Diophantine perspectives to the exponential function and Euler’s factorial series

Doctoral dissertation

Louna Seppälä

Research Unit of Mathematical Sciences
Faculty of Science
University of Oulu Graduate School
University of Oulu
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Abstract

The focus of this thesis is on two functions: the exponential function and Euler’s factorial series. By constructing explicit Padé approximations, we are able to improve lower bounds for linear forms in the values of these functions. In particular, the dependence on the height of the coefficients of the linear form will be sharpened in the lower bound.

The first chapter contains some necessary definitions and auxiliary results needed in later chapters. We give precise definitions for a transcendence measure and Padé approximations of the second type. Siegel’s lemma will be introduced as a fundamental tool in Diophantine approximation. A brief excursion to exterior algebras shows how they can be used to prove determinant expansion formulas. The reader will also be familiarised with valuations of number fields.

In Chapter 2, a new transcendence measure for $e$ is proved using type II Hermite-Padé approximations to the exponential function. An improvement to the previous transcendence measures is achieved by estimating the common factors of the coefficients of the auxiliary polynomials.

The exponential function is the underlying topic of the third chapter as well. Now we study the common factors of the maximal minors of some large block matrices that appear when constructing Padé-type approximations to the exponential function. The factorisation of these minors is of interest both because of Bombieri and Vaaler’s improved version of Siegel’s lemma and because they are connected to finding explicit expressions for the approximation polynomials. In the beginning of Chapter 3, two general theorems concerning factors of Vandermonde-type block determinants are proved.

In the final chapter, we concentrate on Euler’s factorial series which has a positive radius of convergence in $p$-adic fields. We establish some non-vanishing results for a linear form in the values of Euler’s series at algebraic integer points. A lower bound for this linear form is derived as well.

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Oulu, April 2019

Louna
Contents

Some notation .............................................. viii

Introduction .............................................. 1
  Overview of the results .................................. 4
  A transcendence measure for $e$ ......................... 5
  Padé approximations and Siegel’s lemma ............... 5
  Euler’s factorial series ................................ 6

1 Background and preliminaries ......................... 7
  1.1 Lower bounds for linear forms ....................... 7
     1.1.1 Linear forms in complex numbers ............... 7
     1.1.2 Linear forms in $p$-adic numbers ............... 8
  1.2 Padé approximations ................................ 11
  1.3 Siegel’s lemma ..................................... 12
  1.4 Some tools ......................................... 13
     1.4.1 An auxiliary polynomial ......................... 13
     1.4.2 The inverse function of $z \log z$ ............... 15
     1.4.3 Properties of the $n$-factorial ................... 16
     1.4.4 A divisibility lemma for polynomials ............ 17
  1.5 Exterior algebras and determinant expansions ....... 17
     1.5.1 Exterior algebras ................................ 18
     1.5.2 Generalised minor expansion ..................... 20
  1.6 Number fields and valuations ....................... 21
     1.6.1 Extending valuations ............................ 21
     1.6.2 Normalisation .................................. 22
     1.6.3 The product formula ............................ 22
     1.6.4 Global relations ................................ 23
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.7</td>
<td>Lower bound: proof of Theorem 4.5</td>
<td>114</td>
</tr>
<tr>
<td>4.7.1</td>
<td>Product formula again</td>
<td>114</td>
</tr>
<tr>
<td>4.7.2</td>
<td>Deriving contradiction</td>
<td>117</td>
</tr>
<tr>
<td>4.7.3</td>
<td>Bounds for $f$</td>
<td>119</td>
</tr>
<tr>
<td>4.7.4</td>
<td>Measure</td>
<td>121</td>
</tr>
<tr>
<td>4.7.5</td>
<td>Infinitely many intervals</td>
<td>123</td>
</tr>
<tr>
<td>4.8</td>
<td>Corollaries and examples</td>
<td>124</td>
</tr>
<tr>
<td>4.8.1</td>
<td>The field of rationals</td>
<td>124</td>
</tr>
<tr>
<td>4.8.2</td>
<td>Linear recurrences</td>
<td>125</td>
</tr>
<tr>
<td>4.9</td>
<td>Arithmetic progressions</td>
<td>128</td>
</tr>
</tbody>
</table>

Bibliography 131
Some notation

$\mathbb{C}_p$ the topological closure of the algebraic closure of the field of $p$-adic numbers $\mathbb{Q}_p$

$F[[x]]$ the set of formal power series with coefficients in $F$

$K$ an algebraic number field

$K_v$ the $v$-adic completion of the field $K$

$L$ Laplace transform

$[L : K]$ the degree of the field extension $L$ over $K$

$log$ the natural logarithm (base $e$)

$\mathcal{M}_{k \times l}(R)$ the set of $k \times l$ matrices with entries in $R$

$M_\beta(x)$ the minimal polynomial of an algebraic number $\beta$

$\wedge(M)$ the exterior algebra of a module $M$

$(n)_k$ the falling factorial $n(n - 1) \cdots (n - k + 1)$

$O$ 'the big $O$: $f(x) = O(g(x))$ for some positive function $g(x)$ if there exists a constant $A > 0$ such that $|f(x)| < Ag(x)$ for all $x$ large enough

$\text{ord}$ the order of a power series $P(x) = \sum_{n=0}^{\infty} a_n (x - b)^n$:

$\text{ord} \ P(x) = \inf \{n \in \mathbb{Z}_{\geq 0} \mid a_n \neq 0\}$

$\mathbb{P}$ the set of prime numbers

$\mathbb{Q}_p$ the field of $p$-adic numbers: the $p$-adic completion of $\mathbb{Q}$

$R[x]$ the set of one-variable polynomials with coefficients in $R$

$\text{UFD}$ unique factorisation domain

$v|p$ valuation $v$ extends the $p$-adic valuation

$v_p(x)$ the largest exponent $k \in \mathbb{Z}_{\geq 0}$ such that $p^k | x$, where $x \in \mathbb{Z}$

$\mathbb{Z}_{\geq a}$ the set of integers greater than or equal to $a$

$\mathbb{Z}_K$ the ring of integers of the field $K$
Introduction

Diophantine approximation, together with the closely linked topic of Diophantine equations, belongs to the field of Diophantine analysis, named after Diophantus of Alexandria who lived around 250. He was the first writer to systematically investigate integral and rational solutions of polynomial equations. Diophantus’ *Arithmetica* could be viewed as the birth of number theory; its first translation in Europe appeared in the fifteenth century. Hundreds of years later mathematicians started to ask for generalisations of Diophantine problems.

Diophantine approximation is about estimating real numbers with rationals, and it has firm roots in practice: it is inconvenient and often impossible to use infinite decimal expansions that irrational numbers have. Of course we want to have neat estimates with as little error as possible. The problem of how well a real number can be approximated was solved in the eighteenth century by means of continued fractions. In 1842, Peter Dirichlet proved a fundamental result of Diophantine approximation:

**Dirichlet’s approximation theorem.** If \( \alpha \in \mathbb{R} \) is irrational, there exist infinitely many rational numbers \( \frac{p}{q} \) such that

\[
|\alpha - \frac{p}{q}| < \frac{1}{q^2}.
\]

Joseph Liouville then observed that there exists a limit to the accuracy with which any irrational algebraic number can be approximated by rationals.

**Liouville’s theorem.** For any algebraic number \( \alpha \in \mathbb{C} \) of degree \( d \geq 2 \) there exists a constant \( c(\alpha) > 0 \) such that

\[
|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^d}.
\]

for all \( \frac{p}{q} \in \mathbb{Q} \), \( q \geq 1 \).
It follows that a real number which can be better approximated than algebraic numbers has to be transcendental, and so in 1844 Liouville was the first to produce a transcendental number.

From Diophantine approximation we have quickly arrived to transcendental number theory. There is a lot to study, because (in the sense of the Lebesgue measure) almost all numbers are transcendental, as was shown by Georg Cantor in 1874. Regardless of this, deciding whether a given number is algebraic or transcendental is a hard task. In 1873, Charles Hermite \[23\] managed to prove the transcendence of \(e\) by using simultaneous rational approximations to the powers of \(e\), and soon after, in 1882, Ferdinand von Lindemann did the same for \(\pi\) by generalising Hermite’s methods. Their works were later simplified by several other mathematicians. Lindemann’s result showed the ancient Greek problem of constructing, with ruler and compass only, a square with area equal to that of a given circle to be impossible.

The results of Hermite and Lindemann are contained in the following general theorem proved by Lindemann:

**Lindemann’s theorem.** If \(\alpha_1, \ldots, \alpha_m\) are distinct algebraic numbers, and if \(\lambda_1, \ldots, \lambda_m\) are algebraic numbers not all zero, then

\[
\lambda_1 e^{\alpha_1} + \ldots + \lambda_m e^{\alpha_m} \neq 0.
\]

The next important advancements in the theory of transcendental numbers were due to Alexander Gel’fond, Theodor Schneider, and Carl Siegel. In 1900, David Hilbert proposed his famous list of 23 problems, the study of which he considered to be influential and stimulating for science. The seventh of those addressed the transcendence of numbers of the form \(\alpha^\beta\), where \(\alpha\) is an algebraic number not equal to 0 or 1, and \(\beta\) is an algebraic irrational number. Despite of the fact that Hilbert himself regarded this problem as ‘extraordinarily difficult’, already in 1929 Gel’fond was able to give a solution to it in the special case of \(\beta\) being an imaginary quadratic irrationality, thus implying that the number \(e^\pi = i^{-2i}\) is transcendental. Then, in 1934, Gel’fond and Schneider both independently obtained a complete solution to Hilbert’s seventh problem, using different methods.

At the same time, in 1929, Siegel \[48\] developed a new method for proving transcendence and algebraic independence of the values of a class of entire functions at algebraic points, directly generalising the method of Hermite and Lindemann. This paper was a point of departure for an entire branch of transcendental number theory. In 1949, Siegel presented his method in the form of a more general theorem, and in 1954, Andrei Shidlovskii published a theorem
similar to Siegel’s but with less restrictive assumptions, making it possible to apply Siegel’s method more extensively to sets of $E$-functions that are solutions of a non-homogeneous system of linear differential equations.

Siegel’s method can be applied to a class of functions he named $E$-functions, possibly because of their resemblance to the exponential function $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. The class of $E$-functions consists of analytic functions of the form

$$f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!},$$

where the coefficients $c_n$ are from an algebraic number field $\mathbb{K}$ of finite degree and satisfy the conditions

1) $|c_n| = O(c^n)$ for all $n \in \mathbb{Z}_{\geq 0}$ and some positive constant $c$ (here $|\alpha|$ denotes the maximum modulus of the algebraic conjugates of an algebraic number $\alpha$);

2) there exists a sequence of positive integers $(q_n)_{n=1}^{\infty}$, with $q_n = O(c^n)$, such that $q_n c_j \in \mathbb{Z}_K$ for all $n \in \mathbb{Z}_{\geq 1}$ and all $j = 0, 1, \ldots, n$.

The Siegel-Shidlovskii method leads to lower bounds for linear forms in the values of $E$-functions at algebraic points, as in the following general theorem:

**Theorem** (Shidlovskii, 1967). Let $m \in \mathbb{Z}_{\geq 1}$. Suppose that the $E$-functions $f_0(z), f_1(z), \ldots, f_m(z)$ have power series coefficients in an imaginary quadratic field $I$, are linearly independent over $\mathbb{C}(z)$, and form a solution to the system of differential equations

$$y_k' = \sum_{i=1}^{m} Q_{k,i}(z)y_i, \quad k = 0, 1, \ldots, m,$$

where $Q_{k,i}(z) \in \mathbb{C}(z)$. Let $\alpha \in I \setminus 0$ be not equal to a pole of any of the functions $Q_{k,i}$. Then there exists a positive constant $b$ such that for any $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}_I$ satisfying $\max_{0 \leq j \leq m} |\lambda_j| = H$, $H \geq 3$, it holds

$$|\lambda_0 f_0(\alpha) + \lambda_1 f_1(\alpha) + \ldots + \lambda_m f_m(\alpha)| > bH^{-m - \frac{\gamma(m+1)^3}{\sqrt{\log H}}},$$

where $\gamma$ is a constant depending on $\alpha, K$, and the constant $c$ in the above definition of an $E$-function.
Alongside his $E$-functions, Siegel also defined $G$-functions which have a finite radius of convergence and are therefore much harder to study. A third class of functions called $F$-functions was introduced by Vladimir Chirski˘ı [11] in 1989. It consists of analytic functions of the form

$$f(z) = \sum_{n=0}^{\infty} c_n n! z^n,$$

where again the coefficients $c_n$ lie in an algebraic number field $K$ of finite degree and have certain growth conditions—echoing the definition of $E$-functions. $F$-functions are studied in non-Archimedean metrics, where, unlike on the Archimedean side, they have a positive radius of convergence.

Modifying the Siegel-Shidlovski˘ı method to $p$-adic domains, Chirski˘ı [11, 12] showed in the nineties that there are no global relations (see Definition 1.19) among $F$-functions, and gave a bound for the prime number for which there exists a valuation with the property that a given algebraic relation is violated in the $v$-adic closure $K_v$ of the number field $K$ (see [13]). Daniel Bertrand, Chirski˘ı, and Johan Yebbou [5] then made the bound entirely effective and replaced the non-vanishing in the algebraic relation with a lower bound. Their theorem is too technical to be quoted here in its entirety, but, in short, they show that for given $F$-series $f_1(z), f_2(z), \ldots, f_m(z)$, there exists a valuation $v$ on $K$ such that in $K_v$,

$$|\lambda_0 + \lambda_1 f_1(\xi) + \ldots + \lambda_m f_m(\xi)|_v \geq H^{-(m+1) - \frac{m+4+2(m+1)(m+1)^2 c_4+2}{\sqrt{\log \log H}}}$$

(0.1)

where $\xi \in K$, $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}_K$ and $H$ is the maximum height of the coefficients $\lambda_j$, and $c_4$ is a constant depending on the chosen series.

This short historical introduction benefited from the books of Baker [3], Fel’dman and Nesterenko [18], Schmidt [46], Shidlovski˘ı [47], and Steuding [51], where the reader may find more details on the diverse subjects of Diophantine approximation and transcendental number theory.

**Overview of the results**

In this thesis, we focus on two functions: first, on the simplest $E$-function, the exponential function, and then on the simplest $F$-function which is Euler’s series. Instead of the Siegel-Shidlovski˘ı method, we use explicit Padé approximations, which is why we are able to sharpen earlier lower bounds for linear forms in
these functions. In particular, the dependence on $H$ in the error term (which is of the form $\sqrt{\log \log H}$ in the above bounds) will be improved.

The necessary definitions and tools are presented in Chapter 1.

A transcendence measure for $e$

In Chapter 2 we shall prove a new transcendence measure for $e$ by using Hermite-Padé approximations to the exponential function. The improvement to the previous transcendence measures of Masayoshi Hata \[22\] and others is achieved by carefully studying the common factors in the coefficients of the approximation polynomials. Also, the use of Laplace transform helps us to make better estimates of the auxiliary functions. The results are joint work with Anne-Maria Ernvall-Hytönen and Tapani Matala-aho and have appeared in *On Mahler’s transcendence measure for $e* \[16\].

Padé approximations and Siegel’s lemma

The construction of Padé-type approximations to given functions leads to large groups of equations. In some cases, these equations can be solved explicitly, but in general Siegel’s lemma is used to estimate the solution. Enrico Bombieri and Jeffrey Vaaler \[7\] have established an improved version of Siegel’s lemma which involves computing the greatest common factor of the maximal minors of the coefficient matrix of the group of equations. Via Cramer’s rule, these minors are also closely linked to the problem of finding the explicit solution. The corresponding homogeneous matrix equation representing both approaches has an $M \times (L + 1)$ coefficient matrix, where $M \leq L$. If a non-trivial common factor of the maximal minors exists, Bombieri and Vaaler’s result can be applied to improve all results where Siegel’s lemma has been used.

In Chapter 3 we focus on Padé-type approximations to the exponential function and show that there indeed exists a big common factor in the $M \times M$ minors of the coefficient matrix. We begin with some general block determinant factoring considerations which form the basis of our results but are also interesting on their own right (see Section 3.3). This is joint work with Tapani Matala-aho and has appeared as *Hermite-Thue equation: Padé approximations and Siegel’s lemma* \[35\].
Euler’s factorial series

The last chapter is devoted to Euler’s series

$$F(t) := \sum_{n=0}^{\infty} n! t^n$$

which has a positive radius of convergence in the $p$-adic domain. We study a linear form

$$\Lambda_v = \lambda_0 + \lambda_1 F_v(\alpha_1) + \ldots + \lambda_m F_v(\alpha_m), \quad \lambda_i \in \mathbb{Z}_K,$$

in the $v$-adic values of Euler’s series $F_v(t)$ at algebraic integer points $\alpha_j \in \mathbb{Z}_K$, $j = 1, \ldots, m$, belonging to a number field $K$. In the two main results it is shown that in some collection $V$ of non-Archimedean valuations of the field $K$ there exists a valuation $v$ such that the linear form $\Lambda_v$ does not vanish.

In the first of the results, this collection $V$ of valuations is infinite, and the theorem can also be extended to the case of primes in residue classes, generalising the recent result of Ernvall-Hytönen et al. [17]. In the second main theorem, we estimate the interval in which there exists a prime $p$ for which there is a valuation $v|p$ such that $\Lambda_v \neq 0$. We also prove a lower bound for the $v$-adic absolute value of $\Lambda_v$, improving the corresponding bound of Bertrand et al. [5]. On the way to the main results, we shall present explicit Padé approximations to the generalised factorial series $\sum_{n=0}^{\infty} \left( \prod_{k=0}^{n-1} P(k) \right) t^n$, where $P(x)$ is a polynomial of degree one (see Section 4.3.1). The results are independent work of the author.
Chapter 1

Background and preliminaries

1.1 Lower bounds for linear forms

As Fel’dman and Nesterenko [18] put it, very often in problems of transcendental number theory one ends up looking for lower bounds for linear forms in the numbers under consideration with rational or algebraic coefficients. A transcendental number does not satisfy any polynomial equation with integer coefficients, so linear forms in its powers over integers must have a positive lower bound. So do linear forms of any linearly independent numbers, and one may wonder how big can these lower bounds be. This is Diophantine approximation in a more general form.

1.1.1 Linear forms in complex numbers

Definition 1.1. Let \( m, H \geq 1 \) be given and suppose that the numbers \( \alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{C} \) are linearly independent over \( \mathbb{Q} \). Define \( \omega(m, H) \) as the infimum of the numbers \( r > 0 \) satisfying the estimate

\[
|\lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_m \alpha_m| > \frac{1}{H^r}
\]

for all \( \bar{\lambda} = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{Z}^{m+1} \setminus \{0\} \) with \( \max_{0 \leq i \leq m} \{ |\lambda_i| \} \leq H \). Then any function greater than or equal to \( \omega(m, H) \) is called a linear independence measure (exponent) for the numbers \( \alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{C} \).
Chapter 1. Background and preliminaries

The special case $\alpha_0 = 1, \alpha_1 = \alpha, \ldots, \alpha_m = \alpha^m$ gives a transcendence measure (exponent) for a number $\alpha$.

**Theorem 1.2.** \[45\] Suppose that the numbers $1, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$ are linearly independent over the rationals. Then there are infinitely many coprime $(m + 1)$-tuples $(\lambda_0, \ldots, \lambda_m)^T \in \mathbb{Z}^{m+1} \setminus \{0\}$ with $\max_{1 \leq i \leq m} \{|\lambda_i|\} = H$ and

$$|\lambda_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_m \alpha_m| < \frac{1}{H^m}.$$ 

**Corollary 1.3.** If the numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$ and there exist $c, \omega \in \mathbb{R}_{>0}$ such that

$$|\lambda_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_m \alpha_m| \geq \frac{c}{H \omega} \quad (1.1)$$

for all $(\lambda_0, \ldots, \lambda_m)^T \in \mathbb{Z}^{m+1}$ with $H = \max_{1 \leq i \leq m} \{|\lambda_i|\} \geq 1$, then $\omega \geq m$.

**Proof.** Assume on the contrary that $\omega < m$. By Theorem [1.2] there exists an $(m + 1)$-tuple $(\lambda_0, \ldots, \lambda_m)^T \in \mathbb{Z}^{m+1} \setminus \{0\}$ with $\max_{1 \leq i \leq m} \{|\lambda_i|\} = H$, where

$$\log H \geq \frac{\log \left( \frac{1}{m-\omega} \right)}{m-\omega},$$

such that

$$|\lambda_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_m \alpha_m| < \frac{1}{H^m}. \quad (1.2)$$

But (1.1) and (1.2) together imply

$$\log H < \frac{\log \left( \frac{1}{m-\omega} \right)}{m-\omega},$$

a contradiction. \qed

From the previous corollary it follows that the transcendence measure of a transcendental number is always at least $m$.

1.1.2 Linear forms in $p$-adic numbers

On the non-Archimedean side, one may state an analogous definition:
Chapter 1. Background and preliminaries

Definition 1.4. Let \( m, H \geq 1 \) be given and suppose that the \( p \)-adic numbers \( \alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{C}_p \) are linearly independent over \( \mathbb{Q} \). Define \( \psi(m, H) \) as the infimum of the numbers \( r > 0 \) satisfying the estimate

\[
|\lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_m \alpha_m|_p > \frac{1}{H^r}
\]

for all \( \lambda = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{Z}^{m+1} \setminus \{0\} \) with \( \max_{0 \leq i \leq m} \{|\lambda_i|\} \leq H \). Then any function greater than or equal to \( \psi(m, H) \) is called a linear independence measure (exponent) for the numbers \( \alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{C}_p \).

As for corresponding results for linear forms in \( p \)-adic numbers, we have the following theorem which is a slightly modified version of the one given by Mahler [30, Chapter 3].

Theorem 1.5. Let \( p \in \mathbb{P} \) and \( E \in \mathbb{Z}_{\geq 1} \). Let \( \alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{Q}_p \) be such that

\[
0 < \max_{0 \leq j \leq m} \{|\alpha_j|_p\} \leq 1.
\]

Then there exist integers \( \lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z} \), not all zero, for which

\[
0 < \max_{0 \leq j \leq m} \{|\lambda_j|\} \leq p^E
\]

and

\[
|\lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \ldots + \lambda_m \alpha_m|_p \leq \frac{1}{p^{(m+1)E}}.
\]

Proof. Because the numbers \( \alpha_j \) are \( p \)-adic integers, there exist rational integers \( a_0, a_1, \ldots, a_m \in \mathbb{Z} \) such that

\[
|\alpha_j - a_j|_p \leq \frac{1}{p^{(m+1)E}}, \quad 0 \leq a_j \leq p^{(m+1)E} - 1, \quad j = 0, 1, \ldots, m.
\]

Now define

\[
L(\bar{x}) := \frac{a_0}{p^{(m+1)E}} \cdot x_0 + \frac{a_1}{p^{(m+1)E}} \cdot x_1 + \ldots + \frac{a_m}{p^{(m+1)E}} \cdot x_m + x_{m+1},
\]

\[
L_j(\bar{x}) := x_j, \quad j = 0, 1, \ldots, m,
\]

for any \( \bar{x} = (x_0, x_1, \ldots, x_{m+1})^T \in \mathbb{R}^{m+2} \).

From Minkowski’s first convex body theorem it follows that for any \( H \geq 1 \) there exists a vector \( (\lambda_0, \lambda_1, \ldots, \lambda_{m+1}) \in \mathbb{Z}^{m+2} \) with \( \lambda_j \neq 0 \) for some \( j \in \{0, 1, \ldots, m\} \) such that

\[
|L(\lambda)| < \frac{1}{H^{m+1}}.
\]
Chapter 1. Background and preliminaries

and

$$|\lambda_j| \leq H, \quad j = 0, 1, \ldots, m.$$  

(See [33, Theorem 4.1].) Put $H = p^E$. Then

$$|\lambda_0 a_0 + \lambda_1 a_1 + \ldots + \lambda_m a_m + \lambda_{m+1} p^{(m+1)E}| < 1,$$

implying that $\lambda_0 a_0 + \lambda_1 a_1 + \ldots + \lambda_m a_m + \lambda_{m+1} p^{(m+1)E} = 0$. So we have

$$\lambda_0 a_0 + \lambda_1 a_1 + \ldots + \lambda_m a_m \equiv 0 \pmod{p^{(m+1)E}},$$

or equivalently,

$$|\lambda_0 a_0 + \lambda_1 a_1 + \ldots + \lambda_m a_m|_p \leq \frac{1}{p^{(m+1)E}}.$$

Then also

$$\left| \sum_{j=0}^{m} \lambda_j \alpha_j - \sum_{j=0}^{m} \lambda_j a_j \right|_p = \left| \sum_{j=0}^{m} \lambda_j (\alpha_j - a_j) \right|_p \leq \max_{0 \leq j \leq m} |\alpha_j - a_j|_p \leq \frac{1}{p^{(m+1)E}},$$

and finally

$$|\lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \ldots + \lambda_m \alpha_m|_p = \left| \sum_{j=0}^{m} \lambda_j \alpha_j - \sum_{j=0}^{m} \lambda_j a_j + \sum_{j=0}^{m} \lambda_j a_j \right|_p$$

$$\leq \max \left\{ \left| \sum_{j=0}^{m} \lambda_j \alpha_j - \sum_{j=0}^{m} \lambda_j a_j \right|_p, \left| \sum_{j=0}^{m} \lambda_j a_j \right|_p \right\}$$

$$\leq \frac{1}{p^{(m+1)E}}.$$

In this case, too, one can deduce the infinity of solutions:

**Corollary 1.6.** Let $p \in \mathbb{P}$ and suppose that the $p$-adic numbers $\alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{Q}_p$ with $0 < \max_{0 \leq j \leq m} \{\alpha_j\}_p \leq 1$ are linearly independent over $\mathbb{Q}$. Then there exist infinitely many $m$-tuples $[0] \neq (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{Z}^{m+1}$ such that

$$|\lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \ldots + \lambda_m \alpha_m|_p \leq \frac{1}{H^{m+1}}, \quad (1.3)$$

where $H := \max_{0 \leq i \leq m} \{|\lambda_i|\}$.
Proof. Assume on the contrary that there are only finitely many solution vectors \( \mathbf{0} \neq \mathbf{x} \in \mathbb{Z}^{m+1} \) to (1.3). Then there is a minimum

\[
0 < \frac{1}{R} := \min_{\mathbf{x}} \left\{ |\lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \ldots + \lambda_m \alpha_m|_p \right\}.
\]

Pick a number \( E' \in \mathbb{Z}_{\geq 1} \) so that \( p^{(m+1)E'} > R \). Then, by Theorem 1.5, there exists a vector \( \mathbf{0} \neq (\lambda'_0, \lambda'_1, \ldots, \lambda'_m)^T \in \mathbb{Z}^{m+1} \) with \( H' := \max_{0 \leq i \leq m} \{|\lambda'_i|\} \leq p^{E'} \) such that

\[
|\lambda'_0 \alpha_0 + \lambda'_1 \alpha_1 + \ldots + \lambda'_m \alpha_m|_p \leq \frac{1}{p^{(m+1)E'}} \leq \frac{1}{(H')^{m+1}}.
\]

So \( (\lambda'_0, \lambda'_1, \ldots, \lambda'_m)^T \) is a solution to (1.3), and it follows that

\[
\frac{1}{R} \leq |\lambda'_0 \alpha_0 + \lambda'_1 \alpha_1 + \ldots + \lambda'_m \alpha_m|_p \leq \frac{1}{p^{(m+1)E'}} < \frac{1}{R},
\]

which is a contradiction. \( \square \)

Finally, we get a corresponding result to Corollary 1.3.

**Corollary 1.7.** Let \( p \in \mathbb{P} \) and suppose that the numbers \( \alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{Q}_p \) with \( 0 < \max_{0 \leq j \leq m} \{|\alpha_j|_p\} \leq 1 \) are linearly independent over \( \mathbb{Q} \). If there exist \( c, \omega \in \mathbb{R}_{>0} \) such that

\[
|\lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \ldots + \lambda_m \alpha_m|_p \geq \frac{c}{H \omega}
\]

for all \( (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{Z}^{m+1} \) with \( H := \max_{0 \leq i \leq m} \{|\lambda_i|\} \geq 1 \), then \( \omega \geq m + 1 \).

### 1.2 Padé approximations

The method for finding a lower bound for a linear form of \( m + 1 \) numbers \( 1, \beta_1, \ldots, \beta_m \) is to construct small approximating forms for each of the numbers \( \beta_1, \ldots, \beta_m \) separately. This is achieved by using Padé approximations, another important generalisation of Diophantine approximation named after Henri Padé who did the first systematic study at the end of the nineteenth century.

In 1873 Hermite [23] proved the transcendence of \( e \), the base of the natural logarithm. For the proof, Hermite introduced rational function approximations to the exponential function in the following sense:
Chapter 1. Background and preliminaries

Definition 1.8 (Hermite-Padé approximations of the second type). Let \( m \in \mathbb{Z}_{\geq 1} \) and \( \{F_1(t), \ldots, F_m(t)\} \) be a given set of functions. Let \( l_0, l_1, \ldots, l_m \in \mathbb{Z}_{\geq 1} \) and denote \( L_0 := \sum_{j=0}^m l_j \), \( \mathbf{I} = (l_0, l_1, \ldots, l_m)^T \). If there exist polynomials \( A_{l,j}(t) \) and remainders \( R_{l,j}(t) \) such that

\[
A_{l,0}(t)F_j(t) - A_{l,j}(t) = R_{l,j}(t), \quad j = 1, \ldots, m,
\]

then these polynomials are called Hermite-Padé approximations to the functions \( F_1(t), \ldots, F_m(t) \).

The question of finding explicit Padé approximations to a given set of functions \( \{F_1(t), \ldots, F_m(t)\} \) is called a Padé problem. Baker and Graves-Morris [4] cover the general setting; the problem of simultaneous Padé approximations is stated in Chapter 8 of their book. In the case \( l_0 = l_1 = \ldots = l_m = l \) for \( l \in \mathbb{Z}_{\geq 1} \), we use the term diagonal Padé approximations. Diagonal Hermite-Padé approximations to the generalised hypergeometric series are quite well established; see de Bruin [10], Huttner [24], Matala-aho [33], Nesterenko [37], and Nikšin [38] for more details.

When the unknown denominator polynomial \( A_{l,0}(t) \) is denoted by \( c_0 + c_1x + \ldots + c_Lx^L \) and the multiplication \( A_{l,0}(t)F_j(t) \) performed, the gap conditions in (1.4) give us \( L \) equations in the \( L + 1 \) unknowns \( c_i \).

1.3 Siegel’s lemma

The solutions to groups of equations with less equations than unknowns can be estimated by using the Thue-Siegel lemma. In 1909 Thue [53] improved the Liouville bound for algebraic numbers. For that purpose he needed to find a small non-zero integer solution \( (x_1, \ldots, x_N) \) to a system of \( M \) equations in integer coefficients with \( N \) unknowns, \( M < N \). An essential feature is that the small solution is bounded with a non-trivial upper bound depending on the coefficients. Thue’s idea was present already in the 1908 paper [52].

In his celebrated paper [33], Siegel formalised Thue’s idea which has been known since then as the Thue-Siegel lemma or Siegel’s lemma. The variant presented here is from Mahler [31]. From now on, \( \mathcal{M}_{k \times l}(R) \) will denote the set of \( k \times l \) matrices with coefficients in \( R \).
Siegel’s lemma. Let $V = (v_{mn}) \in M_{M \times N}(\mathbb{Z})$, and assume that
\[
\|v_m\|_1 := \sum_{n=1}^{N} |v_{mn}| \in \mathbb{Z}_{\geq 1}, \quad m = 1, \ldots, M,
\] (1.5)
where $v_m$ denotes the $m$th row of the matrix $V$. Suppose that $M < N$; then the equation
\[
V \mathbf{x} = \mathbf{0}
\]
has a non-zero integer solution $\mathbf{x} = (x_1, \ldots, x_N)^T \in \mathbb{Z}^N \setminus \{0\}$ with
\[
1 \leq \|\mathbf{x}\|_\infty := \max_{1 \leq n \leq N} |x_n| \leq \left(\frac{\|v_1\|_1 \cdots \|v_M\|_1}{N-1}\right)^{1/N}.
\]

Starting from Siegel’s work [48], Siegel’s lemma has been greatly appreciated because of its flexibility and power in Diophantine approximation and transcendence proofs.

1.4 Some tools

This section contains several small but useful results that are needed later.

1.4.1 An auxiliary polynomial

The following notation will be used throughout the text when constructing Padé approximation polynomials.

Let $m \in \mathbb{Z}_{\geq 1}$ and $l_0 \in \mathbb{Z}_{\geq 0}$, $l_1, \ldots, l_m \in \mathbb{Z}_{\geq 1}$. Denote $\bar{l} = (l_0, l_1, \ldots, l_m)^T$, $L_0 := l_0 + l_1 + \ldots + l_m$, and $L := l_1 + \ldots + l_m$. Set $\beta_0 = 0$. For a given vector $\bar{\beta} = (\beta_0, \beta_1, \ldots, \beta_m)^T$, define the numbers $\sigma_i = \sigma_i(l, \bar{\beta})$ by the equation
\[
\Omega(w, \bar{\beta}) := \prod_{j=0}^{m} (\beta_j - w)^{l_j} = \sum_{i=l_0}^{L_0} \sigma_i w^i.
\] (1.6)

Then, by the binomial theorem,
\[
\sigma_i(l, \bar{\beta}) = (-1)^i \sum_{l_0+i_1+\ldots+i_m=i} \binom{l_1}{i_1} \cdots \binom{l_m}{i_m} \beta_1^{l_1-i_1} \cdots \beta_m^{l_m-i_m}.
\] (1.7)
Chapter 1. Background and preliminaries

Lemma 1.9. We have

\[ \sum_{i=l_0}^{L_0} \sigma_i^j \beta_j^i = 0 \]  

for all \( j = 1, \ldots, m \) and \( k_j = 0, 1, \ldots, l_j - 1 \). Moreover, when \( \mathfrak{f} \in \mathbb{K}^m \), where \( \mathbb{K} \) is an algebraic number field, and \( | \cdot |_v \) is any Archimedean absolute value of the field \( \mathbb{K} \), it holds

\[ \sum_{i=l_0}^{L_0} |\sigma_i|^v t^i \leq \prod_{j=0}^{m} (|\beta_j |_v + t)^{l_j} \]  

for all \( t \geq 0 \).

Proof. It is not too hard to deduce that

\[ \left( x \frac{d}{dx} \right)^n f(x) = \sum_{i=1}^{n} a_{n,i} x^i \left( \frac{d}{dx} \right)^i f(x), \]  

where the coefficients \( a_{n,i} \) satisfy the recursions

\[ \begin{cases} a_{n,1} = 1; \\ a_{n,i} = a_{n-1,i-1} + ia_{n-1,i}, & i = 2, \ldots, n-1; \\ a_{n,n} = 1 \end{cases} \]

for all \( n \in \mathbb{Z}_{\geq 1} \). Let now \( j \in \{1, \ldots, m\} \). For \( k_j = 0 \), the claim \( (1.8) \) follows directly from the definition \( (1.6) \). For \( k_j \in \{1, \ldots, l_j - 1\} \), we use \( (1.6) \) and \( (1.10) \):

\[ \sum_{i=l_0}^{L_0} \sigma_i^j \beta_j^i = \left. \left( w \frac{d}{dw} \right)^{k_j} m \prod_{i=0}^{m} (\beta_i - w)^{l_i} \right|_{w=\beta_j} = 0 \]

because \( \left. \left( \frac{d}{dw} \right)^h m \prod_{i=0}^{m} (\beta_i - w)^{l_i} \right|_{w=\beta_j} = 0 \) for all \( h = 1, \ldots, k_j \).

Property \( (1.9) \) follows simply from the expansion of \( \sigma_i \) and the triangle
inequality:
\[
\sum_{i=l_0}^{L_0} |\sigma_i| v^i t^i \leq \sum_{i=l_0}^{L_0} \left( \sum_{l_0+i_1+\ldots+i_m=i} \left( \frac{l_1}{i_1} \right) \ldots \left( \frac{l_m}{i_m} \right) \right) \cdot |\beta_1|_{v}^{i_1-1} \ldots |\beta_m|_{v}^{i_m-1} t^i
\]
\[
\leq \sum_{i=l_0}^{L_0} \left( \sum_{l_0+i_1+\ldots+i_m=i} \left( \frac{l_1}{i_1} \right) \ldots \left( \frac{l_m}{i_m} \right) \right) \cdot |\beta_1|_{v}^{i_1-1} \ldots |\beta_m|_{v}^{i_m-1} t^i
\]
\[
= \prod_{j=0}^{m} (|\beta_j|_{v} + t)^{l_j}
\]
when \( t \geq 0 \).

1.4.2 The inverse function of \( z \log z \)

For the purpose of proving sharp bounds for linear forms, we introduce the inverse function of the function \( y(z) = z \log z \), \( z \geq \frac{1}{e} \), considered by Hančl et al. [21].

Lemma 1.10. \([21]\) The inverse function \( z(y) \) of the function \( y(z) = z \log z \), \( z \geq \frac{1}{e} \), is strictly increasing. When \( y > e \), the inverse function may be given by the infinite nested logarithm fraction

\[
z(y) = \lim_{n \to \infty} z_n(y) = \frac{y}{\log \left( \frac{y}{\log y} \right)}.\]

Let \( z_0(y) = y \) and \( z_n(y) = \frac{y}{\log z_{n-1}(y)} \) for all \( n \in \mathbb{Z}_{\geq 1} \). Then we also have

\[
z_1 < z_3 < \cdots < z < \cdots < z_2 < z_0.
\]

The following little lemma gives a useful upper estimate:

Lemma 1.11. If \( y \geq re^r \), where \( r \geq e \), then

\[
z(y) \leq \left( 1 + \frac{\log r}{r} \right) \frac{y}{\log y}.
\]

Proof. Denote \( z := z(y) \) with \( y \geq re^r \). Then

\[
z = \frac{y}{\log z} = \frac{y}{\log y} \frac{\log y}{\log z} = \frac{y}{\log y} \left( 1 + \frac{\log \log z}{\log z} \right) \leq \frac{y}{\log y} \left( 1 + \frac{\log r}{r} \right)
\]

because \( \log z \geq r \geq e \).
1.4.3 Properties of the $n$-factorial

Because of our interest in the exponential function $e^t = \sum_{n=0}^\infty \frac{t^n}{n!}$ and Euler’s series $\sum_{n=0}^\infty n!t^n$, we shall be needing estimates for the $n$-factorial and its $p$-adic valuation at several points.

