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Abstract

We explore dynamics in an OG economy under pure exchange and Stone-Geary preferences with positive level of saving and non-negative public debt and a relatively small second period endowment. We show the existence of a nontrivial steady state, which is either determinate or indeterminate. Furthermore, and importantly, there can be only two cycles, but no cycles of higher order.

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1. Introduction

A large set of goods called necessities are essential for consumers. The necessities are a key part of the linear expenditure system, which can be derived from the Stone-Geary preferences originally developed by Geary (1950-51). These preferences were used by Stone (1954) in empirical work, and they have been utilized also in theoretical work (see e.g. Auerbach and Hines, 2002, and Azariadis, 1996). Interestingly, and importantly, it is argued by Steger (2000) in a recent study that the inclusion of subsistence consumption (i.e. basically Stone-Geary preferences) in a linear growth model improves the model's explanatory power in a substantial way.

Consumer with the Stone-Geary preferences gets utility from that part of consumption, which exceeds the subsistence level. These preferences are closely related to preferences with habit persistence. Lahiri and Puhakka (1998) studied the implications of these preferences in an overlapping generations (OG) model with pure exchange. Among other things they showed that habit persistence preferences can induce the possibility for cycles, with relatively weaker concavity of the utility function than in conventional specifications.

Koskela and Puhakka (2007) introduced logarithmic Stone-Geary preferences into a standard overlapping generations economy with pure exchange, and studied stability, indeterminacy and cycles with positive level of saving and non-negative public debt. They showed that dynamics depend on the relative necessities of current and future consumption. In particular, a stable, and at the same time indeterminate, nontrivial steady state exists for parameter values, for which there is no such equilibrium in the model with purely logarithmic utility function. Moreover, the features of saving behavior lead to the possibility of period-doubling bifurcation (a two-cycle).

In this paper we generalize the model in Koskela and Puhakka (2007) by using more general Stone-Geary preferences. We show that a stable steady state exists, and furthermore and importantly, that there can be only two cycles, but no cycles of higher order. As shown first by Grandmont (1985) (see also Gale, 1973) an overlapping generations economy with pure exchange can exhibit a stable monetary steady states and cycles of any order, if the elasticity of the marginal utility of the second period consumption (or the Arrow-Pratt measure of the relative risk aversion) is greater than one.

Section 2 specifies the more general Stone-Geary utility function and presents its implications for saving behavior. Section 3 explores the dynamic properties of equilibrium. Finally, there is a short conclusion.

2. General Stone-Geary utility function and saving behavior

We analyze the dynamic implications of a perfect foresight overlapping generations economy under pure exchange, where consumers live for two periods. The general Stone-Geary preferences of the consumer are

$$(1) \quad v(c_1^t, c_2^t) = \frac{(c_1^t - a_1)^{1-\sigma}}{1-\sigma} + \beta \frac{(c_2^t - a_2)^{1-\sigma}}{1-\sigma},$$

where σ is the elasticity of the marginal utility of the respective period's consumption calculated for consumption over the subsistence level, $c - a$, β is the discount factor, and a is the exogenous level of subsistence.¹ The measure of concavity of the periodic utility function is $-cu''/u' = \sigma c/(c - a)$. It is decreasing in consumption and increasing in subsistence level. Since the measure is greater

¹ This formulation of preferences was originally not due to Geary and Stone. Literature calls these more general preferences Stone-Geary preferences, since they also lead to linear expenditure system (see e.g. Barro and Sala-i-Martin, 2004, p. 178).

than σ , the Stone-Geary utility function is more concave than the standard iso-elastic function.

A representative consumer maximizes (1) subject to the periodic budget constraints

$$(2i) \quad c_1^t + s_t = y_1$$

$$(2ii) \quad c_2^t = R_{t+1}s_t + y_2,$$

where y_1 (y_2) is the endowment in the first (second) period, s_t the saving, R_{t+1} the interest factor from period t to period $t+1$, and c_1^t (c_2^t) the first (second) period consumption, respectively. For the problem to make sense the following inequality must be satisfied for the equilibrium interest factor

$$(3) \quad a_1 + \frac{a_2}{R} < y_1 + \frac{y_2}{R}.$$

This means that the point (a_1, a_2) should lie inside the budget line. Figure 1 describes the budget set, when $y_1 - a_1 > 0$ and $y_2 - a_2 < 0$ so that saving must be positive. In Figure 1 we need to have $R > \underline{R}$ for the decision problem to make sense. Inspecting Figure 1 reveals that $\underline{R} = (a_2 - y_2)/(y_1 - a_1)$. In what follows we study this case.² The Figure also reveals a crucial difference between this case and the regular case, since the restrictions (in particular, $y_2 - a_2 < 0$) force consumers' savings to be positive for every $R > \underline{R}$. Therefore it is not then surprising that saving is a decreasing function of the interest factor at least for interest factors close to $\underline{R}$. And as pointed out below the resulting equilibrium dynamics will differ substantially from that of the regular case.

