissipation channels in the Lindblad master equation (4.47), one for the resonator and one for the transmon. In the case of \( T = 0 \), both sub-systems relax towards their ground states. Therefore, the equilibrium state of the total system is \( |n = 0,i = 0\rangle \), where \( n \) and \( i \) label the eigenstates of the bare resonator and transmon, respectively. The state \( |0,0\rangle \) is the ground state of the system if the coupling between the resonator and transmon is of the Jaynes–Cummings type, i.e., it conserves the total excitation number \( N \). However, without the Jaynes–Cummings approximation the excitation number is not conserved and the ground state of the coupled system is a linear combination of all the states \( |n,i\rangle \), where \( n + i \) is even, with \( n \in \mathbb{N} \) and \( i = 0,...,K - 1 \). Thus, at zero temperature the Lindblad model does not dissipate the system to its ground state. This is a known
Figure 4.5: Steady-state photon number $N_{\text{osc}}$ with different simulation methods as a function of drive amplitude $A$. (a) $\omega_q/\omega_r = 1.041$, (b) $\omega_q/\omega_r = 1$. The simulations are done using the two-state truncation for the transmon. The used parameter values are $\omega_d/\omega_r = 0.98$, $\kappa/\omega_r = 0.002$, $g_0/\omega_r = 0.04$, $E_J/E_C = 30$, $n_g = 0$, and $k_B T/(\hbar \omega_r) = 0.1$.

problem and there are methods to derive more accurate dissipators using the eigenbasis of $\hat{H}_S$ [52, 56] but this form where the resonator and transmon dissipators are considered separately is commonly used [23].

In the Floquet–Born–Markov model, the dissipation occurs between the eigenstates of the driven system. Therefore, at $T = 0$ this dissipation channel relaxes the undriven system towards its ground state. So, at least in this respect, the Floquet–Born–Markov model produces physically more reliable results than the Lindblad approach introduced above.

The comparison of the steady-state photon numbers $N_{\text{osc}}$, obtained with the Floquet–Born–Markov and the Lindblad master equation methods, are shown in Fig. 4.5. Here, the difference caused by the above mentioned dissipation towards different states is too small to be seen. At zero drive power, the difference in the photon numbers of the Floquet–Born–Markov and the Lindblad methods is about $10^{-4}$.

If the qubit and the resonator frequencies are far from resonance, the differences between the Floquet–Born–Markov and the Lindblad methods are very small, as seen in Fig. 4.5(a), where $(\omega_q - \omega_r)/\omega_r = 0.041$. In Fig. 4.5(b), where $\omega_q - \omega_r = 0$, we see that the largest difference is towards the end of the sudden increase caused by the photon blockade at $A/\kappa \approx 5$, where the relative difference is $(N_{\text{osc}}^{(F)} - N_{\text{osc}}^{(L)})/N_{\text{osc}}^{(F)} \approx 0.5$. Overall, the differences in the results between the two models are not big. The major advantage of the Floquet–Born–Markov model is its shorter calculation time on a computer. For example, the Floquet–Born–Markov model was roughly 20 times faster when calculating the data in Fig. 4.5. This difference increases when dealing with more transmon states.
4.6 Semiclassical model

The above mentioned models, i.e., the Floquet–Born–Markov and the Lindblad model, are completely quantum mechanical. Since we are dealing with a quantum-to-classical transition, a model with some classical elements might prove to be useful. Increasing the driving power excites the resonator to higher states which means that more resonator states need to be included in the quantum models. This increases the calculation times with the quantum systems. Modelling the resonator as a classical system removes the energy state structure from the resonator and there is not a similar increase in calculation times as the drive power is increased. We will introduce a semiclassical model, in which the resonator is treated as a classical harmonic oscillator and the transmon as a quantum system.

According to the correspondence principle, at high energies the bare resonator behaves as a classical harmonic oscillator. Therefore, the semiclassical model is expected to yield accurate results at strong driving when the resonator is highly populated. However, at low drive powers, the resonator dynamics displays quantum dynamics that are not captured by the semiclassical model.

Starting from Hamiltonian (3.33), we introduce the operators

$$\hat{q}_r = \frac{1}{2}(\hat{a}^\dagger + \hat{a}) ,$$
$$\hat{\phi}_r = \frac{i}{2}(\hat{a}^\dagger - \hat{a}) ,$$

and make the semiclassical approximation

$$\hat{n}_J \hat{q}_r \approx n_J \hat{q}_r + \hat{n}_J q_r$$

where

$$n_J = \langle \hat{n}_J(t) \rangle$$
$$q_r = \langle \hat{q}_r(t) \rangle .$$

As a consequence, the Hamiltonian for the coupled and driven system can be written as

$$\hat{H} = \hbar \omega_r \hat{a}^\dagger \hat{a} + \hbar \sum_{n=0}^\infty \omega_n |n\rangle \langle n| + 2\hbar g q_r \sum_{n,m=0}^\infty \Gamma_{nm} + 2\hbar g n_J \hat{q}_r + \hbar A \cos(\omega dt)(\hat{a}^\dagger + \hat{a}) .$$

