TASK POTENTIAL OF REVERSED EQUATION SOLVING

Dimitri Tuomela

University of Oulu

Reversed Equation Solving (RES) is a task which aims to foster understanding and reasoning above rote-based learning. RES is analyzed with a framework formed by combining two earlier frameworks for analyzing the quality of mathematical tasks. The analysis suggests that conceptual emphasis and high degree of freedom in creating equations for one's peers to solve makes Reversed Equation Solving a cognitively demanding, rewarding and mathematically rich task.

INTRODUCTION

Understanding the process of equation solving is an important but challenging point in learning mathematics (Andrews & Sayers, 2012). Finnish teachers often emphasize routines and procedural fluency in algebra (Attorps, 2006) which is connected to learners' poor conceptual knowledge (Hihnala, 2005) and inability to describe and justify their thinking (Andrews, 2013). Seeing equation solving as a set of memorized routines prevents learners from applying algebra (Hiebert et al., 1997) and building a productive disposition towards mathematics (Andrews, 2013). Apparently there is a strong need for reasoning-centered classroom practices and material in algebra.

Flexible Equation Solving project aims to answer to this need as a part of a 6-year national program (LUMA Suomi). While developing material for this project the author immersed himself in reading previous research describing aspects of high-quality mathematical tasks (e.g. Stein, Grover & Henningsen, 1996; Jackson, Garrison, Wilson, Gibbons & Shahan, 2013) and aspects of high-quality learning environments for learning mathematics (e.g. Hiebert et al., 1997; Kilpatrick, Swafford & Findell, 2001). As a consequence the approach of Reversed Equation Solving (RES) was created and it was used as a one-lesson task in the 10-lesson pilot of Flexible Equation Solving.

Briefly described RES is an intervention where learners create, share, solve and compare equations. The approach is intended to have a positive impact on conceptual understanding related to equation solving and to change answer-orientated classroom practices towards process-oriented ones. The intervention also aims to help teachers reflect on their enacted classroom practices.

This study is a theoretical analysis of the task potential of RES to support the research-based development of the approach. A framework for analyzing the task potential is formed by combining a rubric for analyzing cognitive demand of any task (Smith & Stein, 1998) and a framework describing features of high-quality mathematical tasks (Hiebert et al., 1997). This study can be seen as an initial phase of a design study (Cobb, Jackson & Dunlap, 2015) seeking to establish an optimal instructional design of RES. The aim of this paper is to understand what is the task potential of RES for supporting all learners to develop understanding on mathematical ideas related to equation solving?

THE TASK AND CONTEXT

Reversed Equation Solving has been developed in the context of 10-lesson pilot of Flexible Equation Solving –project. RES is the sixth lesson in the middle of the material. RES was preceded by three conceptually oriented lessons on what are equations and transformations, and two procedurally oriented lessons. On one of those lessons all simple transformations (multiplication, division, addition and subtraction on both sides and modifying an expression) were introduced altogether as a cluster and not one by one as separate lessons which is common in books used in Finnish classrooms.

RES begins by having small groups of 2-5 learners decide a number and marking it equal with a symbol of their choice. Hence they have created a simple equation (e.g. 4 = t). The learners grow their equation with several transformations of their choice (e.g. 6+4 = 6+t) and write these steps above the previous. Once they are satisfied, they share their equation (with their names) in the blackboard for everyone to solve. Gradually as they solve each other's equations they compare their work with the creator group. The written instruction of the task on the pilot was the following.

1. Choose together with your group members a number and mark it equal with some variable. (For example 12 = t)
2. Use transformations of your choice to this equation. (For example multiply both sides by 3. Add 2t to each side and combine like terms.)
3. You have formed an equation. Go share it in the blackboard.

			$36+2t=5t$
		$36+2t=3t+2t$	$36+2t=3t+2t$
	$12\cdot 3=3\cdot t$	$12\cdot 3=3\cdot t$	$12\cdot 3=3\cdot t$
$12=\ t$	$12=\ t$	$12=\ t$	$12=\ t$

Picture 1. The provided example of transforming the equation.

FRAMEWORK FOR ANALYZING TASK POTENTIAL

Several researchers argue that high-quality mathematics instruction should be based on fewer but more demanding tasks set up in a way that it encourages high-level mathematical thinking and reasoning (Fennema et al., 1996; Stein, Grover & Henningsen, 1996; Jackson, Garrison, Wilson, Gibbons & Shahan, 2013; Henningsen & Stein, 1997).