**Lemma 1.12** (Stirling’s formula). \cite{42} Let $n \in \mathbb{Z}_{\geq 1}$. There exists a number $\theta(n) \in [0,1]$ such that

$$n! = \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n+\theta(n)}.$$  

When $p$ be is a prime number and $x \in \mathbb{Z}$, we denote by $v_p(x)$ the $p$-adic valuation of the number $x$; in other words,

$$v_p(x) = \begin{cases} \max \{ k \in \mathbb{Z}_{\geq 0} \mid p^k | x \} , & x \neq 0; \\ \infty , & x = 0. \end{cases}$$

**Lemma 1.13.** Let $n \in \mathbb{Z}_{\geq 1}$. Then

$$v_p(n!) = \sum_{i=1}^\infty \left\lfloor \frac{n}{p^i} \right\rfloor,$$

and we have the estimate

$$\frac{n}{p-1} - \frac{\log n}{\log p} - 1 \leq v_p(n!) \leq \frac{n-1}{p-1}. \quad (1.11)$$

**Proof.** See Leppälä et al. \cite{28}, for example. \hfill \square

Yet a third lemma from Ernvall-Hytönen et al. \cite{17} will be quoted here:

**Lemma 1.14.** \cite{17} Let $m \in \mathbb{Z}_{\geq 3}$ be a given integer and suppose that $a$ is an integer such that $\gcd(a,m) = 1$. Then

$$\log \left( \prod_{\substack{p \in \mathbb{P} \\ p \equiv a \pmod{m}}} |n!|_p \right) = -\frac{n \log n}{\varphi(m)} + O(n \log \log n).$$
1.4.4 A divisibility lemma for polynomials

In Chapter 3, we are going to prove the existence of high order factors in generalised Vandermonde-type polynomial block determinants. For the proofs to work, we need to study the determinants inside a polynomial ring. If the ring of coefficients is not a field, then the division algorithm is not available, but luckily the following well-known lemma over a field is valid also over any integral domain of characteristic zero.

**Lemma 1.15.** Let $I$ be an integral domain of characteristic zero, $P(x) \in I[x]$, $x_0 \in I$ and $n \in \mathbb{Z}_{\geq 1}$. If

$$P^{(k)}(x_0) = 0 \quad \text{for all } k \in \{0, 1, \ldots, n-1\}, \quad (1.12)$$

then

$$(x - x_0)^n \mid P(x).$$

**Proof.** Using the binomial expansion we may write

$$P(x) = P(x - x_0 + x_0) = a_0 + a_1(x - x_0) + \ldots + a_d(x - x_0)^d,$$

where

$$a_i \in I, \quad i = 0, 1, \ldots, d, \quad d := \deg P(x).$$

Now condition (1.12) implies $a_0 = a_1 = \ldots = a_{n-1} = 0$, so

$$(x - x_0)^n \mid P(x).$$

\[ \square \]

1.5 Exterior algebras and determinant expansions

The properties of exterior algebras are closely linked to determinant expansion formulas. This section forms the basis to Chapter 3. A recommended source is Rotman’s book [44].
1.5.1 Exterior algebras

Let \( R \) be a commutative ring and \( M \) a free \( R \)-module with \( \text{rank} M = n \). The exterior algebra of \( M \) is denoted by \( \bigwedge(M) \). The ring product \( \wedge \) is called \textit{wedge product} and it has the property \( m \wedge m = 0 \) for all \( m \in M \).

Increasing lists

Let \( n \in \mathbb{Z}_{\geq 1} \) and \( p \in \{0, 1, \ldots, n\} \). The numbers \( i_1, i_2, \ldots, i_p \in \{1, \ldots, n\} \) form an \textit{increasing} \( (0 \leq p \leq n) \)-list \( \sigma_p \) if

\[
1 \leq i_1 < i_2 < \ldots < i_p \leq n.
\]

If \( p = 0 \), then \( \sigma_p = \emptyset \). The set of all increasing \( (0 \leq p \leq n) \)-lists is denoted by

\[
C(n, p) := \{ \{i_1, i_2, \ldots, i_p\} \mid 1 \leq i_1 < i_2 < \ldots < i_p \leq n \}.
\]

Let \( H = \{h_1, \ldots, h_p\} \) and \( K = \{k_1, \ldots, k_q\} \) be increasing \( (0 \leq p \leq n) \)-and \( (0 \leq q \leq n) \)-lists, respectively. When the lists \( H \) and \( K \) are disjoint, we denote by \( \tau_{H,K} \) the permutation in the symmetric group \( S_{p+q} \) which arranges the list \( \{h_1, \ldots, h_p, k_1, \ldots, k_q\} \) into an increasing \( (0 \leq p + q \leq n) \)-list

\[
H * K = \{j_1, \ldots, j_{p+q}\}, \quad 1 \leq j_1 < \ldots < j_{p+q} \leq n.
\]

More generally, let \( H_k = \{h_{k,1}, \ldots, h_{k,n_k}\} \), \( k = 1, \ldots, m \), be \( m \) increasing \( (0 \leq n_k \leq n) \)-lists. When the lists \( H_k \) are pairwise disjoint, we denote by \( \tau_{H_1,\ldots,H_m} \) the permutation in the symmetric group \( S_{n_1+\ldots+n_m} \) which arranges the list

\[
\{h_{1,1}, \ldots, h_{1,n_1}, h_{2,1}, \ldots, h_{2,n_2}, \ldots, h_{m,1}, \ldots, h_{m,n_m}\}
\]

into an increasing \( (0 \leq n_1 + \ldots + n_m \leq n) \)-list

\[
H_1 * \ldots * H_m = \{j_1, \ldots, j_{n_1+\ldots+n_m}\}, \quad 1 \leq j_1 < \ldots < j_{n_1+\ldots+n_m} \leq n.
\]

Basis vectors

When \( I = \{i_1, \ldots, i_p\} \) is an increasing \( (0 \leq p \leq n) \)-list, we use the notation

\[
\overline{A}_I := \overline{a}_{i_1} \wedge \cdots \wedge \overline{a}_{i_p},
\]

where \( \overline{a}_{i_1}, \ldots, \overline{a}_{i_p} \in M \). The element \( \overline{A}_I \) is called a \textit{p-vector}. 
Let \{\tau_1, \ldots, \tau_n\} be a basis of \(M\). For \(p \in \{0, 1, \ldots, n\}\), consider the products

\[
\overline{E}_{\sigma_p} := \tau_{i_1} \wedge \tau_{i_2} \wedge \ldots \wedge \tau_{i_p}, \quad \sigma_p = \{i_1, i_2, \ldots, i_p\} \in C(n, p),
\]

with \(\overline{E}_\varnothing = 1\). There are \(\binom{n}{p}\) of them for each \(p \in \{0, 1, \ldots, n\}\), and together they form a basis for \(\bigwedge(M)\).

Let \(H\) and \(K\) be increasing \((0 \leq p \leq n)\)- and \((0 \leq q \leq n)\)-lists, respectively. Then

\[
\overline{E}_H \wedge \overline{E}_K = \begin{cases} 
0 & \text{if } H \cap K \neq \emptyset; \\
\text{sign}(\tau_{H,K})\overline{E}_{H+K} & \text{if } H \cap K = \emptyset.
\end{cases}
\]

### Grassmann coordinates

When \(A\) is an \(s \times t\) matrix and \(H \in C(s, p), K \in C(t, q)\), we denote by

\[
A_{HK} := (a_{hk}), \quad h \in H, \quad k \in K,
\]

the \(p \times q\) submatrix of \(A\) with rows and columns chosen according to the lists \(H\) and \(K\), respectively.

Let \(\overline{\tau}_{i_1}, \ldots, \overline{\tau}_{i_p} \in M\), where \(\{i_1, \ldots, i_p\} =: I\) is an increasing \((0 \leq p \leq n)\)-list. Then

\[
\overline{\tau}_{i_1} \wedge \ldots \wedge \overline{\tau}_{i_p} = \sum_{L \in C(n, p)} \det(A_{L,I})\overline{E}_L,
\]

where

\[
\det(A_{L,I}) = \det((a_{li}), \quad l \in L, \quad i \in I,
\]

is a \(p \times p\) minor of the \(n \times p\) matrix formed by the vectors \(\overline{\tau}_{i_1}, \ldots, \overline{\tau}_{i_p}\). The determinants \(\det(A_{L,I})\) are the so-called Grassmann or Plücker coordinates of the \(p\)-vector \(\overline{\tau}_{i_1} \wedge \ldots \wedge \overline{\tau}_{i_p}\).

Let \(I = \{i_1, \ldots, i_p\}\) and \(J = \{j_1, \ldots, j_q\}\) be increasing \((0 \leq p \leq n)\)- and \((0 \leq q \leq n)\)-lists, respectively. Then it is immediate that

\[
\overline{A}_I \wedge \overline{A}_J = \sum_{H \in C(n, p), K \in C(n, q)} \text{sign}(\tau_{H,K})\det(A_{H,I})\det(A_{K,J})\overline{E}_{H+K}.
\]

This product of a \(p\)-vector and a \(q\)-vector has a direct generalisation given below.
Chapter 1. Background and preliminaries

Lemma 1.16. Let $I_k = \{i_{k,1}, \ldots, i_{k,n_k}\}$, $k = 1, \ldots, m$, be $m$ increasing $(0 \leq n_k \leq n)$-lists. Then

$$\bigwedge_{k=1}^{m} A_{I_k} = \sum_{H_1 \in C(n,n_1), \ldots, H_m \in C(n,n_m) \atop H_i \cap H_j = \emptyset, i \neq j} \text{sign}(\tau_{H_1, \ldots, H_m}) \cdot \det (A_{H_1, I_1}) \cdots \det (A_{H_m, I_m}) \overline{E}_{H_1 \ast \ldots \ast H_m}.$$

1.5.2 Generalised minor expansion

Since the Grassmann coordinates are determinants, we may use them to prove determinant expansion formulas.

Lemma 1.17. Let $A = (a_{ij}) \in M_{n \times n}(R)$, where $R$ is a commutative ring. Let $I_k = \{i_{k,1}, \ldots, i_{k,n_k}\}$, $k = 1, \ldots, m$, be $m$ increasing $(0 \leq n_k \leq n)$-lists such that $\sum_{k=1}^{m} n_k = n$ and $I_j \cap I_k = \emptyset$ when $j \neq k$. Then

$$\det(A) = \text{sign}(\tau_{I_1, \ldots, I_m}) \cdot \sum_{H_1 \in C(n,n_1), \ldots, H_m \in C(n,n_m) \atop H_i \cap H_j = \emptyset, i \neq j} \text{sign}(\tau_{H_1, \ldots, H_m}) \cdot \det (A_{H_1, I_1}) \cdots \det (A_{H_m, I_m}).$$

Proof. First

$$\bigwedge_{k=1}^{m} (\overline{a}_{i_{k,1}} \wedge \ldots \wedge \overline{a}_{i_{k,n_k}}) = \bigwedge_{k=1}^{m} (A_{\overline{a}_{i_{k,1}} \wedge \ldots \wedge \overline{a}_{i_{k,n_k}}})$$

$$= \det(A) \cdot \bigwedge_{k=1}^{m} (\overline{a}_{i_{k,1}} \wedge \ldots \wedge \overline{a}_{i_{k,n_k}})$$

$$= \det(A) \cdot \text{sign}(\tau_{I_1, \ldots, I_m}) \cdot \overline{a}_1 \wedge \ldots \wedge \overline{a}_n.$$

On the other hand, Lemma 1.16 implies

$$\bigwedge_{k=1}^{m} (\overline{a}_{i_{k,1}} \wedge \ldots \wedge \overline{a}_{i_{k,n_k}})$$

$$= \sum_{H_1 \in C(n,n_1), \ldots, H_m \in C(n,n_m) \atop H_i \cap H_j = \emptyset, i \neq j} \text{sign}(\tau_{H_1, \ldots, H_m}) \cdot \det (A_{H_1, I_1}) \cdots \det (A_{H_m, I_m}) \overline{E}_{H_1 \ast \ldots \ast H_m}.$$
Since \( \sum_{k=1}^m n_k = n \), we have \( \prod_{k \leq 1} H_{1,k} \cdots H_{m,k} = v_1 \cdots v_n \) for all \( H_k \in C(n, n_k) \), \( k = 1, \ldots, m \), such that \( H_i \cap H_j = \emptyset \) when \( i \neq j \). Hence

\[
\det(A) \sign(\tau_{I_1, \ldots, I_m}) = \sum_{\substack{H_1 \in C(n, n_1), \ldots, H_m \in C(n, n_m) \\ H_i \cap H_j = \emptyset, i \neq j}} \sign(\tau_{H_1, \ldots, H_m}) \det(A_{H_1, I_1}) \cdots \det(A_{H_m, I_m}),
\]

and multiplication by \( \sign(\tau_{I_1, \ldots, I_m}) \) proves the claim. \( \Box \)

1.6 Number fields and valuations

Recommended sources for more information are, for example, the books of Bachman [1], Borevich and Safarevich [9], Koblitz [26], and Weiss [57].

Let \( K/k \) be a finite separable field extension of degree \( n \). There is an extension \( \Omega/k \) such that there exist precisely \( n \) isomorphisms \( \sigma_i : K \rightarrow \Omega \), \( i = 1, \ldots, n \), for which \( \sigma_i(x) = x \) for all \( x \in k \) and \( i = 1, \ldots, n \). (See [9, p. 404].) The norm of an element \( \alpha \in K \) is defined as

\[
N_{K/k}(\alpha) = \prod_{i=1}^n \sigma_i(\alpha).
\]

1.6.1 Extending valuations

Let \( K = \mathbb{Q}(\beta) \) be an algebraic number field of degree \( \kappa \) over \( \mathbb{Q} \). All the absolute values of \( K \) are extensions of the absolute values of \( \mathbb{Q} \). The theorem of Ostrowski states that the non-trivial absolute values of \( \mathbb{Q} \) are the Archimedean absolute value \( \cdot \| = \cdot \|_\infty \) and the \( p \)-adic absolute values \( \cdot \|_p \), \( p \in \mathbb{P} \) (with \( \| p \| = \frac{1}{p} \)).

The topological closure of \( \mathbb{Q} \) with respect to the metric \( \cdot \|_p \) with \( p \in \mathbb{P} \cup \{\infty\} \) will be denoted by \( \mathbb{Q}_p \) (so \( \mathbb{Q}_\infty = \mathbb{R} \)), and \( \mathbb{C}_p \) denotes the topological closure of the algebraic closure of \( \mathbb{Q}_p \) (so \( \mathbb{C}_\infty = \mathbb{C} \)). The minimal polynomial of \( \beta \) is \( M_\beta(x) \in \mathbb{Q}[x] \).

When \( p \in \mathbb{P} \cup \{\infty\} \), there are as many extensions of the absolute value \( \cdot \|_p \) as there are irreducible factors of \( M_\beta(x) \) in \( \mathbb{Q}_p[x] \). (See [11, Chapter V.2].) If \( \cdot \|_v \) extends the standard \( p \)-adic metric \( \cdot \|_p \) to \( K \), we write \( v|p \), and when extending the Archimedean absolute value \( \cdot \| = \cdot \|_\infty \), we write \( v|\infty \). The collection of non-Archimedean valuations of \( K \) is denoted by \( V_0 \), and the collection of Archimedean valuations of \( K \) by \( V_\infty \).
Chapter 1. Background and preliminaries

The topological closure of $\mathbb{K}$ with respect to the metric $| \cdot |_v$ is denoted by $\mathbb{K}_v$. We also denote $\kappa_v = [\mathbb{K}_v : \mathbb{Q}_p]$, so that $\sum_v \kappa_v = \kappa = [\mathbb{K} : \mathbb{Q}]$. There are $\kappa$ different monomorphisms $\sigma_i : \mathbb{K} \rightarrow \mathbb{C}_p$, $i = 1, \ldots, \kappa$; denote $\beta_i := \sigma_i(\beta)$.

Assume that $M_\beta(x) = f_1(x) \cdots f_r(x)$ in $\mathbb{Q}_p[x]$, $p \in \mathbb{P} \cup \{\infty\}$, where the factors $f_j(x)$ are irreducible in $\mathbb{Q}_p[x]$ and $\deg f_j(x) = \kappa_j$, $\sum_{j=1}^r \kappa_j = \kappa$. Then we may re-index the conjugates of beta:

$$\{\beta_i \mid i = 1, \ldots, \kappa\} = \{\beta_{jk} \mid j = 1, \ldots, r, k = 1, \ldots, \kappa_j\},$$

where $f_j(\beta_{jk}) = 0$ for $k = 1, \ldots, \kappa_j$. We know that there are $r$ extensions $v_1, \ldots, v_r$ of the $p$-adic valuation to $\mathbb{K}$, and these are defined by the field monomorphisms $\sigma_i$. The monomorphisms $\sigma_{jk}$ corresponding to the same irreducible factor $f_j(x)$ always define the same extension: if $\alpha \in \mathbb{K}$ and $j \in \{1, \ldots, r\}$, then

$$|\alpha|_{v_j} = \sqrt[p]{N_{\mathbb{Q}_p(\beta_{jk}) : \mathbb{Q}_p}(\sigma_{jk}(\alpha))},$$

for any $k \in \{1, \ldots, \kappa_j\}$.

1.6.2 Normalisation

It is convenient to normalise the absolute values $| \cdot |_v$ by setting

$$| \cdot |_{v_j} = | \cdot |_{v_j} = | \cdot |_{v_j}.$$

Because $\frac{v_j}{v} \leq 1$, the triangle inequality still holds for the normalised absolute value also in the case $v|\infty$.

1.6.3 The product formula

The next lemma will be a fundamental tool in Chapter 4.

Lemma 1.18. For any $\alpha \in \mathbb{K} \setminus \{0\}$ we have

$$\prod_v |\alpha|_v = 1,$$

where the product is taken over all normalised, pairwise non-equivalent valuations of $\mathbb{K}$.
Proof. This follows from the product formula for the valuations of \( \mathbb{Q} \):

\[
1 = \prod_{p \in \mathbb{P} \cup \{\infty\}} |N_{\mathbb{K} \mathbb{Q}}(\alpha)|_p \\
= \prod_{p \in \mathbb{P} \cup \{\infty\}} \left| \prod_{i=1}^{\kappa} \sigma_i(\alpha) \right|_p \\
= \prod_{p \in \mathbb{P} \cup \{\infty\}} \prod_{j=1}^{r} |N_{\mathbb{Q}_p(\sigma_j(\alpha)); \mathbb{Q}_p(\sigma_j(\alpha))}|_p \\
= \prod_{p \in \mathbb{P} \cup \{\infty\}} \prod_{v|p} |\alpha|^{\kappa_v}_v \\
= \prod_{v} |x|^{\kappa_v}_v \\
= \prod_{v} \|x\|_v^{\kappa_v}.
\]

\[ \Box \]

Note that if \( \alpha \in \mathbb{Q} \), then

\[ \prod_{v|p} \|\alpha\|_v = |\alpha|_p \]  \hspace{1cm} (1.14)

for any \( p \in \mathbb{P} \cup \{\infty\} \).

1.6.4 Global relations

Bombieri [6] was the first to introduce the concept of a global relation. Here we shall follow Chirskiĭ (see [13], for example).

**Definition 1.19.** Let \( P \in \mathbb{K}[x_1, \ldots, x_m] \) be a polynomial of \( m \) variables and suppose that \( F_1(t), \ldots, F_m(t) \in \mathbb{K}[[t]] \) are power series. Take a \( \xi \in \mathbb{K} \). A relation

\[ P(F_1(\xi), \ldots, F_m(\xi)) = 0 \]

is called global if it holds in all the fields \( \mathbb{K}_v \) where all the series \( F_j(\xi), j = 1, \ldots, m \), converge.
Chapter 2

A transcendence measure for $e$

The results of this chapter are joint work with A-M. Ernvall-Hytönen and T. Matala-aho and have appeared in *On Mahler’s transcendence measure for $e* [16]. The presentation here includes a corrected proof of Lemma 2.2.

2.1 Introduction

Let $m, H \geq 1$. Since the number $e$ is transcendental, we have $P(e) \neq 0$ for any non-zero polynomial $P(x) = \lambda_0 + \lambda_1 x + \ldots + \lambda_m x^m \in \mathbb{Z}[x]$ with $\max_{1 \leq i \leq m} \{|\lambda_i|\} \leq H$. It is then natural to ask, how small can the absolute value $|P(e)|$ actually be in terms of the degree $m$ of the polynomial $P$ and the size $H$ of its coefficients. Due to the method used in this chapter, the lower bound derived in the subsequent sections will not depend on the coefficient $\lambda_0$. Hence we may slightly modify Definition 1.1 and say (like Hata [22] does) that any function greater than or equal to $\omega(m, H)$ is a transcendence measure for $e$, when $\omega(m, H)$ is defined to be the infimum of the numbers $r > 0$ satisfying the estimate

$$|\lambda_0 + \lambda_1 e + \lambda_2 e^2 + \ldots + \lambda_m e^m| > \frac{1}{H^r}$$

for all $\overline{X} = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{Z}^{m+1} \setminus \{0\}$ with $\max_{1 \leq i \leq m} \{|\lambda_i|\} \leq H$.

The quest to obtain good transcendence measures for $e$ dates back to Borel [8]. He proved that $\omega(m, H)$ is smaller than $c \log \log H$ for some positive constant.
Chapter 2. A transcendence measure for $e$

c depending only on $m$. This was considerably improved by Popken [40, 41], who showed that $\omega(m, H) < m + \frac{c \log \log H}{\log \log H}$ for some positive constant $c$ depending on $m$. Soon afterwards, Mahler [29] was able to get the dependence on $m$ explicit:

$$\omega(m, H) < m + \frac{cm^2 \log (m + 1)}{\log \log H},$$

where $c$ is an absolute positive constant. The price he had to pay was that he was only able to prove the validity of the result in some subset of the set consisting of $m, H \in \mathbb{Z}_{\geq 1}$ with $H \geq 3$, unlike the results by Borel and Popken. Finally, in 1991, Khassa and Srinivasan [25] established that the constant can be chosen to be 98 in the set $m, H \in \mathbb{Z}_{\geq 1}$ with $\log \log H \geq d(m + 1)^{6m}$ for some absolute constant $d > e^{950}$. Soon after, in 1995, Hata [22] proved that the constant can be chosen to be 1 in the set of $m$ and $H$ with $\log H \geq \max \{(m!)^3 \log m, e^{24}\}$.

Hata’s observation was that there were big common factors hiding in the auxiliary numerical approximation forms. These numerical approximation forms are closely related to the classical Hermite-Padé approximations to the exponential function used already by Hermite. The impact of the common factors was utilised in an asymptotic manner resulting in Theorem 1.2 in [22]. Hata’s Theorem 1.2 is sharper than Theorem 1.1 in his paper, but it is only valid for $H$ in an asymptotic sense: no explicit lower bound is given, instead, the theorem is formulated for a large enough $H$.

The purpose of this chapter is to present and prove the following theorem which improves Hata’s bound for the function $\omega$ in his Theorem 1.1, and extends the set of values of $H$ for which the result is valid whenever $m \geq 5$. In addition, this result makes Hata’s Theorem 1.2 completely explicit, mainly due to our rigorous treatment of the common factors, giving rise to a more complicated behaviour visible in the term

$$\kappa_m := \frac{1}{m} \sum_{\substack{p \leq \frac{m+1}{2} \atop p \in \mathbb{P}}} \min_{0 \leq j \leq m} \left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right\} \frac{\log p}{p-1} w_p \left(s(m)e^{s(m)}\right),$$

(2.1)

where

$$w_n(x) := 1 - \frac{n}{x} - \frac{n-1}{\log n} \frac{\log x}{x}$$

for any $n \in \mathbb{Z}_{\geq 2}$, and $\mathbb{P}$ is the set of prime numbers.
Chapter 2. A transcendence measure for $e$

**Theorem 2.1.** Suppose that $\log H \geq s(m)e^{s(m)}$, where $s(2) = e$ and $s(m) = m(\log m)^2$ for $m \geq 3$. Then

\[
\omega(m, H) \leq \begin{cases} 
2 + \frac{4.93}{\log \log H}, & m = 2; \\
3 + \frac{6.49}{\log \log H}, & m = 3; \\
4 + \frac{15.7}{\log \log H}, & m = 4; \\
m + \left(1 - \frac{2\kappa_m}{(\log m)^2}\right) \left(1 - \frac{\kappa_m}{\log m}\right) \frac{m^2 \log m}{\log \log H}, & 5 \leq m \leq 14; \\
m + \left(1 - \frac{1+\kappa_m}{(\log m)^2}\right) \left(1 - \frac{\kappa_m}{\log m}\right) \frac{m^2 \log m}{\log \log H}, & m \geq 15.
\end{cases}
\tag{2.2}
\]

Asymptotically, we have

\[
\lim_{m \to \infty} \kappa_m = \kappa = \sum_{p \in \mathbb{P}} \frac{\log p}{p(p - 1)} = 0.75536661083 \ldots. \tag{2.3}
\]

The choice of the function $s(m)$ was made as an attempt to balance between the amount of technical details and the improvement of the function $\omega$ against the size of the set of the values of $H$.

Theorem 2.1 is a consequence of a more extensive result, Theorem 2.14 where we give a completely explicit transcendence measure for $e$, in terms of $m$ and $H$. The proof starts with Lemma 2.2 which gives a suitable criterion for studying lower bounds of linear forms in given numbers. Mainly due to the use of the function $z(y)$ introduced in Section 1.4.2, the functional dependence on $H$ will be improved compared to earlier considerations.

It should be noted that all our results are actually valid over an imaginary quadratic field $\mathbb{I}$.

### 2.2 Preliminaries, lemmas, and notation

Let $z : \left[-\frac{1}{e}, \infty[ \to \left[\frac{1}{e}, \infty[\right.$ denote the inverse function of the function $y(z) = z \log z$, $z \geq \frac{1}{e}$ (see Section 1.4.2). Throughout this chapter, let $\mathbb{I}$ denote an imaginary quadratic field or the field of rational numbers and $\mathbb{Z}_0$ its ring of integers.

Fix now $\Theta_1, \ldots, \Theta_m \in \mathbb{C} \setminus \{0\}$. Assume that we have a sequence of simultaneous linear forms

\[
L_{k,j}(n) = B_{k,0}(n)\Theta_j + B_{k,j}(n), \tag{2.4}
\]

$k = 0, 1, \ldots, m$, $j = 1, \ldots, m$, where the coefficients

\[
B_{k,j} = B_{k,j}(n) \in \mathbb{Z}_0, \quad k, j = 0, 1, \ldots, m,
\]
Chapter 2. A transcendence measure for $e$

satisfy the determinant condition

$$\Delta := \begin{vmatrix} B_{0,0} & B_{0,1} & \cdots & B_{0,m} \\ B_{1,0} & B_{1,1} & \cdots & B_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,0} & B_{m,1} & \cdots & B_{m,m} \end{vmatrix} \neq 0. \quad (2.5)$$

Further, let $a, b, c, d \in \mathbb{R}$, $a, c, d > 0$, $c \leq 1$, and suppose that

$$|B_{k,0}(n)| \leq Q(n) = e^{q(n)}, \quad (2.6)$$

$$\sum_{j=1}^{m} |L_{k,j}(n)| \leq R(n) = e^{-r(n)}, \quad (2.7)$$

where

$$q(n) = an \log n + bn, \quad (2.8)$$

$$-r(n) = -cn \log n + dn \quad (2.9)$$

for all $k \in \{0, 1, \ldots, m\}$. Let the above assumptions be valid for all $n \geq n_0$.

Further, we denote

$$B := b + \frac{ad}{c}, \quad C := a, \quad D := a + b + ae^{-s(m)}, \quad F^{-1} := 2e^D,$$

$$v := c - \frac{d}{s(m)}, \quad n_1 := \max \left\{ n_0, e, e^{s(m)} \right\}, \quad (2.10)$$

where $s(m)$ is a real-valued function of $m$ (later we set $s(m) = m(\log m)^2$).

**Lemma 2.2.** Let $m \geq 2$ and $\log H \geq n_1 \log n_1$. Assume that $v > 0$. Then, under the above assumptions $(2.5)$–$(2.9)$, the bound

$$|\lambda_0 + \lambda_1 \Theta_1 + \ldots + \lambda_m \Theta_m| > F(2H)^{-\frac{a}{v} - \epsilon(H)},$$

$$\epsilon(H) \log(2H) = Bz \left( \frac{\log(2H)}{v} \right) + C \log \left( z \left( \frac{\log(2H)}{v} \right) \right),$$

holds for all $\mathbf{x} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{Z}_1^{m+1} \setminus \{0\}$ with $\max_{1 \leq j \leq m} \{|\lambda_j|\} \leq H$.

**Proof.** We use the notation

$$\Lambda := \lambda_0 + \lambda_1 \Theta_1 + \ldots + \lambda_m \Theta_m, \quad \lambda_j \in \mathbb{Z}_1,$$
Chapter 2. A transcendence measure for \( e \)

for the linear form to be estimated. Using our simultaneous linear forms

\[ L_{k,j}(n) = B_{k,0}(n) \Theta_j + B_{k,j}(n) \]

from (2.4), we get

\[ B_{k,0}(n) \Lambda = W_k + \lambda_1 L_{k,1}(n) + \ldots + \lambda_m L_{k,m}(n), \tag{2.11} \]

where

\[ W_k = W_k(n) = B_{k,0}(n) \lambda_0 - \lambda_1 B_{k,1}(n) - \ldots - \lambda_m B_{k,m}(n) \in \mathbb{Z}. \tag{2.12} \]

If now \( W_k(n) \neq 0 \), then by (2.11) and (2.12) we get

\[
1 \leq |W_k(n)| = |B_{k,0}(n)\Lambda - (\lambda_1 L_{k,1} + \ldots + \lambda_m L_{k,m})| \\
\leq |B_{k,0}(n)||\Lambda| + \sum_{j=1}^{m} |\lambda_j||L_{k,j}(n)| \\
\leq Q(n)|\Lambda| + HR(n).
\]

Define

\[ n_2 := \max \left\{ n \in \mathbb{Z}_{\geq 1} \mid \frac{1}{2} \leq HR(n) \right\}. \]

Consequently, \( HR(n_2 + 1) < \frac{1}{2} \), from which it follows

\[ \log(2H) - c(n_2 + 1) \log(n_2 + 1) + d(n_2 + 1) < 0. \]

Assuming that \( c \leq 1 \) and \( d > 0 \), we get

\[ n_1 \log n_1 \leq \log H < \log(2H) + d(n_2 + 1) < c(n_2 + 1) \log(n_2 + 1) \leq (n_2 + 1) \log(n_2 + 1), \]

implying that \( n_2 + 1 > n_1 \) (Lemma 1.10), so

\[ n_2 \geq n_1 \geq e^{s(m)}. \tag{2.13} \]
Chapter 2. A transcendence measure for $e$

According to the non-vanishing of the determinant (2.5) and the assumption $\lambda \neq 0$, it follows by linear algebra that $W_k(n_2+1) \neq 0$ for some integer $k \in [0, m]$. Hence we get the estimate

$$1 < 2|\Lambda|Q(n_2 + 1)$$

(2.14)

for our linear form $\Lambda$, where we need to write $Q(n_2 + 1)$ in terms of $2H$.

Since $\frac{1}{2} \leq HR(n_2)$, we have

$$\log(2H) \geq r(n_2) = cn_2 \log n_2 - dn_2 = n_2 \log n_2 \left( c - \frac{d}{\log n_2} \right).$$

(2.15)

By (2.13) we have $\log n_2 \geq s(m)$. Thus

$$\log(2H) \geq \left( c - \frac{d}{s(m)} \right) n_2 \log n_2 = vn_2 \log n_2,$$

or equivalently,

$$n_2 \log n_2 \leq \frac{\log(2H)}{v}.$$

Estimate (2.13) further implies

$$\frac{\log(2H)}{v} \geq n_2 \log n_2 \geq s(m)e^{s(m)}.$$

Then, by the properties of the function $z(y)$ given in Lemma 1.10 we get

$$n_2 \leq z \left( \frac{\log(2H)}{v} \right).$$

(2.16)

Now we are ready to estimate $Q(n_2 + 1) = e^{q(n_2 + 1)}$ as follows:

$$q(n_2 + 1) = a(n_2 + 1) \log(n_2 + 1) + b(n_2 + 1)$$

$$< a(n_2 + 1) \left( \log n_2 + \frac{1}{n_2} \right) + b(n_2 + 1)$$

$$= an_2 \log n_2 + a \log n_2 + bn_2 + a + b + \frac{a}{n_2}.$$

(2.17)

By (2.15) we get

$$n_2 \log n_2 \leq \frac{1}{c} \left( \log(2H) + d n_2 \right).$$

(2.18)
Substituting (2.18) into (2.17) gives

\[ q(n_2 + 1) \leq \frac{a}{c} \left( \log(2H) + dn_2 \right) + a \log n_2 + bn_2 + a + b + \frac{a}{n_2} \]

\[ \leq \frac{a}{c} \log(2H) + \left( b + \frac{ad}{c} \right) n_2 + a \log n_2 + a + b + ae^{-s(m)}, \]

where we applied estimate (2.13).

Hence

\[ Q(n_2 + 1) \leq \exp \left( \frac{a}{c} \log(2H) + \left( b + \frac{ad}{c} \right) n_2 + a \log n_2 + a + b + ae^{-s(m)} \right) \]

\[ = (2H)^{\frac{a}{c}e^{Bn_2 + C \log n_2 + D}}, \]

where \( B, C \) and \( D \) are precisely as in the formulation of Lemma 2.2. The claim now follows from (2.14) and (2.16).

**Corollary 2.3.** If \( c \leq 1 + \frac{d}{s(m)}, \log H \geq n_1 \log n_1, v > 0 \), and \( u := 1 + \frac{\log(s(m))}{s(m)} \), then

\[ |\lambda_0 + \lambda_1 \Theta_1 + \ldots + \lambda_m \Theta_m| \geq \frac{v^C}{2eD u^C} \left( \log(2H) \right)^C \cdot (2H)^{-\frac{u}{v} - \frac{B}{v \log \log(2H)}}. \]

**Proof.** Since \( c \leq 1 + \frac{d}{s(m)}, \) Lemma 1.11 gives

\[ z \left( \frac{\log(2H)}{c - \frac{d}{s(m)}} \right) \leq \left( 1 + \frac{\log(s(m))}{s(m)} \right) \frac{\log(2H)}{\log \left( \frac{\log(2H)}{c - \frac{d}{s(m)}} \right)} \leq \frac{u}{v} \cdot \frac{\log(2H)}{\log \log(2H)}, \] (2.19)

when we denote \( u := 1 + \frac{\log(s(m))}{s(m)} \). Lemma 2.2 now implies

\[ |\lambda_0 + \lambda_1 \Theta_1 + \ldots + \lambda_m \Theta_m| \geq \frac{1}{2eD} (2H)^{-\frac{u}{v} - \frac{Bz(\log(2H))}{\log(2H)}} - C \log \left( \frac{\log(2H)}{v} \right) \]

\[ \geq \frac{1}{2eD} (2H)^{-\frac{u}{v} - \frac{Bz(\log(2H))}{\log(2H)}} \left( \frac{u}{v} \cdot \frac{\log(2H)}{\log \log(2H)} \right)^{-C} \]

\[ = \frac{v^C}{2eD u^C} \left( \frac{\log(2H)}{\log(2H)} \right)^C \cdot (2H)^{-\frac{u}{v} - \frac{B}{v \log \log(2H)}}. \]
Chapter 2. A transcendence measure for $e$

2.3 Hermite-Padé approximations to the exponential function

Hermite-Padé approximations to the exponential function date back to Hermite’s proof of the transcendence of $e$ [23]; see also [56]. In the following, the notation introduced in Section 1.4.1 will be used.

