

VOLUME GROWTH OF QUASIHYPHERBOLIC BALLS

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ABSTRACT. The purpose of this paper is to study the notion of the quasihyperbolic volume and to find growth estimates for the quasihyperbolic volume of balls in a domain $G \subset \mathbb{R}^n$, in terms of the radii.

Keywords. Quasihyperbolic volume, uniform porosity, Q -regularity

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1. INTRODUCTION

Since its introduction three decades ago, the quasihyperbolic metric has become a popular tool in many subfields of geometric function theory, for instance, in the study of quasiconformal maps of \mathbb{R}^n , $n \geq 2$, and of Banach spaces [GH, Va1], analysis of metric spaces [H] and hyperbolic type metrics [HIMPS]. A natural question is whether or not and to what extent, the results of hyperbolic geometry have counterparts for the quasihyperbolic geometry. For instance in [K] it was noticed that some facts from hyperbolic trigonometry of the plane have counterparts in the quasihyperbolic setup.

The purpose of this paper is to study the notion of the quasihyperbolic volume and to find growth estimates for the quasihyperbolic volume of balls in a domain $G \subset \mathbb{R}^n$, in terms of the radii. Below we give an explicit growth estimate for the case of domains with Ahlfors regular boundary.

For a compact set with empty interior $E \subset \mathbb{R}^n$, and $0 < s < t$, we consider the layer sets $E(s, t) = \{z \in \mathbb{R}^n : s \leq d(z, E) \leq t\}$ and relate their volume to the metric size of E via its (Hausdorff) dimension under the additional assumption that E be Ahlfors Q -regular for some $0 < Q < n$. It is practically equivalent to formulate this idea in terms of the number of those Whitney cubes of the Whitney decomposition of $\mathbb{R}^n \setminus E$ that meet $E(s, t)$. This idea goes back to [MV]. One of the key ideas of [MV] was to estimate the metric size of E from above in terms of the size of the layer sets. In Lemmas 4.2 and 4.4, we will refine some results of [MV].

In the last section, we estimate from below the growth of the quasihyperbolic volume of balls in the case when $G = \mathbb{R}^n \setminus E$ is ψ -uniform and E is Ahlfors regular. The main result of the paper is Theorem 5.6 which gives a lower bound for the volume growth.

2. NOTATION

We assume $n \geq 2$ and use the notation $B^n(x, r) = \{z \in \mathbb{R}^n : |x - z| < r\}$ for the Euclidean ball, and its boundary is the sphere $S^{n-1}(x, r) = \partial B^n(x, r)$, where the center x can be omitted if $x = 0$.

The *quasihyperbolic distance* between two points x and y in a proper subdomain G of the Euclidean space \mathbb{R}^n , $n \geq 2$, is defined by

$$k_G(x, y) = \inf_{\alpha \in \Gamma_{xy}} \int_{\alpha} \frac{|dz|}{d(z)},$$

where $d(z) = d(z, \partial G)$ is the (Euclidean) distance between the point $z \in G$ and the boundary of G (denoted ∂G), and Γ_{xy} is the collection of all rectifiable curves in G joining x and y . Note that the quasihyperbolic metric in a half-space is just the hyperbolic metric, and that for a ball the two are bilipschitz equivalent. Some basic properties of the quasihyperbolic metric can be found in [V1]. In particular, Martin and Osgood [MO, page 38] showed that for $x, y \in \mathbb{R}^n \setminus \{0\}$ and $n \geq 2$

$$(2.1) \quad k_{\mathbb{R}^n \setminus \{0\}}(x, y) = \sqrt{\theta^2 + \log^2 \frac{|x|}{|y|}},$$

where $\theta = \angle(x, 0, y) \in [0, \pi]$.

For the volume of the unit ball in \mathbb{R}^n we use the notation $\Omega_n := m(B^n(0, 1)) = \pi^{n/2} / \Gamma((n/2) + 1)$ where m is the n -dimensional Lebesgue measure and Γ stands for the usual Γ -function, see [AS, Chapter 6]. The surface area of the unit sphere is $\omega_{n-1} := m_{n-1}(S^{n-1}(0, 1)) = n\Omega_n$.

The quasihyperbolic volume of a Lebesgue measurable set $A \subset G$ is defined by

$$\text{vol}_{k_G}(A) = \int_A \frac{dm(z)}{d(z)^n}.$$

We also use the notation $\text{vol}_k(A)$ for $\text{vol}_{k_G}(A)$ if the domain G is clear from the context.