² For (3) to hold it is not necessary that $a_1 < y_1$ and $a_2 < y_2$ at the same time. If $a_1 < y_1$ and $a_2 < y_2$, we have the regular decision problem with well-known properties. Naturally the decision problem becomes different when either $a_1 > y_1$ or $a_2 > y_2$.

Fig. 1. Endowments.

The first-order condition for consumer's optimum,

$(c_2^t - a_2)^\sigma / \beta(c_1^t - a_1)^\sigma = R_{t+1}$, leads to the saving function

$$(4) \quad s(R; \bullet) = \frac{R^\sigma \beta^{\frac{1}{\sigma}}}{R + R^\sigma \beta^{\frac{1}{\sigma}}} (y_1 - a_1) - \frac{y_2 - a_2}{R + R^\sigma \beta^{\frac{1}{\sigma}}},$$

with the following property

$$(5) \quad \frac{\partial s}{\partial R} = \frac{\left(\frac{1}{\sigma} - 1\right) R^\sigma \beta^{\frac{1}{\sigma}}}{\left[R + R^\sigma \beta^{\frac{1}{\sigma}}\right]^2} (y_1 - a_1) + \frac{1 + \frac{1}{\sigma} R^{\frac{1}{\sigma}-1} \beta^{\frac{1}{\sigma}}}{\left[R + R^\sigma \beta^{\frac{1}{\sigma}}\right]^2} (y_2 - a_2).$$

If $a_1 < y_1$ and $a_2 < y_2$, we have the same result as with typical CRRA (i.e. constant relative risk aversion) preferences ($a_1 = a_2 = 0$). In that case it is necessary to have $\sigma > 1$ for saving to be a decreasing function of the interest factor. If $y_1 > a_1$ and $a_2 > y_2$, saving is also a decreasing function of the interest factor when $\sigma > 1$, while it can be an increasing function of the interest factor when $\sigma < 1$. We see that $s(\underline{R}; \bullet) = y_1 - a_1$, and also quite easily that $\lim_{R \rightarrow \infty} s(R) = 0$, when $\sigma > 1$,

and $\lim_{R \rightarrow \infty} s(R) = y_1 - a_1$, when $\sigma < 1$. It then follows that the offer curve of the consumer must at some point turn backwards, when $\sigma < 1$.

3. Dynamic Equilibrium

To analyze the dynamics of the model we introduce an outside asset into the economy by assuming that government borrows (lends) from (to) the public. Government debt (or assets) at the beginning of the period is denoted by b_t , and the primary deficit by d_t so that government's budget constraint is

$$(6) \quad b_{t+1} = d_t + R_{t+1}b_t.$$

Since we want to concentrate on the fundamental dynamic implications of the model with Stone-Geary preferences, we assume the primary deficit to be zero, i.e. $d_t = 0$ for all t , and study the case with nonnegative government debt, i.e. $b_t \geq 0$. Thus in the asset market equilibrium $b_t = s_t$, which leads to the following difference equation

$$(7) \quad s_{t+1} \equiv s(R_{t+2}) = R_{t+1}s(R_{t+1}) \equiv R_{t+1}s_t.$$

To repeat, we study the case presented in Figure 1, when $y_1 - a_1 > 0$ and $y_2 - a_2 < 0$. This is a more interesting case than the other one (i.e. $a_2 < y_2$) since the restriction ($a_2 > y_2$) seems to be quite plausible e.g. in economies with retirement systems, when the subsistence consumption exceeds the second period endowment. Because saving is always positive, there is no interest factor such that saving is zero. Hence in (7) there is only one steady state such that $R_t = 1$ for all t , which is possible only when $\underline{R} = (a_2 - y_2)/(y_1 - a_1) < 1$. This means that the first period endowment must be sufficiently higher than the subsistence consumption when $a_2 - y_2 > 0$. Below we assume that $\underline{R} < 1$.

It is important to emphasize that we analyze equation (7) by the geometric techniques of the reflected generational offer curves, developed by Cass, Okuno and Zilcha (1979). Inverting the saving function and substituting for R_{t+1} in (7) we obtain the reflected generational offer curve.

To derive the equilibrium dynamics we take into account the periodic budget constraints and the fact that under zero primary deficit $R_{t+1} = s_{t+1} / s_t$. We can then rewrite the first-order condition as an equilibrium condition

$$(8) \quad s_t^{\frac{1}{\sigma}}(s_{t+1} + y_2 - a_2) = \beta^{\frac{1}{\sigma}} s_{t+1}^{\frac{1}{\sigma}}(y_1 - a_1 - s_t),$$

which implicitly defines the reflected generational offer curve. The steady state saving is

$$(9) \quad s^* = \frac{(a_2 - y_2) + \beta^{\frac{1}{\sigma}}(y_1 - a_1)}{1 + \beta^{\frac{1}{\sigma}}}.$$

Totally differentiating equation (8), and using it, gives

$$(10) \quad \frac{ds_{t+1}}{ds_t} = \frac{-\beta^{\frac{1}{\sigma}} s_{t+1}^{\frac{1}{\sigma}} - (s_{t+1} + y_2 - a_2) \frac{1}{\sigma} s_t^{\frac{1}{\sigma}-1}}{\beta^{\frac{1}{\sigma}} s_{t+1}^{\frac{1}{\sigma}}(y_1 - a_1 - s_t) \left[\frac{\sigma s_{t+1} - (s_{t+1} + y_2 - a_2)}{\sigma s_{t+1}(s_{t+1} + y_2 - a_2)} \right]}.$$

The slope of the offer curve (see Figure 2) is thus negative, if $\sigma \geq 1$, and can be positive, if $\sigma < 1$. Now the possibility for a two cycle depends on the magnitude of the curve's slope at the steady state. Below we construct an example of a two cycle.