**Theorem 2.4.** Let $\alpha_0 = 0, \alpha_1, \ldots, \alpha_m$ be $m + 1$ distinct complex numbers, $\vec{l} = (l_0, l_1, \ldots, l_m)^T \in \mathbb{Z}_{\geq 1}^{m+1}$, and $L_0 = l_0 + l_1 + \ldots + l_m$. Put

$$A_{\vec{l}, 0}(t, \vec{\alpha}) = \sum_{i=0}^{L_0} t^{L_0-i} i! \sigma_i(\vec{l}, \vec{\alpha}). \quad (2.20)$$

Then there exist polynomials $A_{\vec{l}, j}(t, \vec{\alpha})$ and remainders $R_{\vec{l}, j}(t, \vec{\alpha})$ such that

$$e^{\alpha_j t} A_{\vec{l}, 0}(t, \vec{\alpha}) - A_{\vec{l}, j}(t, \vec{\alpha}) = R_{\vec{l}, j}(t, \vec{\alpha}), \quad (2.21)$$

where

$$\begin{cases} \deg_i A_{\vec{l}, j}(t, \vec{\alpha}) \leq L_0 - l_j, & j = 0, 1, \ldots, m; \\ L_0 + 1 \leq \text{ord}_{\vec{l}} R_{\vec{l}, j}(t, \vec{\alpha}) < \infty, & j = 1, \ldots, m. \end{cases}$$

**Proof.** First we have

$$A_{\vec{l}, 0}(t, \vec{\alpha}) = \sum_{i=0}^{L_0} t^{L_0-i} i! \sigma_i(\vec{l}, \vec{\alpha}) = \sum_{i=0}^{L_0} t^{L_0-i} i! \sigma_i(\vec{l}, \vec{\alpha}) = t^{L_0+1} \sum_{i=0}^{L_0} \frac{i! \sigma_i(\vec{l}, \vec{\alpha})}{t^{i+1}},$$

since $\sigma_i(\vec{l}, \vec{\alpha}) = 0$ for $0 \leq i < l_0$. Using Laplace transform, this can be written as

$$t^{L_0+1} \sum_{i=0}^{L_0} \frac{i! \sigma_i(\vec{l}, \vec{\alpha})}{t^{i+1}} = t^{L_0+1} \sum_{i=0}^{L_0} \mathcal{L} \left( \sigma_i(\vec{l}, \vec{\alpha}) x^i \right)(t) \quad (2.22)$$

Then

$$e^{\alpha t} A_{\vec{l}, 0}(t, \vec{\alpha}) = t^{L_0+1} \int_0^\infty e^{(\alpha - x)t} \Omega(x, \vec{\alpha}) dx$$

$$= t^{L_0+1} \int_0^\infty e^{-yt} \Omega(y + \alpha, \vec{\alpha}) dy + t^{L_0+1} \int_0^\infty e^{(\alpha - x)t} \Omega(x, \vec{\alpha}) dx.$$
We have
\[
\Omega(x, \overline{\alpha}) = \prod_{j=0}^{m} (\alpha_j - x)^{l_j},
\]
and consequently
\[
\Omega(y + \alpha, \overline{\alpha}) = \prod_{j=0}^{m} (\alpha_j - \alpha - y)^{l_j} = \Omega(y, (\alpha_0 - \alpha, \ldots, \alpha_m - \alpha)^{T}). \tag{2.23}
\]

By setting \(\alpha = \alpha_j, j = 1, \ldots, m\), we get the approximation formula
\[
e^{\alpha_j t} A_{i,j,0}(t, \overline{\alpha}) - A_{i,j}(t, \overline{\alpha}) = R_{i,j}(t, \overline{\alpha}),
\]
where
\[
A_{i,j}(t, \overline{\alpha}) = A_{i,j,0}(t, (\alpha_0 - \alpha_j, \ldots, \alpha_m - \alpha_j)^{T})
= t^{L_0 + 1} \int_{0}^{\infty} e^{-yt} \Omega(y + \alpha_j, \overline{\alpha}) dy \tag{2.24}
\]
and
\[
R_{i,j}(t, \overline{\alpha}) = t^{L_0 + 1} \int_{0}^{\alpha_j} e^{(\alpha_j - x)t} \Omega(x, \overline{\alpha}) dx, \quad j = 1, \ldots, m.
\]

Going backwards in (2.22) with (2.23) in mind, we see that
\[
A_{i,j,0}(t, \overline{\alpha}) = t^{L_0 + 1} \int_{0}^{\alpha_j} e^{(\alpha_j - x)t} \Omega(x, \overline{\alpha}) dx
= t^{L_0 + 1} \sum_{i=0}^{L_0} \mathcal{L} \left( \sigma_i (l_i, (\alpha_0 - \alpha_j, \ldots, \alpha_m - \alpha_j)^T) y^i \right) (t)
= \sum_{i=l_j}^{L_0} t^{L_0 - i} \sigma_i (l_i, (\alpha_0 - \alpha_j, \ldots, \alpha_m - \alpha_j)^T) .
\]

Note that the coordinate \(\alpha_j - \alpha_j = 0\) corresponds to \(\beta_0 = 0\) in the notation of Section 1.4.1 and therefore we now have \(l_j\) in the place of \(l_0\) in the expansion of \(\sigma_i\) [1.7]. Hence \(\sigma_i (l_i, (\alpha_0 - \alpha_j, \ldots, \alpha_m - \alpha_j)^T) = 0\) for \(0 \leq i < l_j\), and \(\deg_t A_{i,j}(t, \overline{\alpha}) = L_0 - l_j\). In addition, \(\text{ord}_{t=0} R_{i,j}(t, \overline{\alpha}) \geq L_0 + 1\) for \(j = 1, \ldots, m\), since the function
\[
t \mapsto \int_{0}^{\alpha_j} e^{(\alpha_j - x)t} \Omega(x, \overline{\alpha}) dx
\]
is analytic at the origin. \(\square\)

**Lemma 2.5.** We have \(\frac{1}{l_j} A_{i,j,0}(t, \overline{\alpha}) \in \mathbb{Z}[t, \alpha_1, \ldots, \alpha_m]\) for all \(j = 0, 1, \ldots, m\).
Proof. In the case \( j = 0 \), we have
\[
\frac{1}{l_0!} A_{l,0}(t, \alpha) = \sum_{i=l_0}^{L_0} t^{L_0-i} \sigma_i(\bar{l}, \alpha) \frac{i!}{l_0!}
\]
by (2.20). The claim clearly holds due to the definition of \( \sigma_i \).

Next write
\[
A_{l,0}(t, \alpha)e^{\alpha_j t} = \sum_{N=0}^{\infty} r_N t^N,
\]
where
\[
r_N = \sum_{N=h+n}^{\infty} \sigma_{L_0-h}(\bar{l}, \alpha) \frac{(L_0-h)!}{n!} \alpha_j^n.
\]
By (2.21) it is sufficient to show that \( \frac{r_N}{l_j!} \in \mathbb{Z}[\alpha_1, \ldots, \alpha_m] \) for \( N = 0, \ldots, L_0 - l_j \).

By (2.25) we have
\[
\frac{1}{l_j!} r_N = \sum_{N=h+n}^{\infty} \sigma_{L_0-h}(\bar{l}, \alpha) \frac{(L_0-h)!}{l_j! n!} \alpha_j^n,
\]
where \( h + n = N \leq L_0 - l_j \) implies \( l_j + n \leq L_0 - h \), thus giving the result.

\[ \square \]

## 2.4 Determinant

In order to fulfil the determinant condition (2.5) we choose
\[
l^{(k)} = (l, l, \ldots, l - 1, \ldots, l)^T, \quad k = 0, 1, \ldots, m;
\]
i.e., \( l_i = l \) for \( i = 0, 1, \ldots, k-1, k+1, \ldots, m \), and \( l_k = l-1 \). Now \( L_0 = (m+1)(l-1) \).

Then we write
\[
\begin{cases}
A_{k,j}^*(t) := A_{l^{(k)},j}(t, \alpha), & j = 0, 1, \ldots, m; \\
R_{k,j}^*(t) := R_{l^{(k)},j}(t, \alpha), & j = 1, \ldots, m,
\end{cases}
\]
for all \( k = 0, 1, \ldots, m \).

The non-vanishing of the determinant \( \Delta \) follows from the next well-known lemma (see for example Mahler [32, p. 232] or Waldschmidt [56, p. 53]).
Lemma 2.6. There exists a constant \( c \neq 0 \) such that

\[
\Delta(t) = \begin{vmatrix}
A_{0,0}^*(t) & A_{0,1}^*(t) & \ldots & A_{0,m}^*(t) \\
A_{1,0}^*(t) & A_{1,1}^*(t) & \ldots & A_{1,m}^*(t) \\
\vdots & \vdots & \ddots & \vdots \\
A_{m,0}^*(t) & A_{m,1}^*(t) & \ldots & A_{m,m}^*(t)
\end{vmatrix} = ct^{m(m+1)t}.
\]

Proof. According to Theorem 2.4 and the equations in (2.27), the degrees of the entries of the matrix defining \( \Delta \) are

\[
\begin{pmatrix}
ml & ml - 1 & \ldots & ml - 1 \\
ml - 1 & ml & \ldots & ml - 1 \\
\vdots & \vdots & \ddots & \vdots \\
ml - 1 & ml - 1 & \ldots & ml
\end{pmatrix}_{(m+1)\times(m+1)}.
\]

We see that \( \deg \Delta(t) = (m + 1)ml \) and the leading coefficient \( c \) is a product of the leading coefficients of \( A_{0,0}^*(t), A_{1,1}^*(t), \ldots, A_{m,m}^*(t) \), which are non-zero.

On the other hand, column operations yield

\[
\Delta(t) = \begin{vmatrix}
A_{0,0}^*(t) & -R_{0,1}^*(t) & \ldots & -R_{0,m}^*(t) \\
A_{1,0}^*(t) & -R_{1,1}^*(t) & \ldots & -R_{1,m}^*(t) \\
\vdots & \vdots & \ddots & \vdots \\
A_{m,0}^*(t) & -R_{m,1}^*(t) & \ldots & -R_{m,m}^*(t)
\end{vmatrix}
\]

as \( R_{k,j}^*(t) = e^{\alpha_j t}A_{k,0}^*(t) - A_{k,j}^*(t) \). By Theorem 2.4, the order of each element in columns 1, \ldots, m is at least \( L_0 + 1 = (m + 1)l \). Therefore \( \text{ord}_{t=0} \Delta(t) \geq m(m + 1)l \).

\[\square\]

2.5 Common factors

From now on we set \( \alpha_j = j \) for \( j = 0, 1, \ldots, m \) and denote

\[
B_{k,j}^*(t) := \frac{1}{(l - 1)!} A_{k,j}^*(t)
\]

for \( j = 0, 1, \ldots, m, k = 0, 1, \ldots, m \) and

\[
L_{k,j}^*(t) := \frac{1}{(l - 1)!} R_{k,j}^*(t)
\]

35
Chapter 2. A transcendence measure for $e$

for $j = 1, \ldots, m$, $k = 0, 1, \ldots, m$. Then, by Theorem 2.4, we have a system of linear forms

$$B^*_{k,0}(t)e^{\alpha_j t} + B^*_{k,j}(t) = L^*_{k,j}(t), \quad j = 1, \ldots, m; \quad k = 0, 1, \ldots, m,$$  \hspace{1cm} (2.28)

where

$$B^*_{k,j}(t) = \frac{t^{L_0+1}}{(l-1)!} \int_0^\infty e^{-yt}(0 - j - y)^l(1 - j - y)^l \cdots (k - j - y)^{l-1} \cdots (m - j - y)^l dy$$  \hspace{1cm} (2.29)

for all $j, k = 0, 1, \ldots, m$, and

$$L^*_{k,j}(t) = \frac{t^{L_0+1}}{(l-1)!} \int_0^j e^{(j-x)t}(0 - x)^l(1 - x)^l \cdots (k - x)^{l-1} \cdots (m - x)^l dx$$  \hspace{1cm} (2.30)

for all $j = 1, \ldots, m$, $k = 0, 1, \ldots, m$.

Further, by Lemma 2.5 it holds $B^*_{k,j}(t) \in \mathbb{Z}[t]$ for all $j = 0, 1, \ldots, m$, $k = 0, 1, \ldots, m$. Next we try to find a common factor from the integer coefficients of the new polynomials $B^*_{k,j}(t)$.

Let $m \in \mathbb{Z}_{\geq 1}$ in this section. Recall the definition of the $p$-adic valuation $v_p$ from Section 1.4.3.

**Theorem 2.7.** For $k = 0, 1, \ldots, m$, we have

$$\left( \prod_{p \leq m} p^{\left\lfloor \frac{m}{p^k} \right\rfloor v_p(t) - v_p(t)} \right)^{-1} \cdot B^*_{k,0}(t) \in \mathbb{Z}[t].$$

**Proof.** Let us start by writing the polynomial $B^*_{k,0}(t)$ from (2.29) in a different way, using the representation (2.20):

$$B^*_{k,0}(t) = \frac{1}{(l-1)!} A^{p^k}_{l^0} (t, (0, 1, \ldots, m)^T)$$

$$= \sum_{i=0}^{L_0} t^{L_0-i} \sigma_i \left( \bar{l}^{(k)}, (0, 1, \ldots, m)^T \right) \cdot \frac{i!}{(l-1)!}$$

$$= \sum_{r=0}^{L_0-r} t^r \sigma_{L_0-r} \left( \bar{l}^{(k)}, (0, 1, \ldots, m)^T \right) \cdot \frac{(L_0-r)!}{(l-1)!}.$$
where $L_0 = (m+1)l - 1$ and, by (1.7),

$$\sigma_{L_0-r}\left(l^{(k)}, (0, 1, \ldots, m)^T\right)$$

$$= (-1)^{L_0-r} \sum_{h_1+\ldots+h_m=r} \frac{l_1!}{(l_1-h_1)!h_1!} \cdots \frac{l_m!}{(l_m-h_m)!h_m!} \cdot 1^{h_1}2^{h_2} \cdots m^{h_m}.$$  

So

$$B_{k,0}^r(t) = \sum_{r=0}^{L_0-l_0} t^r (-1)^{L_0-r} \sum_{h_1+\ldots+h_m=r} \frac{(L_0-r)!}{(l_1-h_1)! \cdots (l_m-h_m)!} \cdot \frac{l_1!}{h_1!} \cdots \frac{l_m!}{h_m!} \cdot 1^{h_1} \cdots m^{h_m}.$$  

Here

$$\frac{(L_0-r)!}{(l_1-h_1)! \cdots (l_m-h_m)!} \in \mathbb{Z}$$

because

$$(l-1)+(l_1-h_1)+\ldots+(l_m-h_m) \leq l_0+l_1+\ldots+l_m-(h_1+\ldots+h_m) = L_0-r.$$  

So, we may expect some common factors from the terms

$$\frac{l_1!}{h_1!} \cdots \frac{l_m!}{h_m!} \cdot 1^{h_1}2^{h_2} \cdots m^{h_m}.$$  

Let $p \leq m$ be a prime number. Now, using estimate (1.11),

$$v_p\left(\frac{l_1!}{h_1!} \cdot \frac{l_2!}{h_2!} \cdots \frac{l_m!}{h_m!} \cdot 1^{h_1}2^{h_2} \cdots m^{h_m}\right)$$

$$= \sum_{i=1}^{m} (v_p(l_i!) - v_p(h_i!)) + \sum_{i=1}^{m} h_iv_p(i)$$

$$\geq \sum_{i=1}^{m} \left( v_p(l_i!) + h_i \left( v_p(i) - \frac{1}{p-1} \right) \right)$$ 

$$\geq \begin{cases} \left( \left[ \frac{m}{p} \right] - 1 \right) v_p(l!) + v_p((l-1)!), & k \in \{1, \ldots, m\}; \\ v_p(l!), & k = 0 \end{cases}$$

$$\geq \left( \left[ \frac{m}{p} \right] - 1 \right) v_p(l!) + v_p((l-1)!).$$  

37
Recall from (2.26) that $l_k = l - 1$ while $l_j = l$ for $j \neq k$. Since $v_p(l!) = v_p(l) + v_p((l - 1)!)$, the result (2.31) can be written as

$$v_p \left( \frac{l_1!}{h_1!} \cdot \frac{l_2!}{h_2!} \cdots \frac{l_m!}{h_m!} \cdot 1^{h_1}2^{h_2} \cdots m^{h_m} \right) \geq \left\lfloor \frac{m}{p} \right\rfloor v_p(l!) - v_p(l).$$

Hence, there is a factor

$$p^{\left\lfloor \frac{m}{p} \right\rfloor v_p(l!)-v_p(l)} \left( \frac{l!}{h_1!} \cdot \frac{l!}{h_2!} \cdots \frac{l!}{h_m!} \cdot 1^{h_1}2^{h_2} \cdots m^{h_m} \right),$$

which is a common divisor of all the coefficients of $B_{k,0}^*(t)$. The proof is complete. \hfill \Box

Now we need to find a common factor dividing all $B_{k,j}^*(t)$.

**Theorem 2.8.** Assume $j \in \{1, \ldots, m\}$. Then there exists a positive integer

$$D_{m,l} := \prod_{p \leq \frac{m+1}{p} \in \mathbb{F}} p^{v_p}$$

with

$$v_p \geq \left( \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right) v_p((l - 1)!),$$

satisfying

$$D_{m,l}^{-1} \cdot B_{k,j}^*(t) \in \mathbb{Z}[t]$$

for all $k = 0, 1, \ldots, m$.

**Proof.** From our assumption $\alpha_i = i, i = 1, \ldots, m$, and equations (2.24) and (2.20) it follows

$$B_{k,j}^*(t, \bar{\alpha}) = \frac{1}{(l-1)!} A_{k,0}^{(k)}(t,(0-j,1-j,\ldots,m-j)^T)$$

$$= \sum_{r=0}^{L_0-l_j} t^r \sigma_{L_0-r} \left( (l^{(k)},(0-j,1-j,\ldots,m-j)^T) \left( \frac{(L_0-r)!}{(l-1)!} \right)^{r} - \right).$$
where, by (1.7),

\[
\sigma_{L_0-r} \left( l^{(k)}; (0 - j, 1 - j, \ldots, m - j)^T \right) \\
= (-1)^{L_0-r} \sum_{h_0 + \ldots + h_{j-1} + h_{j+1} \ldots + h_m = r} \frac{l_0! \ldots l_{j-1}!}{(l_0 - h_0)! \ldots (l_{j-1} - h_{j-1})! h_{j+1}! \ldots h_m!} \\
\cdot \frac{l_{j+1}! \ldots l_m!}{(l_{j+1} - h_{j+1})! \ldots (l_m - h_m)!}
\]

\[
= \sum_{r=0}^{L_0-l_j} t^r \left( -1 \right)^{L_0-r} \sum_{h_0 + \ldots + h_{j-1} + h_{j+1} \ldots + h_m = r} \frac{(L_0 - r)!}{(l_0 - h_0)! \ldots (l_{j-1} - h_{j-1})! \ldots (l_m - h_m)!} \\
\cdot \frac{l_0! \ldots l_{j-1}! l_{j+1}! \ldots l_m!}{h_0! \ldots h_{j-1}! h_{j+1}! \ldots h_m!} \\
\cdot (0 - j)^{h_0} (1 - j)^{h_1} \ldots (-1)^{h_{j-1}} 1^{h_{j+1}} \ldots (m - j)^{h_m}.
\]

So

\[
B^*_{k,j}(t) = \sum_{r=0}^{L_0-l_j} t^r (-1)^{L_0-r} \sum_{h_0 + \ldots + h_{j-1} + h_{j+1} \ldots + h_m = r} \frac{(L_0 - r)!}{(l_0 - h_0)! \ldots (l_{j-1} - h_{j-1})! \ldots (l_m - h_m)!} \\
\cdot \frac{l_0! \ldots l_{j-1}! l_{j+1}! \ldots l_m!}{h_0! \ldots h_{j-1}! h_{j+1}! \ldots h_m!} \\
\cdot (0 - j)^{h_0} (1 - j)^{h_1} \ldots (-1)^{h_{j-1}} 1^{h_{j+1}} \ldots (m - j)^{h_m}.
\]

As before, we may expect some common factors from the terms

\[
T_j := \frac{l_0! \ldots l_{j-1}! l_{j+1}! \ldots l_m!}{h_0! \ldots h_{j-1}! h_{j+1}! \ldots h_m!} \\
\cdot (0 - j)^{h_0} (1 - j)^{h_1} \ldots (-1)^{h_{j-1}} 1^{h_{j+1}} \ldots (m - j)^{h_m}.
\]

Let \( p \leq \frac{m+1}{2} \). With considerations similar to those in (2.31), we get

\[
v_p(T_j) \geq \begin{cases} 
\left( \left\lfloor \frac{j}{p} \right\rfloor - 1 \right) v_p(\ell!) + v_p((l-1)!) + \left\lfloor \frac{m-j}{p} \right\rfloor v_p(\ell!), & k \in \{0, 1, \ldots, j-1\}; \\
\left\lfloor \frac{j}{p} \right\rfloor v_p(\ell!) + \left( \left\lfloor \frac{m-j}{p} \right\rfloor - 1 \right) v_p(\ell!) + v_p((l-1)!), & k \in \{j+1, \ldots, m-j\}; \\
\left\lfloor \frac{j}{p} \right\rfloor v_p(\ell!) + \left\lfloor \frac{m-j}{p} \right\rfloor v_p(\ell!), & k = j
\end{cases}
\]

\[
\geq \left( \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right) v_p(\ell!) - v_p(\ell)
\]

\[
\geq \left( \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right) v_p((l-1)!).
\]
which completes the proof.

Combining Theorems 2.7 and 2.8 gives us the complete result:

**Corollary 2.9.** For all $k, j = 0, 1, \ldots, m$ we have

$$D_{m,l}^{-1} \cdot B_{k,j}^e(t) \in \mathbb{Z}[t].$$

**Theorem 2.10.** Let $l \geq s(m)e^{s(m)}$. Then the common factor $D_{m,l}$ satisfies the bound

$$D_{m,l} \geq e^{\kappa_{m}m},$$

where

$$\kappa_m := \frac{1}{m} \sum_{p \leq \frac{m+1}{2}} \min_{0 \leq j \leq m} \left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lceil \frac{m-j}{p} \right\rceil \right\} \frac{\log p}{p-1} w_p\left(s(m)e^{s(m)}\right)$$

and

$$w_p(x) := 1 - \frac{p}{x} - \frac{p-1}{\log p} \cdot \frac{\log x}{x}.$$  

Further,

$$\kappa_m \geq \frac{w_{\frac{m+1}{2}}\left(s(m)e^{s(m)}\right)}{m} \sum_{p \leq \frac{m+1}{2}} \left( \left\lfloor \frac{m+1}{p} \right\rfloor - 1 \right) \frac{\log p}{p-1},$$

and asymptotically we have

$$\lim_{m \to \infty} \kappa_m = \kappa = \sum_{p \in \mathbb{P}} \frac{\log p}{p(p-1)} = 0.75536661083\ldots$$

**Proof.** We begin with the estimate of Theorem 2.8

$$\nu_p \geq \left( \left\lfloor \frac{j}{p} \right\rfloor + \left\lceil \frac{m-j}{p} \right\rceil \right) v_p((l-1)!).$$

Then

$$\prod_{p \leq \frac{m+1}{2}} \nu_p^{v_p} \geq \exp \left( \sum_{p \leq \frac{m+1}{2}} \left( \left\lfloor \frac{j}{p} \right\rfloor + \left\lceil \frac{m-j}{p} \right\rceil \right) v_p((l-1)! \log p) \right).$$
Chapter 2. A transcendence measure for $e$

Next we use property (1.11) and the assumption $l \geq s(m)e^{s(m)}$ in order to estimate $v_p((l-1)! \log p)$:

$$v_p((l-1)! \log p) \geq \left( \frac{l-1}{p-1} - \frac{\log(l-1)}{\log p} - 1 \right) \log p$$

$$\geq l \cdot \frac{\log p}{p-1} \left( 1 - \frac{p}{s(m)e^{s(m)}} - \frac{p-1}{\log p} \frac{\log(s(m)e^{s(m)})}{s(m)e^{s(m)}} \right).$$

 Altogether,

$$\prod_{p \leq \frac{m+1}{2}} p^{\nu_p} \geq e^{\kappa_m m l},$$

where

$$\kappa_m := \frac{1}{m} \sum_{p \leq \frac{m+1}{2}} \min_{0 \leq j \leq m} \left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right\} \log p$$

$$\cdot \left( 1 - \frac{p}{s(m)e^{s(m)}} - \frac{p-1}{\log p} \frac{\log(s(m)e^{s(m)})}{s(m)e^{s(m)}} \right),$$

proving estimate (2.32).

Let us move to the bound (2.34). Denote

$$w_n(x) := 1 - \frac{n}{x} - \frac{n-1}{\log n} \cdot \frac{\log x}{x}, \quad n \geq 2.$$ 

For a fixed $x \in \mathbb{R}_{>1}$, we have

$$w_y(x) > w_z(x)$$

when $2 \leq y < z$. This can be seen by differentiating the function $w_y(x)$:

$$\frac{\partial}{\partial y} w_y(x) = -\frac{1}{x} - \frac{1}{\log y} \frac{\log x}{x} + \frac{y-1}{y(\log y)^2} \frac{\log x}{x}$$

$$= -\frac{1}{x} - \frac{\log x}{x \log y} \left( 1 - \left( 1 - \frac{1}{y} \right) \frac{1}{\log y} \right) < 0$$

since $\log y > 1 - \frac{1}{y}$ when $y \geq 2$. Next write

$$m + 1 = hp + \bar{m}, \quad j = lp + \bar{j}, \quad h, l, \bar{m}, \bar{j}, \in \mathbb{Z}_{\geq 0}, \quad 0 \leq \bar{m}, \bar{j} \leq p - 1.$$ (2.37)
Chapter 2. A transcendence measure for \(e\)

Then

\[
\left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor = \left\lfloor \frac{l+j}{p} \right\rfloor + \left\lfloor \frac{m-lp-j}{p} \right\rfloor \\
= l + \left\lfloor \frac{m+1}{p} - l - 1 + \frac{p-1-j}{p} \right\rfloor \\
\geq \left\lfloor \frac{m+1}{p} \right\rfloor - 1 \\
= h - 1.
\]

Thus, the bound

\[
\min_{0 \leq j \leq m} \left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right\} \geq \left\lfloor \frac{m+1}{p} \right\rfloor - 1 \geq \frac{m}{p} - 2
\]

together with (2.36) verifies the estimate

\[
\kappa_m \geq w_{\frac{m+1}{2}} \left( s(m) e^{s(m)} \right) \frac{1}{m} \sum_{p \leq \frac{m+1}{2}} \left( \left\lfloor \frac{m+1}{p} \right\rfloor - 1 \right) \log \frac{p}{p-1} \\
\geq w_{\frac{m+1}{2}} \left( s(m) e^{s(m)} \right) \sum_{p \leq \frac{m+1}{2}} \left( 1 - \frac{2p}{m} \right) \log \frac{p}{p(p-1)}.
\]

By restricting the sum to primes \(p \leq \sqrt{m}\), we get

\[
\kappa_m \geq w_{\frac{m+1}{2}} \left( s(m) e^{s(m)} \right) \left( 1 - \frac{2}{\sqrt{m}} \right) \sum_{p \leq \sqrt{m}} \frac{\log p}{p(p-1)} \xrightarrow{m \to \infty} \sum_{p \in B} \frac{\log p}{p(p-1)}.
\]

On the other hand,

\[
\kappa_m = \frac{1}{m} \sum_{p \leq \frac{m+1}{2}} \min_{0 \leq j \leq m} \left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right\} \frac{\log p}{p-1} w_p \left( s(m) e^{s(m)} \right) \\
\leq \frac{1}{m} \sum_{p \leq \frac{m+1}{2}} \left\lfloor \frac{m}{p} \right\rfloor \frac{\log p}{p-1} \\
\leq \sum_{p \leq \frac{m+1}{2}} \frac{\log p}{p(p-1)} \xrightarrow{m \to \infty} \sum_{p \in B} \frac{\log p}{p(p-1)}.
\]

This proves the asymptotic behaviour (2.35). As for the numerical value in (2.35), see the sequence \[A138312\] in [50].
Chapter 2. A transcendence measure for \( e \)

With \( s(m) = m(\log m)^2 \), for instance (2.33) gives

\[
\kappa_m \geq \begin{cases} 
0, & m = 2; \\
0.215544, & m = 3; \\
0.173121, & m = 4; \\
0.387118, & m = 5; \\
0.322600, & m = 6; \\
0.375535, & m = 7; \\
0.397256, & m = 8; \\
0.474840, & m = 9; \\
0.427356, & m = 10; \\
0.501455, & m = 11; \\
0.459667, & m = 12; \\
0.502575, & m = 13; \\
0.534653, & m = 14.
\end{cases}
\] (2.39)

Note that to simplify numerical computations for large \( m \), estimate (2.34) is already rather sharp, where in addition the factor \( w_{m+\frac{1}{2}} (s(m)e^{s(m)}) \) is very close to 1.

**Lemma 2.11.** It holds that \( \kappa_m \geq 0.5 \) for all \( m \geq 13 \).

**Proof.** By (2.37) and (2.38) we get

\[
\frac{1}{m} \left( \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right) \geq h - 1 \geq \frac{1}{p} \left( 1 - \frac{2p-2}{m} \right).
\]

Let us suppose, for example, that \( 1 - \frac{2p-2}{m} \geq \frac{9}{10} \) which is equivalent to \( p \leq \frac{m}{20} + 1 \). Then

\[
\kappa_m \geq \frac{9}{10} w_{m+\frac{1}{2}} (s(m)e^{s(m)}) \sum_{p \leq \frac{m}{20} + 1} \frac{\log p}{p(p-1)}.
\] (2.40)

Now

\[
w_{m+\frac{1}{2}} (s(m)e^{s(m)}) = 1 - \frac{m+1}{2} + \frac{\frac{m+1}{2} - 1}{\log \left( \frac{m+1}{2} \right)} \cdot \log \left( m(\log m)^2 e^{m(\log m)^2} \right)
\]

\[
> 1 - 10^{-666}
\]

43
Chapter 2. A transcendence measure for $e$

when $m \geq 80$. In (2.40) we have an increasing lower bound for $\kappa_m$, and therefore

$$\kappa_m \geq \frac{9}{10} w_{80+1} \left( s(80) e^{s(80)} \right) \sum_{p \leq 80+1} \frac{\log p}{p(p-1)}$$

$$> \frac{9}{10} \cdot (1 - 10^{-666}) \cdot \sum_{p \leq 5} \frac{\log p}{p(p-1)}$$

$$\geq 0.549133$$

when $m \geq 80$. As for $13 \leq m \leq 79$, the estimate $\kappa_m \geq 0.5$ is quickly verified by computer using (2.34).

2.6 Numerical linear forms

By extracting the common factor $D_{m,l}$ from the linear forms (2.28) we are led to the numerical linear forms

$$B_{k,0} e^j + B_{k,j} = L_{k,j}, \quad j = 1, \ldots, m; \quad k = 0, 1, \ldots, m; \quad (2.41)$$

where

$$B_{k,j} := \frac{1}{D_{m,l}} B_{k,j}(1), \quad j = 0, 1, \ldots, m; \quad k = 0, 1, \ldots, m;$$

are integers and

$$L_{k,j} := \frac{1}{D_{m,l}} L_{k,j}(1), \quad j = 1, \ldots, m; \quad k = 0, 1, \ldots, m.$$

Note the dependence $B_{k,j} = B_{k,j}(l)$ and $L_{k,j} = L_{k,j}(l)$.

According to (2.26), now $L_0 = (m + 1)l - 1$. We have $s(m) = m(log m)^2$ for $m \geq 3$, $s(2) = e$. Because of the condition (2.13) we have the assumption $l \geq e^{s(m)}$.

The following two lemmas give the necessary estimates for the coefficients $B_{k,0}$ and the remainders $L_{k,j}$ of the linear forms (2.41). Note that from Stirling’s formula (Lemma 1.12) it follows that

$$\frac{1}{(l-1)!} \leq \exp \left( -l \log l + l(\log l - \log(l-1)) + l - 1 + \frac{1}{2} \log(l-1) - \log \sqrt{2\pi} \right). \quad (2.42)$$

44
Lemma 2.12. Let \( l \geq e^{m(\log m)^2} \) when \( m \geq 3 \), and \( l \geq e^e \) when \( m = 2 \). Then

\[
|B_{k,0}| \leq \exp(ml \log l + l((m + 1) \log(m + 1) - (1 + \kappa_m)m + 0.0000525));
\]

for \( k = 0, 1, \ldots, m \) and \( m \geq 5 \). When \( m = 2, 3, 4 \), we have the bounds

\[
\begin{align*}
|B_{k,0}| &\leq \exp(2l \log l + 1.6791l), \quad \text{when } m = 2; \\
|B_{k,0}| &\leq \exp(3l \log l + 2.1016l), \quad \text{when } m = 3; \\
|B_{k,0}| &\leq \exp(4l \log l + 3.3612l), \quad \text{when } m = 4.
\end{align*}
\]

(2.43)

Proof. The structure of the proof is the following: First we treat the term \( B_{k,0}^*(1) \) by using the earlier integral representation to obtain a bound. After that, the common divisor will be factored out, yielding the bound \( Q(l) \) (2.6) for \( |B_{k,0}| \).

By (2.29) we have

\[
B_{k,0}^*(1) = \frac{1}{(l-1)!} \int_0^\infty e^{-x} \frac{\prod_{j=0}^{m} (j - x)^l}{k - x} \, dx.
\]

Let us split the integral into the following parts:

\[
\int_0^\infty e^{-x} \frac{\prod_{j=0}^{m} (j - x)^l}{k - x} \, dx = \left( \int_0^m + \int_m^{2(m+1)l} + \int_{2(m+1)l}^\infty \right) e^{-x} \frac{\prod_{j=0}^{m} (j - x)^l}{k - x} \, dx.
\]

When \( x \geq m \), it holds

\[
\left| \frac{\prod_{j=0}^{m} (j - x)^l}{k - x} \right| = x^l(x - 1)^l \cdots (x - k)^l \cdots (x - m)^l \leq x^{(m+1)l-1} \leq x^{(m+1)l},
\]

so

\[
\left| \left( \int_m^{2(m+1)l} + \int_{2(m+1)l}^\infty \right) e^{-x} \frac{\prod_{j=0}^{m} (j - x)^l}{k - x} \, dx \right| \leq \left( \int_m^{2(m+1)l} + \int_{2(m+1)l}^\infty \right) e^{-x} x^{(m+1)l} \, dx.
\]

Write

\[
f(x) := e^{-x} x^{(m+1)l}.
\]
The derivative
\[ f'(x) = (-x + (m+1)l)e^{-x}x^{(m+1)l-1} \]
has a unique zero at \( x = (m+1)l \). The function \( f(x) \) is increasing for \( x \leq (m+1)l \) and decreasing for \( x \geq (m+1)l \), and obtains its maximum at \( x = (m+1)l \). Thus we can bound the second integral as follows:
\[
\int_{m}^{2(m+1)l} e^{-x}x^{(m+1)l} \, dx \leq 2(m+1)l e^{-(m+1)l}((m+1)l)^{(m+1)l}.
\]

When \( x \geq 2(m+1)l \), the function \( f(x) \) is decreasing. The aim is to find an upper bound for the third integral using a geometric sum. Let us first write
\[
\int_{2(m+1)l}^{\infty} e^{-x}x^{(m+1)l} \, dx \leq \sum_{h=0}^{\infty} e^{-2(m+1)l-h}(2(m+1)l+h)^{(m+1)l}.
\]
Notice that
\[
\frac{f(x+1)}{f(x)} = e^{-1} \left( 1 + \frac{1}{x} \right)^{(m+1)l} \leq e^{-1/2}
\]
when \( x \geq 2(m+1)l \). It follows that
\[
f(x+h) \leq \left( e^{-\frac{1}{2}} \right)^h f(x).
\]
Hence
\[
\sum_{h=0}^{\infty} e^{-2(m+1)l-h}(2(m+1)l+h)^{(m+1)l} \leq \frac{e^{-2(m+1)l}(2(m+1)l)^{(m+1)l}}{(1 - e^{-1/2})} < 2.55e^{-2(m+1)l}(2(m+1)l)^{(m+1)l}.
\]

Finally, we have to estimate the first integral. We have
\[
\max_{n \leq x \leq n+1} \prod_{j=0}^{m} |j - x|^{l-1} \leq \max_{0 \leq x \leq 1} \prod_{j=0}^{m} |j - x|^{l-1}
\]
for \( 0 \leq n \leq m - 1 \). Now
\[
\left| \frac{\prod_{j=0}^{m} (j - x)}{k - x} \right| \leq m! \max_{0 \leq x \leq 1} \prod_{j=0}^{m} |j - x|^{l-1}.
\]
Chapter 2. A transcendence measure for $e$

Hence

$$\left| \int_0^m e^{-x} \frac{\prod_{j=0}^m (j - x)}{k - x} \, dx \right| \leq \frac{m!(ml)^{l-1}}{(5!)^{l-1}} \int_0^m e^{-x} \max_{0 \leq x \leq 1} \prod_{j=0}^5 |j - x|^{l-1} \, dx \leq \frac{(ml)!16.91^{l-1}}{120^{l-1}} \quad (2.44)$$

when $m \geq 5$. When $m < 5$, we can estimate

$$\left| \int_0^m e^{-x} \frac{\prod_{j=0}^m (j - x)}{k - x} \, dx \right| \leq \begin{cases} \frac{2^{l+1}}{3^{m(l-1)/2}}, & \text{when } m = 2; \\ 6, & \text{when } m = 3; \\ 24 \cdot 3.632^{l-1}, & \text{when } m = 4. \end{cases}$$

We may conclude that

$$|B_{k,0}^*(1)| \leq \frac{(ml)!16.91^{l-1}}{(l-1)!120^{l-1}} + \frac{2(m+1)l((m+1)l)^{(m+1)l}}{(l-1)!e^{(m+1)l}} \right.$$  

$$+ \frac{2.55e^{-2(m+1)l}(2(m+1)l)^{(m+1)l}}{(l-1)!}$$  

$$\leq \frac{6(m+1)l}{(l-1)!}e^{-(m+1)l((m+1)l)^{(m+1)l}}$$  

$$\leq \exp \left( ml \log l + l((m+1) \log (m+1) - m + \log l - \log (l-1)) \right.$$  

$$+ \log l + \frac{1}{2} \log (l-1) + \log (m+1) + \log (6 - 1 - \log 2\pi) \right).$$

Next we take into account the common factor $D_{m,l} \geq e^{\kappa_m ml}$. Remember that $B_{k,0}$ will be the number obtained when $B_{k,0}^*(1)$ is divided by the common factor. Now

$$|B_{k,0}| \leq \exp \left( ml \log l + l((m+1) \log (m+1) - (1 + \kappa_m)m \right.$$  

$$+ \log l - \log (l-1)) + \log l + \frac{1}{2} \log (l-1) \quad (2.45)$$  

$$+ \log (m+1) + \log (6 - 1 - \log 2\pi) \right).$$

Since $m \geq 5$ and $l \geq e^{m(\log m)^2} \geq e^{5(\log 5)^2}$, we have

$$\log l - \log (l-1) \leq 0.000002373 \quad (2.46)$$
Chapter 2. A transcendence measure for $e$

and

$$\frac{\log l}{l} + \frac{\log(l - 1)}{2l} + \frac{\log(m + 1)}{l} + \frac{\log 6 - 1 - \log \sqrt{2\pi}}{l} \leq 0.00005005. \quad (2.47)$$

At last, estimate (2.45) with the bounds in (2.46) and (2.47) yields

$$|B_{k,0}| \leq \exp(ml \log l + (m + 1) \log(m + 1) - (1 + \kappa_m)m + 0.0000525)).$$

For the small values of $m$, we have

$$\kappa_2 \geq 0, \quad \kappa_3 \geq 0.215544, \quad \kappa_4 \geq 0.173121$$

by (2.39). Thus, when $m = 2$, we have $|B_{k,0}| \geq |B^*_k(1)|$ and $l \geq \lceil e^e \rceil = 16$.