Below we use the Whitney decomposition of the complement of a closed set in \mathbb{R}^n . If $E \subset \mathbb{R}^n$ is non-empty and closed, then $\mathbb{R}^n \setminus E$ can be presented [S, page 16] as a union of closed dyadic cubes Q_j^i

$$(2.2) \quad \mathbb{R}^n \setminus E = \bigcup_{i \in \mathbb{Z}} \bigcup_{j=1}^{N_i} Q_j^i,$$

where the cubes Q_j^i have the following properties:

- (i) each Q_j^i has faces parallel to the coordinate planes and edges of length 2^{-i} ,
- (ii) the interiors of the cubes Q_j^i and Q_l^k are mutually disjoint provided $l \neq j$ or $i \neq k$,
- (iii) the distance between the cube Q_j^i and E satisfies the following inequalities

$$(2.3) \quad 2^{-i} \sqrt{n} \leq d(Q_j^i, E) \leq 2^{2-i} \sqrt{n}.$$

The decomposition (2.2) is called the *Whitney decomposition*, the cubes Q_j^i are called *Whitney cubes* and the set $\{Q_j^i: j = 1, \dots, N_i\}$ is called the i^{th} generation of cubes.

For $0 < s \leq t < \infty$ and $E \subset \mathbb{R}^n$, we define

$$E(s) = \{x \in \mathbb{R}^n: d(x, E) \leq s\}$$

and

$$(2.4) \quad E(s, t) = \{x \in \mathbb{R}^n: s \leq d(x, E) \leq t\}.$$

We set $E(s, t) = \emptyset$ for $s > t$. Because $\text{diam}(Q_j^i) = \sqrt{n}2^{-i}$ we have by (2.3)

$$(2.5) \quad Q_j^i \subset E(\sqrt{n}2^{-i}, 5\sqrt{n}2^{-i}).$$

We say that a set $E \subset \mathbb{R}^n$ is (p, r_0) -uniformly porous [Va2] if for all $x \in E$ and $0 < r < r_0$ there is $y \in B(x, r)$ with $d(y, E) \geq \alpha r$. A set E is uniformly porous, if it is (p, r_0) -uniformly porous for some $p, r_0 > 0$. We refer to p, r_0 and n as the *uniform porosity data*.

A set $E \subset \mathbb{R}^n$ is Q -regular for $0 < Q < n$, if there is a (Borel regular, outer-) measure μ with $\text{spt}(\mu) = E$ and constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha r^Q \leq \mu(B^n(x, r)) \leq \beta r^Q, \text{ for all } x \in E \text{ and } 0 < r < \text{diam}(E).$$

Here $\text{spt}(\mu)$ denotes the smallest closed set with full μ -measure. We refer to n, Q, α, β and $\text{diam}(E)$ as the *Q -regularity data*.

There is a close connection between the notions of uniform porosity and Q -regularity. Indeed, it is well known and easy to see (e.g. [BHR, Lemma 3.12]) that if E is Q -regular for some $0 < Q < n$, then it is uniformly porous. Conversely, if E is uniformly porous, then it is a subset of some Q -regular set for some $0 < Q < n$. See [JJKRRS, Theorem 5.3] for a more precise quantitative statement.

Let $\psi: [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism with $\psi(0) = 0$. A domain $G \subset \mathbb{R}^n$ is said to be ψ -uniform [V2] if

$$(2.6) \quad k_G(x, y) \leq \psi(|x - y| / \min\{d(x, \partial G), d(y, \partial G)\})$$

for all $x, y \in G$. In particular, the domain G is L -uniform if $\psi(t) = L \log(1 + t)$ with $L > 1$. A domain is called *uniform* if it is a L -uniform domain for some $L > 1$. These domains have been studied by many authors, see [GO, KLSV, M, Va3, V2].

3. BASIC EXAMPLES FOR THE QUASIHYPERSBOLIC VOLUME

As mentioned in Introduction, for a given domain G and a fixed point $x \in G$, our goal is to find estimates for the asymptotic behaviour of $\text{vol}_k(B_k(x, r))$ when

$$B_k(x, r) = \{z \in G: k_G(x, z) < r\}$$

is the quasihyperbolic ball. As an example, we first consider very regular domains for which the correct asymptotic behavior can be obtained by direct calculation.