Using Eq. (4.47) and

$$\frac{d\langle \hat{A} \rangle}{dt} = \text{Tr} \left( \hat{A} \frac{d\hat{\rho}}{dt} \right) ,$$

one can write

$$\dot{q}_r = \omega_r \phi_r - \frac{\kappa}{2} q_r ,$$
$$\dot{\phi}_r = -\omega_r q_r - A \cos(\omega dt) + g n_J - \frac{\kappa}{2} \phi_r ,$$
$$\dot{\hat{\rho}}_q = -\frac{i}{\hbar} [\hat{H}_q, \hat{\rho}_q] + \sum_{\omega_{nm}>0} \Gamma_{nm} \mathcal{L}[|n\rangle \langle m|] \hat{\rho}$$
$$+ \sum_{\omega_{nm}<0} \Gamma_{nm} \mathcal{L}[|n\rangle \langle m|] \hat{\rho} + \sum_{\omega_{nm}=0} \Gamma_{nm} \mathcal{L}[|n\rangle \langle m|] \hat{\rho} ,$$

(4.57)
where
\[ \hat{H}_q = \hbar \sum_{n=0}^{\infty} \omega_n |n\rangle\langle n| + 2\hbar g_{q_{\tau}} \sum_{n,m=0}^{\infty} \hat{\Pi}_{nm}, \]  
\( \hat{\Pi}_{nm} = \langle n|\hat{n}_m|m\rangle|n\rangle\langle m|, \) and \( n_{\tau} = \text{Tr}(\hat{n}_1\hat{\rho}_q). \) The resonator occupation number \( N_{\text{osc}} \) is given by the energy of the resonator \( \hbar \omega_q (q_{\tau}^2 + \phi_{\tau}^2) \) divided by the energy of one photon \( \hbar \omega_{\tau}, \) i.e.,
\[ N_{\text{osc}} = q_{\tau}^2 + \phi_{\tau}^2. \]

The resonator occupation \( N_{\text{osc}} \) calculated with the semiclassical model is plotted in Fig. 4.5 alongside with the corresponding results obtained with the other models. At weak driving, the semiclassical model is not in agreement with the other simulations. With zero temperature and zero drive amplitude, the resonator occupation in the semiclassical model is zero, contrary to the quantum models. In resonance \( \omega_q = \omega_{\tau}, \) the semiclassical model does not agree with the quantum models, as seen in Fig. 4.5(b). An interesting point is that the sudden increase in the resonator occupation number in the quantum models is replaced by a discreet jump at \( A/\kappa = 7 \) in the semiclassical model. In the non-resonant case in Fig. 4.5(a), the semiclassical and quantum models agree if the drive amplitude is strong enough. If the amplitude is increased further, beyond the range of Fig. 4.5 \((A/\kappa \leq 30),\) the relative difference between the semiclassical model and the quantum models will decrease.
Chapter 5
Conclusions and discussion

In this thesis the interaction between radiation and matter was studied with a system consisting of a transmon coupled to a transmission-line resonator. We studied a region where the system experiences a quantum-to-classical transition as the drive power is increased. We focused on the region where the coupling and driving are so strong that the two-state approximation for the transmon is not sufficient but the driving is still weak enough that the system cannot be regarded as completely classical.

We showed the insufficiencies of the Jaynes–Cummings model and the two-transmon-state approximation. To fully capture the Bloch–Siegert shift observed in the experimental realization of our setup, we needed to include seven transmon states in our simulations. Also, we studied photon blockade in the case where the resonator was in resonance with the lowest transition of the transmon, and showed that the multi-photon blockade phenomenon is significantly different between the two-state and the multistate transmons.

We simulated the behavior of the resonator-transmon system with three different methods: the Floquet–Born–Markov, the Lindblad master equations, and the semiclassical model. The results with the first two methods differed only slightly for our parameters but the Floquet–Born–Markov method is much faster to numerically simulate. The semiclassical approximation was faster to calculate than the others but it was shown to be insufficient at low drive powers or if the resonator and transmon are in resonance.

In this research, we have only studied the coupling strength up to $g_0/\omega_r = 0.04$. An important expansion to this study would be the increase of the coupling all the way to the ultrastrong coupling regime ($g_0/\omega_r \approx 1$). With our parameters, this would require many more transmon and resonator states than we are able to include in our current implementations of the used models. The behavior of the multistate transmon was studied using the ratio $E_J/E_C \approx 30$ between the Josephson energy and the charging energy. This is a relatively small value. A detailed study about the effects of the ratio $E_J/E_C$ on the multistate phenomena should also be studied in the future.
Bibliography