Smith and Stein (1998) described a four-level (memorization, procedures without connections, procedures with connections, doing mathematics) rubric for analyzing the cognitive demand of tasks. Characteristics of tasks with the highest cognitive demand (doing mathematics) are similar with characteristics of high-quality mathematical tasks described by Hiebert and colleagues (1997). They fit to each other so well that their work could be combined resulting in a 4-dimension framework that describes characteristics of tasks which have the potential to engage students in "doing mathematics" and provide insights into the structure of mathematics. In the following table Smith and Stein's framework will be referred to with (S) and Hiebert and colleagues's framework with (H).

Table 1. Framework for analyzing task potential.

Makes mathematics problematic • Learners see the task as an interesting problem and see that there is something to find out, something to make sense of (H) • Learners don't have memorized rules for the task, nor do they perceive there is one right solution method (H) • Complex and non-algorithmic thinking is required instead of following a prescription (S) • Considerable cognitive effort is required and the unpredictable nature of the solution process may involve some level of anxiety (S)
Connects with where students are • Task is accessible to all learners regardless of their mathematical ability (H) • Opportunities to apply previous skills and knowledge arise (H) • Students are required to access relevant knowledge and experiences and make appropriate use of them in working through the task (S)
Leaves behind something of mathematical value • Task engages learners in reflecting important mathematical ideas in a way that leaves behind something of mathematical value (H) • Task offers opportunities to explore mathematics and reasonable solution methods (H) • Task requires learners to explore and understand the nature of mathematical concepts, processes, or relationships (S) • Task requires learners to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions (S)
Demands reflection and regulation • Task creates opportunities to reflect and communicate (H) • Task demands self-monitoring or self-regulation of one's own cognitive processes (S)

ANALYZING THE TASK POTENTIAL OF RES

The potential of RES for being a high-quality mathematical task is analyzed against the previously described theoretical framework.

Does RES make mathematics interesting and problematic?

In RES learners create something to think about for their peers and work on each other's creations. This collaborative element coupled with the need to "work backwards" has the potential to make the task interesting and meaningful for the learners. They have considerable freedom in picking the transformations instead of having clear instructions what to do next. Thus they cannot apply any memorized rule for the task and it is very probable that different kind of ideas emerges and groups end up having different kind of equation creation paths. Often learners focus on just getting the right answer which has been found to be related to doing procedures without connections (Henningsen & Stein, 1997). When creating equations the learners cannot focus on getting the correct answer in any other way than checking if their equation is still true with the same value of variable. This unpredictable reverse approach can initially cause anxiety, especially for learners who are used to strictly following instructions and copying the procedures from a similar example. These elements of the task very well correspond to the aspects described in the first dimension "Makes mathematics problematic".

Does RES connect with where learners are?

Although the task may initially seem complex for the learners, they will soon notice that everyone can succeed in the task. When the initial anxiety is overcome the learners have an opportunity to gain a positive experience in doing mathematics as they notice they can do something they thought as difficult. For procedure-oriented learners it can be relieving to see that the process of creating an equation has the following simple steps "choose a transformation, choose a number (unless modifying an expression), execute". This is a straightforward advice teachers can give while leaving room for learners' own reasoning because the way transformations and numbers are chosen effect significantly what kind of conceptual and procedural challenges come forth. The choices learners do effect on the difficulty of the task making it accessible as well as challenging enough for everyone. The nature of the task is such that it affords learners to work on a level suitable for them by deciding how they will create their equation and what kind of equations they choose to solve. All learners need to modify expressions, they need to reflect on the concept of variable and they may start thinking what kind of numbers are they familiar with. All this demands reflection on how their previous knowledge connects to equation solving and requires learners to use their existing skills in a meaningful way. Briefly, RES shows for the learners as a difficult and complex task but turns out to be an achievable task which connects to learners' previous skills regardless of their mathematical ability. This fits well with the second category of the framework "connects with where learners are".

Does RES leave behind something of mathematical value?

In RES learners cannot initially focus on the answer which is connected to doing procedures without conceptual reflection according to Henningsen and Stein (1997). Instead the task has the potential to direct learners' attention to creating equivalent equations thus forcing learners to reflect on conceptual ideas relevant in equation solving (adapted from Kieran, 2004):

- seeing relations instead of calculating a numerical answer,
- letters as unknowns and variables,
- reflecting the meaning of equal sign,
- equation as a statement that has a truth-value,
- transformations as a process of changing the equation without changing its truth value,
- operations and their inverses, doing and undoing.