3.1. Unit ball. Let us find the quasihyperbolic volume of $B_k(0, r)$ in the unit ball $G = B^n = B^n(0, 1)$. Observe first that

$$B_k(0, t) = B^n(0, 1 - e^{-t}).$$

We need the following integral representation for the hypergeometric function: for $c > b > 0$,

$$F(a, b; c; x) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt$$

where $B(b, c-b) = \Gamma(b)\Gamma(c-b)/\Gamma(c)$. By definition

$$\begin{aligned} \text{vol}_k(B^n(0, r)) &= \int_{B^n(r)} \frac{dm(x)}{(1-|x|)^n} = \omega_{n-1} \int_0^r \frac{t^{n-1}}{(1-t)^n} dt \\ &= \omega_{n-1} r^n \int_0^1 t^{n-1} (1-rt)^{-n} dt \\ &= \frac{\omega_{n-1} r^n}{n} F(n, n; n+1; r) \\ (3.1) \quad &= \frac{\omega_{n-1} r^n}{n(1-r)^{n-1}} F(1, 1; n+1; r), \end{aligned}$$

where the last equality follows from [AS, 15.3.3]

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x).$$

Now $\text{vol}_k(B_k(0, r)) = \text{vol}_k(B^n(0, 1 - e^{-r}))$ and hence for $0 < r < \infty$

$$\begin{aligned} \text{vol}_k(B_k(0, r)) &= \frac{\omega_{n-1} (1 - e^{-r})^n}{n (1 - (1 - e^{-r}))^{n-1}} F(1, 1; n+1; 1 - e^{-r}) \\ &\sim \frac{\omega_{n-1}}{n-1} e^{(n-1)r} \quad (r \rightarrow \infty), \end{aligned}$$

since by [AS, 15.1.20]

$$F(1, 1; n+1; 1) = \frac{\Gamma(n+1)\Gamma(n-1)}{\Gamma(n)^2} = \frac{n}{n-1}.$$

Proposition 3.2. For $s > 0$, $\lambda > 1$, $G = B^n$ and $E = \mathbb{R}^n \setminus G$

$$\frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} \rightarrow 1 - \lambda^{1-n}$$

as $s \rightarrow 0^+$.

Proof. Let $0 < s < 1/\lambda$. Then the claim is equivalent to

$$\frac{\text{vol}_k(B^n(1-s) \setminus B^n(1-\lambda s))}{\text{vol}_k(B^n(1-s))} \rightarrow 1 - \lambda^{1-n}$$

as $s \rightarrow 0^+$.

By (3.1)

$$\begin{aligned} & \frac{\text{vol}_k(B^n(1-s) \setminus B^n(1-\lambda s))}{\text{vol}_k(B^n(1-s))} \\ &= 1 - \frac{\lambda^{1-n}(1-\lambda s)^n F(1, 1; n+1; 1-\lambda s)}{(1-s)^n F(1, 1, n+1, 1-s)} \\ &\rightarrow 1 - \lambda^{1-n} \quad (s \rightarrow 0). \end{aligned}$$

□

Remark 3.3. For $G = B^n$, let $m_h(B^n(r))$ be the hyperbolic volume of the Euclidean ball $B^n(r)$. For a Lebesgue measurable set $A \subset B^n$ its hyperbolic volume is defined by

$$m_h(A) = \int_A \frac{2^n dm(x)}{(1-|x|^2)^n}.$$

It follows that

$$\begin{aligned} \left(\frac{2}{1+r}\right)^n \text{vol}_k(B^n(r)) &= \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1+r)^n (1-t)^n} dt \\ &\leq \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1-t^2)^n} dt = m_h(B^n(r)) \end{aligned}$$

and

$$m_h(B^n(r)) \leq \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1-t)^n} dt = 2^n \text{vol}_k(B^n(r)).$$

It is easy to see that

$$m_h(B^n(r)) = \omega_{n-1} 2^n r^n F(n/2, n; 1+n/2; r^2)/n.$$

Corollary 3.2 is true for the hyperbolic metric with the same limiting value $1 - \lambda^{n-1}$.