Now

$$|B_{k,0}| \leq \frac{1}{(l - 1)!} \left(2^{l+1}3^{-3(l-1)/2} + 2 \cdot 3 \cdot 6l - 3l^3 + 2.55 \cdot e^{-6l / (6l)} \right)$$

$$\leq \frac{3 \cdot 2 \cdot 3! \cdot e^{-3l / (6l)} \cdot 3l}{(l - 1)!}$$

$$\leq \exp \left( - l \log l + l(\log l - \log(l - 1)) + l - 1 + \frac{1}{2} \log(l - 1) 
- \log \sqrt{2\pi} + \log 18 + \log l - 3l + 3l \log 3 + 3l \log l \right)$$

$$= \exp \left( 2l \log l + l \left(3 \log 3 - 2 + \log \frac{l}{l - 1} + \log \frac{l}{l} 
+ \frac{\log(l - 1)}{2l} + \frac{\log 18 \sqrt{2\pi} - 1}{l} \right) \right)$$

$$\leq \exp (2l \log l + 1.6791l).$$

When $m = 3$, we have $l \geq \lceil e^{3(\log 3)^2} \rceil = 38$, and we need to divide $|B^*_k(1)|$ by 48.
Chapter 2. A transcendence measure for $e$

the common factor to get the correct bound for the term $|B_{k,0}|$. Hence

$$|B_{k,0}| \leq \frac{e^{-0.215544-3l}}{(l-1)!} \left( 6 + 8l e^{-4l}(4l)^{4l} + 2.55e^{-2-4l}(8l)^{4l} \right)$$

$$\leq \frac{3e^{-0.215544-3l}}{(l-1)!} \cdot 8l e^{-4l}(4l)^{4l}$$

$$\leq \exp \left( -l \log l + l(\log l - \log(l-1)) + l - 1 + \frac{1}{2} \log(l-1) \\
- \log \sqrt{2\pi} - 3 \cdot 0.215544l + \log 24 + \log l - 4l + 4l \log 4 + 4l \log l \right)$$

$$= \exp \left( 3l \log l + l \left( 4 \log 4 - 3 - 3 \cdot 0.215544 + \log \frac{l}{l-1} \\
+ \frac{\log l}{l} + \frac{\log(l-1)}{2l} + \frac{\log \frac{24}{\sqrt{2\pi}} - 1}{l} \right) \right)$$

$$\leq \exp \left( 3l \log l + 2.1016l \right).$$

Similarly, when $m = 4$, it holds $l \geq \left[ e^{4(\log 4)^2} \right] = 2181$, and we get

$$|B_{k,0}| \leq \frac{e^{-0.173121-4l}}{(l-1)!} \left( 2 \cdot 5le^{-5l}(5l)^{5l} + 2.55e^{-2-5l}(2 \cdot 5l)^{5l} + 24 \cdot 3.632^{l-1} \right)$$

$$\leq \frac{3e^{-0.173121-4l}}{(l-1)!} \cdot 10le^{-5l}(5l)^{5l}$$

$$\leq \exp \left( -l \log l + l(\log l - \log(l-1)) + l - 1 \\
+ \frac{1}{2} \log(l-1) - \log \sqrt{2\pi} - 4 \cdot 0.173121l \\
+ \log 3 + \log 10 + \log l - 5l + 5l \log 5 + 5l \log l \right)$$

$$= \exp \left( 4l \log l + l \left( 5 \log 5 - 4 - 4 \cdot 0.173121 + \log \frac{l}{l-1} \\
+ \frac{\log l}{l} + \frac{\log(l-1)}{2l} + \frac{\log \frac{30}{\sqrt{2\pi}} - 1}{l} \right) \right)$$

$$\leq \exp(-l \log l + 3.3612l).$$

49
Chapter 2. A transcendence measure for $e$

(In all three cases, the coefficient of $l$ is a decreasing function in $l$.)

\( \square \)

**Lemma 2.13.** Let \( l \geq e^{m(\log m)^2} \) when \( m \geq 3 \), and \( l \geq e^e \) when \( m = 2 \). Then
\[
\sum_{j=1}^{m} |L_{k,j}| \leq \exp \left( -l \log l + l \left( \left( m + \frac{1}{2} \right) \log m - (\kappa_m + 1)m - 0.02394 \right) \right)
\]
for \( j = 1, \ldots, m \), \( k = 0, 1, \ldots, m \), and \( m \geq 5 \). When \( m = 2, 3, 4 \), we have the bounds
\[
\begin{align*}
\sum_{j=1}^{2} |L_{k,j}| &\leq \exp \left( -l \log l + 0.3654 l \right), \quad \text{when } m=2; \\
\sum_{j=1}^{3} |L_{k,j}| &\leq \exp \left( -l \log l + 0.5139 l \right), \quad \text{when } m=3; \\
\sum_{j=1}^{4} |L_{k,j}| &\leq \exp \left( -l \log l + 1.6016 l \right), \quad \text{when } m=4.
\end{align*}
\] (2.48)

**Proof.** We proceed as in the proof of Lemma 2.12: first we bound the terms \( L_{k,j}^*(1) \), then sum them, and finally factor out the common divisor to obtain the bound \( R(l) \) (2.7). According to equation (2.30), we have the representation
\[
L_{k,j}^*(t) = \frac{t^{L_0+1}}{(l-1)!} \int_0^t e^{(j-x)t}(0-x)^l(1-x)^l \cdots (k-x)^{l-1} \cdots (m-x)^l \, dx.
\]
The expression \( |x(1-x) \cdots (m-x)| \) attains its maximum in the interval \([0, m]\) for the first time when \( 0 < x < 1 \), so
\[
\max_{0 < x < m} |x(1-x) \cdots (m-x)| \leq \frac{m!}{5!} \max_{0 < x < 1} \prod_{j=0}^{5} |j - x| = \frac{m! 16.91}{120}.
\]

Much like in (2.44), it follows that
\[
|L_{k,j}^*(1)| \leq \frac{e^j}{(l-1)!} \int_0^j e^{-x} \frac{\prod_{r=0}^{m} |r - x|^l}{|k - x|} \, dx \leq \frac{(ml)^l 16.91^{l-1}}{120^{l-1}(l-1)!} \left( e^j - 1 \right)
\]
when \( m \geq 5 \). Using the estimate
\[
\sum_{j=1}^{m} (e^j - 1) < e^m \frac{e}{e - 1}
\]
and summing together the terms \( L_{j,k}^*(1) \), we get
\[
\sum_{j=1}^{m} |L_{j,k}^*(1)| < \frac{1.582 e^m (ml)^l 16.91^{l-1}}{120^{l-1}(l-1)!}.
\]
Again we divide by the common factor $D_{m,l}$. Thus the new values $L_{k,j}$ satisfy:

$$
\sum_{j=1}^{m} |L_{k,j}| < \frac{1.582(m!)^l 16.91^{l-1}}{120^{l-1}(l-1)!} \cdot e^{m - \kappa_m m l} \\
< \exp \left( - \left( l - \frac{1}{2} \right) \log(l-1) + l - \log \sqrt{2\pi} \right) \\
+ l \left( \left( m + \frac{1}{2} \right) \log m - m + \log \sqrt{2\pi} + \frac{1}{12m} + \frac{m}{l} \\
- \kappa_m m + \log 16.91 - \log 120 \right) \\
+ \log 120 - \log 16.91 + \log 1.582 \right).
$$

Now

$$
\frac{m}{l} + \log(l) - \log(l-1) + \frac{\log(l-1)}{2l} + 1 - \frac{\log(\sqrt{2\pi})}{l} \leq 1.00003,
$$

$$\log(\sqrt{2\pi}) + \frac{1}{12m} + \log 16.91 - \log 120 \leq -1.02398,$$

and

$$\frac{\log 120 - \log 16.91 + \log 1.582}{l} \leq 5.73802 \cdot 10^{-6}$$

because $m \geq 5$ and $l \geq e^{5(\log 5)^2}$. Together these estimates yield

$$
\sum_{j=1}^{m} |L_{k,j}| \leq \exp \left( -l \log l + l \left( \left( m + \frac{1}{2} \right) \log m - (\kappa_m + 1)m - 0.02394 \right) \right).
$$

When $m = 2, 3, 4$, we can bound the terms $L_{k,j}$ in the following way:

$$
\frac{(l-1)!}{e^j} |L_{k,j}^*| \leq \int_0^j e^{-x} \prod_{r=0}^{m} \frac{|r - x|^l}{|k - x|} \, dx \leq \begin{cases} 
\frac{2^{l+1}}{3^{3(l-1)/2}}, & \text{when } m = 2; \\
6, & \text{when } m = 3; \\
24 \cdot 3.632^{l-1}, & \text{when } m = 4.
\end{cases}
$$

We may now move to estimating the sums $\sum_{j=1}^{m} |L_{k,j}|$ for small $m$. When $m = 2,$
we have $l \geq 16$ and

$$
\sum_{j=1}^{2} |L_{k,j}| \leq \sum_{j=1}^{2} |L^*_{k,j}|
$$

$$
\leq \frac{1}{(l-1)!} \cdot \frac{2^{l+1}}{3^{3(l-1)/2}} (e + e^2)
$$

$$
\leq \exp \left(-l \log l + l(\log l - \log(l-1)) + l - 1 + \frac{1}{2} \log(l-1) + \log \sqrt{2\pi} + \log (e + e^2) + (l + 1) \log 2 - \frac{3(l-1)}{2} \log 3 \right)
$$

$$
= \exp \left(-l \log l + l \left( \log 2 - \frac{3}{2} \log 3 + 1 + \log \frac{l}{l-1}
\right.
\left. + \log \frac{(l-1)}{2l} + \log \frac{e + e^2}{l} + \frac{2}{\sqrt{2\pi}} + \frac{3}{2} \log 3 - 1 \right)\right)
$$

$$
\leq \exp (-l \log l + 0.3654l).
$$

When $m = 3$, we have $l \geq 38$ and we need to divide by the common factor. Then

$$
\sum_{j=1}^{3} |L_{k,j}| \leq \frac{e^{-0.215544\cdot3l}}{(l-1)!} \cdot 6 (e + e^2 + e^3)
$$

$$
\leq \exp \left(-l \log l + l(\log l - \log(l-1)) + l - 1 + \frac{1}{2} \log(l-1)
\right.
\left. - \log \sqrt{2\pi} - 3 \cdot 0.215544l + \log (e + e^2 + e^3) \right)
$$

$$
\leq \exp \left(-l \log l + l \left( 1 - 3 \cdot 0.215544 + \log \frac{l}{l-1} + \frac{\log(l-1)}{2l}
\right.
\left. + \frac{\log(e + e^2 + e^3)}{l} + \log \frac{6}{\sqrt{2\pi}} - 1 \right)\right)
$$

$$
\leq \exp (-l \log l + 0.5139l).
$$

When $m = 4$, we have $l \geq 2181$ and again we divide by the common factor.
Thus
\[
\sum_{j=1}^{m} |L_{k,j}| \leq e^{-0.173121\cdot 4l} \cdot (l - 1)! \cdot 24 \cdot 3.6^{l-1} \left( e + e^2 + e^3 + e^4 \right) \\
\leq \exp \left( -l \log l + l(\log l - \log(l-1)) + l - 1 + \frac{1}{2} \log(l-1) \\
- \log \sqrt{2\pi} - 4 \cdot 0.173121l + \log 24 + (l - 1) \log 3.632 \\
+ \log (e + e^2 + e^3 + e^4) \right) \\
= \exp \left( -l \log l + l \left( 1 - 4 \cdot 0.173121 + \log 3.632 + \log \frac{l}{l-1} \\
+ \frac{\log(l-1)}{2l} + \frac{\log(e + e^2 + e^3 + e^4) + \log \frac{24}{3.632\sqrt{2\pi}} - 1}{l} \right) \right) \\
\leq \exp(-l \log l + 1.6016l).
\]
(Again, in all three cases, the coefficient of \(l\) is a decreasing function in \(l\).) \(\Box\)

### 2.7 Measure

First we shall apply Lemma 2.2 to derive a general lower bound for the expression \(\lambda_0 + \lambda_1 e + \ldots + \lambda_m e^m\) before moving to the proof of Theorem 2.1. The determinant condition (2.5) is certainly satisfied by Lemma 2.6 and (2.41). According to Lemmas 2.12 and 2.13 of the previous section, we have

\[
|B_{k,0}(l)| \leq Q(l) = e^{q(l)} \\
\sum_{j=1}^{m} |L_{k,j}| \leq R(l) = e^{-r(l)},
\]

where

\[
q(l) = ml \log l + l((m+1) \log(m+1) - (1 + \kappa_m)m + 0.0000525),
\]

\[
- r(l) = -l \log l + l \left( \left( m + \frac{1}{2} \right) \log m - (1 + \kappa_m)m - 0.02394 \right),
\]

(2.49)

(2.50)
for all $k, j = 0, 1, \ldots, m$, $m \geq 5$. Comparison of formulas (2.49) and (2.50) to (2.8) and (2.9) yields

$$
\begin{align*}
    a &= m; \\
    b &= (m + 1) \log(m + 1) - (1 + \kappa_m)m + \delta, \quad \delta = 0.0000525; \\
    c &= 1; \\
    d &= (m + \frac{1}{2}) \log m - (1 + \kappa_m)m - 0.02394.
\end{align*}
$$

Now, with $s(m) = m(\log m)^2$, the formulas in (2.10) give

$$
\begin{align*}
    B &= b + \frac{ad}{c} \\
    &= m^2 \log m - (1 + \kappa_m)m^2 + (m + 1) \log(m + 1) \\
    &\quad + \frac{1}{2} m \log m - (1.02394 + \kappa_m + \delta) m + \delta; \\
    C &= a = m; \\
    D &= a + b + ae^{-s(m)} = (m + 1) \log(m + 1) - \kappa_m m + \delta + \frac{m}{e^{m(\log m)^2}}
\end{align*}
$$

for all $m \geq 5$. Recall also the shorthand notation

$$
    u = 1 + \frac{\log(s(m))}{s(m)}, \quad v = 1 - \frac{d}{s(m)}.
$$

For the small values $m = 2, 3, 4$, we compare equations (2.43) and (2.48) to equations (2.8) and (2.9). Again $a = m$ and $c = 1$, and moreover

$$
\begin{align*}
    \begin{cases}
    b = 1.6791; & \text{when } m = 2; \\
    d = 0.3654 \\
    \end{cases} \\
    \begin{cases}
    b = 2.1016; & \text{when } m = 3; \\
    d = 0.5139 \\
    \end{cases} \\
    \begin{cases}
    b = 3.3612; & \text{when } m = 4. \\
    d = 1.6016
    \end{cases}
\end{align*}
$$

(2.51)
Hence, with \( s(2) = e \) and \( s(m) = m(\log m)^2 \) for \( m = 3, 4 \), we get

\[
\begin{align*}
B &= 2.4099; \\
C &= 2, \quad \text{when } m = 2; \\
D &= 3.8111 \\
B &= 3.6433; \\
C &= 3; \quad \text{when } m = 3; \\
D &= 5.1819 \\
B &= 9.7676; \\
C &= 4; \quad \text{when } m = 4. \\
D &= 7.3631
\end{align*}
\] (2.52)

**Theorem 2.14.** Let \( m \geq 2 \) and \( \log H \geq s(m)e^{s(m)} \). With the above notation, the bound

\[
|\lambda_0 + \lambda_1 e + \ldots + \lambda_m e^m| > \frac{1}{2e^D} (2H)^{-m-\epsilon(H)}, \tag{2.53}
\]

where

\[
\epsilon(H) \log(2H) = B z \left( \frac{\log(2H)}{1 - \frac{d}{s(m)}} \right) + m \log \left( z \left( \frac{\log(2H)}{1 - \frac{d}{s(m)}} \right) \right),
\]
holds for all \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1} \setminus \{0\} \) with \( \max_{1 \leq j \leq m} |\lambda_j| \leq H \).

**Proof.** The values above have been achieved with the choice \( \Theta_j = e^j \). Combining them with Lemma 2.2 leads straight to the result (2.53).

**Corollary 2.15.** With the assumptions of Theorem 2.14, we have

\[
|\lambda_0 + \lambda_1 e + \ldots + \lambda_m e^m| \geq \frac{1}{2e^D} \left( \frac{s(m) - d}{s(m) + \log(s(m))} \cdot \frac{\log(2H)}{\log(2H)} \right)^m (2H)^{-m - \frac{\hat{B}}{\log(2H)}},
\]

where

\[
\hat{B} := \left( 1 + \frac{\log(s(m))}{s(m)} \right) \left( 1 - \frac{d}{s(m)} \right)^{-1} B.
\]

**Proof.** Follows likewise by plugging the above values into Corollary 2.3.

The more neat-looking estimate (2.2) still requires some work.
Chapter 2. A transcendence measure for \( e \)

**Proof of Theorem 2.1.** Let us first consider the case with \( m \geq 5 \). According to Theorem 2.14, we have

\[
1 < |\Lambda| \cdot 2e^D \cdot (2H)^{m+\epsilon(H)} = |\Lambda|H^{m+Y},
\]

where

\[
Y := \frac{1}{\log H} (\epsilon(H) \log(2H) + D + (m + 1) \log 2)
\]

\[
= \frac{1}{\log H} \left( Bz \left( \frac{\log(2H)}{1 - \frac{d}{s(m)}} \right) + m \log \left( z \left( \frac{\log(2H)}{1 - \frac{d}{s(m)}} \right) \right) + D + (m + 1) \log 2 \right) \quad (2.54)
\]

\[
\leq \frac{1}{\log H} \left( \frac{uB}{v} \frac{\log(2H)}{\log(2H)} + m \log \left( \frac{u}{v} \frac{\log(2H)}{\log(2H)} \right) + D + (m + 1) \log 2 \right)
\]

by estimate (2.19). By recalling our assumption \( \log H \geq s(m)e^s(m) \), it is obvious from the expression (2.54) that the terms corresponding to the parameters \( C \) and \( D \) contribute much less than the term corresponding to the parameter \( B \). The first task is to bound the smaller terms in such a way that they only slightly increase the constant term in the expression for the parameter \( B \). Let us start with the terms \( D \) and \( (m + 1) \log 2 \). We have

\[
D + (m + 1) \log 2
= (m + 1) \log(m + 1) - \kappa_m m + \delta + \frac{m}{e^m(\log m)^2} + (m + 1) \log 2
= (m + 1) \log(m + 1) + m \left( \frac{\delta}{m} + \frac{1}{e^m(\log m)^2} + \log 2 + \frac{\log 2}{m} - \kappa_m \right)
\leq (m + 1) \log(m + 1) + \frac{1}{2} m.
\]

Since \( v \log \log(2H) \geq 1 \), we may estimate

\[
m \log \left( \frac{u \log(2H)}{v \log \log(2H)} \right) \leq m \log(u \log(2H)).
\]
Hence, estimate (2.54) becomes

\[
Y \leq \frac{1}{\log H} \left( \frac{uB \log(2H)}{v \log \log(2H)} + m \log \left( (u \log(2H)) + (m + 1) \log(m + 1) + \frac{m}{2} \right) \right)
\]

\[
\leq \frac{1}{\log H} \left( \frac{uB \log(2H)}{v \log \log(2H)} + m \log \left( 2 \log H \right) + (m + 1) \log(m + 1) + \frac{m}{2} \right)
\]

\[
\leq \frac{1}{\log H} \left( \frac{uB \log(2H)}{v \log \log(2H)} \right) + \frac{5}{4} \frac{m \log(2 \log H)}{\log H}
\]

\[
= \frac{u}{v \log \log H} \left( B + \frac{1}{\log H} \left( \log(2) \log(2) + \frac{5vm(m \log(2H) \log(2 \log H))}{4u} \right) \right).
\]

When \( m = 5 \), the above formulation gives

\[
Y \leq \frac{u}{v \log \log H} (B + 0.0002069).
\]

When \( m \geq 6 \), we proceed as follows. Notice now that roughly estimating we have

\[
B \leq m^2 \log m - \kappa_m m^2
\]

because

\[
\log(m + 1) - (1.02394 + \kappa_m) m + \delta \leq 0
\]

and

\[
-m^2 + m \log(m + 1) + \frac{1}{2} m \log m \leq 0.
\]

Furthermore, \( \kappa_m \geq 0.32 \) when \( m \geq 6 \). Since \( 0 < v \leq 1 \leq u \), we have now derived the inequality

\[
Y \leq \frac{u}{v \log \log H} \left( B + \frac{m^2}{\log H} \left( \log(2) \log m - \kappa_m \log(2) \right) + \frac{5vm(m \log(2 \log H))}{4m} \right)
\]

\[
\leq \frac{u}{v \log \log H} (B + 10^{-6}).
\]
Chapter 2. A transcendence measure for $e$

Let us take a closer look at the expression

$$f(m) := \frac{u(B+10^{-6})}{vm^2 \log m}$$

$$= \left( 1 + \frac{1}{m \log m} + \frac{2 \log \log m}{m(m \log m)^2} \right) \frac{1 - \frac{1+\kappa_m}{\log m} + \frac{(m+1) \log (m+1)}{m^2 \log m}}{1 - \frac{1}{\log m} - \frac{1}{2m \log m} + \frac{1+\kappa_m}{(\log m)^2}} + 0.0002069 + 10^{-6},$$

with $m \geq 6$. When $m = 5$, define the value $f(5)$ using the same formula but with $0.0002069$ in the place of $10^{-6}$. Before moving any further, notice that the value of the expression $f(m)$ can be estimated and compared against the value of $\left( 1 - \frac{\kappa_m}{\log m} \right) \left( 1 - \frac{2\kappa_m}{(\log m)^2} \right)$ when $5 \leq m \leq 14$. The values of both functions are presented in the following table:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$f(m)$</th>
<th>$\left( 1 - \frac{\kappa_m}{\log m} \right) \left( 1 - \frac{2\kappa_m}{(\log m)^2} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.4638...</td>
<td>0.5324...</td>
</tr>
<tr>
<td>6</td>
<td>0.6159...</td>
<td>0.6551...</td>
</tr>
<tr>
<td>7</td>
<td>0.6032...</td>
<td>0.6469...</td>
</tr>
<tr>
<td>8</td>
<td>0.6158...</td>
<td>0.6603...</td>
</tr>
<tr>
<td>9</td>
<td>0.5768...</td>
<td>0.6296...</td>
</tr>
<tr>
<td>10</td>
<td>0.6366...</td>
<td>0.6831...</td>
</tr>
<tr>
<td>11</td>
<td>0.5995...</td>
<td>0.6529...</td>
</tr>
<tr>
<td>12</td>
<td>0.6444...</td>
<td>0.6936...</td>
</tr>
<tr>
<td>13</td>
<td>0.6286...</td>
<td>0.6812...</td>
</tr>
<tr>
<td>14</td>
<td>0.6203...</td>
<td>0.6749...</td>
</tr>
</tbody>
</table>

It is evident from these values that

$$f(m) \leq \left( 1 - \frac{\kappa_m}{\log m} \right) \left( 1 - \frac{2\kappa_m}{(\log m)^2} \right)$$

when $5 \leq m \leq 14$. Actually, when $m \neq 6$, the coefficient 2 could be replaced by the better coefficient 2.5. The proof is ready for $5 \leq m \leq 14$.

For the rest of the proof we assume that $m \geq 15$, meaning also that $0.5 \leq \kappa_m \leq 0.756$. Let us continue by writing

$$f(m) = g(m)h(m),$$
Chapter 2. A transcendence measure for $e$

where

$$g(m) := 1 - \frac{1}{\log m} - \frac{1}{2m \log m} + \frac{1 + \kappa_m}{(\log m)^2} \cdot \frac{1 - \frac{\log m}{2m(1 + \kappa_m)}}{1 - \frac{1}{\log m} - \frac{1}{2m \log m} + \frac{1 + \kappa_m}{(\log m)^2}}$$

and

$$h(m) := \left(1 + \frac{1}{m \log m} + \frac{2 \log \log m}{m(\log m)^2}\right) \cdot \frac{1 - \frac{1 + \kappa_m}{\log m} + \frac{(m+1) \log (m+1)}{m^2 \log m} + \frac{1}{2m} - \frac{1.02394 + \kappa_m}{m \log m} + \frac{0.0000535}{m^2 \log m}}{1 - \frac{1}{\log m}}.$$

First we show that

$$g(m) \leq 1 - \frac{1 + \kappa_m}{(\log m)^2}.$$

This claim is equivalent to

$$2 + \frac{1}{m} - \frac{2(1 + \kappa_m)}{\log m} - \frac{(\log m)^2}{(1 + \kappa_m)m} \geq 0$$

which is true when $m \geq 15$ because

$$\frac{2(1 + \kappa_m)}{\log m} < \frac{4}{\log 15} < 1.48$$

and

$$\frac{(\log m)^2}{(1 + \kappa_m)m} < \frac{(\log m)^2}{m} < 0.49.$$

We still need to prove that

$$h(m) \leq 1 - \frac{\kappa_m}{\log m}.$$

First,

$$\frac{(m+1) \log (m+1)}{m^2 \log m} \leq \frac{1}{m} + \frac{1}{m^2 \log m} + \frac{1}{m^2} + \frac{1}{m^3 \log m}.$$
Chapter 2. A transcendence measure for $e$

It follows that

$$1 - \frac{1 + \kappa_m}{\log m} + \frac{(m + 1) \log (m + 1)}{m^2 \log m} + \frac{1}{2m} - \frac{1.02394 + \kappa_m}{m \log m} + \frac{0.0000535}{m^2 \log m}$$

$$\leq 1 - \frac{1 + \kappa_m}{\log m} + \frac{1}{m} + \frac{1}{m^2 \log m} + \frac{1}{m^2} + \frac{1}{m^3 \log m}$$

$$+ \frac{1}{2m} - \frac{1.02394 + \kappa_m}{m \log m} + \frac{0.0000535}{m^2 \log m}$$

$$< 1 - \frac{1 + \kappa_m}{\log m} + \frac{3}{2m}$$

since

$$\frac{1}{m^2} + \frac{1.0000535}{m^2 \log m} + \frac{1}{m^3 \log m} - \frac{1.02394 + \kappa_m}{m \log m} < 0.$$

Thus, we have

$$h(m) = \left(1 + \frac{1}{m \log m} + \frac{2 \log \log m}{m (\log m)^2}\right) \cdot \left(1 - \frac{1 + \kappa_m}{\log m} + \frac{(m + 1) \log (m + 1)}{m^2 \log m} + \frac{1}{2m} - \frac{1.02394 + \kappa_m}{m \log m} + \frac{0.0000535}{m^2 \log m}\right)$$

$$< \left(1 + \frac{1}{m \log m} + \frac{2 \log \log m}{m (\log m)^2}\right) \left(1 - \frac{\kappa_m}{\log m} - \frac{3}{2m} - \frac{3}{2m}\right).$$

Let us continue by proving that

$$\frac{\left(1 + \frac{1}{m \log m} + \frac{2 \log \log m}{m (\log m)^2}\right) \left(1 - \frac{1 + \kappa_m}{\log m} + \frac{3}{2m}\right)}{1 - \frac{1}{\log m}} < 1 - \frac{\kappa_m}{\log m}.$$ 

This is done by showing that

$$1 - \frac{1 + \kappa_m}{\log m} + \frac{3}{2m} < \left(1 - \frac{\kappa_m}{\log m}\right) \left(1 - \frac{1}{\log m}\right) \left(1 - \frac{1}{m \log m} - \frac{2 \log \log m}{m (\log m)^2}\right).$$
because then
\[
\frac{1 + \frac{1}{m \log m} + \frac{2 \log \log m}{m(\log m)^2}}{1 - \frac{1}{\log m}} \left( 1 - \frac{1 + \kappa_m}{\log m} + \frac{3}{2m} \right)
\]
\[
< \frac{1 + \frac{1}{m \log m} + \frac{2 \log \log m}{m(\log m)^2}}{1 - \frac{1}{\log m}} \left( 1 - \frac{1}{\log m} \right) \left( 1 - \frac{1}{m \log m} - \frac{2 \log \log m}{m(\log m)^2} \right)
\]
\[
< 1 - \frac{\kappa_m}{\log m}.
\]

Notice first that
\[
\left( 1 - \frac{\kappa_m}{\log m} \right) \left( 1 - \frac{1}{\log m} \right) \left( 1 - \frac{1}{m \log m} - \frac{2 \log \log m}{m(\log m)^2} \right)
\]
\[
> 1 - \frac{1 + \kappa_m}{\log m} + \frac{\kappa_m}{(\log m)^2} - \frac{1}{m \log m} + \frac{1 + \kappa_m}{m(\log m)^2} - \frac{2 \log \log m}{m(\log m)^2}
\]
\[
> 1 - \frac{1 + \kappa_m}{\log m} + \frac{\kappa_m}{(\log m)^2} - \frac{2}{m \log m} + \frac{1 + \kappa_m}{m(\log m)^2},
\]
so we have to show that
\[
\frac{3}{2m} < \frac{\kappa_m}{(\log m)^2} - \frac{2}{m \log m} + \frac{1 + \kappa_m}{m(\log m)^2}.
\]

This is equivalent to
\[
3(\log m)^2 + 4 \log m < 2\kappa_m m + 2 + 2\kappa_m.
\] (2.55)

When \( m \geq 15 \), the right-hand side of (2.55) is at least \( 2m + 3 \) since \( \kappa_m \geq 0.5 \). The inequality
\[
3(\log m)^2 + 4 \log m < 2m + 3
\]
is true when \( m \geq 14.74 \), and hence for all integer values \( m \geq 15 \). The proof is complete for all \( m \geq 5 \).

Let us now move to the small values of \( m \). We use estimate (2.54) with the values in (2.51) and (2.52). When \( m = 2 \), we have \( \log H \geq s(2)e^{s(2)} = e^{e+1} \),
Chapter 2. A transcendence measure for \( e \)

\[
\frac{u}{v} \leq 1.5804, \text{ and hence } \\
Y \leq \frac{1}{\log H} \left( \frac{uB}{v} \frac{\log(2H)}{\log \log(2H)} + 2 \log \left( \frac{u}{v} \frac{\log(2H)}{\log \log(2H)} \right) + D + 3 \log 2 \right) \\
\leq \frac{1}{\log \log H} \left( 1.5804 \cdot 2.4099 \frac{\log(2H)}{\log H} + 2 \cdot 0.7732 \frac{(\log \log H)^2}{\log H} \\
+ (3.8111 + 3 \log 2) \frac{\log \log H}{\log H} \right) \\
\leq \frac{4.93}{\log \log H}.
\]

When \( m = 3 \), we have \( \log H \geq s(3)e^{s(3)} = 3(\log 3)^2 e^{3(\log 3)^2} \), \( \frac{u}{v} \leq 1.5796 \), and hence

\[
Y \leq \frac{1}{\log H} \left( \frac{uB}{v} \frac{\log(2H)}{\log \log(2H)} + 3 \log \left( \frac{u}{v} \frac{\log(2H)}{\log \log(2H)} \right) + D + 4 \log 2 \right) \\
\leq \frac{1}{\log \log H} \left( 1.5796 \cdot 3.6433 \frac{\log(2H)}{\log H} + 3 \cdot 0.7699 \frac{(\log \log H)^2}{\log H} \\
+ (5.1819 + 4 \log 2) \frac{\log \log H}{\log H} \right) \\
\leq \frac{6.49}{\log \log H}.
\]

Finally, when \( m = 4 \), we have \( \log H \geq s(4)e^{s(4)} = 4(\log 4)^2 e^{4(\log 4)^2} \), \( \frac{u}{v} \leq 1.5984 \), and hence

\[
Y \leq \frac{1}{\log H} \left( \frac{uB}{v} \frac{\log(2H)}{\log \log(2H)} + 4 \log \left( \frac{u}{v} \frac{\log(2H)}{\log \log(2H)} \right) + D + 5 \log 2 \right) \\
\leq \frac{1}{\log \log H} \left( 1.5984 \cdot 9.7676 \frac{\log(2H)}{\log H} + 4 \cdot 0.8144 \frac{(\log \log H)^2}{\log H} \\
+ (7.3631 + 5 \log 2) \frac{\log \log H}{\log H} \right) \\
\leq \frac{15.7}{\log \log H}.
\]
Chapter 3

Hermite-Thue equation

The results of this chapter are joint work with T. Matala-aho and have appeared as \textit{Hermite-Thue equation: Padé approximations and Siegel’s lemma} \cite{35}.

3.1 Introduction

3.1.1 The twin problem

The explicit type II Hermite-Padé approximations to the exponential function have been known since the times of Hermite. In this chapter we shall focus on the problem of finding explicit type II Padé approximations to the exponential function in the case where the degrees of the approximation polynomials are the same but the orders of the remainders are free parameters—the ‘twin problem’, as we shall call it.

Let $l_1, \ldots, l_m$ be positive integers and let $\alpha_1, \ldots, \alpha_m$ be distinct variables. Denote $\alpha = (\alpha_1, \ldots, \alpha_m)^T$, $\tilde{l} = (l_1, \ldots, l_m)^T$, and $L := l_1 + \ldots + l_m$. Then the twin problem may be stated as follows: Find an explicit polynomial $B_{\tilde{l}, 0}(t, \alpha)$, polynomials $B_{\tilde{l}, j}(t, \alpha)$ and remainders $S_{\tilde{l}, j}(t, \alpha)$, $j = 1, \ldots, m$, satisfying the equations

\begin{equation}
B_{\tilde{l}, 0}(t, \alpha)e^{\alpha_j t} - B_{\tilde{l}, j}(t, \alpha) = S_{\tilde{l}, j}(t, \alpha), \quad j = 1, \ldots, m, \tag{3.1}
\end{equation}

with

\[
\begin{align*}
\deg_t B_{\tilde{l}, j}(t, \alpha) & \leq L, \quad j = 0, \ldots, m; \\
L + l_j + 1 & \leq \sup_{t \to \infty} S_{\tilde{l}, j}(t, \alpha) < \infty, \quad j = 1, \ldots, m.
\end{align*}
\]
In the diagonal case \((l_1 = l_2 = \ldots = l_m)\), the twin approximations (3.1) and the classical Hermite-Padé approximations (1.4) coincide. The fact that Padé approximations to a given function are unique up to a non-zero constant is expressed in the homogeneous vector specified by the coefficients of the denominator polynomial \(B_{l,0}(t)\) (or \(A_{l,0}(t)\) in (1.4)).

The motivation for finding the explicit twin approximations (3.1) comes from their possible applicability to arithmetical questions. As we saw in Chapter 2, the known Hermite-Padé approximations (1.4) are well suited for proving sharp transcendence measures for rational powers of \(e\). On the other hand, the following type Padé approximations

\[
B_{l,0}(t, \overline{\alpha}) e^{\alpha_j t} - B_{l,j}(t, \overline{\alpha}) = S_{\sigma,j}(t, \overline{\alpha}), \quad j = 1, \ldots, m,
\]

where

\[
\overline{\nu} = (\nu_1, \ldots, \nu_m)^T \in \mathbb{Z}_{\geq 1}^m, \quad \nu_j \leq l_j, \quad \nu_1 + \ldots + \nu_m =: M \leq L,
\]

and

\[
\begin{cases}
\text{deg}_t B_{l,j}(t, \overline{\alpha}) \leq L, & j = 0, \ldots, m; \\
L + \nu_j + 1 \leq \text{ord}_{t=0} S_{\sigma,j}(t, \overline{\alpha}) < \infty, & j = 1, \ldots, m
\end{cases}
\]

(of which (3.1) is a special case) have been successfully used in proving a Baker-type transcendence measure for \(e\) (see [15]). In Baker-type bounds the dependence on the individual heights of the coefficients of the linear form is visible in the bound. This provides an additional challenge compared to settling for the maximum height only.

Due to the lack of explicit twin approximations (3.1), the works considering Baker-type lower bounds for linear forms have relied on Siegel’s lemma; see Baker [2], Mahler [31], and, for a generalised transcendence measure of \(e\), Ernvall-Hytyönen et al. [15]. In these Baker-type lower bounds the error terms are weaker than the corresponding ones in those transcendence measures that depend on the maximum height only (see Chapter 2).

For example, Ernvall-Hytyönen et al. [15] present an explicit Baker-type lower bound

\[
|\lambda_0 + \lambda_1 e + \lambda_2 e^2 + \ldots + \lambda_m e^m| > \frac{1}{h^{1+\epsilon(h)}},
\]

\[
\epsilon(h) = \frac{(4 + 7m)\sqrt{\log(m + 1)}}{\sqrt{\log \log h}},
\]

(3.3)
Chapter 3. Hermite-Thue equation

valid for all
\[ \vec{\lambda} = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{Z}_2^{m+1}, \quad h_i = \max\{1, |\lambda_i|\}, \quad h = h_1 \cdots h_m, \]
with
\[ \log h \geq m^2(41 \log(m + 1) + 10)e^{m^2(81 \log(m+1)+20)}. \]

On the other hand, the corresponding error term in Theorem 2.1 is roughly
\[ \frac{cm^2 \log m}{\log \log H}, \quad c = c(m) \leq 1, \quad \max_{1 \leq i \leq m} \{1, |\lambda_i|\} \leq H, \]
which is smaller than \( m\epsilon(H^m) \) (above \( \log H \) is very large compared to \( m \)). Thus, it is reasonable to suppose that explicit twin approximations (3.1) could perhaps imply a sharper Baker-type transcendence measure for \( e \).

3.1.2 The Bombieri-Vaaler version of Siegel’s lemma

If we write
\[ B_{\sigma,0}(t, \vec{\alpha}) = \sum_{h=0}^{L} c_h \frac{L!}{h!} l^h, \quad (3.4) \]
then (3.2) yields a group of \( M \) equations with integer coefficients in the \( L + 1 \) unknowns \( c_h \). The use of Siegel’s lemma in estimating the solution vector \( (c_0, c_1, \ldots, c_L)^T \) is the reason for introducing the parameters \( \nu_1, \ldots, \nu_m \) into the approximation problem (3.2). Since we don’t know the explicit solution to (3.1), the parameters \( \nu_1, \ldots, \nu_m \) give more freedom to optimise the estimate. In the Hermite-Padé case (1.4), the explicit solution is known and therefore the problem is presented only with the parameters \( l_0, l_1, \ldots, l_m \).

As explained in the previous section, the lower bounds for linear forms coming from Siegel’s lemma are not always as sharp as those coming from Padé approximations. In the lack of explicit twin approximations the already existing results might be improved by some refined version of Siegel’s lemma. Such an improvement indeed exists and is given by Bombieri and Vaaler [7]; for a shorter proof in the integer case, see [34, Theorem 14.3]. Below we present the Bombieri-Vaaler version in the integer case.

Lemma 3.1. [7, Theorem 2] Let \( \mathcal{V} \in \mathcal{M}_{M \times N}(\mathbb{Z}), M < N, \) and rank \( \mathcal{V} = M \). Then the equation
\[ \mathcal{V} \bar{x} = \vec{0} \]
Chapter 3. Hermite-Thue equation

has $N - M$ linearly independent integer solutions $\mathfrak{x}_1, \ldots, \mathfrak{x}_{N-M} \in \mathbb{Z}^N \setminus \{0\}$ such that

$$\|\mathfrak{x}_1\|_\infty \cdots \|\mathfrak{x}_{N-M}\|_\infty \leq \frac{\sqrt{\det(VV^T)}}{D},$$

where $D$ is the greatest common divisor of all the $M \times M$ minors of $V$.

Bombieri and Vaaler actually proved a more general result over the algebraic numbers by using geometry of numbers over the adèles; see also Fukshansky’s [20] generalisation of the problem.

An important feature of Siegel’s lemma is that it only implies the existence of a non-trivial integer solution with an appropriate upper bound. A priori, no information about the explicit expressions of the solutions to, say, equations (1.4) or (3.1) is given. However, there is a deep connection between Siegel’s lemma and the solution, as Lemma 3.1 reveals. Namely, the Grassmann coordinates of the exterior product of the row vectors of the matrix $V$ are precisely the $M \times M$ minors of $V$; for details, see [34, Section 14.2]. On the other hand, the solution to the Padé problem (3.1) is a homogeneous vector of the $L \times L$ minors of the coefficient matrix of the group of equations formed from (3.2) and (3.4) in the case $\nu_j = l_j$, $M = L$. So the same minors that one needs to study to be able to apply Lemma 3.1 actually form the solution vector to the Padé problem. Hence the title Hermite-Thue equation.