When the learners think about which transformations to use they run into important questions: Why most transformations operate on both sides whereas the transformation of modifying one expression does not? Can we divide if we do not get whole numbers as a result? Can variables be added or subtracted in each side? How about multiplying or dividing by variables? When learners are not in a hurry to get the correct answer they have time to consider these mathematically important questions. The arising mathematical questions and challenges are likely to be different from each other in different groups. This diversity provides a rich ground for learner-driven reflection. Especially considering the question "How can I make the equation as tricky as possible" leads to exploration of new mathematical ideas in a creative manner.

Cuoco, Goldenberg and Mark (1996) described important generic mathematical skills needed to be considered in curricular development and named them as mathematical habits of mind. They described that learners should be pattern sniffers, experimenters, describers, tinkerers, inventors, visualizers, conjecturers and guessers. As tinkerers, learners should develop the habit of taking ideas apart and putting them back together. They should want to see what happens if something is left out or if the pieces are put back in a different way. The open nature of RES creates an opportunity to explore mathematics and practice these generic mathematical habits which can be transferred and applied to multiple contexts. Next, some of these potential opportunities are described.

RES has the potential to have learners trying out different combinations of transformations and numbers to see what happens (Experimenters). They may reflect what happens if they make a small change such as change the order of two transformations or use rational numbers instead of whole numbers (Tinkerers). Is there a general pattern when multiplication follows addition compared to addition following multiplication (Pattern Sniffers)? Learners may conjecture that addition or substraction followed by multiplication or division results in an equation with brackets (Conjecturers). Learners can explore mathematical structure and start generalizing: ”Multiplication and division are somehow similar as well as addition and substraction.” They can invent creative alternatives such as “could I use pi” or “what happens if I add a second variable” (Inventors). Learners can explore important relationships such as finding out how reverse operations are connected to the creation and solving phases (Pattern sniffers). The task has the potential to have learners describing the process of creating and solving equations - what transformations did they use and why (Describers).

Before answering thoroughly to the question if the task leaves behind something of mathematical value it needs to be asked: what is mathematically valuable? One way to answer to this question is to lean on Kilpatrick and colleagues' description of mathematical proficiency comprising five intertwined strands (Kilpatrick, Swafford & Findell, 2001): Conceptual understanding - comprehension of mathematical concepts, operations, and relations; Procedural fluency - skill in carrying out mathematical procedures flexibly, accurately, efficiently, and appropriately; Strategic competence - ability to formulate, represent, and solve mathematical problems; Adaptive reasoning - the capacity for logical thought and for reflection on, explanation of, and justification of mathematical arguments; Productive disposition - habitual inclination to see mathematics as a sensible, useful, and worthwhile subject to be learned, coupled with a belief in the value of diligent work and in one's own efficacy as a doer of mathematics. Developing these strands of mathematical proficiency can be considered as mathematically valuable.

When implementing the task, it is likely that the need for conceptual understanding and procedural fluency alternate in a way which provides opportunities to build a stronger link between conceptual and procedural knowledge of equation solving. The consecutive creation, solving and comparing phases may provide several opportunities to develop adaptive reasoning. Explorative nature of the task may provide opportunities to develop strategic competence through acquiring generic mathematical thinking skills. The task may look difficult but is likely to turn out into manageable task providing positive experiences in doing mathematics thus raising an opportunity to build productive disposition. Therefore, RES seems to have the potential to support the development of mathematical proficiency (Kilpatrick, Swafford & Findell, 2001).

Implementing the task certainly has the potential to have students exploring mathematical structure, ideas, concepts, processes and relationships in a way that leaves behind something of mathematical value.

Does RES demand reflection and regulation?

The novel approach of the task may cause an initial challenge which necessitates perseverance and self-encouragement from the group to start thinking. Unpredictability of the task requires groups to reflect on the task and share their thinking when making suggestions about what to do next. The learners need to collaborate and regulate their work and potentially group members will negotiate some kind of roles. The blackboard filled with student-generated equations offers an excellent chance to wonder if they are mathematically different or similar in structure. Are some equations more difficult than others? If so, why? This kind of negotiation on what counts as mathematically different or more sophisticated has been claimed to develop student autonomy and argumentation in mathematics (Yackel & Cobb, 1996). Especially questions like ”How can we make the equations novel and tricky?” lead to rich discussions and to mathematically rich reflection. When a group solves an equation from the blackboard and compares their work with the creator group it is common that different kind of approaches, mistakes and misconceptions emerge and require reflection. In these situations learners need to communicate and reflect on each others' work in a productive manner. Definitely RES is a task which has the potential to demand reflection and regulation when it is implemented.