3.2. Punctured space. We will consider the growth of quasihyperbolic volume in multiply connected domains. It can be shown [K, Lemma 4.56, Theorem 4.57, Corollary 4.58] that in $\mathbb{R}^2 \setminus \{0\}$ the quasihyperbolic area of $B_k(x, r)$ is equal to πr^2 , if $r \leq \pi$, and $2\pi\sqrt{r^2 - \pi^2} + 2\pi^2 \arctan(\pi/\sqrt{r^2 - \pi^2})$, if $r > \pi$. Therefore

$$2\pi\sqrt{r^2 - \pi^2} \leq \text{vol}_k(B_k(x, r)) \leq 4\pi r$$

if $r > \pi$. We will now find similar lower and upper bounds for the quasihyperbolic volume of $B_k(x, r)$ in $\mathbb{R}^n \setminus \{0\}$ for $n > 2$.

Proposition 3.4. For $x \in \mathbb{R}^n \setminus \{0\}$, $n > 2$ and $r > \pi$

$$2\omega_{n-1}\sqrt{r^2 - \pi^2} \leq \text{vol}_k(B_k(x, r)) \leq 2\omega_{n-1}r.$$

Proof. Since $B_k(x, r)$ is invariant in the inversion in $S^{n-1}(|x|)$ we have $\text{vol}_k(B_k(x, r)) = 2\text{vol}_k(B_k(x, r) \cap B^n(|x|))$. By [V1, (3.9)] $B_k(x, r) \subset \mathbb{R}^n \setminus B^n(|x|e^{-r})$. Let $S = S^{n-1}(|x|e^{-\sqrt{r^2 - \pi^2}})$, then by (2.1), $\max_{y \in S} k_{\mathbb{R}^n \setminus \{0\}}(x, y) = r$ implies

$$B^n(|x|) \setminus B^n(|x|e^{-\sqrt{r^2 - \pi^2}}) \subset B_k(x, r).$$

Hence we have

$$(3.5) \quad \text{vol}_k(B^n(|x|) \setminus B^n(|x|e^{-\sqrt{r^2-\pi^2}})) \leq \text{vol}_k(B_k(x, r) \cap B^n(|x|))$$

and

$$(3.6) \quad \text{vol}_k(B_k(x, r) \cap B^n(|x|)) \leq \text{vol}_k(B^n(|x|) \setminus B^n(|x|e^{-r})).$$

Next let us find the quasihyperbolic volume of annulus $E(a, b) = \{x \in \mathbb{R}^n \mid a < |x| < b\}$, $0 < a < b < \infty$. By definition

$$(3.7) \quad \text{vol}_k(E(a, b)) = \int_{E(a, b)} \frac{dm(x)}{|x|^n} = \omega_{n-1} \int_a^b \frac{t^{n-1}}{t^n} dt = \omega_{n-1} \log \frac{b}{a}.$$

Now the assertion follows from the equations (3.5) and (3.6). \square

Corollary 3.8. *For $s > 0$, $\lambda > 1$, $G = B^n \setminus \{0\}$ and $E = \mathbb{R}^n \setminus G$*

$$\frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} \rightarrow 1 - \lambda^{1-n}$$

as $s \rightarrow 0$.

Proof. Let $s < 1/\lambda$. Then $E(s, \lambda s) = E_1 \cup E_2$ where $E_1 = B^n(1-s) \setminus B^n(1-\lambda s)$ and $E_2 = B^n(\lambda s) \setminus B^n(s)$. Then by (3.1) and (3.7)

$$\begin{aligned} \text{vol}_{k_G}(E(s, \lambda s)) &= \text{vol}_{k_{B^n}}(E_1) + \text{vol}_{k_{\mathbb{R}^n \setminus \{0\}}}(E_2) \\ &= \frac{\omega_{n-1}(1-s)^n}{ns^{n-1}} F(1, 1; n+1, 1-s) \\ &\quad - \frac{\omega_{n-1}(1-\lambda s)^n}{n(\lambda s)^{n-1}} F(1, 1; n+1, 1-\lambda s) + \omega_{n-1} \log \lambda \\ &\sim \frac{\omega_{n-1}}{n-1} (1-\lambda^{1-n}) s^{1-n} \quad (s \rightarrow 0). \end{aligned}$$