3.2 Results

We shall first prove some general factoring theorems for Vandermonde-type block determinants, and then use these as tools when studying the determinants related to the Padé approximation equations. We are able to give a new proof of the well-known type II Hermite-Padé approximations to the exponential function presented in Theorem 2.4.

In Section 3.5 we shall examine case (3.2) which turns out to be much tougher. Choose $\mathfrak{a} = \mathfrak{a} := (a_1, \ldots, a_m)^T$, where $a_1, \ldots, a_m$ are pairwise different, non-zero integers. Write

$$B_{\mathfrak{a}, 0}(t, \mathfrak{a}) = \sum_{h=0}^L c_h \frac{L!}{h!} t^h,$$

where $\mathfrak{a} := (\nu_1, \ldots, \nu_m)^T$, and the numbers $\nu_1, \ldots, \nu_m, l_1, \ldots, l_m \in \mathbb{Z}_{\geq 1}$ satisfy

$$1 \leq \nu_j \leq l_j, \quad M := \nu_1 + \ldots + \nu_m \leq L := l_1 + \ldots + l_m.$$
Then (3.2) yields the matrix equation

$$V \bar{c} = \overline{0}, \quad \bar{c} := (c_0, c_1, \ldots, c_L)^T,$$

with

$$V = V(a) := \begin{pmatrix}
(L+1)a^L_0 & (L+1)a^{L-1}_1 & \cdots & (L+1)a_1 & (L+1)a_L \\
(L+2)a^L_0 & (L+2)a^{L-1}_1 & \cdots & (L+2)a_1 & (L+2)a_L \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(L+\nu_1)a^L_m & (L+\nu_1)a^{L-1}_m & \cdots & (L+\nu_1)a_1 & (L+\nu_1)a_L \\
(L+2)a^L_m & (L+2)a^{L-1}_m & \cdots & (L+2)a_1 & (L+2)a_L \\
(L+\nu_m)a^L_m & (L+\nu_m)a^{L-1}_m & \cdots & (L+\nu_m)a_1 & (L+\nu_m)a_L \\
\end{pmatrix}_{M \times (L+1)}$$

for which Siegel’s lemma produces a non-zero integer solution with a non-trivial upper bound.

In order to be able to study the common factors of the minors of the integer matrix $V(a)$, we shall switch to the corresponding polynomial matrix $V(\overline{a}) \in M_{M \times (L+1)}(\mathbb{Z}[\alpha_1, \ldots, \alpha_m])$. We are able find two different high order polynomial factors from the $M \times M$ minors of $V(\overline{a})$, as will be presented in Theorem 3.13. Choosing $\overline{a} = \overline{a} = (a_1, \ldots, a_m)^T$ then leads us to

**Theorem 3.2.** Let $a_1, \ldots, a_m \in \mathbb{Z}$. Then

$$\left( \prod_{1 \leq j \leq m} a_j^{(\nu_j)} \right) \prod_{1 \leq i < j \leq m} (a_i - a_j)^{\min\{\nu_i^2, \nu_j^2\}} D(\overline{a}),$$

where $D(\overline{a})$ is the greatest common divisor of all the $M \times M$ minors of the matrix $V(\overline{a}) \in M_{M \times (L+1)}(\mathbb{Z})$.

In addition, we can prove that the rank of the polynomial matrix $V(\overline{a})$ over $\mathbb{Z}[\alpha_1, \ldots, \alpha_m]$ is $M$ (see Lemma 3.14), but the rank of the coefficient matrix $V(\overline{a})$ over the integers remains an open question. This is a problem that needs answering before the Bombieri-Vaaler version of Siegel’s lemma (Lemma 3.1) can be applied.
Chapter 3. Hermite-Thue equation

In the particular case \( \nu_j = l_j, M = L \), the wild case (3.2) reduces to the twin Padé problem (3.1), for which we give a partial answer in the following theorem:

**Theorem 3.3.** Let \( l_1, \ldots, l_m \) be positive integers and let \( \alpha_1, \ldots, \alpha_m \) be distinct variables. Denote \( \overline{\alpha} = (\alpha_1, \ldots, \alpha_m)^T, \overline{l} = (l_1, \ldots, l_m)^T, L := l_1 + \cdots + l_m \). Then there exist non-zero polynomials \( B_{\overline{l}, j}(t, \overline{\alpha}) \in \mathbb{Q}[t, \overline{\alpha}] \) and remainders \( S_{\overline{l}, j}(t, \overline{\alpha}) \) such that

\[
B_{\overline{l}, 0}(t, \overline{\alpha})e^{\alpha_j t} - B_{\overline{l}, j}(t, \overline{\alpha}) = S_{\overline{l}, j}(t, \overline{\alpha}), \quad j = 1, \ldots, m,
\]

where

\[
\begin{align*}
& \deg_t B_{\overline{l}, j}(t, \overline{\alpha}) \leq L, \quad j = 0, \ldots, m; \\
& L + l_j + 1 \leq \text{ord}_t S_{\overline{l}, j}(t, \overline{\alpha}) < \infty, \quad j = 1, \ldots, m.
\end{align*}
\]

Moreover, we have

\[
B_{\overline{l}, 0}(t, \overline{\alpha}) = \sum_{i=0}^{L} \frac{L!}{i!} \tau_i(\overline{l}, \overline{\alpha}) t^i, \quad \tau_i(\overline{l}, \overline{\alpha}) = \frac{(-1)^i V[i]}{T(\overline{l}, \overline{\alpha})} \in \mathbb{Z}[\overline{\alpha}],
\]

where

\[
T(\overline{l}, \overline{\alpha}) := \alpha_1^{(\overline{l}_1)} \cdots \alpha_m^{(\overline{l}_m)} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{t_i^2, t_j^2\}} \quad (3.6)
\]

is a common factor of the \( L \times L \) minors \( V[i], i = 0, \ldots, L, \) of the matrix \( V(\overline{\alpha}) \).

Investigation of some special cases by computer indicates that the common factor (3.6) of Theorem 3.3 may be the best achievable primitive polynomial factor. As opposite to the tame case (1.4) however, after dividing by the common factor, we are now left with the non-explicit polynomials \( \frac{L!}{i!} \tau_i(\overline{l}, \overline{\alpha}) \), \( i = 0, 1, \ldots, L \), representing the coordinates of the homogeneous solution vector.

The following considerations have been inspired by the papers of Flowe and Harris [19], Krattenthaler [27], and van der Poorten [39].

### 3.3 Generalised Vandermonde-type block matrices

Let \( D \) be an integral domain of characteristic zero. In the following, we shall work in the polynomial ring denoted by \( D[x_1, \ldots, x_m] \) which is a free commutative \( D \)-algebra generated by \( \{x_1, \ldots, x_m\} \). (As usual, the elements \( x_1, \ldots, x_m \)
are called indeterminates or variables.) Note also that if $D$ is a UFD, then $D[x_1, \ldots, x_m]$ is also UFD, where $x_1, \ldots, x_m$ are prime elements. Further, a polynomial $P \in D[x_1, \ldots, x_m]$ is called primitive, if the gcd of its coefficients is 1. By setting $\deg 0(x) := -\infty$ for the zero-polynomial $0(x) \in D[x]$, the degree formula $\deg(a(x)b(x)) = \deg a(x) + \deg b(x)$ holds for all single variable polynomials $a(x), b(x) \in D[x]$.

The notation $(x_1, \ldots, x_i, \ldots, x_m)$ will be used for the $(m-1)$-tuple where the component $x_i$ has been left out. Before going further, let us recall the definition of multinomials:

$$\binom{k}{k_1, \ldots, k_m} := \frac{k!}{k_1! \cdots k_m!}, \quad k_1 + \ldots + k_m = k, \quad k_1, \ldots, k_m, k \in \mathbb{Z}_{\geq 0}.$$

Let $m \in \mathbb{Z}_{\geq 1}$. Pick then $m$ positive integers $n_1, n_2, \ldots, n_m \in \mathbb{Z}_{\geq 1}$ and set $n := n_1 + n_2 + \ldots + n_m$. Let $p_0(x), p_1(x), \ldots, p_{n-1}(x) \in D[x]$ be $n$ single variable polynomials.

### 3.3.1 Case A

Our method is based on Lemma 1.15, along with the idea already used, for instance, by Flowe and Harris [19]. However, no combinatorial argument will be needed here, and there are no restrictions on the polynomials.

Denote

$$A_j := \begin{pmatrix} p_0(x_j) & p_1(x_j) & \cdots & p_{n-1}(x_j) \\ p_0'(x_j) & p_1'(x_j) & \cdots & p_{n-1}'(x_j) \\ \vdots & \vdots & \ddots & \vdots \\ p_0^{(n-1)}(x_j) & p_1^{(n-1)}(x_j) & \cdots & p_{n-1}^{(n-1)}(x_j) \end{pmatrix}_{n \times n}, \quad j = 1, \ldots, m,$$

and let

$$\mathcal{A} := \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}_{n \times n}.$$

**Lemma 3.4.** With the above notation, we have

$$\prod_{1 \leq i < j \leq m} (x_i - x_j)^{n_i n_j} \bigg|_{D[x_1, \ldots, x_m]} \det \mathcal{A}.$$
Chapter 3. Hermite-Thue equation

Proof. Let us write

$$\tau^{(k)}(x) := \left( \left( \frac{d}{dx} \right)^k p_0(x) \quad \left( \frac{d}{dx} \right)^k p_1(x) \quad \cdots \quad \left( \frac{d}{dx} \right)^k p_{n-1}(x) \right)^T,$$

$$\tau^{(k)}_j := \tau^{(k)}(x_j),$$

for all $k \in \mathbb{Z}_{\geq 0}$, so that

$$\det A = \det \left[ \tau_1 \quad \tau'_1 \quad \cdots \quad \tau^{(n_1-1)}_1 \quad \cdots \quad \tau_m \quad \tau'_m \quad \cdots \quad \tau^{(n_m-1)}_m \right].$$

(We use the transpose of $A$ only to save some space.) Let $i \in \{1, \ldots, m\}$ and consider the determinant $\det A$ as a polynomial in $x_i$ with other variables as coefficients, i.e., $\det A =: P(x_i) \in \mathbb{D}[x_1, \ldots, \hat{x}_i, \ldots, x_m][x_i]$. In the following, the shorthand notation $D_i := \frac{d}{dx_i}$ will be used.

According to the general Leibniz rule for derivatives,

$$P^{(k)}(x_i) = D_i^k \det A$$

$$= \sum_{k_{i,1} + \cdots + k_{i,n_1} + k_{i,1} + \cdots + k_{i,n_m} = k} \binom{k}{k_{i,1}, \ldots, k_{i,n_m}}$$

$$\cdot \det \left[ D_i^{k_{i,1}} \tau_1 \quad D_i^{k_{i,1}+1} \tau'_1 \quad \cdots \quad D_i^{k_{i,1}+n_1-1} \tau^{(n_1-1)}_1 \quad \tau_m \quad \tau'_m \quad \cdots \quad \tau^{(n_m-1)}_m \right]$$

$$= \sum_{k_{i,1} + k_{i,2} + \cdots + k_{i,n_i} = k} \binom{k}{0, \ldots, 0, k_{i,1}, \ldots, k_{i,n_i}, 0, \ldots, 0}$$

$$\cdot \Delta(k_{i,1}, k_{i,2}, \ldots, k_{i,n_i})(x_i),$$

where

$$\Delta(k_{i,1}, k_{i,2}, \ldots, k_{i,n_i})(x_i)$$

$$:= \det \left[ * \quad \tau^{(k_{i,1})}(x_i) \quad \tau^{(k_{i,1}+1)}(x_i) \quad \cdots \quad \tau^{(k_{i,n_i}+n_i-1)}(x_i) \quad * \right].$$

(The star symbol (*) denotes the rest of the blocks which are unchanged.) Now we evaluate the derivative $P^{(k)}$ at $x_j$, $j \neq i$. First we see that

$$\Delta(k_{i,1}, k_{i,2}, \ldots, k_{i,n_i})(x_j) = 0$$

70
Chapter 3. Hermite-Thue equation

if
\[
\begin{cases}
k_{i,s} + s - 1 = k_{i,t} + t - 1, & s \neq t, \quad s, t \in \{1, \ldots, n_i\}; \\
\text{or} \\
\text{there exists an } s \in \{1, \ldots, n_i\} \text{ such that } k_{i,s} + s - 1 \leq n_j - 1.
\end{cases}
\]

Therefore, supposing
\[
\Delta(k_{i,1}, k_{i,2}, \ldots, k_{i,n_i})(x_j) \neq 0
\]
implies that the numbers \(k_{i,1}, k_{i,2} + 1, \ldots, k_{i,n_i} + n_i - 1\) are distinct and strictly greater than \(n_j - 1\). Hence
\[
\sum_{s=1}^{n_i} (k_{i,s} + s - 1) \geq \sum_{t=0}^{n_i-1} (n_j + t),
\]
which gives
\[
k = \sum_{s=1}^{n_i} k_{i,s} \geq \sum_{t=0}^{n_i-1} n_j = n_i n_j.
\]

Thus
\[
P^{(k)}(x_j) = 0, \quad k = 0, 1, \ldots, n_i n_j - 1, \quad j \in \{1, \ldots, m\} \setminus \{i\}.
\]

Applying Lemma 1.15 with \(I = D[x_1, \ldots, x_i, \ldots, x_m]\), we get
\[
(x_i - x_j)^{n_i n_j} \quad \bigg| \quad P(x_i), \quad j \in \{1, \ldots, m\} \setminus \{i\}.
\]

Noting that the elements \(x_g - x_h\) and \(x_i - x_j\) are relatively prime when \((g, h) \neq (i, j)\), we arrive at the common factor
\[
\prod_{1 \leq i < j \leq m} (x_i - x_j)^{n_i n_j} \quad \bigg| \quad \det A.
\]

Corollary 3.5. If \(\deg p_i = i\), then
\[
\det A = F \cdot \prod_{1 \leq i < j \leq m} (x_i - x_j)^{n_i n_j}
\]
for some \(F \in D\).
Chapter 3. Hermite-Thue equation

Proof. Lemma 3.4 shows that
\[ \det A = F \cdot \prod_{1 \leq i < j \leq m} (x_i - x_j)^{n_i n_j} \]
for some \( F \in D[x_1, \ldots, x_m] \). Again, we shall consider \( \det A \) as a polynomial \( P(x_k) \) in \( x_k \) for some \( k \in \{1, \ldots, m\} \). The degrees of the entries in block \( A_k \) form the following matrix:

\[
\begin{pmatrix}
0 & 1 & 2 & \cdots & n-2 & n-1 \\
-\infty & 0 & 1 & \cdots & n-3 & n-2 \\
-\infty & -\infty & 0 & \cdots & n-4 & n-3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\infty & -\infty & -\infty & \cdots & n-n_k & n-(n_k-1) \\
-\infty & -\infty & -\infty & \cdots & n-(n_k+1) & n-n_k \\
\end{pmatrix}_{n_k \times n}
\]

Above, an element in position \((v, n-w)\) is of the form
\[ n - v - w, \quad v \in \{1, \ldots, n_k\}, \quad w \in \{0, 1, \ldots, n-1\}, \]
where a negative number corresponds to a zero polynomial of the lower left corner. When \( \det A \) is expanded using the Leibniz formula, the degree of a non-zero term is a sum
\[ \sum_{i=1}^{n_k} (n - i - w_i), \quad w_i \in \{0, 1, \ldots, n-1\}, \]
where the numbers \( w_i \) are pairwise distinct, so that
\[ \sum_{i=1}^{n_k} w_i \geq \sum_{i=0}^{n_k-1} i. \]

Therefore, we get the upper bound
\[ \deg_{x_k} P(x_k) \leq n_k n - \sum_{i=1}^{n_k} i - \sum_{i=0}^{n_k-1} i = n_k(n - n_k). \quad (3.9) \]

On the other hand, it is easy to see that
\[ \deg_{x_k} \left( \prod_{1 \leq i < j \leq m} (x_i - x_j)^{n_i n_j} \right) = \sum_{j=1}^{m} n_k n_j = n_k(n - n_k). \]
The degree formula and estimate \[3.9\] give

\[
\deg_{x_k} \det A = \deg_{x_k} F + n_k(n - n_k) = \deg_{x_k} P(x_k) \leq n_k(n - n_k),
\]

implying \(\deg_{x_k} F \leq 0\). Hence

\[
\det A = F \cdot \prod_{1 \leq i < j \leq m} (x_i - x_j)^{n_i n_j},
\]

where \(F\) does not depend on \(x_k\) with an arbitrary \(k\). Thus \(F \in \mathbb{D}\). \(\square\)

Corollary 3.5 is a generalisation of the well-known polynomial Vandermonde matrix (see [27, Proposition 1]):

**Corollary 3.6.** If \(\deg p_i = i, i \in \{0, 1, \ldots, n - 1\}\), and \(n_j = 1\) for all \(j = 1, \ldots, m\), then

\[
\det A = a_{0,0} a_{1,1} \cdots a_{n-1,n-1} \prod_{1 \leq i < j \leq m} (x_j - x_i),
\]

where \(a_{i,i}\) are the leading coefficients of the polynomials \(p_i\).

Flowe and Harris’ [19] Theorem 1.1 is a special case of Corollary 3.5 too. In some cases it is possible to compute the constant \(F\). This was done by Flowe and Harris [19] in their case, where the polynomials \(p_i(x)\) are powers \(x^i\) for \(i = 0, 1, \ldots, n - 1\). Later we shall compute \(F\) in a different case (see Lemma 3.11).

### 3.3.2 Case B

In Lemma 3.4 we studied a block matrix in which each block starts with the same row of polynomials which are then differentiated so that the last rows of the blocks are not necessarily the same (in terms of the polynomials).

Now we look at a matrix where rows are created by differentiating to the other direction, starting from the last row of each block. Again the first rows are the same, but last rows may differ between blocks.

Let \(n_{\max} = \max_{1 \leq i \leq m} \{n_i\}\). Denote

\[
B_j := \begin{pmatrix}
\binom{n_{\max} - 1}{n_j} p_0(x_j) & \binom{n_{\max} - 1}{n_j} p_1(x_j) & \cdots & \binom{n_{\max} - 1}{n_j} p_{n-1}(x_j) \\
\binom{n_{\max} - 2}{n_j} p_0(x_j) & \binom{n_{\max} - 2}{n_j} p_1(x_j) & \cdots & \binom{n_{\max} - 2}{n_j} p_{n-1}(x_j) \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n_{\max} - n_1}{n_j} p_0(x_j) & \binom{n_{\max} - n_1}{n_j} p_1(x_j) & \cdots & \binom{n_{\max} - n_1}{n_j} p_{n-1}(x_j)
\end{pmatrix}_{n_j \times n},
\]
Chapter 3. Hermite-Thue equation

\( j = 1, \ldots, m, \) and let

\[
B := \begin{pmatrix}
B_1 \\
B_2 \\
\vdots \\
B_m
\end{pmatrix}_{n \times n}.
\]

Lemma 3.7. With the above notation, we have

\[
\prod_{1 \leq i < j \leq m} (x_i - x_j)^{\min\{n_i^2, n_j^2\}} \det B.
\]

Proof. With the notation in (3.7), block \( B_j \) becomes

\[
B_j^T = \begin{bmatrix}
\tau_j^{(n_{\max} - 1)} \\
\tau_j^{(n_{\max} - 2)} \\
\vdots \\
\tau_j^{(n_{\max} - n_j)}
\end{bmatrix}.
\]

Let \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, m\} \setminus \{i\} \). Without loss of generality we may assume that \( n_i \leq n_j \). Consider the determinant \( \det B \) as a polynomial \( P \) in \( x_i \):

\[
\det B =: P(x_i).
\]

Again we use the general Leibniz rule for derivatives. Differentiating \( P \) means we are differentiating block \( B_i \), since all the other blocks are constants with respect to \( x_i \) (compare to (3.8)):

\[
D_i^k \det B = \sum_{k_1 + k_2 + \ldots + k_i, n_i = k} \begin{pmatrix}
k \\
0, \ldots, 0, k_i, 1, \ldots, k_i, n_i, 0, \ldots, 0
\end{pmatrix} \cdot \det \left[ \begin{array}{c}
\tau_i^{(n_{\max} - 1)} \\
\tau_i^{(n_{\max} - 2)} \\
\vdots \\
\tau_i^{(n_{\max} - n_i)}
\end{array} \right] = \sum_{k_1 + k_2 + \ldots + k_i, n_i = k} \begin{pmatrix}
k \\
0, \ldots, 0, k_i, 1, \ldots, k_i, n_i, 0, \ldots, 0
\end{pmatrix} \cdot \Delta(k_i, 1, k_i, 2, \ldots, k_i, n_i)(x_i),
\]

where

\[
\Delta(k_i, 1, k_i, 2, \ldots, k_i, n_i)(x_i) := \det \begin{bmatrix}
\tau^{(k_i, 1, n_{\max} - 1)}(x_i) \\
\vdots \\
\tau^{(k_i, n_i, + n_{\max} - n_i)}(x_i)
\end{bmatrix},
\]

and the stars are again used to denote the other, unchanged blocks.

We evaluate the derivative \( P^{(k)} \) at \( x_j \). First we see that

\[
\Delta(k_i, 1, k_i, 2, \ldots, k_i, n_i)(x_j) = 0
\]
if
\[
\begin{cases}
  k_{i,s} + n_{\text{max}} - s = k_{i,t} + n_{\text{max}} - t, & s \neq t, \quad s, t \in \{1, \ldots, n_i\}; \\
  \text{or} \\
  \text{there exists an } s \in \{1, \ldots, n_i\} \text{ such that} \\
  k_{i,s} + n_{\text{max}} - s \in [n_{\text{max}} - n_j, n_{\text{max}} - 1].
\end{cases}
\]

Therefore, supposing
\[
\Delta(k_{i,1}, k_{i,2}, \ldots, k_{i,n_i})(x_j) \neq 0
\]
implies that the numbers \(k_{i,1} + n_{\text{max}} - 1, \ldots, k_{i,n_i} + n_{\text{max}} - n_i\) are distinct and do not belong into the interval \([n_{\text{max}} - n_j, n_{\text{max}} - 1]\). However, the case \(k_{i,s} + n_{\text{max}} - s < n_{\text{max}} - n_j\) never happens because \(n_i \leq n_j\) by our assumption.

Hence
\[
\sum_{s=1}^{n_i} (k_{i,s} + n_{\text{max}} - s) \geq \sum_{t=0}^{n_i-1} (n_{\text{max}} + t),
\]

which gives
\[
k = \sum_{s=1}^{n_i} k_{i,s} \geq n_i^2 = \min \{n_i^2, n_j^2\}.
\]

Hence
\[
P^{(k)}(x_j) = 0, \quad k = 0, 1, \ldots, \min \{n_i^2, n_j^2\} - 1.
\]

Applying Lemma 1.15 with \(I = D[x_1, \ldots, \hat{x}_i, \ldots, x_m]\), we get
\[
(x_i - x_j)^{\min \{n_i^2, n_j^2\}} \bigg|_{D[x_1, \ldots, \hat{x}_i, \ldots, x_m][x_i]} P(x_i),
\]

which proves the claim since the elements \(x_g - x_h\) and \(x_i - x_j\) are relatively prime when \((g, h) \neq (i, j)\).

3.4 Hermite-Padé approximations to the exponential function revisited: tame case

Let us first use the tools of Section 3.3 to give a new proof for the well-known Hermite-Padé approximations to the exponential function. The following theorem is just Theorem 2.4 rephrased.
Chapter 3. Hermite-Thue equation

Recall the notation introduced in Section 1.4.1: Let \( \alpha_0, \alpha_1, \ldots, \alpha_m \) be distinct variables and denote \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \). For given \( l_0, l_1, \ldots, l_m \in \mathbb{Z}_{\geq 1} \) we have

\[
\sigma_i(\vec{l}, \vec{\alpha}) = (-1)^i \sum_{l_0+i_1+\ldots+i_m = i} \binom{l_1}{i_1} \cdots \binom{l_m}{i_m} \cdot \alpha_1^{i_1} \cdots \alpha_m^{l_m-i_m}.
\]

**Theorem 3.8.** Let \( \vec{l} = (l_0, l_1, \ldots, l_m)^T \in \mathbb{Z}^{m+1}_{\geq 1} \), \( L_0 := l_0 + l_1 + \ldots + l_m \), and \( L := l_1 + \ldots + l_m \). Then there exist polynomials \( A_{\vec{l}, j}(t, \vec{\alpha}) \in \mathbb{Q}[t, \vec{\alpha}] \) and remainders \( R_{\vec{l}, j}(t, \vec{\alpha}) \) such that

\[
A_{\vec{l}, 0}(t, \vec{\alpha})e^{\alpha_j t} - A_{\vec{l}, j}(t, \vec{\alpha}) = R_{\vec{l}, j}(t, \vec{\alpha}), \quad j = 1, \ldots, m,
\]

where

\[
\begin{align*}
\text{deg}_t A_{\vec{l}, j}(t, \vec{\alpha}) &\leq L_0 - l_j, \quad j = 0, 1, \ldots, m; \\
L_0 + 1 &\leq \lim_{t \to 0} R_{\vec{l}, j}(t, \vec{\alpha}) < \infty, \quad j = 1, \ldots, m.
\end{align*}
\]

Moreover, the polynomial \( A_{\vec{l}, 0}(t, \vec{\alpha}) \) has the explicit expression

\[
A_{\vec{l}, 0}(t, \vec{\alpha}) = \sum_{h=0}^L (L_0 - h)! \sigma_{L_0-h}(\vec{l}, \vec{\alpha}) t^h.
\]

### 3.4.1 A new proof of Theorem 3.8

Write

\[
A_{\vec{l}, 0}(t, \vec{\alpha}) =: \sum_{h=0}^L b_h t^h
\]

and

\[
e^{\alpha_j t} A_{\vec{l}, 0}(t, \vec{\alpha}) =: \sum_{N=0}^\infty r_{N, j} t^N.
\]

Set \( r_{L_0-k_j, j} = 0 \) for \( k_j = 0, \ldots, l_j - 1, j = 1, \ldots, m \). So

\[
\sum_{h+n=L_0-k_j} \frac{\alpha_j^n}{n!} b_h = \sum_{h=0}^{\min\{L_0-k_j, L\}} \frac{\alpha_j^{L_0-k_j-h}}{(L_0-k_j-h)!} b_h = 0
\]

There is a tiny difference between the notation here and in the published version because of the desire to use the same definition throughout this dissertation.
for all \( k_j = 0, \ldots, l_j - 1, \ j = 1, \ldots, m \), meaning that we have \( L \) equations in \( L + 1 \) unknowns \( b, h = 0, 1, \ldots, L \). The equations in (3.12) can be written in matrix form:

\[
\mathcal{U} \mathbf{b} = \mathbf{0}, \quad \mathbf{b} := (b_0, b_1, \ldots, b_L)^T,
\]

with

\[
\mathcal{U} := \begin{pmatrix}
U_1 \\
U_2 \\
\vdots \\
U_m
\end{pmatrix}_{L \times (L+1)}, \quad
U_j := \begin{pmatrix}
\frac{\alpha_j^{L_0}}{L_0!} & \frac{\alpha_j^{L_0-1}}{(L_0-1)!} & \ldots & \frac{\alpha_j^0}{L_0^1!} \\
\frac{\alpha_j^{L_0-1}}{(L_0-1)!} & \frac{\alpha_j^{L_0-2}}{(L_0-2)!} & \ldots & \frac{\alpha_j^{0}}{(L_0-1)!} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\alpha_j^{L_0-l_j+1}}{(L_0-l_j+1)!} & \frac{\alpha_j^{L_0-l_j}}{(L_0-l_j)!} & \ldots & \frac{\alpha_j^0}{(L_0-1)!}
\end{pmatrix}_{l_j \times (L+1)},
\]

\( j = 1, \ldots, m \), where the typical element

\[
\alpha_j^{L_0-k_j-h} \left( \frac{1}{(L_0-k_j-h)!} \right), \quad k_j = 0, \ldots, l_j - 1, \quad j = 1, \ldots, m, \quad h = 0, 1, \ldots, L \quad (3.13)
\]

is zero whenever \( L_0 - k_j - h < 0 \).

In the following, the notation \( \mathcal{U}[j] \) is used for the minor obtained by removing the \( j \)th column of the matrix \( \mathcal{U} \), \( j = 0, 1, \ldots, L \). The rightmost \( L \times L \) minor of \( \mathcal{U} \in \mathcal{M}_{L \times (L+1)}(\mathbb{Q}[\alpha_1, \ldots, \alpha_m]) \) is therefore denoted by \( \mathcal{U}[0] \), and it will be considered as a polynomial in \( \mathbb{Q}[\alpha_1, \ldots, \alpha_m] \). (For technical simplicity, the numbering of columns starts from 0.)

Factors of \( \mathcal{U}[0] \)

**Lemma 3.9.** With the above notation, we have

\[
\left( \prod_{i=1}^{m} \alpha_i^{l_i} \right) \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_i l_j} \mid \mathcal{U}[0], \quad (3.14)
\]

77
Chapter 3. Hermite-Thue equation

Proof. Now

\[
U[0] = \begin{pmatrix}
\alpha_1^{L_0-1} & \alpha_1^{L_0-2} & \cdots & \alpha_1^0 \\
\frac{(L_0-1)!}{(L_0-2)!} & \frac{(L_0-2)!}{(L_0-3)!} & \cdots & \frac{(L_0-1)!}{(L_0-0)!} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{L_0-l_1} & \alpha_1^{L_0-l_1-1} & \cdots & \alpha_1^{L_0-l_1-l_1} \\
\frac{(L_0-l_1)!}{(L_0-l_1-1)!} & \frac{(L_0-l_1-1)!}{(L_0-l_1-2)!} & \cdots & \frac{(L_0-l_1-l_1)!}{(L_0-l_1-l_1-1)!} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_m^{L_0-t_m} & \alpha_m^{L_0-t_m-1} & \cdots & \alpha_m^{L_0-t_m-t_m} \\
\frac{(L_0-t_m)!}{(L_0-t_m-1)!} & \frac{(L_0-t_m-1)!}{(L_0-t_m-2)!} & \cdots & \frac{(L_0-t_m-t_m)!}{(L_0-t_m-t_m-1)!} \\
\end{pmatrix}
\]

where the \( j \)th block of \( U[0] \) looks like

\[
\begin{pmatrix}
D_j^0 \alpha_j^{L_0-1} & \cdots & D_j^0 \alpha_j^0 \\
\frac{(L_0-1)!}{(L_0-2)!} & \cdots & \frac{(L_0-1)!}{(L_0-0)!} \\
\vdots & \ddots & \vdots \\
D_j^{l_1-1} \alpha_j^{L_0-l_1} & \cdots & D_j^{l_1-1} \alpha_j^0 \\
\frac{(L_0-l_1)!}{(L_0-l_1-1)!} & \cdots & \frac{(L_0-l_1-l_1)!}{(L_0-l_1-l_1-1)!} \\
\end{pmatrix}
\]

\( i < j \)

\[
\prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_i l_j}
\]

\( U[0] \)

\( Q[\alpha_1, \ldots, \alpha_m] \)

follows thus directly from Lemma 3.4.

It remains to show the powers of alphas. We can now use the \((l_1 \times l_1, \ldots, l_m \times l_m)\)
minor expansion (Lemma 1.17). Any \( l_j \times l_j \) minor of the \( j \)th block

\[
\begin{pmatrix}
\frac{\alpha_j^{L_0-1}}{(L_0-1)!} & \frac{\alpha_j^{L_0-2}}{(L_0-2)!} & \cdots & \frac{\alpha_j^{L_0}}{(L_0-1)!} \\
\frac{\alpha_j^{L_0-2}}{(L_0-2)!} & \frac{\alpha_j^{L_0-3}}{(L_0-3)!} & \cdots & \frac{\alpha_j^{L_0-1}}{(L_0-2)!} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\alpha_j^{L_0-i_j}}{(L_0-i_j)!} & \frac{\alpha_j^{L_0-i_j-1}}{(L_0-i_j-1)!} & \cdots & \frac{\alpha_j}{(L_0-i_j)!}
\end{pmatrix}
\]

contains the factor

\[
\prod_{i=0}^{l_j-1} \alpha_j^{L_0-i-w_i}, \quad w_i \in \{1, \ldots, L\},
\]

where the numbers \( w_i \) are pairwise distinct, so that

\[
\sum_{i=0}^{l_j-1} w_i \leq \sum_{i=1}^{l_j-1} (L - i).
\]

Now

\[
\sum_{i=0}^{l_j-1} (L_0 - i - w_i) \geq l_j L_0 - \sum_{i=0}^{l_j-1} i - \sum_{i=0}^{l_j-1} (L - i)
\]

\[
\geq l_j (L_0 - L) - \frac{(l_j - 1)l_j}{2} + \frac{(l_j - 1)l_j}{2} = l_0 l_j.
\]

Therefore we get

\[
\alpha_1^{l_0 l_1} \cdots \alpha_m^{l_0 l_m} \bigg| \mathcal{U}[0].
\]

Since the polynomials \( \alpha_1^{l_0 l_1} \cdots \alpha_m^{l_0 l_m} \) and \( \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_i l_j} \) are relatively prime, we have proved the claim (3.14). \( \square \)

Next we show that the remaining factor is a constant, a rational number.
Lemma 3.10. There exists a rational number $F_m = F_m(l_0, \ldots, l_m)$ such that

$$U[0] = F_m \left( \prod_{i=1}^{m} \alpha_i^{l_i} \right) \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_i l_j}. \quad (3.16)$$

Proof. By Lemma 3.9 we know that

$$U[0] = F_m \cdot \alpha_1^{l_1} \cdots \alpha_m^{l_m} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_i l_j} \quad (3.17)$$

for some $F_m \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m]$. Let $k \in \{1, \ldots, m\}$. Just as in the proof of Corollary 3.5 we see that when (3.15) is treated as a polynomial in $\alpha_k$ and expanded using the Leibniz formula, the degree of a non-zero term is a sum

$$\sum_{i=0}^{l_k-1} (L_0 - i - w_i), \quad w_i \in \{1, \ldots, L\},$$

where the numbers $w_i$ are pairwise disjoint, so that

$$\sum_{i=0}^{l_k-1} w_i \geq \sum_{i=1}^{l_k} i.$$

Hence we get the upper bound

$$\deg_{\alpha_k} U[0] \leq l_k L_0 - \frac{(l_k - 1)l_k}{2} - \frac{l_k(l_k + 1)}{2} = l_k(L_0 - l_k).$$

On the other hand,

$$\deg_{\alpha_k} U[0] = \deg_{\alpha_k} F_m + l_0 l_k + \sum_{j=1 \atop j \neq k}^{m} l_k l_j = \deg_{\alpha_k} F_m + l_k(L_0 - l_k),$$

by (3.17). Thus $\deg_{\alpha_k} F_m \leq 0$, and since $k$ was arbitrary, we must have $F_m = F_m(l_0, \ldots, l_m) \in \mathbb{Q}$. \hfill \Box

Rank

Lemma 3.11. We have \(\text{rank}_{\mathbb{Q}[\alpha_1, \ldots, \alpha_m]} U = L.\)
**Chapter 3. Hermite-Thue equation**

**Proof.** It is enough to show that the matrix \( \mathcal{U} \) has one non-zero \( L \times L \) minor, in this case the rightmost one, \( \mathcal{U}[0] \). Therefore, it remains to show that the constant \( F_m \) is non-zero.

First we multiply both sides of equation (3.16) with the term

\[
\frac{l_1(l_1+1)}{\alpha_1^2} \div \frac{l_2(l_2+1)}{\alpha_2^2} \div \cdots \div \frac{l_m(l_m+1)}{\alpha_m^2}.
\]

Then, on the right-hand side of (3.16), the coefficient of the monomial

\[
\frac{l_1(l_1+1)}{\alpha_1^2}l_0l_1 + l_1l_2 + \cdots + l_1l_m + \frac{l_2(l_2+1)}{\alpha_2^2}l_0l_2 + l_2l_3 + \cdots + l_2l_m + \cdots + \frac{l_m(l_m+1)}{\alpha_m^2}l_0l_{m-1} + l_{m-1}l_m + \frac{l_m(l_m+1)}{\alpha_m^2}l_0l_m
\]

is \( F_m \). On the left-hand side of (3.16) this monomial arises from the block diagonal when using the generalised minor expansion (Lemma 1.17) with \( l_j \times l_j \) minors, \( j = 1, \ldots, m \), to expand \( \mathcal{U}[0] \), where the first row of block \( j, j = 1, \ldots, m \), has been multiplied by \( \alpha_j \), the second by \( \alpha_j^2 \) and so on, until the \( l_j \)th row:

\[
F_m \cdot \prod_{j=1}^{m} \frac{l_j(l_j+1)}{\alpha_j^2}l_0l_j + l_j(l_{j-1}+\cdots+l_m)
\]

\[
= \prod_{j=1}^{m} \left( \frac{\alpha_j}{l_0-l_1-\cdots-i_j-1} \right) \left( \frac{L_0-l_1-\cdots-l_j-1}{\alpha_j} \right) \cdots \left( \frac{L_0-l_1-\cdots-l_j-1}{\alpha_j} \right)
\]

\[
= \left( \prod_{j=1}^{m} \frac{\alpha_j}{l_0-l_1-\cdots-l_j-1} \right) \cdot \prod_{j=1}^{m} f_j
\]

with

\[
f_j := \left( \frac{1}{l_0-l_1-\cdots-l_j-1} \right) \left( \frac{1}{l_0-l_1-\cdots-l_j-1} \right) \cdots \left( \frac{1}{l_0-l_1-\cdots-l_j-1} \right)
\]

81
(Note that $l_1 + \ldots + l_{j-1} = 0$ when $j = 1$.) We get an expression for $F_m$ as a product of determinants:

$$F_m(l_0, \ldots, l_m) = \prod_{j=1}^{m} f_j,$$

(3.18)

where the typical element

$$f_j = \frac{1}{(L_0 - l_1 - \ldots - l_{j-1} - 1)!} \cdot \frac{1}{(L_0 - l_1 - \ldots - l_{j-1} - 2)!} \cdot \ldots \cdot \frac{1}{(L_0 - l_1 - \ldots - l_{j-1} - l_j)!},$$

$$\left| \begin{array}{cccc} (L_0 - \left( \sum_{i=1}^{j-1} l_i + 1 \right))_0 & \ldots & (L_0 - \left( \sum_{i=1}^{j-1} l_i + 1 \right))_{l_i-1} \\ \vdots & \ddots & \vdots \\ (L_0 - \left( \sum_{i=1}^{j-1} l_i \right))_0 & \ldots & (L_0 - \left( \sum_{i=1}^{j-1} l_i \right))_{l_j-1} \end{array} \right|$$

(3.19)

using the falling factorials defined by

$$(n)_0 := 1; \quad (n)_k := n(n-1)\ldots(n-k+1), \quad k = 1, \ldots, n.$$  

This notation works for the zero elements as well since now our typical element is

$$(L_0 - c_j)_{r_j},$$

$r_j \in \{0, 1, \ldots, l_j - 1\}, \quad c_j \in \{l_1 + \ldots + l_{j-1} + 1, \ldots, l_1 + \ldots + l_j\}, \quad j = 1, \ldots, m,$

and if $L_0 - r_j - c_j < 0$, then

$$(L_0 - c_j)_{r_j} = (L_0 - c_j)(L_0 - c_j - 1) \ldots (L_0 - c_j - r_j + 1) = 0.$$ 

Most importantly, the falling factorial $(n)_k$ is a polynomial in $n$ of degree $k$. This means that the determinants $f_j$ are essentially polynomial Vandermonde
determinants, the polynomials being in this case \((x)_0, (x)_1, \ldots, (x)_{l_j-1}\). The evaluation of (3.19) follows from Corollary 3.6: let us further denote
\[
c_{j,k} := l_1 + \ldots + l_{j-1} + k,
\]
so that
\[
f_j = \frac{1}{(L_0 - c_{j,1})!} \cdot \frac{1}{(L_0 - c_{j,2})!} \cdot \ldots \cdot \frac{1}{(L_0 - c_{j,l_j})!} \cdot \prod_{1 \leq h < k \leq l_j} ((L_0 - c_{j,k}) - (L_0 - c_{j,h}))
\]
\[
= \frac{1}{(L_0 - c_{j,1})!} \cdot \frac{1}{(L_0 - c_{j,2})!} \cdot \ldots \cdot \frac{1}{(L_0 - c_{j,l_j})!} \cdot \prod_{1 \leq h < k \leq l_j} (h - k)
\]
\[
\neq 0
\]
for all \(j = 1, \ldots, m\).