Fig. 2. The offer curve, when $\sigma > 1$.

We rewrite (10) in a slightly different way and evaluate it at the steady state

$$(11) \quad \left. \frac{ds_{t+1}}{ds_t} \right|_{s_t=s^*} = \frac{\frac{1}{\beta^\sigma (\sigma - 1)(a_2 - y_2)} + \frac{1}{\beta^\sigma (\beta^\sigma \sigma + 1)(y_1 - a_1)}}{\frac{1}{\beta^\sigma (1 - \sigma)(y_1 - a_1)} - \frac{1}{(\beta^\sigma + \sigma)(a_2 - y_2)}}.$$

For stability, and thus for indeterminacy, this slope should be greater than minus unity.³

Suppose for a moment that this slope is positive, i.e. σ must be small enough. For $\sigma = 0$, the numerator of (11) is positive and thus it is positive also for any $\sigma < 1$. Then the stability condition can be written as

$$(12) \quad (\beta^\sigma + 1)(a_2 - y_2) + \beta^\sigma (\beta^\sigma + 1)(y_1 - a_1) < 0,$$

which can obviously never hold, since $a_2 - y_2 > 0$. The offer curve with a small σ and the positive slope at the steady state is drawn in Figure 3. Thus the steady state in this case is unstable, i.e. determinate.

³ Guesnerie and Woodford (1992) discuss thoroughly the concept of indeterminacy in OG models, see their chapter 5.

Fig. 3. The offer curve with small sigma.

In this framework cycles can emerge in equilibrium under preferences without the subsistence level of consumption, if the elasticity of the marginal utility of the second period consumption is greater than unity (see Grandmont 1985). For the existence of a two cycle (or a periodic point with period two) the offer curve must be downward sloping. That property, however, is not sufficient for periodic solutions of higher order. To have periodic points with period three and more, it is necessary that the offer curve must be hump-shaped. If there are at least a three cycle, there must be periodic solutions of any order higher than three as implied by Sarkovskii's theorem.⁴ Since the offer curves in Figures 2 and 3 are not hump-shaped, only two cycles are possible. Even though cycles of higher order than two and chaotic behavior are not possible, there can still be stable equilibria, and thus indeterminacy.

Next we present an example of a two-cycle by fixing the parameters such that the slope of the reflected generational offer curve is slightly less than minus

⁴ An elementary discussion and elaboration of Sarkovskii's theorem can be found e.g. in Holmgren (1996), see in particular chapter 5. On the conditions for the existence of cycles of more than two periods in economic models, see Grandmont (1986).

unity at the steady state. Then we find a pair of numbers, x (the starting value for saving) and z (the value for saving in the next period), which are both different from s^* , and fulfill (8) for $1/\sigma > 1$ such that the following equations (derived from (8)) hold

$$(13) \quad \frac{z + y_2 - a_2}{y_1 - a_1 - x} = \beta^{\frac{1}{\sigma}} \left(\frac{z}{x} \right)^{\frac{1}{\sigma}} \quad \text{and} \quad \frac{x + y_2 - a_2}{y_1 - a_1 - z} = \beta^{\frac{1}{\sigma}} \left(\frac{x}{z} \right)^{\frac{1}{\sigma}}.$$

We choose the parameters as follows: $\beta = 1/2$, $\sigma = .95$, $y_1 = 1$ and $y_2 = 0$. We know from (11) that the slope is sensitive to the values of necessities. We set the slope to -1.001 , and choose the value of a_1 such that the slope indeed equals -1.001 . Such a value for a_1 (and indeed we assume the same value for a_2) is .332782. $\underline{R}$ (in this case $a_2/(y_1 - a_1)$) is .4987. Using these values the steady state, s^* , will be .441566. Solving equations in (13) for these parameter values we get $x = .441564$ and $z = .441568$. The respective equilibrium interest factors are .99999 and 1.000009 so that the computed two-cycle is feasible. Of course the steady state is one solution for (13). Other examples of two cycles with different parameter values can obviously be generated in a similar way.

4. Conclusion

We have used an overlapping generations model with pure exchange and Stone-Geary preferences to study the dynamic properties of equilibrium in economies with positive levels of private saving and non-negative public debt. We have shown that a stable nontrivial steady state exists, which is either determinate or indeterminate. Furthermore, and importantly, there can be a multitude of period-doubling bifurcations (a two-cycle) in equilibrium, but no cycles of higher order as in models without Stone-Geary preferences.

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