Similarly $E(s, \infty) = B^n(1-s) \setminus B^n(s) = E'_1 \cup E'_2$ where $E'_1 = B^n(1-s) \setminus B^n(1/2)$ and $E'_2 = B^n(1/2) \setminus B^n(s)$, and

$$\begin{aligned} \text{vol}_{k_G}(E(s, \infty)) &= \text{vol}_{k_{B^n}}(E'_1) + \text{vol}_{k_{\mathbb{R}^n \setminus \{0\}}}(E'_2) \\ &= \frac{\omega_{n-1}(1-s)^n}{ns^{n-1}} F(1, 1; n+1, 1-s) \\ &\quad - \frac{\omega_{n-1}}{2n} F(1, 1; n+1, 1/2) + \omega_{n-1} \log(1/2s) \\ &\sim \frac{\omega_{n-1}}{n-1} s^{1-n} + \omega_{n-1} \log(1/s) \quad (s \rightarrow 0). \end{aligned}$$

Now we have

$$\lim_{s \rightarrow 0} \frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} = \lim_{s \rightarrow 0} \frac{\frac{\omega_{n-1}}{n-1} (1-\lambda^{1-n}) s^{1-n}}{\frac{\omega_{n-1}}{n-1} s^{1-n} + \omega_{n-1} \log(1/s)} = 1 - \lambda^{1-n}.$$

\square

3.3. Half space. The next two propositions concern the case of the half space, in which case the quasihyperbolic volume coincides with the classical hyperbolic volume. The hyperbolic volume of hyperbolic simplexes in the upper half space has been considered by J. Milnor in [Mi].

Proposition 3.9. *For $x \in \mathbb{H}^2$ and $r > 0$ we have*

$$\text{vol}_k(B_k(x, r)) = 2\pi(\cosh r - 1).$$

Proof. Let us choose $x = te_2 = (0, t) \in \mathbb{H}^2$ implying that

$$B_k(x, r) = B^2(e_2 t \cosh r, t \sinh r).$$

Now

$$\begin{aligned} \text{vol}_k(B_k(x, r)) &= \int_{t \cosh r - t \sinh r}^{t \cosh r + t \sinh r} \frac{\sqrt{t^2 \sinh^2 r - (\cosh r - h)^2}}{h^2} dh \\ &= 2\pi(\cosh r - 1). \end{aligned}$$

□

Proposition 3.10. *For $x \in \mathbb{H}^3$ and $r > 0$ we have*

$$\text{vol}_k(B_k(x, r)) = \pi(\sinh(2r) - 2r).$$

Proof. Let us choose $x = te_3 = (0, 0, t) \in \mathbb{H}^3$ implying that

$$B_k(x, r) = B^3(e_3 t \cosh r, t \sinh r).$$

Now

$$\begin{aligned} \text{vol}_k(B_k(x, r)) &= \int_{t \cosh r - t \sinh r}^{t \cosh r + t \sinh r} \frac{\pi(t^2 \sinh^2 r - (\cosh r - h)^2)}{h^3} dh \\ &= \pi(\sinh(2r) - 2r). \end{aligned}$$

□

4. NUMBER OF WHITNEY CUBES

Recall that N_i denotes the number of the i^{th} generation Whitney cubes of $\mathbb{R}^n \setminus E$. We also define the number

$$(4.1) \quad \tilde{N}_i = N_{i+k_0} + \dots + N_{i+k_1},$$

where k_0 is the smallest integer satisfying $8\sqrt{n}2^{-k_0} \leq 1$ and k_1 is the largest integer with $80\sqrt{n}2^{-k_1} \geq 1$.

Lemma 4.2. *If $E \subset \mathbb{R}^n$ is compact and uniformly porous, then there are constants $0 < c < C < \infty$ and $i_0 \in \mathbb{N}$ (depending only on the uniform porosity data) such that for all $i \geq i_0$, then it holds that*

$$c2^{in}m(E(2^{-i})) \leq \tilde{N}_i \leq C2^{in}m(E(2^{-i})).$$

Proof. Let $0 < \lambda < 1 < \Lambda < \infty$. Choose $0 < p < (1 + \lambda)/(2\Lambda)$ and $r_0 > 0$ such that E is $(2p, r_0)$ -uniformly porous. Given $r > 0$ such that $R = \frac{1}{2p}(1 + \lambda)r < r_0$, let B_1, \dots, B_N be a maximal collection of disjoint balls of radius R centered at E . Then by elementary geometry