Cramer’s rule

Remember that \(U[j]\) denotes the minor obtained by removing the \(j\)th column of \(U\), \(j = 0, 1, \ldots, L\). By Cramer’s rule, the equation
\[
U\bar{b} = \bar{0}, \quad \bar{b} = (b_0, b_1, \ldots, b_L)^T,
\]
has the solution
\[
[b_0, b_1, \ldots, b_L] = [U[0] : -U[1] : \ldots : (-1)^L U[L]],
\]
where one of the minors is appropriately non-zero, as was shown in Lemma 3.11. Here we use the notation of homogeneous coordinates:
\[
[h_1 : h_2 : \ldots : h_n] = \{t(h_1, h_2, \ldots, h_n)^T \mid t \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m] \setminus \{0\}\}
\]
for any \((h_1, h_2, \ldots, h_n)^T \in (\mathbb{Q}[\alpha_1, \ldots, \alpha_m])^n \setminus \{\bar{0}\}\).

To solve our Hermite-Padé approximation problem, we complete the matrix \(U\) into an \((L + 1) \times (L + 1)\) square matrix \(S\) by adding on bottom of it the row
\[
\begin{pmatrix}
x_0^{L_0} & x_0^{L_0-1} & \cdots & x_0^1 \\
L_0! & (L_0-1)! & \cdots & L_0!
\end{pmatrix},
\]
where $x = \alpha_{m+1}$ is a new, technical variable. Now the Laplace expansion along the last row implies

$$\det S = (-1)^k \sum_{h=0}^{L} (-1)^{h} U[h] \frac{x^{L_0-h}}{(L_0-h)!}. \quad (3.22)$$

On the other hand, we know how to compute $\det S$: just replace $L_0$ with $L_0 + 1$ and $m$ with $m + 1$ in the proof of Lemma 3.11 ($l_m+1 = 1$). It tells us that

$$\det S = F_{m+1}(l_0, \ldots, l_{m+1}) \cdot \alpha_{l_0l_1} \cdots \alpha_{l_ml_m} x^{l_0l_{m+1}}$$

$$\cdot \prod_{1 \leq i < j \leq m+1} (\alpha_i - \alpha_j)^{l_il_j}$$

$$= F_{m+1}(l_0, \ldots, l_{m+1}) \cdot \alpha_{l_0l_1} \cdots \alpha_{l_ml_m}$$

$$\cdot (-1)^{l_0} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_il_j} (-x)^{l_0} \prod_{i=0}^{m} (\alpha_i - x)^{l_i},$$

where $F_{m+1}(l_0, \ldots, l_{m+1})$ is given by (3.18) and (3.20) (with $L_0 + 1$ instead of $L_0$ and $m + 1$ instead of $m$).

Now

$$(-x)^{l_0} \prod_{i=1}^{m} (\alpha_i - x)^{l_i} = \prod_{i=0}^{m} (\alpha_i - x)^{l_i} = \sum_{i=l_0}^{L_0} \sigma_i(\overline{l}, \overline{\alpha}) x^i,$$

where

$$\sigma_i(\overline{l}, \overline{\alpha}) = (-1)^i \sum_{l_0+i_1+\ldots+i_m=i} \binom{l_1}{i_1} \cdots \binom{l_m}{i_m} \alpha_{l_1-i_1} \cdots \alpha_{l_m-i_m}$$

and $l_0 + l_1 + \ldots + l_m = L_0$. Thus

$$\det S = F_{m+1}(l_0, \ldots, l_{m+1}) \cdot \alpha_{l_0l_1} \cdots \alpha_{l_ml_m}$$

$$\cdot (-1)^{l_0} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_il_j} \sum_{i=l_0}^{L_0} \sigma_i(\overline{l}, \overline{\alpha}) x^i$$

$$= F_{m+1}(l_0, \ldots, l_{m+1}) \cdot \alpha_{l_0l_1} \cdots \alpha_{l_ml_m}$$

$$\cdot (-1)^{l_0} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_il_j} \sum_{h=0}^{L} \sigma_{L_0-h}(\overline{l}, \overline{\alpha}) x^{L_0-h}. \quad (3.23)$$
Comparison of the two representations (3.22) and (3.23) (as polynomials in $x$) yields

$$U[h] = (-1)^h (L_0 - h)! \sigma_{L_0 - h}(\overline{l}, \overline{\alpha}) \cdot (-1)^{L_0} F_{m+1} \cdot \alpha_1^{l_0 l_1} \cdots \alpha_m^{l_0 l_m} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{l_i l_j} \quad (3.24)$$

for $h = 0, 1, \ldots, L$. Hence, (3.24) shows the complete factorisation of all the $L \times L$ minors of the matrix $U$. Secondly, we see that all the above minors are non-zero polynomials in $\mathbb{Q}[\alpha_1, \ldots, \alpha_m]$.

Because of the homogeneous coordinates in (3.21), we get

$$[b_0, b_1, \ldots, b_L] = [L_0! \sigma_{L_0}(\overline{l}, \overline{\alpha}) : (L_0 - 1)! \sigma_{L_0 - 1}(\overline{l}, \overline{\alpha}) : \cdots : l_0! \sigma_{l_0}(\overline{l}, \overline{\alpha})].$$

Then the polynomial $A_{l,0}(t, \overline{\alpha})$ in (3.10) becomes

$$A_{l,0}(t, \overline{\alpha}) = \sum_{h=0}^L (L_0 - h)! \sigma_{L_0 - h}(\overline{l}, \overline{\alpha}) t^h.$$

Before going to the ‘twin problem’ we remark that the seemingly extra parameter $l_0$ is crucial in the applications of type II Hermite-Padé approximations (3.10) (see [23] and Chapter 2).

### 3.5 Padé-type approximations to the exponential function: wild case

#### 3.5.1 The twin problem

As explained in Section 3.1.1 the problem of finding explicit type II Hermite-Padé approximations in the case where the degrees of the polynomials are the same but the orders of the remainders are free parameters is yet unsolved. We called it the ‘twin problem’, stated as follows: Find an explicit polynomial $B_{l,0}(t, \overline{\alpha})$, polynomials $B_{l,j}(t, \overline{\alpha})$ and remainders $S_{l,j}(t, \overline{\alpha})$, $j = 1, \ldots, m$, satisfying

$$B_{l,0}(t, \overline{\alpha}) e^{\alpha_j t} - B_{l,j}(t, \overline{\alpha}) = S_{l,j}(t, \overline{\alpha}), \quad j = 1, \ldots, m, \quad (3.25)$$
Chapter 3. Hermite-Thue equation

with \( \vec{l} = (l_1, \ldots, l_m)^T \in \mathbb{Z}_{\geq 1}^m \), \( L := l_1 + \ldots + l_m \) and

\[
\begin{align*}
\{ \deg_t B_{t,j}(t, \vec{a}) &\leq L, & j = 0, \ldots, m; \\
L + l_j + 1 &\leq \text{ord}_{t=0} S_{t,j}(t, \vec{a}) < \infty, & j = 1, \ldots, m.
\}
\tag{3.26}
\end{align*}
\]

We note that in the setting of the twin problem it is not possible to use the parameter \( L_0 \) in a similar way as in the tame case. Namely, replacing \( L \) with \( L_0 \) in (3.26) yields \( L = l_1 + \ldots + l_m \) equations with \( L_0 + 1 \) unknowns. If \( L_0 > L \), the resulting Padé polynomial \( B_{t,0}(t, \vec{a}) \) would not be unique.

The twin approximations (3.25) are a special case of the following, more general Padé-type approximations:

\[
B_{\vec{\nu},0}(t, \vec{a}) e^{\alpha_j t} - B_{\vec{\nu},j}(t, \vec{a}) = S_{\vec{\nu},j}(t, \vec{a}), \quad j = 1, \ldots, m, \tag{3.27}
\]

where

\[
\vec{\nu} = (\nu_1, \ldots, \nu_m)^T \in \mathbb{Z}_{\geq 1}^m, \quad \nu_1 \leq l_1, \ldots, \nu_m \leq l_m, \quad \nu_1 + \ldots + \nu_m =: M \leq L,
\]

and

\[
\begin{align*}
\{ \deg_t B_{\vec{\nu},j}(t, \vec{a}) &\leq L, & j = 0, \ldots, m; \\
L + \nu_j + 1 &\leq \text{ord}_{t=0} S_{\vec{\nu},j}(t, \vec{a}) < \infty, & j = 1, \ldots, m.
\}
\end{align*}
\]

3.5.2 Siegel’s lemma

If \( M < L \), there is no unique solution \( B_{\vec{\nu},0}(t, \vec{a}) \) to the Padé-type approximation equations (3.27), neither is it known how to find an explicit solution. Therefore, we now switch to integers and apply Siegel’s lemma.

Choose \( \vec{\nu} = \vec{a} := (a_1, \ldots, a_m)^T \), where \( a_1, \ldots, a_m \) are pairwise different, non-zero integers. Write

\[
B_{\vec{\nu},0}(t, \vec{a}) = \sum_{h=0}^{L} c_h \frac{L!}{h!} t^h,
\tag{3.28}
\]

where \( \vec{\nu} = (\nu_1, \ldots, \nu_m)^T \), and the numbers \( \nu_1, \ldots, \nu_m, l_1, \ldots, l_m \in \mathbb{Z}_{\geq 1} \) satisfy

\[
1 \leq \nu_j \leq l_j, \quad M := \nu_1 + \ldots + \nu_m \leq L := l_1 + \ldots + l_m.
\]

Then (3.27) yields the matrix equation

\[
\begin{pmatrix} \vec{\nu} \end{pmatrix} \vec{c} = \vec{0}, \quad \vec{c} := (c_0, c_1, \ldots, c_L)^T, \tag{3.29}
\]

86
with

$$
\begin{pmatrix}
(L+1)a_1^L & (L+1)a_1^{L-1} & \cdots & (L+1)a_1 & (L+1) \\
L+2 & L+2 & \cdots & L+2 & L+2 \\
L_0 & L_1 & \cdots & L_m & M(L+1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(L+\nu_0)a_m^L & (L+\nu_1)a_m^{L-1} & \cdots & (L+\nu_m)a_1 & (L+\nu_m) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(L+\nu_0)a_m^L & (L+\nu_1)a_m^{L-1} & \cdots & (L+\nu_m)a_1 & (L+\nu_m) \\
\end{pmatrix}
\begin{array}{c}
L_0 \\
L_1 \\
\vdots \\
L_m \\
M(L+1)
\end{array}
$$

for which Siegel’s lemma produces a non-zero integer solution with a non-trivial upper bound (see (3.31) below). The term \( \frac{L}{h} \) has been included in (3.28) just to get an integer matrix in (3.29) and thus make the use of Siegel’s lemma possible. A standard application of Siegel’s lemma is the following proposition, a particular case of [15, Lemma 4.1].

**Proposition 3.12.** There exist a non-zero polynomial

$$
B_{\tau,0}(t, \overline{a}) = \sum_{h=0}^{L} c_h \frac{L!}{h!} t^h \in \mathbb{Z}[t]
$$

and non-zero polynomials \( B_{\tau,j}(t, \overline{a}) \in \mathbb{Z}[t, \overline{a}], j = 1, \ldots, m \), such that

$$
B_{\tau,0}(t, \overline{a}) e^{a_j t} - B_{\tau,j}(t, \overline{a}) = S_{\tau,j}(t, \overline{a}), \quad j = 1, \ldots, m,
$$

where

$$
\begin{cases}
\text{deg}_t \tau_{\tau,j}(t, \overline{a}) \leq L, & j = 0, \ldots, m; \\
L + \nu_j + 1 \leq \text{ord}_{t=0} S_{\tau,j}(t, \overline{a}) < \infty, & j = 1, \ldots, m.
\end{cases}
$$

Moreover, we have \((c_0, c_1, \ldots, c_L)^T \in \mathbb{Z}^{L+1} \setminus \{0\}\) and

$$
|c_h| \leq \left(f M L g^{M^2} \right)^{\frac{L+1}{L+1-M}}, \quad h = 0, 1, \ldots, L,
$$

where

$$
f = f(\overline{a}) = \max_{1 \leq j \leq m} \{|a_j| + 1\}, \quad g = g(\overline{a}) = \max_{1 \leq j \leq m} \left\{1 + \frac{1}{|a_j|}\right\}.
$$
Proof. For the sake of completeness, we shall reproduce the proof from [15]. Let
\begin{equation}
B_{\nu,0}(t, \bar{a})e^{a_j t} = \sum_{N=0}^{\infty} r_{N,j} t^N, \quad j = 1, \ldots, m,
\end{equation}
where
\begin{equation}
r_{N,j} := \sum_{\substack{h+n=N \\ 0 \leq h \leq L \\ h+n=N}} c_h \frac{L! a_{ij}^n}{h! n!}.
\end{equation}
Cut the series (3.32) after \(L+1\) terms and let
\begin{equation}
B_{\nu,j}(t, \bar{a}) := \sum_{N=0}^{L} r_{N,j} t^N, \quad j = 1, \ldots, m.
\end{equation}
Set \(r_{L+i,j} = 0\) for \(i_j = 1, \ldots, \nu_j, \ j = 1, \ldots, m\). Then also
\begin{equation}
0 = \frac{(L+i_j)!}{L!} \frac{1}{a_{ij}^j} r_{L+i,j}
= \sum_{\substack{h+n=L+i_j \\ 0 \leq h \leq L \\ h+n=L+i_j}} \frac{(L+i_j)!}{h! n!} a_{ij}^{n-i_j} c_h
= \sum_{h=0}^{L} \left( \frac{L+i_j}{h} \right) a_{ij}^{L-h} c_h
\end{equation}
for all \(i_j = 1, \ldots, \nu_j, \ j = 1, \ldots, m\), so we have \(M\) equations in \(L+1\) unknowns \(c_h, \ h = 0, 1, \ldots, L\), with coefficients \(\left( \frac{L+i_j}{h} \right) a_{ij}^{L-h} \in \mathbb{Z}\). These coefficients satisfy the estimate
\begin{equation}
A_{j,i_j} := \sum_{h=0}^{L} \left| \left( \frac{L+i_j}{h} \right) a_{ij}^{L-h} \right|
\leq \frac{1}{\left|a_{ij}^j\right|} \sum_{h=0}^{L+i_j} \left( \frac{L+i_j}{h} \right) \left|a_{ij}\right|^{L+i_j-h}
= \frac{1}{\left|a_{ij}^j\right|} \left( \left|a_{ij}\right| + 1 \right)^{L+i_j}
= \left( \left|a_{ij}\right| + 1 \right)^L \left( 1 + \frac{1}{\left|a_{ij}\right|} \right)^{i_j},
\end{equation}
88
when \( i_j = 1, \ldots, \nu_j, j = 1, \ldots, m \). (Recall that in order to apply Siegel’s lemma, the product of the row sums (1.5) is needed.) Then
\[
\prod_{j,i_j} A_{j,i_j} < \left( \prod_{j,i_j} (|a_j| + 1)^L \right) \prod_{j,i_j} \left( 1 + \frac{1}{|a_j|} \right)^{i_j} \\
= \left( \prod_{j} (|a_j| + 1)^{\nu_j L} \right) \prod_{j} \left( 1 + \frac{1}{|a_j|} \right)^{\nu_j(\nu_j + 1)} \\
\leq f^{ML} g^L M^2.
\]
The last step follows by employing the definition \( M = \nu_1 + \ldots + \nu_m \) which implies
\[
\sum_{j=1}^{m} \nu_j(\nu_j + 1) = \sum_{j=1}^{m} \nu_j^2 + \sum_{j=1}^{m} \nu_j \leq M^2.
\]
By Siegel’s lemma, there exists a solution \( \vec{c} = (c_0, c_1, \ldots, c_L)^T \in \mathbb{Z}^{L+1} \setminus \{0\} \) to the group of \( M \) equations derived in (3.33) with
\[
|c_h| \leq \left( f^{ML} g^L M^2 \right)^{\frac{1}{L+1-M}}, \quad h = 0, 1, \ldots, L.
\]
Writing
\[
S_{\nu_j}(t, \vec{a}) := \sum_{N=L+\nu_j+1}^{\infty} r_{N,j} t^N
\]
we get
\[
B_{\nu_0}(t, \vec{a}) \cong_j - B_{\nu_j}(t, \vec{a}) = S_{\nu_j}(t, \vec{a}), \quad j = 1, \ldots, m.
\]
Here \( B_{\nu,j}(t, \vec{a}) \) are non-zero polynomials for all \( j = 0, 1, \ldots, m \) since the solution \( \vec{c} \) is a non-zero vector. Conditions (3.30) are also satisfied, as the series \( S_{\nu,j}(t) \) is non-zero too.

### 3.5.3 The Bombieri-Vaaler version of Siegel’s lemma

Next we are going to examine, would it be possible to improve estimate (3.31) by using the Bombieri-Vaaler version of Siegel’s lemma (Lemma 3.1). Assuming that the rank of \( \mathcal{V}(\vec{a}) \) over \( \mathbb{Z} \) is \( M \), it follows from Lemma 3.1 that equation
Chapter 3. Hermite-Thue equation

(3.29) has a solution \( \bar{c} = (c_0, \ldots, c_L)^T \in \mathbb{Z}^{L+1} \setminus \{0\} \) with

\[
\|\bar{c}\|_\infty = \max_{0 \leq k \leq L} |c_k| \leq \left( \frac{\sqrt{\det(V(\bar{a})V(\bar{a})^T))}}{D(\bar{a})} \right)^{\frac{1}{M-1}} \leq \left( \frac{\prod_{m=1}^{M} \|w_m\|_1}{D(\bar{a})} \right)^{\frac{1}{M-1}},
\]

where \( D(\bar{a}) \) is the greatest common divisor of all the \( M \times M \) minors of \( V(\bar{a}) \), \( w_m \) denotes the \( m \)th row of the matrix \( V(\bar{a}) \), and

\[
\|(v_1, \ldots, v_N)^T\|_1 := |v_1| + \ldots + |v_N|.
\]

Therefore, the bound in (3.31) can be improved if we can find a relatively big common factor from the \( M \times M \) minors of the matrix \( V(\bar{a}) \). Such a factor indeed exists, as was stated in Theorem 3.2. Theorem 3.2 is a direct corollary of Theorem 3.13 which will be presented and proved in the following section.

3.5.4 Common factor

To prove Theorem 3.2 for an arbitrary \( m \)-tuple of integers, we need to treat the determinants as polynomials. Let us therefore consider the matrix \( V(\bar{a}) \in \mathcal{M}_{M \times (L+1)}(\mathbb{Z}[\alpha_1, \ldots, \alpha_m]) \).

Let \( \hat{D}(\bar{a}) \in \mathbb{Z}[\alpha_1, \ldots, \alpha_m] \) be the greatest common divisor of the \( M \times M \) minors of the matrix \( V(\bar{a}) \) (we assume that \( \hat{D}(\bar{a}) \) is a primitive polynomial). We are interested in the divisors of an arbitrary \( M \times M \) minor of \( V(\bar{a}) \), denoted by

\[
\det W := \begin{vmatrix} (L+1)_{e_1} & (L+1)_{e_2} & \ldots & (L+1)_{e_M} \\ (L+\nu_1)_{e_1} & (L+\nu_1)_{e_2} & \ldots & (L+\nu_1)_{e_M} \\ \vdots & \vdots & \ddots & \vdots \\ (L+\nu_m)_{e_1} & (L+\nu_m)_{e_2} & \ldots & (L+\nu_m)_{e_M} \end{vmatrix} = \det \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_m \end{pmatrix},
\]
Chapter 3. Hermite-Thue equation

where \( 0 \leq e_1 < e_2 < \ldots < e_M \leq L \) and

\[
W_j := \begin{pmatrix}
\binom{L+1}{e_1} \alpha_j^{L-e_1} & \binom{L+1}{e_2} \alpha_j^{L-e_2} & \ldots & \binom{L+1}{e_M} \alpha_j^{L-e_M} \\
\vdots & \ddots & \ddots & \vdots \\
\binom{L+\nu_j}{e_1} \alpha_j^{L-e_1} & \binom{L+\nu_j}{e_2} \alpha_j^{L-e_2} & \ldots & \binom{L+\nu_j}{e_M} \alpha_j^{L-e_M}
\end{pmatrix}_{\nu_j \times M},
\]

(3.34)

\( j = 1, \ldots, m \), is the \( j \)th block of the matrix \( W \) defining our arbitrary minor. Clearly \( \det W \in \mathbb{Z}[\alpha_1, \ldots, \alpha_m] \).

**Theorem 3.13.** We have

\[
\left( \prod_{1 \leq j \leq m} \alpha_j^{\binom{\nu_j}{2}} \right) \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}} \mid \hat{D}(\overline{x}).
\]

(3.35)

**Proof.** The factor

\[
\alpha_1^{\binom{\nu_1}{2}} \cdots \alpha_m^{\binom{\nu_m}{2}}
\]

follows by using the generalised minor expansion (Lemma 1.17) with \( \nu_j \times \nu_j \), \( j = 1, \ldots, m \), minors. According to Lemma 1.17, then \( \det W \) is a sum of products of \( \nu_j \times \nu_j \) minors, each taken from block \( W_j \). Looking at (3.34), we see that any \( \nu_j \times \nu_j \) minor of \( W_j \) contains at least the factor

\[
\alpha_j^{(L-e_M)+(L-e_{M-1})+\ldots+(L-e_{M-\nu_j+1})} \geq \alpha_j^{0+1+\ldots+\nu_j-1} = \alpha_j^{\binom{\nu_j}{2}}.
\]

Thus

\[
\prod_{1 \leq j \leq m} \alpha_j^{\binom{\nu_j}{2}} \mid \det W.
\]

It remains to show the Vandermonde-type factor. If each row \((j, i_j)\) of \( \det W \) is multiplied by \( \frac{\alpha_j^{i_j}}{(L+i_j)!} \) and column \( k \) by \( e_k! \), we get a new determinant

\[
\det \widetilde{W} := \left( \prod_{j=1}^{M} \prod_{i_j=1}^{\nu_j} \frac{\alpha_j^{i_j}}{(L+i_j)!} \right) \prod_{k=1}^{M} e_k! \det W
\]

\[
= \begin{vmatrix}
\frac{\alpha_1^{L+1-e_1}}{(L+1-e_1)!} & \frac{\alpha_1^{L+1-e_2}}{(L+1-e_2)!} & \ldots & \frac{\alpha_1^{L+1-e_M}}{(L+1-e_M)!} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{\alpha_m^{L+\nu_m-e_1}}{(L+\nu_m-e_1)!} & \frac{\alpha_m^{L+\nu_m-e_2}}{(L+\nu_m-e_2)!} & \ldots & \frac{\alpha_m^{L+\nu_m-e_M}}{(L+\nu_m-e_M)!}
\end{vmatrix}.
\]

91
Chapter 3. Hermite-Thue equation

The $j$th block of $\hat{W}$ looks like

$$\hat{W}_j := \begin{pmatrix}
\frac{\alpha_j^{L+1-e_1}}{(L+1-e_1)!} & \frac{\alpha_j^{L+1-e_2}}{(L+1-e_2)!} & \cdots & \frac{\alpha_j^{L+1-e_M}}{(L+1-e_M)!} \\
\frac{\alpha_j^{L+2-e_1}}{(L+2-e_1)!} & \frac{\alpha_j^{L+2-e_2}}{(L+2-e_2)!} & \cdots & \frac{\alpha_j^{L+2-e_M}}{(L+2-e_M)!} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\alpha_j^{L+\nu_j-e_1}}{(L+\nu_j-e_1)!} & \frac{\alpha_j^{L+\nu_j-e_2}}{(L+\nu_j-e_2)!} & \cdots & \frac{\alpha_j^{L+\nu_j-e_M}}{(L+\nu_j-e_M)!}
\end{pmatrix}_{\nu_j \times M}.$$ 

Notice that the upper rows are derivatives of the last row, whence application of Lemma 3.7 results in the common factor

$$\prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}} \mid Q[\alpha_1, \ldots, \alpha_m]$$

implies

$$\det \hat{W} = \left( \prod_{j=1}^{m} \prod_{i_j=1}^{\nu_j} \alpha_j^{i_j} \right) \frac{1}{M} \prod_{k=1}^{M} e_k! \det W,$$

$$\text{(3.37)}$$

Since the polynomials

$$\prod_{j=1}^{m} \prod_{i_j=1}^{\nu_j} \alpha_j^{i_j} = \prod_{j=1}^{m} \alpha_j^{\frac{(\nu_j+1)}{2}} \quad \text{and} \quad \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}}$$

are relatively prime, property (3.36) and equation (3.38) now imply that

$$\prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}} \mid Q[\alpha_1, \ldots, \alpha_m]$$

implies

$$\det \hat{W} = \left( \prod_{j=1}^{m} \prod_{i_j=1}^{\nu_j} \alpha_j^{i_j} \right) \frac{1}{M} \prod_{k=1}^{M} e_k! \det W,$$

$$\text{(3.38)}$$

implies

$$\prod_{j=1}^{m} \prod_{i_j=1}^{\nu_j} \alpha_j^{i_j} = \prod_{j=1}^{m} \alpha_j^{\frac{(\nu_j+1)}{2}} \quad \text{and} \quad \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}}$$

are relatively prime, property (3.36) and equation (3.38) now imply that

$$\prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}} \mid Q[\alpha_1, \ldots, \alpha_m]$$

implies

$$\det \hat{W} = \left( \prod_{j=1}^{m} \prod_{i_j=1}^{\nu_j} \alpha_j^{i_j} \right) \frac{1}{M} \prod_{k=1}^{M} e_k! \det W,$$
Hence
\[
\det \mathcal{W} = Q \cdot \left( \prod_{1 \leq j \leq m} \alpha_j^{(\nu_j^2)} \right) \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}}
\]
for some polynomial
\[
Q = Q(\alpha_1, \ldots, \alpha_m) \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m].
\]
But since \( \det \mathcal{W} \in \mathbb{Z}[\alpha_1, \ldots, \alpha_m] \) and the polynomials
\[
\prod_{1 \leq j \leq m} \alpha_j^{(\nu_j^2)}, \quad \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}} \in \mathbb{Z}[\alpha_1, \ldots, \alpha_m]
\]
are primitive, we have \( Q \in \mathbb{Z}[\alpha_1, \ldots, \alpha_m] \). Hence
\[
\prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{\nu_i^2, \nu_j^2\}} \bigg| \quad \det \mathcal{W}, \quad \mathbb{Z}[\alpha_1, \ldots, \alpha_m]
\]

**Proof of Theorem 3.2** The integer factor (3.5) follows from Theorem 3.13 by choosing \( \overline{\alpha} = (a_1, \ldots, a_m)^T \) and noticing that \( D(\overline{\alpha}) | D(\overline{\alpha}) \).

### 3.5.5 Rank

In order to benefit from Theorem 3.2, we should have rank \( \mathcal{V}(\overline{\alpha}) = M \) over \( \mathbb{Z} \). Proving this seems to be out of reach at the moment. However, as a step towards the solution, we shall show that rank \( \mathcal{V}(\overline{\alpha}) = M \) over the polynomial ring \( \mathbb{Q}[\alpha_1, \ldots, \alpha_m] \).

**Lemma 3.14.** We have rank \( \mathcal{V}(\overline{\alpha}) = M \) over the ring \( \mathbb{Q}[\alpha_1, \ldots, \alpha_m] \).

**Proof.** We show that one of the \( M \times M \) minors of \( \mathcal{V}(\overline{\alpha}) \) is not the zero poly-
mial, in this case the rightmost one:

\[
\begin{vmatrix}
\frac{L+1}{L-(M-1)} & \alpha_1^{M-1} & \cdots & \frac{L+1}{L-(M-1)} & \alpha_1 \\
\frac{L+2}{L-(M-1)} & \alpha_1^{M-1} & \cdots & \frac{L+2}{L-(M-1)} & \alpha_1 \\
\vdots & \ddots & \ddots & \vdots & \ddots \\
\frac{L+\nu_1}{L-(M-1)} & \alpha_1^{M-1} & \cdots & \frac{L+\nu_1}{L-(M-1)} & \alpha_1 \\
\vdots & \ddots & \ddots & \vdots & \ddots \\
\frac{L+\nu_m}{L-(M-1)} & \alpha_1^{M-1} & \cdots & \frac{L+\nu_m}{L-(M-1)} & \alpha_1 \\
\end{vmatrix}
\] (3.39)

The monomial

\[\alpha_1^{(\nu_1^2)} + \nu_1 \nu_2 + \nu_1 \nu_3 + \cdots + \nu_1 \nu_m, \quad \alpha_2^{(\nu_2^2)} + \nu_2 \nu_3 + \cdots + \nu_2 \nu_m, \quad \cdots \quad \alpha_m^{(\nu_m^2)} \]

arises uniquely from the block diagonal when using the generalised minor expansion with \(\nu_j \times \nu_j, \ j = 1, \ldots, m\), minors to expand (3.39). We show that its coefficient, denoted by \(G\), is non-zero, therefore implying that the determinant (3.39) is a non-zero polynomial.

The generalised minor expansion (Lemma 1.17) gives

\[
G \cdot \alpha_1^{(\nu_1^2)} + \nu_1 \nu_2 + \nu_1 \nu_3 + \cdots + \nu_1 \nu_m \cdot \alpha_2^{(\nu_2^2)} + \nu_2 \nu_3 + \cdots + \nu_2 \nu_m \cdot \cdots \cdot \alpha_m^{(\nu_m^2)}
\]

\[
= \prod_{j=1}^{m} \alpha_j^{(\nu_j^2) + \nu_j (\nu_{j+1} + \cdots + \nu_m)} \cdot \prod_{j=1}^{m} g_j
\]

94
where

\[
g_j := \begin{bmatrix}
(L - M + (\nu_1 + \ldots + \nu_{j-1} + 1)) & \cdots & (L - M + (\nu_1 + \ldots + \nu_j)) \\
(L - M + (\nu_1 + \ldots + \nu_{j-1} + 1)) & \cdots & (L - M + (\nu_1 + \ldots + \nu_j)) \\
\vdots & \ddots & \vdots \\
(L - M + (\nu_1 + \ldots + \nu_{j-1} + 1)) & \cdots & (L - M + (\nu_1 + \ldots + \nu_j))
\end{bmatrix}_{\nu_j \times \nu_j}.
\] (3.40)

(Note that \(\nu_1 + \ldots + \nu_{j-1} = 0\) when \(j = 1\).) We get an expression for \(G\) as a product of determinants: \(G = \prod_{j=1}^{m} g_j\). To see the structure of these determinants more clearly, we write the binomial coefficients in different form:

\[
\binom{n}{k} = \frac{(n)_k}{k!},
\]

where

\[
(n)_0 := 1; \quad (n)_k := (n-1) \cdots (n-k+1), \quad k = 1, \ldots, n.
\]

As stated earlier, the falling factorial \((n)_k\) is a polynomial in \(n\) of degree \(k\). Just as in the proof of Lemma 3.11, the determinants in (3.40) are thus essentially polynomial Vandermonde matrices, the polynomials being \((x)_0, (x)_1, \ldots, (x)_{\nu_j-1}\).

Now

\[
g_j = \begin{bmatrix}
(L - M + (\nu_1 + \ldots + \nu_{j-1} + 1)) & \cdots & (L - M + (\nu_1 + \ldots + \nu_j)) \\
(L - M + (\nu_1 + \ldots + \nu_{j-1} + 1)) & \cdots & (L - M + (\nu_1 + \ldots + \nu_j)) \\
\vdots & \ddots & \vdots \\
(L - M + (\nu_1 + \ldots + \nu_{j-1} + 1)) & \cdots & (L - M + (\nu_1 + \ldots + \nu_j))
\end{bmatrix}_{\nu_j \times \nu_j} = \frac{1}{(L - M + (\nu_1 + \ldots + \nu_{j-1} + 1))!} \cdots \frac{1}{(L - M + (\nu_1 + \ldots + \nu_j))!}
\]

\[
\begin{bmatrix}
(L + 1)_{L-M+(\nu_1+\ldots+\nu_{j-1}+1)} & \cdots & (L + 1)_{L-M+(\nu_1+\ldots+\nu_j)} \\
(L + 2)_{L-M+(\nu_1+\ldots+\nu_{j-1}+1)} & \cdots & (L + 2)_{L-M+(\nu_1+\ldots+\nu_j)} \\
\vdots & \ddots & \vdots \\
(L + \nu_j)_{L-M+(\nu_1+\ldots+\nu_{j-1}+1)} & \cdots & (L + \nu_j)_{L-M+(\nu_1+\ldots+\nu_j)}
\end{bmatrix}_{\nu_j \times \nu_j},
\]

95
where, by Corollary 3.6,
\[
\begin{vmatrix}
(L + 1)_{L-M+(v_1+\ldots+v_{j-1}+1)} & \cdots & (L + 1)_{L-M+(v_1+\ldots+v_j)} \\
(L + 2)_{L-M+(v_2+\ldots+v_{j-1}+1)} & \cdots & (L + 2)_{L-M+(v_2+\ldots+v_j)} \\
\vdots & \ddots & \vdots \\
(L + v_j)_{L-M+(v_1+\ldots+v_{j-1}+1)} & \cdots & (L + v_j)_{L-M+(v_1+\ldots+v_j)} \\
\end{vmatrix}_{v_j \times v_j}
\]
\[
= (L + 1)_{L-M+(v_1+\ldots+v_{j-1}+1)} \cdots (L + v_j)_{L-M+(v_1+\ldots+v_{j-1}+1)} \\
\begin{vmatrix}
1_{M - \sum_{i=1}^{j-1} v_i} & \cdots & 1_{M - \sum_{i=1}^{j-1} v_i} \\
1_{M + 1 - \sum_{i=1}^{j-1} v_i} & \cdots & 1_{M + 1 - \sum_{i=1}^{j-1} v_i} \\
\vdots & \ddots & \vdots \\
1_{M + v_j - 1 - \sum_{i=1}^{j-1} v_i} & \cdots & 1_{M + v_j - 1 - \sum_{i=1}^{j-1} v_i} \\
\end{vmatrix}_{v_j \times v_j}
\]
\[
= (L + 1)_{L-M+(v_1+\ldots+v_{j-1}+1)} \cdots (L + v_j)_{L-M+(v_1+\ldots+v_{j-1}+1)} \\
\cdot (v_j - 1)!(v_j - 2)! \cdots 1!.
\]
So
\[
g_j = \frac{1}{(L - M + (v_1 + \ldots + v_{j-1} + 1))! \cdots (L - M + (v_1 + \ldots + v_j))!} \\
\cdot (L + 1)_{L-M+(v_1+\ldots+v_{j-1}+1)} \cdots (L + v_j)_{L-M+(v_1+\ldots+v_{j-1}+1)} \\
\cdot (v_j - 1)!(v_j - 2)! \cdots 1!
\]
\[
\neq 0
\]
for all \( j = 1, \ldots, m \). Hence also \( G = \prod_{j=1}^{m} g_j \neq 0 \).

### 3.5.6 Twin type II Padé approximations

Bombieri and Vaaler’s theorem 3.1 made us look for a big common factor from the \( M \times M \) minors of the matrix \( V(\overline{\pi}) \in \mathcal{M}_{M \times (L+1)}(\mathbb{Z}) \). This common factor is a result from the general polynomial common factor \( (3.35) \) of the \( M \times M \) minors of the matrix \( V(\overline{\pi}) \) proved in Theorem 3.13. As it happens, the explicit solution to the Padé approximation problem \( (3.25) \) requires calculating those same minors of \( V(\overline{\pi}) \) in the special case \( M = L \). (This has been explained in Section 3.4.1, where the procedure was successfully carried out in the ‘tame’ case.)
Chapter 3. Hermite-Thue equation

So, as a special case \((\nu_i = l_i, M = L)\) of Theorem 3.13 and Lemma 3.14 we get Theorem 3.3, a partial solution to the twin problem (3.25).

Proof of Theorem 3.3 Recall again that by Cramer’s rule, equation (3.29) has a solution

\[
[c_0, c_1, \ldots, c_L] = [\mathcal{V}[0] : -\mathcal{V}[1] : \ldots : (-1)^L \mathcal{V}[L]],
\]

where one of the \(L \times L\) minors \(\mathcal{V}[i] \in \mathbb{Z}[\alpha_1, \ldots, \alpha_m]\) is a non-zero polynomial by Lemma 3.14. Because of the homogeneous coordinates, we may divide by the common factor

\[
T(l, \alpha) = \alpha_1^{(i_1)} \cdots \alpha_m^{(i_m)} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j)^{\min\{i_i, i_j\}}
\]

(Theorem 3.13 with \(\nu_i = l_i\)), so that

\[
c_i = \frac{(-1)^i \mathcal{V}[i]}{T(l, \alpha)} =: \tau_i(l, \alpha), \quad i = 0, 1, \ldots, L.
\]

Now (3.28) gives

\[
B_{l,0}(t, \alpha) = \sum_{i=0}^{L} \frac{L!}{i!} \tau_i(l, \alpha) t^i \in \mathbb{Z}[t, \alpha].
\]

The proof of Theorem 3.3 could be carried out in the same way as the proof of Theorem 3.8, leaving out the term \(\frac{L!}{i!}\) in (3.28). This would result in a slightly different matrix, but essentially the same common factor. This is due to the connection between the minors of these matrices, apparent in (3.37). Since our starting point was the Bombieri-Vaaler version of Siegel’s lemma, we had to manipulate the equations so that the coefficient matrix has integer elements. This technicality only is the reason to the presence of \(\frac{L!}{i!}\) in Theorem 3.3.