DISCUSSION

Analyzing Reversed Equation Solving against the theoretical framework shows that it has characteristics of high-quality mathematical tasks which have the potential to engage learners in "doing mathematics" and which provide insights into the structure of mathematics. These results can inform the development of an instructional approach which could support learners in the difficult transition (Andrews & Sayers, 2012) from arithmetic to abstract algebra. Design research like this can partly answer to the need for more reasoning-centered classroom practices in algebra. In a broader sense this study contributes to the reflection of student-created tasks as a method for increasing student engagement and creating learning process -oriented classroom culture.

Critique and implications

This study is of theoretical nature and empirical data is needed to understand how elements such as classroom social culture and teacher's role affect on the learning opportunities created when implementing the task. Possibly RES provides more learning opportunities for learners who have previous experience in reasoning about mathematics together and discussing concepts. It needs to be studied how to support learners and teachers in using the task to maximize learning opportunities. Previous research of how to support teachers in using challenging tasks should be adapted to the development of instructional design of RES. For example Sullivan, Mousley & Zevenbergen (2006) argued the following three teacher actions to be relevant in maximizing learning opportunities in heterogeneous classrooms:

> Using open-ended tasks, preparing prompts to support students experiencing difficulty, and posing extension tasks to students who finish the set tasks quickly; as well as actions that address the socio-mathematical goals by making classroom processes explicit.

Stein, Grover and Henningsen (1996) found that five most important factors assisting maintenance of high-level cognitive activity were i) task builds on students' prior knowledge, ii) appropriate amount of time, iii) high-level performance modeled, iv) sustained pressure for explanation and meaning and v) scaffolding. Depending on the learners in the classroom different kind of timing and supportive actions from the teacher is needed during RES. It is challenging for the teacher to make these on-the-fly decisions for example how to use learners work and when to use whole-class discussion. Thus a detailed guide for using RES is needed.

Suggestions for developing Reversed Equation Solving

RES could be developed further by contextualizing the task as an encryption and decryption activity and describing how these skills are increasingly important as data is digitalized and needs to be safely encrypted or how these skills were in a key role in the second world war. Having a secret that needs to be encrypted works as a metaphor of transformations as the the kind of changes to the equation which doesn't change its truth value (the solution). Encryption and decryption is also similar to the conceptually important idea of operations and their inverses, doing and undoing. Considering the framework for analyzing task potential this addition would make the task more interesting (first dimension) and increase the potential for leaving behind mathematically valuable conceptual understanding (third dimension).

By adding a phrase "When you are satisfied with your equation, find out which value of variable makes the created equation true. Prove it!" in the instruction sheet before sharing equations in the blackboard would give more opportunities for conceptual reflection. This way it can be checked if the learners understand what stays the same when transformations are used to create equivalent equations (again increasing the task potential in the third dimension). A section in the instruction sheet for classifying the diversity of encountered equations and to reflect on ones own learning could increase the task potential in the forth dimension (demands reflection and regulation). This section would include the following kind of questions: Classify what kind of equations a) exists b) you can create c) you can solve. What kind of methods did you find to make the equation trickier? What similarities and differences you found in creating and solving equations? Did you find some common patterns? Could your findings be generalized?

It should be considered whether the task should be used quickly after introducing transformations thus providing an engaging way to start developing intertwined procedural and conceptual knowledge or later on to launch mathematical exploration and classification of different kind of equations thus providing more room for creativity and opportunities to deepen and summarize knowledge related to equation solving. Alternatively RES could be used as a project combining the previous two approaches. Extending the task from one lesson to a longer period could give more room for mathematical exploration.

CONCLUSION

Conceptual emphasis and the high degree of freedom in Reversed Equation Solving may result in a cognitively demanding but mathematically rich and rewarding implementation of the task. It has potential to engage diverse learners in sharing their reasoning and developing their mathematical proficiency. Using RES as a part of a conceptually oriented set of tasks could ease transitioning from arithmetic to abstract algebra. This study only analyzed the theoretical potential of the task. Naturally the environment it is implemented in, specifically the social culture of the classroom, is likely to have a significant effect on what kind of learning opportunities emerge. It may be a challenging task for the teacher to manage because of its unpredictability and potentially diverse student interactions. It would be important to develop an instructional design of RES which supports teachers and learners in creating fruitful learning opportunities.

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