$$(4.3) \quad N\Omega_n R^n \leq m(E(R)) \leq 3^n N\Omega_n R^n.$$

For each B_i , we can find $y_i \in \frac{1}{2}B_i$ with $d(y_i, E) \geq pR = \frac{1}{2}(1 + \lambda)r$. Moreover (since we live in the Euclidean space), we can in fact find such y_i with $d(y_i, E) = \frac{1}{2}(1 + \lambda)r$. Then $B(y_i, \frac{1}{2}(1 - \lambda)r) \subset E(r) \setminus E(\lambda r)$ and thus, combined with (4.3),

$$\begin{aligned} m(E(r) \setminus E(\lambda r)) &\geq N\Omega_n \left(\frac{1 - \lambda}{2}\right)^n r^n \geq \left(\frac{p(1 - \lambda)}{3(1 + \lambda)}\right)^n 3^n N\Omega_n R^n \\ &\geq \left(\frac{p(1 - \lambda)}{3(1 + \lambda)}\right)^n m(E(\Lambda r)), \end{aligned}$$

recall that $R \geq \Lambda r$.

Applying the above estimate with $r = 2^{-i-3}$, $\lambda = \frac{1}{2}$ and $\Lambda = 8$ and using (2.5) implies

$$\begin{aligned} m(E(2^{-i})) &\leq cm(E(2^{-i-3}) \setminus E(2^{-i-4})) \\ &\leq c \sum_{j=k_0}^{k_1} N_{i+j} 2^{-(i+j)n} \leq cm(E(2^{-i})) \end{aligned}$$

where $c < +\infty$ depends only on n and p . □

We also need the following simple fact

Lemma 4.4. *If $E \subset \mathbb{R}^n$ is compact and Q -regular for some $0 < Q < n$, then there are constants $0 < c < C < \infty$ (depending only on the Q -regularity data) such that $cr^{n-Q} \leq m(E(r)) \leq Cr^{n-Q}$ for $0 < r < \text{diam}(E)$.*

Proof. Let $r \in (0, \text{diam}(E))$. Let B_1, \dots, B_N be a maximal collection of disjoint balls each of radius r centered at points in E . Then, as in (4.3),

$$\bigcup_{i=1}^N B_i \subset E(r) \subset \bigcup_{i=1}^N 3B_i,$$

and in particular,

$$(4.5) \quad N\Omega_n r^n \leq m(E(r)) \leq N\Omega_n 3^n r^n.$$

Using the Q -regularity, we have

$$(4.6) \quad N\alpha r^Q \leq \sum_i \mu(B_i) \leq \mu(E) = \mu(E(r)) \leq \sum_i \mu(3B_i) \leq N\beta 3^Q r^Q.$$

The claim follows by combining (4.5) and (4.6). □

Putting the two previous Lemmas together, we obtain:

Corollary 4.7. *If $E \subset \mathbb{R}^n$ is compact and Q -regular for some $0 < Q < n$, then there are constants $0 < c < C < \infty$ and $i_0 \in \mathbb{N}$ (depending only on the Q -regularity data) such that for all $i \geq i_0$,*

$$c2^{iQ} \leq \tilde{N}_i \leq C2^{iQ}.$$

Remark 4.8. 1. The porosity and regularity assumptions in the above results are essential because without them, we cannot guarantee that $m(E(2r) \setminus E(r))$ is comparable to $m(E(r))$ for small $r > 0$.

2. In [KLV] the authors considered the relations between the number of Whitney balls and Minkowski dimension, in an Ahlfors regular space, which also can be reformulated equivalently in terms of the measures of the layer sets. They also showed a close connection between the 'surface area' of the layer sets and the number of Whitney balls.

5. GROWTH OF THE QUASIHYPERSBOLIC VOLUME

Our next goal is to prove growth estimates of the above type for a larger class of domains, the so-called φ -uniform domains. We first observe that the quasihyperbolic volume of any Whitney cube is essentially a constant.