Near the diagonal the common factor \(T(l, \alpha)\) is big, implying that the coefficients \(\tau_i(l, \alpha)\) are relatively small. However, the next example of the case \(m = 2, l_0 = 0, l_1 = 1, l_2 = 3\) in Table 3.5 already shows the big difference between the Padé polynomials

\[
A_{l,0}(t, \alpha) = \sum_{i=0}^{L} b_i t^i = \sum_{h=0}^{L} (L_0 - h)\sigma_{L_0 - h}(l, \alpha) t^h
\]

97
Chapter 3. Hermite-Thue equation

and

$$B_{l,0}(t, \alpha) = \sum_{i=0}^{L} \frac{L!}{i!} \tau_i(l, \alpha) t^i,$$

illustrating our decision to call the latter case ‘wild’.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$b_i$</th>
<th>$\frac{L!}{i!} \tau_i(l, \alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>24</td>
<td>$24 \cdot 35(5\alpha_1^2 - 5\alpha_1 \alpha_2 + \alpha_2^2)$</td>
</tr>
<tr>
<td>1</td>
<td>$-6(\alpha_1 + 3\alpha_2)$</td>
<td>$-24 \cdot 5(7\alpha_1^3 + 7\alpha_1^2 \alpha_2 - 13\alpha_1 \alpha_2^2 + 3\alpha_2^3)$</td>
</tr>
<tr>
<td>2</td>
<td>$6\alpha_2(\alpha_1 + \alpha_2)$</td>
<td>$12 \cdot 5\alpha_2(7\alpha_1^3 - 3\alpha_1^2 \alpha_2 - 3\alpha_1 \alpha_2^2 + \alpha_2^3)$</td>
</tr>
<tr>
<td>3</td>
<td>$-\alpha_2^2(3\alpha_1 + \alpha_2)$</td>
<td>$-4\alpha_2^2(3\alpha_1 - \alpha_2)(7\alpha_1^2 - 4\alpha_1 \alpha_2 - \alpha_2^2)$</td>
</tr>
<tr>
<td>4</td>
<td>$\alpha_1 \alpha_2^3$</td>
<td>$\alpha_1 \alpha_2^3(7\alpha_1^2 - 8\alpha_1 \alpha_2 + 2\alpha_2^2)$</td>
</tr>
</tbody>
</table>

Table 3.1: Case $m = 2$, $l_0 = 0$, $l_1 = 1$, $l_2 = 3$.  

98
Chapter 4

Euler’s factorial series

4.1 Introduction

Leonhard Euler laid the foundations for the study of diverging series in the eighteenth century. It is because of him that we are still today interested in the series

\[ F(t) := 2F_0(1, 1 | t) = \sum_{n=0}^{\infty} n!t^n. \tag{4.1} \]

(The substitution \( t = -1 \) yields Euler’s object of interest, the series he called *divergent series par excellence*. On how he managed to associate the value 0.5963... to it, see [54].) In the \( v \)-adic metric (where \( v \) extends \( p \) for some prime \( p \)) of a number field \( K \), the series \( F(t) \) converges to a point in the \( v \)-adic closure \( K_v \) when \( t \in K \) is such that \( |t|_v < p^{\frac{1}{2}} \). Thus we write \( \sum_{n=0}^{\infty} n!t^n =: F_v(t) \) when treating the series as a function in the \( v \)-adic domain \( K \).

Euler’s series (4.1) is a member of the class of \( F \)-series (series of the form \( \sum_{n=0}^{\infty} a_n n!z^n \), with a couple of growth conditions on the coefficients \( a_n \)) introduced by V. G. Chirski˘ı in [11, 12]. In those papers he answered the problem of the existence of global relations between members of the class of \( F \)-functions. As he points out in [13], the results can be refined in terms of estimating the prime \( p \) for which there exists a valuation \( v | p \) breaking the global relation. These estimates were made entirely effective by Bertrand, Chirski˘ı, and Yebbou [5]. In [5, Theorem 1.1] they describe an infinite collection of intervals each containing
Chapter 4. Euler’s factorial series

a prime number $p$ such that for some valuation $v|p$ it holds

$$\lambda_0 + \lambda_1 f_1(\xi) + \ldots + \lambda_m f_m(\xi) \neq 0,$$

(4.2)

where $\lambda_i \in \mathbb{Z}_K$ and $1, f_1(t), f_2(t), \ldots, f_m(t)$ are $F$-series that are linearly independent over $\mathbb{K}(z)$ and constitute a solution to a differential system $D$, and $\xi \in \mathbb{K}\{0\}$ is an ordinary point of the system $D$. What is more, the non-vanishing in (4.2) is replaced by a lower bound for the expression $|\lambda_0 + \lambda_1 f_1(\xi) + \ldots + \lambda_m f_m(\xi)|_v$.

In their recent paper [36], T. Matala-aho and W. Zudilin studied the irrationality of the $p$-adic value of Euler’s series (4.1), $F_p(\xi)$, at a point $\xi \in \mathbb{Z}\{0\}$ (i.e., global relations of the numbers $1$ and $F_p(\xi)$). They proved

**Proposition 4.1.** [36] Given $\xi \in \mathbb{Z}\{0\}$, let $\mathcal{P}$ be a subset of the prime numbers such that

$$\limsup_{n \to \infty} c^n n! \prod_{p \in \mathcal{P}} |n!|_p^2 = 0,$$

where

$$c = c(\xi, \mathcal{P}) := 4|\xi| \prod_{p \in \mathcal{P}} |\xi|_p^2.$$

Then either there exists a prime $p \in \mathcal{P}$ for which the value $F_p(\xi)$ is irrational, or there are two distinct primes $p, q \in \mathcal{P}$ such that $F_p(\xi) \neq F_q(\xi)$ (while $F_p(\xi), F_q(\xi) \in \mathbb{Q}$).

In Theorem 4.2 of this chapter we generalise their idea to a linear form

$$\Lambda_v := \lambda_0 + \lambda_1 F_v(\alpha_1) + \ldots + \lambda_m F_v(\alpha_m), \quad \lambda_i \in \mathbb{Z}_K,$$

in the values of Euler’s series (4.1) at $m$ given pairwise distinct algebraic integer points $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}_K \{0\}$. Theorem 4.2 states that in any collection $V$ of non-Archimedean valuations of $\mathbb{K}$ satisfying a certain condition, there exists a valuation $v' \in V$ such that $\Lambda_{v'} \neq 0$. The result can also be extended to the case of primes in arithmetic progressions, generalising the recent result of Ernvall-Hytönen et al. [17]. This is done in Theorem 4.18 of Section 4.9.

In the second main result, Theorem 4.5, we characterise an interval $I(m, H)$ (where $H$ is an upper bound for the height of the coefficients $\lambda_i$) from which one can find a prime $p$ such that there exists a valuation $v'|p$ for which

$$||\Lambda_{v'}||_{v'} > H^{-(m+1) - 114m^2 \log \log H \log \log H}.$$
Our method is grounded on explicit Padé approximations, whereas Bertrand, Chirskii, and Yebbou [5] rely on Siegel’s lemma. In addition, the functional dependence on $H$ in the error term of our lower bound is improved compared to that of Bertrand et al. [5] visible in (0.1).

The proofs of both our main results are based on Padé approximations which are used to construct small approximation forms for the values $F_v(\alpha_j)$, $j = 1, \ldots, m$. Therefore, before moving to the proofs of the theorems, we shall present explicit Padé approximations (with the orders of the remainders as free parameters) to the generalised factorial series

$$G(t) = \sum_{n=0}^{\infty} [P]_n t^n,$$

where $P(x)$ is a polynomial of degree one and $[P]_n := \prod_{k=0}^{n-1} P(k)$ (see Theorem 4.7).

4.2 Results

Let $K$ be any number field and let $F_v(t)$ denote the value of the series (4.1) at a point $t$ in the $v$-adic domain $K$.

Let $m \in \mathbb{Z}_{\geq 1}$ and choose $m$ pairwise distinct, non-zero algebraic integers $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}_K \setminus \{0\}$. Denote $\overline{\alpha} = (\alpha_1, \ldots, \alpha_m)^T$. We define

$$c_1 = c_1(\overline{\alpha}) = \prod_{v \in V_{\infty}} \left( \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right) \prod_{i=1}^{m} \left( \|\alpha_i\|_v + \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)$$

and

$$c_2 = c_2(\overline{\alpha}, V) = c_1 \prod_{v \in V} \max_{1 \leq j \leq m} \{\|\alpha_j\|_v\}$$

for any $V \subseteq V_0$.

**Theorem 4.2.** Let $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}_K$ be such that $\lambda_j \neq 0$ for at least one $j$. Suppose $V \subseteq V_0$ is a collection of non-Archimedean valuations of $K$ such that

$$\limsup_{l \to \infty} c_2^l (ml + m)^{\nu}(ml + m)! \prod_{v \in V} \| (ml)! \|_v = 0. \quad (4.4)$$

Then there exists a valuation $v' \in V$ for which

$$\lambda_0 + \lambda_1 F_{v'}(\alpha_1) + \ldots + \lambda_m F_{v'}(\alpha_m) \neq 0.$$
Chapter 4. Euler’s factorial series

Remark 4.3. Let us show that any collection $V \subseteq V_0$ whose complement in $V_0$ is finite satisfies condition [4.4]. Choose $v_1, \ldots, v_k \in V_0$ and let $V = V_0 \setminus \{v_1, \ldots, v_k\}$. Suppose in addition that $v_i|p_i$ for some $p_i \in \mathbb{P}$, $i = 1, \ldots, k$. Then, by recalling that $|n!|_p \geq p^{-\frac{n}{p^r}}$ (4.5) (see estimate (1.11)) and using the product formula (1.13), we get

$$c_2^l (ml + m)^\kappa (ml + m)! \prod_{v \in V} \| (ml)!\|_v$$

$$= \frac{c_2^l (ml + m)^\kappa (ml + m)!}{\left( \prod_{i=1}^{k} \| (ml)!\|_{v_i} \right) \prod_{v \in V_0} \| (ml)!\|_v}$$

$$= \frac{c_2^l (ml + m)^\kappa (ml + m)!}{\left( \prod_{i=1}^{k} \| (ml)!\|_{v_i} \right) (ml)!}$$

$$\leq \frac{c_2^l \left( \prod_{i=1}^{k} \frac{\nu_{v_i} \cdot ml + m}{p_i} \right) (ml)!}{\prod_{i=1}^{k} \frac{\nu_{v_i} \cdot ml + m}{p_i}}$$

$$= \frac{c_2^l \left( \prod_{i=1}^{k} \frac{\nu_{v_i} \cdot ml + m}{p_i} \right) (ml + m)!}{\prod_{i=1}^{k} \frac{\nu_{v_i} \cdot ml + m}{p_i}}$$

$$\to 0 \text{ as } l \to \infty.$$

Remark 4.4. From the previous remark it follows that there are infinitely many valuations $v \in V_0$ such that $\Lambda_v \neq 0$.

Theorem 4.5. Let $\log H \geq \sigma^s$ with $s = \max \{ e^{e^k} + 1, c_1 + 1, (m + 3)^2 + 1 \}$, $\kappa = [K : \mathbb{Q}]$. Suppose that $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}_K$ are such that at least one of them is non-zero and

$$\prod_{v \in V_0} \max_{0 \leq i \leq m} \{ \| \lambda_i \|_v \} \leq H.$$

Then there exists a prime

$$p \in \left\lceil \frac{\log H}{\log \log H} \right\rceil, \quad \frac{17m \log H}{\log \log H}$$

and a valuation $v'|p$ for which

$$\| \lambda_0 + \lambda_1 F_{v'}(\alpha_1) + \ldots + \lambda_m F_{v'}(\alpha_m) \|_{v'} > H^{-(m+1)-114m^2} \frac{\log \log \log H}{\log \log H}. \quad (4.6)$$

Remark 4.6. Letting $H$ have values in a very rapidly increasing sequence, something like $H_{i+1} = e^{\epsilon H_i}$, the intervals $I(m, H_i)$ will be distinct.
Chapter 4. Euler’s factorial series

4.2.1 Outline of the proofs

The idea behind the following proofs is to use Padé approximations to construct small linear forms

\[ s_{l,\mu,j} = b_{l,\mu,0} F_v(\alpha_j) - b_{l,\mu,j}, \quad b_{l,\mu,0}, b_{l,\mu,j} \in \mathbb{Z}_K, \quad j = 1, \ldots, m, \]

in the numbers \( F(\alpha_j) \). (Here \( l \in \mathbb{Z}_{\geq 1} \) and \( \mu \in \{0, 1, \ldots, m\} \) are auxiliary parameters.) With these equations the linear form

\[ \Lambda_v = \lambda_0 + \lambda_1 F_v(\alpha_1) + \ldots + \lambda_m F_v(\alpha_m) \]

under study can be written as

\[ b_{l,\mu,0} \Lambda_v = W + \lambda_1 s_{l,\mu,1} + \ldots + \lambda_m s_{l,\mu,m}, \tag{4.7} \]

where \( W = W(l, \mu) = \sum_{i=0}^m \lambda_i b_{l,\mu,i} \) is an integer element in \( K \). In case it is non-zero, the product formula implies

\[ 1 = \prod_v ||W||_v. \tag{4.8} \]

In the proof of Theorem 4.2, we shall assume that \( \Lambda_v = 0 \) for all \( v \in V \) (whence equation (4.7) gives \( W \) another representation as a linear combination of \( s_{l,\mu,i} \)), and then aim at a contradiction by estimating the product \( \prod_v ||W||_v \) from above. For this, we need estimates for the Padé coefficients \( b_{l,\mu,i}, s_{l,\mu,i} \), expressed in terms of the auxiliary parameter \( l \). These are very roughly

\[ ||b_{l,\mu,i}||_v \approx ||(ml)!||_v, \quad i = 0, 1, \ldots, m, \quad v \mid \infty, \]

\[ ||s_{l,\mu,j}||_v \approx ||(ml)!||_v, \quad j = 1, \ldots, m, \quad v \in V_0. \]

The contradiction with (4.8) is reached via the condition (4.4) when \( l \) is taken to infinity.

When the target is a precise lower bound for \( ||\Lambda_v||_v \), the use of the parameter \( l \) also becomes more subtle: We define the number \( \ell \) so that it is the largest \( l \) for which the expression

\[ N(l) \sim \log H + ml \log \log l - l \log l \]

is still positive. Then we make the assumption that

\[ ||b_{\ell+1,\mu,0} \Lambda_v||_v < ||\lambda_1 s_{\ell+1,\mu,1} + \ldots + \lambda_m s_{\ell+1,\mu,m}||_v \]

103
Chapter 4. Euler’s factorial series

for all \( v|p, p \in [\log(\ell + 1), m(\ell + 2)] \cap \mathbb{P} \). This leads to the estimate

\[
0 \leq \log \left( \prod_{v} \|W(\ell + 1, \mu)\|_{v} \right) \\
\approx \log H + m(\ell + 1) \log \log(\ell + 1) - (\ell + 1) \log(\ell + 1) < 0,
\]

giving the desired contradiction. It follows that there exists a prime

\[
p \in [\log(\ell + 1), m(\ell + 2)]
\]

and a valuation \( v'|p \) such that

\[
\|W(\ell + 1, \mu)\|_{v'} \leq \|\Lambda_{v'}\|_{v'},
\]

leading to

\[
1 \leq \left( \prod_{v \in V_{\infty}} \|W(\ell + 1, \mu)\|_{v} \right) \|\Lambda_{v'}\|_{v'}.
\]

This is the key to the lower bound for \( \|\Lambda_{v'}\|_{v'} \), and the final step is to give an estimate for the product \( \prod_{v \in V_{\infty}} \|W(\ell + 1, \mu)\|_{v} \). Approximately it is

\[
\log \left( \prod_{v \in V_{\infty}} \|W(\ell + 1, \mu)\|_{v} \right) \approx (m + 1) \log H + m^2 \ell \log \log \ell. \tag{4.10}
\]

The definition of \( \ell \) gives a connection between \( \ell \) and \( H \), enabling us to write the bound (4.10) and the interval (4.9) solely in terms of \( H \):

\[
\ell \log \log \ell \approx \frac{\log \log \log H}{\log \log H} \cdot \log H.
\]

As the attentive reader may have noted, one crucial point in the proofs is the non-vanishing of the quantity \( W(l, \mu) \). This is the part where the auxiliary parameter \( \mu \) is needed. A non-vanishing determinant of the Padé polynomials will ensure that for each \( l \in \mathbb{Z}_{\geq 1} \), there exists a \( \mu \in \{0, 1, \ldots, m\} \) such that \( W(l, \mu) \neq 0 \).

4.3 Padé approximations

Once again we shall be needing Padé approximation formulas. Set \( l_0 = 0, \beta_0 = 0 \) and let \( \overline{l} = (l_1, \ldots, l_m)^{T} \in \mathbb{Z}_{\geq 1}^{m}, \overline{\beta} = (\beta_1, \ldots, \beta_m)^{T} \) (omitting the zero coordi-
nates for convenience). Now (1.7) becomes
\[
\sigma_i(l, \beta) = (-1)^i \sum_{i_1 + \ldots + i_m = i} \binom{i_1}{i_1} \ldots \binom{i_m}{i_m} \beta_{1^{i_1-1}} \ldots \beta_{m^{i_m-1}}.
\]

4.3.1 Generalised factorial series

When \(l_1 = l_2 = \ldots = l_m\), the following theorem is a particular case of Theorem 2.2 in [33]. Due to the special nature of the function (4.3), however, we don’t need to restrict the parameters \(l_j\).

**Theorem 4.7.** Let \(G(t) = \sum_{n=0}^{\infty} [P]_n t^n\), where \(P(x)\) is a polynomial of degree one and \([P]_n = \prod_{k=0}^{n-1} P(k)\). Let \(\mu \in \mathbb{Z}_{\geq 0}\) and set
\[
A_{l,\mu,0}(t) = \sum_{i=0}^{L} \sigma_i(l, \beta) t^{L-i}.
\]

Then there exist polynomials \(A_{l,\mu,j}(t)\) and remainders \(R_{l,\mu,j}(t)\), \(j = 1, \ldots, m\), such that
\[
A_{l,\mu,0}(t)G(\beta_j t) - A_{l,\mu,j}(t) = R_{l,\mu,j}(t),
\]
where
\[
\begin{cases}
\deg A_{l,\mu,0}(t) = L, \\
\deg A_{l,\mu,j}(t) \leq L + \mu - 1, \\
\text{ord}_{t=0} R_{l,\mu,j}(t) \geq L + \mu + l_j.
\end{cases}
\]

**Proof.** Writing
\[
A_{l,\mu,0}(t) = \sum_{h=0}^{L} \sigma_{L-h}(l, \beta) [P]_{L-h+\mu} t^h,
\]
we have
\[
A_{l,\mu,0}(t)G(\beta_j t) = \sum_{N=0}^{\infty} r_{N,j} t^N,
\]
where
\[
r_{N,j} = \sum_{n+h=N}^{\min(L,N)} \sigma_{L-h}(l, \beta) \cdot \frac{[P]_n}{[P]_{L-h+\mu}} \cdot \beta_j^n
\]
\[
= \sum_{h=0}^{L} \sigma_{L-h}(l, \beta) \cdot \frac{[P]_{N-h}}{[P]_{L-h+\mu}} \cdot \beta_j^{N-h}.
\]
Chapter 4. Euler’s factorial series

When \( N = L + \mu + a, \ 0 \leq a \leq l_j - 1 \), then

\[
\begin{align*}
 r_{N,j} &= \beta_j^{\mu+a} \sum_{h=0}^{L} \sigma_{L-h}(l, \beta) \left( \prod_{k=1}^{a} P(L + \mu - h - 1 + k) \right) \beta_j^{L-h} \\
&= \beta_j^{\mu+a} \sum_{i=0}^{L} \sigma_{i}(l, \beta) \left( \prod_{k=1}^{a} P(i + \mu - 1 + k) \right) \beta_j^{i}.
\end{align*}
\]

(Note that the product above equals 1 when \( a = 0 \).) Since \( \deg P(x) = 1 \), we may write

\[
\prod_{k=1}^{a} P(i + \mu - 1 + k) = \sum_{k=0}^{a} p_k i^k,
\]

where the coefficients \( p_k \) do not depend on \( i \). Hence

\[
\begin{align*}
 r_{N,j} &= \beta_j^{\mu+a} \sum_{i=0}^{L} \sigma_{i}(l, \beta) \left( \sum_{k=0}^{a} p_k i^k \right) \beta_j^{i} = \beta_j^{\mu+a} \sum_{k=0}^{a} p_k \sum_{i=0}^{L} \sigma_{i}(l, \beta) i^k \beta_j^{i} = 0
\end{align*}
\]

due to (1.8). Thus we can choose

\[
A_{l,\mu,j}(t) = \sum_{N=0}^{L+\mu-1} r_{N,j} t^N
\]

(4.14)

and

\[
R_{l,\mu,j}(t) = \sum_{N=L+\mu+l_j}^{\infty} r_{N,j} t^N.
\]

(4.15)

4.3.2 Euler’s series

To prove Theorem 4.2, we need approximations to the series \( F(\alpha_j t) \). Thus we choose \( P(x) = 1 + x \) and \( \beta = \bar{\alpha} = (\alpha_1, \ldots, \alpha_m)^T \), and set \( l_j = l \in \mathbb{Z}_{\geq 1} \) for all \( j = 1, \ldots, m \). Theorem 4.7 gives

\[
A_{l,\mu,0}(t) = \sum_{i=0}^{m_l} \frac{\sigma_i}{(i + \mu)!} t^{m_l-i}, \quad \sigma_i = \sigma_i(l, \bar{\alpha}),
\]

106
and, directly by (4.14) and (4.13),

$$A_{l,j}(t) = \sum_{N=0}^{m^l+m-1} t^N \sum_{h=0}^{\min\{ml,N\}} \sigma_{ml-h} \cdot \frac{(N-h)!}{(ml-h+\mu)!} \cdot \alpha_j^{N-h}$$

when \( j = 1, \ldots, m \). Similarly by (4.15) and (4.13), for \( N = (m+1)l + \mu + k \), \( k \in \mathbb{N} \), we have

$$r_{N,j} = \sum_{h=0}^{ml} \sigma_{ml-h} \cdot \frac{(m+1)l + \mu + k - h)!}{(ml-h+\mu)!} \cdot \alpha_j^{(m+1)l+\mu+k-h}$$

$$= \alpha_j^{l+\mu+k} \sum_{i=0}^{ml} \sigma_i \cdot \frac{(i+\mu+l+k)!}{(i+\mu)!} \cdot \alpha_j^i$$

$$= \frac{l! k!}{k!} \alpha_j^{l+\mu+k} \sum_{i=0}^{ml} \sigma_i \binom{i+\mu+l+k}{i+\mu} \alpha_j^i,$$

so that

$$R_{l,j}(t) = lt^{(m+1)l+\mu} \sum_{k=0}^{\infty} t^k k! \binom{l+k}{k} \alpha_j^{l+\mu+k} \sum_{i=0}^{ml} \sigma_i \binom{i+\mu+l+k}{i+\mu} \alpha_j^i,$$

when \( j = 1, \ldots, m \).

In order to make the polynomials belong to \( \mathbb{Z}_k[t] \), we multiply everything by \( (ml+\mu)! \) and denote

$$B_{l,\mu,0}(t) := (ml+\mu)! A_{l,\mu,0}(t) = \sum_{i=0}^{ml} \sigma_i \cdot \frac{(ml+\mu)!}{(i+\mu)!} \cdot t^{ml-i},$$

$$B_{l,\mu,j}(t) := (ml+\mu)! A_{l,\mu,j}(t)$$

$$= (ml+\mu)! \sum_{N=0}^{ml+m-1} t^N \sum_{h=0}^{\min\{ml,N\}} \sigma_{ml-h} \cdot \frac{(N-h)!}{(ml-h+\mu)!} \cdot \alpha_j^{N-h},$$

$$S_{l,\mu,j}(t) := (ml+\mu)! R_{l,\mu,j}(t)$$

$$= (ml+\mu)! \frac{l! t^{(m+1)l+\mu}}{l} \sum_{k=0}^{\infty} k! \binom{l+k}{k} \alpha_j^{l+k+\mu} \sum_{i=0}^{ml} \sigma_i \binom{i+\mu+l+k}{i+\mu} \alpha_j^i.$$
Chapter 4. Euler’s factorial series

In this notation, the Padé approximation formula in \((4.11)\) may be rewritten as

\[
B_{l;0}(t)F(\alpha_j t) - B_{l;j}(t) = S_{l;j}(t), \quad j = 1, \ldots, m. \tag{4.17}
\]

4.4 Linear form and product formula

Let \(\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}_K\) be such that at least one of them is non-zero, and denote

\[
\Lambda_v := \lambda_0 + \lambda_1 F_v(\alpha_1) + \ldots + \lambda_m F_v(\alpha_m)
\]

when \(v \in V_0\). Equation \((4.17)\) gives

\[
s_{l;\mu,i} = b_{l;\mu,0}F_v(\alpha_i) - b_{l;\mu,i},
\]

where

\[
b_{l;\mu,i} = B_{l;\mu,i}(1), \quad i = 0, 1, \ldots, m; \quad s_{l;\mu,i} = S_{l;\mu,i}(1), \quad i = 1, \ldots, m.
\]

Assume that \(\Lambda_v = 0\) for all \(v \in V\), where the collection \(V\) satisfies condition \((4.4)\). Then also

\[
0 = b_{l;\mu,0}\Lambda_v = W + \lambda_1 s_{l;\mu,1} + \ldots + \lambda_m s_{l;\mu,m},
\]

where

\[
W = W(l, \mu) := \lambda_0 b_{l;\mu,0} + \lambda_1 b_{l;\mu,1} + \ldots + \lambda_m b_{l;\mu,m} \in \mathbb{Z}_K.
\]

If \(W \neq 0\), then

\[
1 = \prod_v \|W\|_v
\]

\[
\leq \left( \prod_{v \in V_\infty} \|W\|_v \right) \prod_v \|W\|_v
\]

\[
\leq \left( \prod_{v \in V_\infty} \|\lambda_0 b_{l;\mu,0} + \lambda_1 b_{l;\mu,1} + \ldots + \lambda_m b_{l;\mu,m}\|_v \right)
\]

\[
\cdot \prod_v \| - \lambda_1 s_{l;\mu,1} - \ldots - \lambda_m s_{l;\mu,m}\|_v
\]

\[
\leq \left( \prod_{v \in V_\infty} \left( \sum_{i=0}^m \|\lambda_i\|_v \right)^\max_{0 \leq i \leq m} \{\|b_{l;\mu,i}\|_v\} \right) \prod_{v \in V} \max_{1 \leq i \leq m} \{\|s_{l;\mu,i}\|_v\}.
\]

Next we shall see that such a non-zero \(W(l, \mu)\) actually exists.
4.5 Determinant

Lemma 4.8. When the numbers $\alpha_j$, $j = 1, \ldots, m$, are pairwise different and non-zero, we have

$$
\Delta(t) := \begin{vmatrix}
B_{l,0,0}(t) & B_{l,0,1}(t) & \cdots & B_{l,0,m}(t) \\
B_{l,1,0}(t) & B_{l,1,1}(t) & \cdots & B_{l,1,m}(t) \\
\vdots & \vdots & \ddots & \vdots \\
B_{l,m,0}(t) & B_{l,m,1}(t) & \cdots & B_{l,m,m}(t)
\end{vmatrix} \neq 0.
$$

Proof. By (4.12), the degrees of the entries are at most

$$
\begin{pmatrix}
ml & ml - 1 & \cdots & ml - 1 \\
ml & ml & \cdots & ml \\
\vdots & \vdots & \ddots & \vdots \\
ml & ml + m - 1 & \cdots & ml + m - 1
\end{pmatrix}.
$$

Hence

$$
\deg \Delta(t) \leq (m + 1)ml + \frac{(m - 1)m}{2}.
$$

Column operations together with (4.17) yield the representation

$$
\Delta(t) = \begin{vmatrix}
B_{l,0,0}(t) & -S_{l,0,1}(t) & \cdots & -S_{l,0,m}(t) \\
B_{l,1,0}(t) & -S_{l,1,1}(t) & \cdots & -S_{l,1,m}(t) \\
\vdots & \vdots & \ddots & \vdots \\
B_{l,m,0}(t) & -S_{l,m,1}(t) & \cdots & -S_{l,m,m}(t)
\end{vmatrix}.
$$

According to (4.12), the orders of the entries in (4.19) are at least

$$
\begin{pmatrix}
0 & (m + 1)l & \cdots & (m + 1)l \\
0 & (m + 1)l + 1 & \cdots & (m + 1)l + 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & (m + 1)l + m & \cdots & (m + 1)l + m
\end{pmatrix}.
$$

By expanding (4.19) by the first column we see that

$$
\ord \Delta(t) \geq \sum_{i=0}^{m-1} ((m + 1)l + i) = m(m + 1)l + \frac{(m - 1)m}{2}.
$$
Chapter 4. Euler’s factorial series

Thus \( \Delta(t) = bt^{m(m+1)t+\frac{(m-1)m}{2}} \), where the coefficient \( b \) is an \( m \times m \) determinant formed from the lowest term coefficients of the remainders \( -S_{l,\mu,j} \), \( \mu = 0, 1, \ldots, m-1 \), \( j = 1, \ldots, m \) (corresponding to \( k = 0 \) in (4.16)), multiplied by the lowest term coefficient of the polynomial \( B_{l,m,0}(t) \) which is \( \sigma_{ml} = (-1)^{ml} \):

\[
b = (-1)^{ml} \cdot (-1)^{(ml)!} \left( \prod_{\mu=0}^{m-1} (ml + \mu)! \right) \left( \prod_{j=1}^{m} \alpha_j^j \right) \cdot \det \left( \alpha_j^\mu \sum_{i=0}^{ml} \sigma_i (i + \mu + l) \alpha_j^i \right)_{j=1,\ldots,m; \mu=0,\ldots,m-1}.
\]

It remains to show that \( b \neq 0 \).

Since

\[
\binom{i + \mu + l}{i + \mu} = \frac{(i + \mu + l)!}{(i + \mu)! l!} = \frac{1}{l!} (i + \mu + 1) \cdots (i + \mu + l) = \frac{1}{l!} \left( i^l + \sum_{k=0}^{l-1} p_k i^k \right)
\]

for any \( \mu \in \{0, 1, \ldots, m - 1\} \), where the coefficients \( p_k \) do not depend on \( i \), we get

\[
\sum_{i=0}^{ml} \sigma_i (i + \mu + l) \alpha_j^i = \frac{1}{l!} \left( \sum_{i=0}^{ml} \sigma_i i^l \alpha_j^i + \sum_{k=0}^{l-1} p_k \sum_{i=0}^{ml} \sigma_i i^k \alpha_j^i \right) = \frac{1}{l!} \sum_{i=0}^{ml} \sigma_i i^l \alpha_j^i
\]

for all \( j = 1, \ldots, m \), \( \mu = 0, 1, \ldots, m - 1 \) by property (1.8). Hence

\[
b = (-1)^{m(l+1)} \left( \prod_{\mu=0}^{m-1} (ml + \mu)! \right) \left( \prod_{j=1}^{m} \alpha_j^j \right) \cdot \prod_{1 \leq i < j \leq m} (\alpha_j - \alpha_i)
\]

110
by the Vandermonde determinant formula. Here, using (1.6) and (1.10),
\[
\sum_{i=0}^{ml} \sigma_i \alpha_j^i = \left( \frac{d}{dw} \right)^l \prod_{i=1}^{m} (\alpha_i - w)^i \bigg|_{w=\alpha_j} = (-1)^l l! \alpha_j^l \prod_{i=1 \atop i \neq j}^{m} (\alpha_i - \alpha_j)^l \neq 0
\]
for all \( j = 1, \ldots, m \).

Lemma 4.9. For any given \( l \in \mathbb{Z}_{\geq 1} \), there exists a \( \mu \in \{0, 1, \ldots, m\} \) such that \( W(l, \mu) \neq 0 \).

Proof. From Lemma 4.8 it follows in particular that
\[
\begin{vmatrix}
 b_{l,0,0} & b_{l,0,1} & \cdots & b_{l,0,m} \\
 b_{l,1,0} & b_{l,1,1} & \cdots & b_{l,1,m} \\
 \vdots & \vdots & \ddots & \vdots \\
 b_{l,m,0} & b_{l,m,1} & \cdots & b_{l,m,m}
\end{vmatrix} = \Delta(1) \neq 0.
\]
We assumed that \((\lambda_0, \lambda_1, \ldots, \lambda_m)^T \neq \overline{0}\), so by linear algebra it follows that the quantity \( W(l, \mu) = \lambda_0 b_{l,\mu,0} + \lambda_1 b_{l,\mu,1} + \ldots + \lambda_m b_{l,\mu,m} \) must be non-zero for some \( \mu \in \{0, 1, \ldots, m\} \). \( \square \)

4.6 Estimates for the polynomials and remainders and proof of Theorem 4.2

As the last step in proving Theorem 4.2 we give upper bounds for the Padé polynomials and remainders. Now, using the triangle inequality and property (1.9) with \( v|\infty \),
\[
\|b_{l,\mu,0}\|_v = \|B_{l,\mu,0}(1)\|_v \\
= \left\| \sum_{i=0}^{ml} \sigma_i \left( \frac{(ml+\mu)!}{(i+\mu)!} \right) \right\|_v \\
\leq \left\| (ml)! \binom{ml+\mu}{\mu} \right\|_v \sum_{i=0}^{ml} \|\sigma_i\|_v \\
\leq \left\| (ml)! \binom{ml+\mu}{\mu} \right\|_v \prod_{j=1}^{m} (\|\alpha_j\|_v + 1)^j
\]
Chapter 4. Euler’s factorial series

and

\[ \|b_{l,\mu,j}\|_v = \|B_{l,\mu,j}(1)\|_v \]
\[ = \left\| \sum_{N=0}^{ml+\mu-h} \sum_{h=0}^{ml+\mu-h} \frac{(N-h)!}{(ml-h+\mu)!} \sigma_{ml-h} \alpha_j^{N-h} \right\|_v \]
\[ \leq \left\| \sum_{N=0}^{ml+\mu-h} \sum_{h=0}^{ml+\mu-h} \frac{(N-h)!}{(ml-h+\mu)!} \sigma_{ml-h} \alpha_j^{N-h} \right\|_v \]
\[ \leq \left\| \sum_{N=0}^{ml+\mu-h} \sum_{h=0}^{ml+\mu-h} \sigma_{ml-h} \left( \max \left\{ 1, \|\alpha_j\|_v \right\} \right)^{N-h} \right\|_v \]
\[ \leq \left\| \sum_{N=0}^{ml+\mu-h} \sum_{h=0}^{ml+\mu-h} \sigma_{ml-h} \left( \max \left\{ 1, \|\alpha_j\|_v \right\} \right)^{ml+m-1} \right\|_v \]
\[ \leq \left\| \sum_{N=0}^{ml+\mu-h} \sum_{h=0}^{ml+\mu-h} \sigma_{ml-h} \left( \max \left\{ 1, \|\alpha_j\|_v \right\} \right)^{ml+m-1} \right\|_v \]
\[ \leq \left\| \prod_{i=1}^{m} \left( \|\alpha_i\|_v + \max \left\{ 1, \|\alpha_j\|_v \right\} \right)^l \right\|_v \]

for all \( j = 1, \ldots, m, \mu = 0, 1, \ldots, m. \)

We still need non-Archimedean estimates for the remainders, so let now \( v \in V_0. \) Then

\[ \|s_{l,\mu,j}\|_v = \|S_{l,\mu,j}(1)\|_v \]
\[ = \left\| \sum_{k=0}^{\infty} \frac{(l+k)!}{k!} \sigma_{\alpha_j}^{l+k+\mu} \sum_{i=0}^{ml} \alpha_j^i \right\|_v \]
\[ \leq \left\| (ml+\mu)! \|\alpha_j\|_v^l \right\|_v \]

for all \( j = 1, \ldots, m, \mu = 0, 1, \ldots, m. \)

So, recalling property (1.14) of our normalised valuations, the expression in
Chapter 4. Euler’s factorial series

(4.18) becomes

$$\left( \prod_{v \in V} \left( \sum_{i=0}^{m} \|\lambda_i\|_v \right) \max_{0 \leq i \leq m} \{ \|b_{l,\mu, i}\|_v \} \right) \prod_{v \in V} \max_{1 \leq i \leq m} \{ \|s_{l,\mu, i}\|_v \} \right)$$

$$\leq \left( \prod_{v \in V} \left( \sum_{i=0}^{m} \|\lambda_i\|_v \right) (ml + m) \|(ml + m)!\|_v \right)$$

$$\cdot \left( \max_{1 \leq j \leq m} \{ 1, \|\alpha_j\|_v \} \right)^m \cdot \prod_{i=1}^{m} \left( \|\alpha_i\|_v + \max_{1 \leq j \leq m} \{ 1, \|\alpha_j\|_v \} \right)^l$$

$$\cdot \prod_{v \in V} \|(ml)!!\|_v \left( \max_{1 \leq j \leq m} \{ \|\alpha_j\|_v \} \right)^l$$

$$\leq \left( \prod_{v \in V} \left( \sum_{i=0}^{m} \|\lambda_i\|_v \right) \right) c_2 (ml + m)^{\kappa} (ml + m)! \cdot \prod_{v \in V} \|(ml)!!\|_v,$$

where

$$c_2 = \left( \prod_{v \in V} \left( \max_{1 \leq j \leq m} \{ 1, \|\alpha_j\|_v \} \right)^m \right) \prod_{i=1}^{m} \left( \|\alpha_i\|_v + \max_{1 \leq j \leq m} \{ 1, \|\alpha_j\|_v \} \right)^l \cdot \prod_{v \in V} \max_{1 \leq j \leq m} \{ \|\alpha_j\|_v \}.$$

**Proof of Theorem 4.2.** In Section 4.5 we saw that for every $l \in \mathbb{Z}_{\geq 1}$, there exists a $\mu \in \{ 0, 1, \ldots, m \}$ such that $W = W(l, \mu) \neq 0$. Hence estimate (4.18) holds for infinitely many $W(l, \mu)$, so that our assumption $\Lambda_v = 0$ for all $v \in V$ and estimates (4.18) and (4.20) lead to

$$1 \leq \left( \prod_{v \in V} \left( \sum_{i=0}^{m} \|\lambda_i\|_v \right) \right) c_2 (ml + m)^{\kappa} (ml + m)! \prod_{v \in V} \|(ml)!!\|_v$$

which holds for infinitely many $l$. This is a contradiction with condition (4.4), and thus there must exist a valuation $v' \in V$ such that $\Lambda_{v'} \neq 0$. 