Lemma 5.1. *Let E be a closed subset of \mathbb{R}^n and let G be a component of $\mathbb{R}^n \setminus E$. Then for the Whitney cubes of $\mathbb{R}^n \setminus E$ contained in G we have*

$$2^{-2n}n^{-n/2} \leq \text{vol}_k(Q_j^i) \leq n^{-n/2}.$$

Proof. Since $m(Q_j^i) = 2^{-in}$ we have by (2.3)

$$\text{vol}_k(Q_j^i) \geq \frac{2^{(i-2)n}}{\sqrt{n}^n} 2^{-in} = 2^{-2n}n^{-n/2},$$

and

$$\text{vol}_k(Q_j^i) \leq \frac{2^{in}}{\sqrt{n}^n} 2^{-in} = n^{-n/2}.$$

□

Theorem 5.2. *Let G be a proper subdomain of \mathbb{R}^n with compact and Q -regular boundary for some $Q \in (0, n)$. For each $x \in G$, there exists a constant $C < \infty$ which depends only on the Q -regularity data and on $d(x)$ such that for all sufficiently large r , we have*

$$\text{vol}_k(B_k(x, r)) \leq Ce^{Qr}.$$

Proof. Let $E = \partial G$. We have

$$B_k(x, r) \subset \{z \in G : e^{-r}d(x) < d(z) < e^r d(x)\} =: D$$

and

$$Q_j^i \subset E(2^{-i}\sqrt{n}, 5 \cdot 2^{-i}\sqrt{n}).$$

If $5 \cdot 2^{-i}\sqrt{n} < e^{-r}d(x)$ or $2^{-i}\sqrt{n} > e^r d(x)$, i.e.

$$i > \frac{r + \log(5\sqrt{n}/d(x))}{\log 2} =: K, \quad \text{or} \quad i < -\frac{r - \log(\sqrt{n}/d(x))}{\log 2} =: K'$$

then $Q_j^i \cap D = \emptyset$, and thus

$$(5.3) \quad B_k(x, r) \subset \bigcup_{i=[K']}^{[K]} \bigcup_{j=1}^{N_i} Q_j^i.$$

Note that the inequality $N_i \leq M(\text{diam}(E))2^{ni}$ always holds for all $i \in \mathbb{N}$ [MV] and therefore $N_i \leq C(\text{diam}(E), i_0)$ for $i \leq i_0$, where i_0 is as in Lemma 4.2. Hence for all sufficiently large r , by (5.3) and Lemma 5.1, we have

$$\text{vol}_k(B_k(x, r)) \leq \sum_{i=[K']}^{[K]} \sum_{j=1}^{N_i} \text{vol}_k(Q_j^i) \leq n^{-n/2} \sum_{i=[K']}^{[K]} N_i.$$

Using the estimates for the number of Whitney cubes from Lemma 4.2 and for the measure of level sets from Lemma 4.4, we get

$$\begin{aligned} \text{vol}_k(B_k(x, r)) &\leq n^{-n/2}(i_0 - [K'])C(\text{diam}(E), i_0) \\ &\quad + n^{-n/2} \sum_{i=i_0}^{[K]} c2^{in} m(E(2^{-i})) \\ &\leq n^{-n/2}(i_0 - [K'])C(\text{diam}(E), i_0) \\ &\quad + n^{-n/2} \sum_{i=i_0}^{[K]} c2^{in} c'(2^{-i})^{n-Q} \\ &\leq n^{-n/2}(i_0 - [K'])C(\text{diam}(E), i_0) + \frac{cc'n^{-n/2}}{2^Q - 1} 2^{KQ} \\ &\leq Ce^{Qr}, \end{aligned}$$

where the last inequality holds for sufficiently large r by the definition of K and K' . \square

Lemma 5.4. *Let $\emptyset \neq E \subset \mathbb{R}^n$ be a compact set such that $G = \mathbb{R}^n \setminus E$ is a ψ -uniform domain and let $x \in G$. Then for all $r > 0$,*

$$E \left(\frac{\text{diam}(E) + 2d(x)}{\psi^{-1}(r)}, d(x) \right) \subset B_k(x, r).$$

Proof. We only need to consider for $r \geq \psi((\text{diam}(E) + 2d(x))/d(x))$. Otherwise, it is trivial. For arbitrary

$$y \in E \left(\frac{\text{diam}(E) + 2d(x)}{\psi^{-1}(r)}, d(x) \right),$$

we have that

$$(5.5) \quad d(y) < d(x) \quad \text{and} \quad \psi \left(\frac{\text{diam}(E) + 2d(x)}{d(y)} \right) < r.$$

By (5.5) and the ψ -uniformity of G we get

$$\begin{aligned} k_G(x, y) &\leq \psi \left(\frac{|x - y|}{\min\{d(x), d(y)\}} \right) \\ &\leq \psi \left(\frac{\text{diam}(E) + d(x) + d(y)}{\min\{d(x), d(y)\}} \right) \\ &\leq \psi \left(\frac{\text{diam}(E) + 2d(x)}{d(y)} \right) \\ &< r, \end{aligned}$$

which implies the desired inclusion. \square

Theorem 5.6. *Let $E \subset \mathbb{R}^n$ be a compact Q -regular set with $0 < Q < n$ such that $G = \mathbb{R}^n \setminus E$ is a ψ -uniform domain. Then for each $x \in G$ there exist $c > 0$ and $r_1 > 0$ such that*

$$\text{vol}_k(B_k(x, r)) \geq c(\psi^{-1}(r))^Q,$$

for all $r > r_1$, where c and r_1 depend only on the Q -regularity data and $d(x)$.

Proof. Let K be the largest integer such that

$$(5.7) \quad 2^{-(K+k_1)}\sqrt{n} \geq \frac{\text{diam}(E) + 2d(x)}{\psi^{-1}(r)},$$

and let

$$r_1 = \psi \left((\text{diam}(E) + 2d(x)) \max \left\{ \frac{10 \cdot 2^{k_1 - k_0}}{d(x)}, \frac{2^{i_0 + k_1}}{\sqrt{n}} \right\} \right),$$

where i_0 is as in Lemma 4.2, and k_1 and k_0 are as in (4.1). Then for all $r > r_1$, we have that

$$(5.8) \quad 2^{-K} < \frac{C}{\psi^{-1}(r)}, \quad C := \frac{2^{k_1+1}(\text{diam}(E) + 2d(x))}{\sqrt{n}},$$

and

$$(5.9) \quad 5\sqrt{n}2^{-(K+k_0)} \leq d(x).$$

For $r > r_1$, it follows from (2.5), (5.7), (5.9), and Lemma 5.4 that for $K + k_0 \leq m \leq K + k_1$, all the m^{th} -generation Whitney cubes belong to $B_k(x, r)$. Together with Corollary 4.7 and Lemma 5.1 this yields for large values of r the required estimate

$$\text{vol}_k(B_k(x, r)) \geq c_0 \tilde{N}_K \geq c_1 2^{KQ} \geq c(\psi^{-1}(r))^Q,$$

where the last inequality follows from (5.8), and the constant $c > 0$ only depends on the Q -regularity data and $d(x)$. \square

By combining Theorem 5.2, Theorem 5.6 and the definition of uniform domain we get the following corollary.

Corollary 5.10. *Let $E \subset \mathbb{R}^n$ be a compact Q -regular set with $0 < Q < n$ such that $G = \mathbb{R}^n \setminus E$ is a uniform domain with the uniformity constant $L > 1$. Then for each $x \in G$ and sufficiently large $r > 0$,*

$$ce^{Qr/L} \leq \text{vol}_k(B_k(x, r)) \leq Ce^{Qr},$$

where $C < \infty$ depends only on the Q -regularity data and L and $c > 0$ depends only on the Q -regularity data, L , and $d(x)$.

Example 5.11. Let $E \subset \mathbb{R}^n$ be a self-similar set whose complement is a uniform domain. For instance, E can be a Cantor set on a hyperplane, the $\frac{1}{4}$ -Cantor set in the plane, or more generally any self-similar set satisfying the strong separation condition (See e.g. [F] for the definitions). Then E is Q -regular in its dimension and Corollary 5.10 can be applied.

Remark 5.12. Although we stated the Theorem 5.6 for unbounded domains, essentially the same proof works for all ψ -uniform domains $G \subset \mathbb{R}^n$ whose boundary is Q -regular and uniformly porous in G . The last assumption means that the porosity holes in the definition of the uniform porosity lie completely inside G . For instance, in \mathbb{R}^2 , ∂G can be the Von-Koch snowflake curve. More generally, the boundary could be the union of a finite number of such Q_i -regular pieces, and in this case we could apply the lower bound with the exponent $r \min\{Q_i\}/L$ and the upper bound with the exponent $r \max\{Q_i\}$.

It is not known what is the best possible lower bound in Corollary 5.10. It is an interesting open problem if the exponent Qr/L could be replaced by cr for some uniform constant $c > 0$ independent of L .

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