113
Chapter 4. Euler’s factorial series

4.7 Lower bound: proof of Theorem 4.5

4.7.1 Product formula again

The fundamental product formula (1.13) is the starting point for the proof of our second theorem as well. We repeat Section 4.4 with a slightly more refined assumption. First we need some notation though.

Let \( m \in \mathbb{Z}_{\geq 1} \) and \( \log H \geq s e^s \), where

\[
s = \max \{ e^\kappa + 1, c_1 + 1, (m + 3)^2 + 1 \},
\]

\[
\kappa = [K : \mathbb{Q}],
\]

\[
c_1 = \prod_{v \in V_\infty} \left( \max_{1 \leq j \leq m} \{ 1, \| \alpha_j \|_v \} \right)^{m} \prod_{i=1}^{m} \left( \| \alpha_i \|_v + \max_{1 \leq j \leq m} \{ 1, \| \alpha_j \|_v \} \right).
\]

Suppose that \( \lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}_K \) are such that at least one of them is non-zero and

\[
\prod_{v \in V_\infty} \max_{0 \leq i \leq m} \{ \| \lambda_i \|_v \} \leq H.
\]

Define

\[
N(l) := \log H + \left( \frac{2(m + 1)}{l} + \frac{2m}{\log \log l} \right)
\]

\[
+ \frac{1}{\log \log l} + \frac{(\kappa - \frac{1}{2}) \log l}{\log \log l} + \frac{\kappa \log m}{\log \log l}
\]

\[
+ \frac{\kappa \log(m + 1)}{l \log \log l} + \frac{\kappa}{l^2 \log \log l} \right) l \log \log l - l \log l.
\]

and let

\[
\ell := \max \{ l \in \mathbb{Z}_{\geq 2} \mid N(l) \geq 0 \}.
\]

Denote, as before,

\[
\Lambda_v = \lambda_0 + \lambda_1 F_v(\alpha_1) + \ldots + \lambda_m F_v(\alpha_m).
\]

We saw in Section 4.4 that

\[
b_{l, \mu, 0} \Lambda_v = W + \lambda_1 s_{l, \mu, 1} + \ldots + \lambda_m s_{l, \mu, m},
\]

114
where

\[ W = W(l, \mu) = \lambda_0 b_{l,0} + \lambda_1 b_{l,1} + \ldots + \lambda_m b_{l,m} \in \mathbb{Z}_K. \quad (4.24) \]

By Lemma 4.9 we know that \( W(\ell+1, \mu) \neq 0 \) for some \( \mu \in \{0, 1, \ldots, m\} \). Assume that

\[ \|b_{\ell+1,0}A_v\|_v < \|\lambda_1 s_{\ell+1,1} + \ldots + \lambda_m s_{\ell+1,m}\|_v \]

for all \( v|p, p \in [\log(\ell+1), m(\ell+2)] \cap \mathbb{P} \). (The intersection certainly is non-empty due to Bertrand’s postulate. As for the choice of this interval, see Remark 4.12.) Then

\[ \|W(\ell+1, \mu)\|_v = \|b_{\ell+1,0}A_v - (\lambda_1 s_{\ell+1,1} + \ldots + \lambda_m s_{\ell+1,m})\|_v \]

\[ = \|\lambda_1 s_{\ell+1,1} + \ldots + \lambda_m s_{\ell+1,m}\|_v \quad (4.25) \]

for all \( v|p, p \in [\log(\ell+1), m(\ell+2)] \cap \mathbb{P} \).

Now we use the product formula and plug in the two representations (4.24) and (4.25):

\[
1 = \prod_v \|W(\ell+1, \mu)\|_v \\
\leq \left( \prod_{v \in V_\infty} \|W\|_v \right) \prod_{p \in [\log(\ell+1), m(\ell+2)] \cap \mathbb{P}} \|W\|_v \\
= \left( \prod_{v \in V_\infty} \left\| \sum_{i=0}^m \lambda_i b_{\ell+1,i} \right\|_v \right) \prod_{p \in [\log(\ell+1), m(\ell+2)] \cap \mathbb{P}} \|\sum_{i=1}^m \lambda_i s_{\ell+1,i}\|_v. \quad (4.26)
\]

On the Archimedean side we have

\[
\left\| \sum_{i=0}^m \lambda_i b_{\ell+1,i} \right\|_v \leq (m+1) \max_{0 \leq i \leq m} \{\|\lambda_i\|_v\} \max_{0 \leq i \leq m} \{\|b_{\ell+1,i}\|_v\}, \quad v \in V_\infty,
\]
so, using the estimates made in Section 4.6 together with property (1.14),

\[
\prod_{v \in V_\infty} \left| \sum_{i=0}^{m} \lambda_i b_{\ell+1,\mu, i} \right|_v \\
\leq (m + 1)^\kappa H \left( \prod_{v \in V_\infty} \left| (m(\ell + 1) + m)!(m(\ell + 1) + \mu)! \right|_v \\
\cdot \left( \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^{m(\ell + 1)} \prod_{i=1}^{m} \left( \|\alpha_i\|_v + \max_{1 \leq j \leq m} \{1, \|\alpha_j\|_v\} \right)^{\ell + 1} \right) \\
\leq (m + 1)^\kappa (m(\ell + 1) + m)^\kappa H \ell^{\ell + 1} (m(\ell + 1) + \mu)!. 
\]

(4.27)

(Recall that \#V_\infty \leq \kappa = [K : \mathbb{Q}].)

As for the case \(v \mid p\), we get

\[
\left| \sum_{i=1}^{m} \lambda_i s_{\ell+1,\mu, i} \right|_v \leq \max_{1 \leq i \leq m} \{\|s_{\ell+1,\mu, i}\|_v\},
\]

so, by the estimates of Section 4.6 and property (1.14),

\[
\prod_{p \in [\log(\ell + 1), m(\ell + 2)]} \prod_{v \mid p} \left| \sum_{i=1}^{m} \lambda_i s_{\ell+1,\mu, i} \right|_v \\
\leq \prod_{p \in [\log(\ell + 1), m(\ell + 2)]} \prod_{v \mid p} \left| (m(\ell + 1) + \mu)! (\ell + 1)! \right|_v \\
= \prod_{p \in [\log(\ell + 1), m(\ell + 2)]} |(m(\ell + 1) + \mu)! (\ell + 1)!|_p. 
\]

(4.28)

Let us now combine estimates (4.26), (4.27), and (4.28):

\[
1 \leq \left( \prod_{v \in V_\infty} \left| \sum_{i=0}^{m} \lambda_i b_{\ell+1,\mu, i} \right|_v \right) \prod_{p \in [\log(\ell + 1), m(\ell + 2)]} \prod_{v \mid p} \left| \sum_{i=1}^{m} \lambda_i s_{\ell+1,\mu, i} \right|_v \\
\leq (m + 1)^\kappa (m(\ell + 1) + m)^\kappa H \ell^{\ell + 1} (m(\ell + 1) + \mu)! \\
\cdot \prod_{p \in [\log(\ell + 1), m(\ell + 2)]} |(m(\ell + 1) + \mu)! (\ell + 1)!|_p \\
= \frac{(m + 1)^\kappa (m(\ell + 2))^\kappa H \ell^{\ell + 1} \prod_{p \in [\log(\ell + 1), m(\ell + 2)]} |(\ell + 1)!|_p}{\prod_{p < \log(\ell + 1)} |(m(\ell + 1) + \mu)!|_p} =: \Omega. 
\]

116
Chapter 4. Euler’s factorial series

Note that $m(\ell + 1) + \mu \leq m(\ell + 2)$, whence the equality on the last row.

4.7.2 Deriving contradiction

We are working to establish a contradiction with (4.29), so let us study the expression $\log \Omega$ more closely. First of all, we have

$$
\Omega = \frac{(m + 1)^\kappa (m(\ell + 2))^\kappa H_{1}^{\ell + 1} \prod_{p < \log(\ell + 1)} |(m(\ell + 1) + \mu)!|_p}{\prod_{p < \log(\ell + 1)} |(m(\ell + 1) + \mu)!|_p}
$$

$$
= \frac{(m + 1)^\kappa (m(\ell + 2))^\kappa H_{1}^{\ell + 1}}{(\ell + 1)! \prod_{p < \log(\ell + 1)} |(m(\ell + 1) + \mu)!|_p}
$$

$$
\leq \frac{(m + 1)^\kappa (m(\ell + 2))^\kappa H_{1}^{\ell + 1} \prod_{p < \log(\ell + 1)} p^{m(\ell + 1) + \mu + (\ell + 1)} p^{-1}}{(\ell + 1)!}
$$

because of the product formula and property (4.5). With Stirling’s formula (Lemma 1.12) and estimate $\mu \leq m$ we get

$$
\log \Omega \leq \log \left( \frac{(m + 1)^\kappa (m(\ell + 2))^\kappa H_{1}^{\ell + 1} \prod_{p < \log(\ell + 1)} p^{m(\ell + 1) + \mu + (\ell + 1)} p^{-1}}{(\ell + 1)!} \right)
$$

$$
\leq \kappa \log(m + 1) + \kappa \log(m(\ell + 2)) + \log H + (\ell + 1) \log c_1 +
$$

$$
\sum_{p < \log(\ell + 1)} \log p \frac{m(\ell + 2) + (\ell + 1)}{p - 1} - (\ell + 1) + \frac{1}{2} \log(\ell + 1) + (\ell + 1)
$$

$$
\leq \kappa \log(m + 1) + \kappa \log m + \kappa \log(\ell + 2) + \log H + (\ell + 1) \log c_1
$$

$$
+ (m(\ell + 2) + (\ell + 1)) \sum_{p < \log(\ell + 1)} \frac{\log p}{p - 1} - (\ell + 1) \log(\ell + 1) - \frac{1}{2} \log(\ell + 1) + (\ell + 1)
$$

$$
\leq \log H + \kappa \log m + \kappa \log(m + 1) + \frac{\kappa}{\ell + 1} + \left( \kappa - \frac{1}{2} \right) \log(\ell + 1)
$$

$$
+ \left( \log c_1 + \left( m + 1 + \frac{m}{\ell + 1} \right) \sum_{p < \log(\ell + 1)} \frac{\log p}{p - 1} + 1 \right) (\ell + 1)
$$

- (\ell + 1) \log(\ell + 1),

117
where \( \log(\ell + 2) < \log(\ell + 1) + \frac{1}{\ell + 1} \) by the mean value theorem.

To be able to continue, we need to know how the sum \( \sum_{p < x} \frac{\log p}{p-1} \) behaves. Help is found from [43] (see the corollary of Theorem 6):

**Lemma 4.10.** [43]  
\[
\sum_{p \leq x} \frac{\log p}{p} < \log x, \quad x > 1.
\]

Since \( p - 1 \geq \frac{x}{2} \) for all primes \( p \), it follows that

\[
\sum_{p < x} \frac{\log p}{p-1} \leq 2 \sum_{p < x} \frac{\log p}{p} < 2 \log x. \tag{4.31}
\]

Combining estimates (4.29), (4.30), and (4.31), we have

\[
0 \leq \log \Omega
\]

\[
\leq \log H + \kappa \log m + \kappa \log(m + 1) + \frac{\kappa}{\ell + 1} + \left( \kappa - \frac{1}{2} \right) \log(\ell + 1)
\]

\[
+ \left( \log c_1 + \left( m + 1 + \frac{m}{\ell + 1} \right) \sum_{p < \log(\ell + 1)} \frac{\log p}{p-1} + 1 \right) (\ell + 1)
\]

\[
- (\ell + 1) \log(\ell + 1)
\]

\[
< \log H + \kappa \log m + \kappa \log(m + 1) + \frac{\kappa}{\ell + 1} + \left( \kappa - \frac{1}{2} \right) \log(\ell + 1)
\]

\[
+ \left( \log c_1 + 2 \left( m + 1 + \frac{m}{\ell + 1} \right) \log \log(\ell + 1) + 1 \right) (\ell + 1)
\]

\[
- (\ell + 1) \log(\ell + 1)
\]

\[
< \log H + \left( 2(m + \frac{2m}{\ell + 1} + \frac{\log c_1}{\log(\ell + 1)} + \frac{1}{\log(\ell + 1)}
\]

\[
+ \left( \kappa - \frac{1}{2} \right) \log(\ell + 1) \right) \log(\ell + 1) + \frac{\kappa \log m}{(\ell + 1) \log(\ell + 1)} + \frac{\kappa \log(m + 1)}{(\ell + 1) \log(\ell + 1)}
\]

\[
+ \frac{\kappa}{(\ell + 1)^2 \log(\ell + 1)} \right) (\ell + 1) \log(\ell + 1) - (\ell + 1) \log(\ell + 1)
\]

\[
= N(\ell + 1) < 0,
\]

118
a contradiction with (4.23). Thus there must exist a prime

\[ p \in [\log(\ell + 1), m(\ell + 2)] \]  

(4.32)

and a valuation \( v' \mid p \) such that

\[ \| b_{\ell+1,\mu,0} \Lambda_{v'} \|_{v'} \geq \| \lambda_1 s_{\ell+1,\mu,1} + \ldots + \lambda_m s_{\ell+1,\mu,m} \|_{v'} . \]

Then, for this valuation \( v' \),

\[ \| W \|_{v'} = \| b_{\ell+1,\mu,0} \Lambda_{v'} - (\lambda_1 s_{\ell+1,\mu,1} + \ldots + \lambda_m s_{\ell+1,\mu,m}) \|_{v'} \]
\[ \leq \| b_{\ell+1,\mu,0} \Lambda_{v'} \|_{v'} \]
\[ \leq \| \Lambda_{v'} \|_{v'} , \]

and

\[ 1 = \prod_v \| W \|_v \leq \left( \prod_{v \in V_\infty} \| W \|_v \right) \| W \|_{v'} \leq \left( \prod_{v \in V_\infty} \| W \|_v \right) \| \Lambda_{v'} \|_{v'} . \]  

(4.33)

### 4.7.3 Bounds for \( \ell \)

For the final stages of the proof, we need to express the number \( \ell \) in terms of the height \( H \). This is done with the help of Lemmas 1.10 and 1.11.

Now, \( N(\ell + 1) < 0 \) implies

\[ (\ell + 1) \log(\ell + 1) \geq \log H \geq se^s , \]  

(4.34)

and further, by applying the \( z \)-function, \( \ell + 1 > e^s \). According to (4.21), we have

\[ \ell > e^s - 1 \geq \max \left\{ e^{e^n}, e^{e^1}, e^{(m+3)^2} \right\} . \]  

(4.35)

Hence, using the lower bound (4.35) and the fact that \( m \geq 1 \), we may estimate
from the definition of $N(l)$ in (4.22):

$$0 \leq N(l) < \log H + \left(2(m + 1) + \frac{2m}{e^{(m+3)^2}} + 1 + \frac{1}{2 \log(m + 3)} \right.$$

$$+ \left(\kappa - \frac{1}{2}\right) \frac{(m + 3)^2}{e^{(m+3)^2} \cdot \kappa} + \frac{\kappa \log m}{e^{(m+3)^2} \cdot \kappa} + \frac{\kappa \log(m + 1)}{e^{(m+3)^2} \cdot \kappa}$$

$$+ \left(\kappa \frac{e^{2/(m+3)^2} \cdot \kappa}\right) \ell \log \log \ell - \ell \log \ell$$

$$\leq \log H + \left(2(m + 1) + 1 + 0.360674 + 3 \cdot 10^{-6}\right) \ell \log \log \ell - \ell \log \ell$$

$$< \log H + (2m + 3.361) \ell \log \log \ell - \ell \log \ell.$$  

Thus

$$\ell \log \ell \left(1 - \frac{(2m + 3.361) \log \log \ell}{\log \ell}\right) \leq \log H,$$

where

$$\frac{\log \log \ell}{\log \ell} < \frac{2 \log(m + 3)}{(m + 3)^2}$$

by (4.35), and so

$$1 - \frac{(2m + 3.361) \log \log \ell}{\log \ell} > 1 - \frac{(2m + 3.361) \cdot 2 \log(m + 3)}{(m + 3)^2} > 0$$

(4.38)

for all $m \geq 1$.

By inequalities (4.37) and (4.38) and the lower bound in (4.35), we have

$$\frac{(m + 3)^2}{(m + 3)^2 - (2m + 3.361) \cdot 2 \log(m + 3)} \cdot \log H > \ell \log \ell > (m + 3)^2 e^{(m+3)^2},$$

so we may apply Lemma 1.11 with $r = (m + 3)^2$:

$$\ell < z \left(\frac{(m + 3)^2}{(m + 3)^2 - (2m + 3.361) \cdot 2 \log(m + 3) \cdot \log H}\right)$$

$$\leq \left(1 + \frac{2 \log(m + 3)}{(m + 3)^2}\right) \frac{(m + 3)^2}{\log \log H - (m + 3)^2 \cdot 2 \log(m + 3)} \cdot \log H.$$
4.7.4 Measure

To get the measure from (4.33), we need an upper bound for the product
\[ \prod_{v \in V} \|W\|_v \]. Back in (4.27) we estimated that
\[ \|Y\|_v \leq (m + 1)^\kappa (m(\ell + 2))^{\kappa} H c_1^{\ell + 1} (m(\ell + 2))! \]
(taking into account that \( \mu \leq m \)). From estimate (4.36) it follows that
\[ \ell \log \ell < (2m + 3.361) \ell \log \ell + \log H \]
and by the mean value theorem, we have \( \log(\ell + 2) < \frac{2}{\ell} + \log \ell \). With these estimates we get
\[
\log \left( \prod_{v \in V} \|W\|_v \right) \\
\leq \log ((m + 1)^\kappa (m(\ell + 2))^{\kappa} H c_1^{\ell + 1} (m(\ell + 2))!) \\
\leq \kappa \log(m + 1) + \kappa \log m + \kappa \log(\ell + 2) + \log H + (\ell + 1) \log c_1 \\
+ (m(\ell + 2)) \log(m(\ell + 2)) \\
= \kappa \log(m + 1) + \kappa \log m + \kappa \log(\ell + 2) + \log H + \ell \log c_1 + \log c_1 \\
+ (m \log m) \ell + m \ell \log(\ell + 2) + 2m \log m + 2m \log(\ell + 2) \\
\leq \kappa \log(m + 1) + \kappa \log m + \frac{2\kappa}{\ell} + \kappa \log \ell + \log H + \ell \log c_1 + \log c_1 \\
+ (m \log m) \ell + m \ell \log(\ell + 2) + 4m \frac{m}{\ell} \\
< \kappa \log(m + 1) + \kappa \log m + \frac{2\kappa}{\ell} + \kappa \log \ell + \log H \\
+ \ell \log c_1 + \log c_1 + (m \log m) \ell + m ((2m + 3.361) \ell \log \ell + \log H) \\
+ 2m + 2m \log m + 2m \log \ell + \frac{4m}{\ell} \\
= (m + 1) \log H + \left( \frac{\kappa \log(m + 1)}{\ell \log \ell} + \frac{\kappa \log m}{\ell \log \ell} + \frac{2\kappa}{\ell^2 \log \ell} + \frac{\kappa \log \ell}{\ell \log \ell} \\
+ \frac{\log c_1}{\log \ell} + \frac{\log c_1}{\ell \log \ell} + \frac{m \log m}{\log \ell} + 2m^2 + 3.361m + \frac{2m}{\ell \log \ell} \\
+ \frac{2m \log m}{\ell \log \ell} + \frac{2m \log \ell}{\ell \log \ell} + \frac{4m}{\ell^2 \log \ell} \right) \ell \log \ell.
\]
Chapter 4. Euler’s factorial series

In the coefficient of $\ell \log \log \ell$, we have (using the bound (4.35) and the fact that $m \geq 1$)

$$\frac{m \log m}{\log \log \ell} < \frac{m \log m}{2 \log(m + 3)} < \frac{m}{2}, \quad \frac{\log c_1}{\log \log \ell} < 1,$$

and the rest of the fractions together are less than 0.0000034. Hence

$$\log \left( \prod_{v \in V_{\infty}} ||W||_v \right) \leq (m + 1) \log H$$

$$+ (2m^2 + 3.861m + 1.0000034) \ell \log \log \ell.$$ 

By (4.39) and the assumption $\log H \geq se^a > (m + 3)^2 e^{(m+3)^2}$, we have

$$\ell < \frac{(1 + \frac{2 \log(m+3)}{(m+3)^2}) (m + 3)^2}{(m + 3)^2 - (2m + 3.361) \cdot 2 \log(m + 3)} \cdot \frac{\log H}{2 \log(m + 3) + (m + 3)^2}$$

$$= \frac{1}{(m + 3)^2 - (2m + 3.361) \cdot 2 \log(m + 3)} \cdot \log H$$

$$< \log H.$$

Thus

$$\log \log \ell < \log \log \log H.$$  

(4.41)

Let us next estimate $(2m^2 + 3.861m + 1.0000034) \ell$, again using (4.39):

$$(2m^2 + 3.861m + 1.0000034) \ell$$

$$\leq \frac{(2m^2 + 3.861m + 1.0000034) (1 + \frac{2 \log(m+3)}{(m+3)^2}) (m + 3)^2}{(m + 3)^2 - (2m + 3.361) \cdot 2 \log(m + 3)} \cdot \frac{\log H}{\log \log H}$$

$$= \frac{m^2 (2 + \frac{3.861}{m} + \frac{1.0000034}{m^2}) (1 + \frac{2 \log(m+3)}{(m+3)^2})}{1 - \frac{2 \cdot 2m \log(m+3)}{(m+3)^2} - \frac{2 \cdot 3.361 \log(m+3)}{(m+3)^2}} \cdot \frac{\log H}{\log \log H}$$

$$< 114m^2 \cdot \frac{\log H}{\log \log H}.$$  

(4.42)

since $m \geq 1$.

Combining estimates (4.40), (4.41), and (4.42), yields

$$\log \left( \prod_{v \in V_{\infty}} ||W||_v \right) < \left( (m + 1) + 114m^2 \cdot \frac{\log \log \log H}{\log \log H} \right) \log H,$$

122
so that inequality (4.33) implies

$$\|\Lambda \|_{\psi} \geq \prod_{v \in V_{\infty}} \|W\|_{v} > H^{-(m+1) - 114m^2} \frac{\log \log \log H}{\log \log H}.$$ 

### 4.7.5 Infinitely many intervals

We still need an upper estimate for $m(\ell + 1)$ in terms of the height $H$ in order to write the interval (4.32) with respect to $H$. Once more we use the bound in (4.39) and the assumption $\log H > (m + 3)^2 e^{(m+3)^2}$:

$$m(\ell + 2) \leq m \cdot \frac{1 + 2 \log(m+3)}{(m+3)^2 - (2m + 3.361) \cdot 2 \log(m + 3)} \cdot \frac{\log H}{\log \log H} + 2m$$

$$= m \left( 1 - \frac{2.2m \log(m+3)}{(m+3)^2} - \frac{2.361 \log(m+3)}{(m+3)^2} + \frac{2 \log \log H}{\log \log H} \cdot \log \log H \cdot \right) \cdot \frac{\log H}{\log \log H}$$

$$\leq m \left( 1 - \frac{2.2m \log(m+3)}{(m+3)^2} - \frac{2.361 \log(m+3)}{(m+3)^2} + \frac{4 \log(m + 3) + 2(m + 3)^2}{(m + 3)^2 e^{(m+3)^2}} \right) \cdot \frac{\log H}{\log \log H}$$

$$< 17m \cdot \frac{\log H}{\log \log H}$$

since $m \geq 1$.

By (4.34) and Lemma 1.11, we have

$$\log(\ell + 1) > \log(z(\log H)) > \log(z_1(\log H)) = \log \left( \frac{\log H}{\log \log H} \right).$$

Combining this with (4.43) above leads to

$$[\log(\ell + 1), m(\ell + 2)] \subseteq \log \left( \frac{\log H}{\log \log H} \right) \cdot \frac{17m \log H}{\log \log H} =: I(m, H).$$

This ends the proof of Theorem 4.5.
Remark 4.11. The constants 114 and 17 can be improved by adjusting the lower bound of $\log H$, i.e., the choice of $s$ in (4.21). For instance, taking $(m + 3)^3$ instead of $(m + 3)^2$ will reduce them considerably.

Remark 4.12. There is a connection between the width of the interval $I(m, H)$ and the error term in the lower bound (4.6). Our choice of $\log(\ell + 1)$ in the interval (4.32) results in the term $\log \log \log H$ in (4.6) (see (4.41)), improving the corresponding lower bound of Bertrand et al. [5] for this function. This is done at a cost, though, since our interval $I(m, H)$ is wider than theirs. Had we chosen $e^{\sqrt{\log(\ell + 1)}}$ instead of $\log(\ell + 1)$, we would have ended up with $\sqrt{\log \log H}$ instead of $\log \log \log H$. Then the dependence on $H$ in the error term of (4.6) would have been $\frac{1}{\log \log H}$, just as it is in [5], and the interval $I(m, H)$ would have had $\exp\left(\sqrt{\log \frac{H}{\log \log H}}\right)$ as its lower bound, very much like in [5] and [55].

The best lower bound (in terms of $H$) would have been achieved by considering an interval of the form $[2, ml]$ with no dependence on $l$ in the lower bound, because the empty sum $\sum_{p < 2} \frac{\log p}{p - 1}$ would not then cause an extra term in our estimates. The disappearing of $\log \log l$ from the estimates would mean that we would have had $\frac{1}{\log \log H}$ instead of $\frac{1}{\log \log \log H}$ in the error term. This is in line with the exponential function (see Chapter 2). However, this result won’t give us infinitely many distinct primes when $H$ grows, like Theorem 4.5 does.

4.8 Corollaries and examples

4.8.1 The field of rationals

When $\mathbb{K} = \mathbb{Q}$, Theorem 4.12 reduces to:

**Corollary 4.13.** Let $m \in \mathbb{Z}_{\geq 1}$ and $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}$, where $\lambda_j \neq 0$ for at least one $j$. Choose $m$ pairwise distinct, non-zero integers $\alpha_j \in \mathbb{Z} \setminus \{0\}$, $j = 1, \ldots, m$. Suppose $P$ is a subset of the prime numbers such that

$$\limsup_{l \to \infty} c_2(ml + m)(ml + m)! \prod_{p \in P} (|ml|!!)_p = 0,$$

where

$$c_2 = \left(\max_{1 \leq j \leq m} \{1, |\alpha_j|\}\right)^m \left(\prod_{i=1}^{m} (|\alpha_i| + \max_{1 \leq j \leq m} \{1, |\alpha_j|\})\right) \prod_{p \in P} \max_{1 \leq j \leq m} \{ |\alpha_j|_p \}.$$
Then there exists a prime \( p' \in P \) for which

\[
\lambda_0 + \lambda_1 F_{p'}(\alpha_1) + \ldots + \lambda_m F_{p'}(\alpha_m) \neq 0.
\]

**Example 4.14.** For instance, take \( \alpha_1 = 1 \) and \( \alpha_2 = -1 \). Then, if \( P \subseteq \mathbb{P} \) is such that

\[
\limsup_{l \to \infty} 4^l (2l + 2)(2l + 2)! \prod_{p \in P} |(2l)!!|_p = 0,
\]

there exists a prime \( p' \in P \) for which

\[
\lambda_0 + \lambda_1 F_{p'}(1) + \lambda_2 F_{p'}(-1) \neq 0.
\]

In particular, taking \( \alpha_0 = 2a \in \mathbb{Z} \) and \( \alpha_1 = -b \in \mathbb{Z} \), it follows that there exists a prime \( p' \in P \) such that

\[
a - b \sum_{n=0}^{\infty} (2n)! \neq 0,
\]

i.e., \( \sum_{n=0}^{\infty} (2n)! \neq \frac{a}{b} \) for some \( p' \in P \).

### 4.8.2 Linear recurrences

A sequence \( (x_n)_{n=0}^{\infty} \) satisfies a **kth order homogeneous linear recurrence with constant coefficients**, if, for all \( n \in \mathbb{Z}_{\geq k} \),

\[
x_n = c_1 x_{n-1} + c_2 x_{n-2} + \ldots + c_k x_{n-k}
\]

for some \( c_1, \ldots, c_k \in \mathbb{C} \) with \( c_k \neq 0 \). If the characteristic polynomial \( x^k - c_1 x^{k-1} - \ldots - c_k \in \mathbb{C}[x] \) of this recurrence has \( k \) distinct zeros \( \alpha_1, \ldots, \alpha_k \in \mathbb{C} \), then the solution \( (x_n)_{n=0}^{\infty} \) is given by the linear combination

\[
x_n = a_1 \alpha_1^n + \ldots + a_k \alpha_k^n, \quad n \in \mathbb{Z}_{\geq 0},
\]

where the coefficients \( a_1, \ldots, a_k \in \mathbb{C} \) are determined by given initial conditions. (More about recurrences in [14].)

Suppose now that \( c_1, \ldots, c_k \in \mathbb{Z} \). Then the roots \( \alpha_1, \ldots, \alpha_k \) lie in a number field \( \mathbb{K} \) of degree at most \( k \), and so do the coefficients \( a_1, \ldots, a_k \). Furthermore, if \( \alpha_1, \ldots, \alpha_k \in \mathbb{Z}_K \), then \( F(\alpha_i), i = 1, \ldots, k, \) converges for any non-Archimedean valuation \( v \) of \( \mathbb{K} \), and we have

\[
\sum_{i=1}^{k} a_i F_v(\alpha_i) = \sum_{i=1}^{k} a_i \sum_{n=0}^{\infty} n! \alpha_i^n = \sum_{n=0}^{\infty} n! \sum_{i=1}^{k} a_i \alpha_i^n = \sum_{n=0}^{\infty} n! x_n.
\]
Chapter 4. Euler’s factorial series

Multiplying both sides by \( d := \text{lcm}_{1 \leq i \leq k} \{ \text{den } a_i \} \) we get a linear form with coefficients \( b_i := da_i \in \mathbb{Z}_K \):

\[
\sum_{i=1}^{k} b_i F_v(\alpha_i) = d \sum_{n=0}^{\infty} n!x_n.
\]

If at least one of the coefficients \( a_i \) is non-zero, it follows from Theorem 4.2 that for any \( a, b \in \mathbb{Z}_K \), there exists a non-Archimedean valuation \( v' \) of \( K \) such that

\[
\sum_{n=0}^{\infty} n!x_n \neq \frac{a}{b}.
\]

Example 4.15 (The Fibonacci numbers). The Fibonacci numbers are given by the sequence

\[
f_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n), \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad n \in \mathbb{Z}_{\geq 0}.
\]

Let us work in \( \mathbb{Q}(\sqrt{5}) \) and study the series \( \sum_{n=0}^{\infty} n!f_n \). The minimal polynomial of \( \alpha \) and \( \beta \) is \( x^2 - x - 1 \), so \( \alpha \) and \( \beta \) are algebraic integers and thus \( \| \alpha \|_v, \| \beta \|_v \leq 1 \) for any non-Archimedean valuation \( v \) of the field \( \mathbb{Q}(\sqrt{5}) \). Actually, as \( \alpha \beta = -1 \), we get \( \| \alpha \|_v = \| \beta \|_v = 1 \) for all \( v \in V_0 \). Hence both series \( \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} n!\alpha^n \) and \( -\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} n!\beta^n \) converge \( v' \)-adically and their sum is

\[
\frac{1}{\sqrt{5}} (F_v(\alpha) - F_v(\beta)) = \frac{1}{\sqrt{5}} \left( \sum_{n=0}^{\infty} n!\alpha^n - \sum_{n=0}^{\infty} n!\beta^n \right)
= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} n!(\alpha^n - \beta^n) \tag{4.44}
= \sum_{n=0}^{\infty} n!f_n.
\]

Because \( x^2 - 5 = (x - \sqrt{5})(x + \sqrt{5}) \) in \( \mathbb{R}[x] \), the Archimedean absolute value of \( \mathbb{Q} \) has two extensions to \( \mathbb{Q}(\sqrt{5}) \). These are given by

\[
|a + b\sqrt{5}|_1 = |a + b\sqrt{5}|, \quad |a + b\sqrt{5}|_2 = |a - b\sqrt{5}|,
\]

\(^1\)The denominator \( \text{den } \alpha \) of an algebraic number \( \alpha \in K \) is the smallest positive rational integer \( n \) such that \( n\alpha \) is an algebraic integer.
where now $|\cdot|$ is the unique Archimedean extension of the Archimedean absolute value of $\mathbb{Q}$ to $\mathbb{C}$, the algebraic closure of the Archimedean completion of $\mathbb{Q}$. Further,

\[
\left\| a + b\sqrt{5} \right\|_1 = \left\| a + b\sqrt{5} \right\|_2 = \sqrt{a + b\sqrt{5}},
\]

\[
\left\| a + b\sqrt{5} \right\|_2 = \left\| a + b\sqrt{5} \right\|_2 = \sqrt{a - b\sqrt{5}}.
\]

Let $a, b \in \mathbb{Z}$, $b \neq 0$ and choose $\alpha_1 = \alpha$, $\alpha_2 = \beta$. Then

\[
c_2 (\alpha, \beta, V) = (\max \{1, \|\alpha\|_1, \|\beta\|_1\})^2 (\max \{1, \|\alpha\|_2, \|\beta\|_2\})^2 \cdot (\|\alpha\|_1 + \max \{1, \|\alpha\|_1, \|\beta\|_1\}) (\|\beta\|_1 + \max \{1, \|\alpha\|_1, \|\beta\|_1\}) \\
\cdot (\|\alpha\|_2 + \max \{1, \|\alpha\|_2, \|\beta\|_2\}) (\|\beta\|_2 + \max \{1, \|\alpha\|_2, \|\beta\|_2\}) \\
\cdot \prod_{v \in V} \max \{\|\alpha\|_v, \|\beta\|_v\}
\]

\[
= 4 \left(1 + \frac{\sqrt{5}}{2}\right)^3 \left(\sqrt{\frac{-1 + \sqrt{5}}{2}} + \sqrt{\frac{1 + \sqrt{5}}{2}}\right)^2 \approx 72.
\]

By taking $\lambda_0 = 5a \in \mathbb{Z}$, $\lambda_1 = -b\sqrt{5} \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}$, $\lambda_2 = b\sqrt{5} \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}$, and considering the linear form

\[
\lambda_0 + \lambda_1 F_v(\alpha) + \lambda_2 F_v(\beta), \quad v \in V_0,
\]

equation (4.44) and Theorem 4.2 give:

**Corollary 4.16.** If $V$ is any collection of non-Archimedean valuations of $\mathbb{Q}(\sqrt{5})$ such that

\[
\limsup_{l \to \infty} c_2^l(2l + 2)(2l + 2)! \prod_{v \in V} \| (2l)! \|_v = 0,
\]

then there exists a valuation $v' \in V$ for which

\[
a - b \sum_{n=0}^{\infty} n! f_n \neq 0.
\]
Chapter 4. Euler’s factorial series

4.9 Arithmetic progressions

The last theorem of this chapter will be a generalisation of the following result of Ernvall-Hytönen et al. [17].

**Proposition 4.17.** [17, Theorem 5] Let $a \in \mathbb{Z}$, $b, \xi \in \mathbb{Z} \setminus \{0\}$, and $n \in \mathbb{Z}_{\geq 3}$ be given. Assume that $R$ is any union of the primes in $r$ residue classes in the reduced residue system modulo $n$, where $r > \frac{\varphi(n)}{2}$. Then there are infinitely many primes $p \in R$ such that $a - bF_p(\xi) \neq 0$.

Because each non-Archimedean valuation of the number field $\mathbb{K}$ is attached to the prime it extends, the division of primes into $\varphi(n)$ residue classes induces a division of the non-Archimedean valuations into $\varphi(n)$ classes. How many of these classes are needed to fulfil condition (4.4)?

**Theorem 4.18.** Let $m \in \mathbb{Z}_{\geq 1}$ and $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}_\mathbb{K}$ where $\lambda_j \neq 0$ for at least one $j$. Choose $m$ pairwise distinct, non-zero algebraic integers $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}_\mathbb{K}$. Let $n \in \mathbb{Z}_{\geq 3}$ be given. Assume that $R$ is a union of the primes in $r$ residue classes in the reduced residue system modulo $n$, where $r > \frac{m\varphi(n)}{m+1}$, and let $V = \{v \in V_0 \mid v|p \text{ for some } p \in R\}$. Then there exists a valuation $v' \in V$ such that

$$\lambda_0 + \lambda_1 F_{v'}(\alpha_1) + \ldots + \lambda_m F_{v'}(\alpha_m) \neq 0.$$

**Proof.** Let us show that the collection $V$ satisfies condition (4.4). We shall follow the method of Ernvall-Hytönen at al. [17]. By Lemma 1.14, we have

$$\log \left( \prod_{p \equiv a \pmod{n}} \frac{|l|_p}{p} \right) = -\frac{l \log l}{\varphi(n)} + O(l \log \log l)$$

when $n \in \mathbb{Z}_{\geq 3}$ and $\gcd(a, n) = 1$. Using this, the fact

$$\log((ml + m)!) = ml \log l + O(l),$$

128
and property (1.14), we get

$$\log \left( c_l^2 (ml + m)^\kappa (ml + m)! \prod_{v \in V} \| (ml)! \|_v \right)$$

$$= l \log c_2 + \kappa \log (m(l + 1)) + \log((ml + m)!) + \sum_{v \in V} \log \| (ml)! \|_v$$

$$= ml \log l + O(l) + \sum_{p \in R} \sum_{v \mid p} \log \| (ml)! \|_v$$

$$= ml \log l + O(l) + \sum_{p \in R} \log \| (ml)! \|_p$$

$$= ml \log l + O(l) - \frac{rml \log l}{\varphi(n)} - \frac{rl \log l}{\varphi(n)} + O(l \log \log l)$$

$$= \left( m - \frac{r(m + 1)}{\varphi(n)} \right) l \log l + O(l \log \log l)$$

$$\lim_{l \to \infty} = -\infty$$

because the coefficient \( m - \frac{r(m + 1)}{\varphi(n)} \) is negative. The claim follows from Theorem 4.2. \qed
Bibliography


131
Bibliography


132
Bibliography


