

# Elliptic boundary value problems with Gaussian white noise loads

Sari Lasanen<sup>a,\*</sup>, Lassi Roininen<sup>a,b</sup>, Janne M.J. Huttunen<sup>c</sup>

<sup>a</sup>*Sodankylä Geophysical Observatory, Tähteläntie 62 FI-99600, Finland*

<sup>b</sup>*Tallinn University of Technology, Department of Mathematics, Ehitajate tee 5, 19086 Tallinn, Estonia*

<sup>c</sup>*University of Eastern Finland, Department of Applied Physics, Yliopistoranta 1 F, FI-70211 Kuopio, Finland*

Received 2 October 2016; received in revised form 15 November 2017; accepted 18 November 2017

Available online 2 December 2017

---

## Abstract

Linear second order elliptic boundary value problems (BVP) on bounded Lipschitz domains are studied in the case of Gaussian white noise loads. The challenging cases of Neumann and Robin BVPs are considered.

The main obstacle for usual variational methods is the irregularity of the load. In particular, the Neumann boundary values are not well-defined.

In this work, the BVP is formulated by replacing the continuity of boundary trace mappings with measurability. Instead of variational methods alone, the novel BVP derives also from Cameron–Martin space techniques.

The new BVP returns the study of irregular white noise to the study of  $L^2$ -loads.

© 2017 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

MSC: 34B05; 60H15; 60H35; 65N30

Keywords: Stochastic partial differential equations; Boundary value problems; Gaussian measures; Finite element methods; White noise

---

## 1. Introduction

Let  $D$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ , where the dimension  $d \geq 2$ , and denote with  $\partial_n$  the normal derivative on the boundary of  $D$ . We study a stochastic counterpart of a boundary

---

\* Corresponding author.

E-mail address: [sari.lasanen@oulu.fi](mailto:sari.lasanen@oulu.fi) (S. Lasanen).

value problem

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } D \\ Bu = 0 & \text{on } \partial D, \end{cases} \quad (1)$$

where  $f \in H^{-1}(D)$  is replaced with the Gaussian zero mean white noise  $\dot{W}$  on  $D$  and the boundary operator  $B$  is either of the Dirichlet type ( $Bu = u|_{\partial D}$ ), the Neumann type ( $Bu = \partial_n u|_{\partial D}$ ), or the Robin type ( $Bu = \partial_n u|_{\partial D} + \beta u|_{\partial D}$  for  $\beta \in \mathbb{R}$ ). The constant  $\lambda$  in (1) is positive for simplicity.

The study of stochastic elliptic boundary value problems (1) initiated from the works of Walsh [28,29] who considered solvability of the Poisson equation with zero Dirichlet boundary condition and the white noise source. Walsh studied the very weak formulation of (1) in the sense of generalized functions, that is, distributions.

In the case of homogeneous Dirichlet boundary condition  $Bu = u|_{\partial D} = 0$  and white noise load  $f = \dot{W}$ , the existence and uniqueness of pathwise continuous solution of (1) is well-known for  $d = 1, 2, 3$ , even for nonlinear equations by the results of Buckdahn and Pardoux [11]. The corresponding Neumann and Robin problems are less extensively studied, although there are numerous studies on elliptic BVPs with more regular deterministic loads. This leaves a gap in the literature which appears for example in connection with Bayesian statistical inverse problems, where the solutions of stochastic BVPs serve as priors [23]. The aim of this work is to provide a rigorous description of the stochastic BVP with white noise load that utilizes both the stochastic nature of the problem and the existing literature on more regular problems.

The main difference between stochastic Dirichlet and Neumann problems is specification of the solution space. In [11], one seeks a stochastic field  $X$  with continuous realizations that satisfies  $-\Delta X + \lambda X = \dot{W}$  in the sense of distributions and  $X|_{\partial D} = 0$ . In corresponding Neumann and Robin problems the normal derivative on the boundary is not well-defined when only the continuity of realizations has been verified, which is the main obstacle for formulating the problem. Indeed, the weak definition of the (co)normal derivative  $\partial_n u$  of the variational solution  $u$  of (1) on the boundary  $\partial D$  requires that functionals

$$\int_{\partial D} \phi \partial_n u \, d\sigma := \int_D \nabla u \cdot \nabla \phi + \lambda u \phi - f \phi \, dx \quad (2)$$

are well-defined for all  $\phi$  in a suitable function space  $H$ . This is for example satisfied when  $u \in H^1(D) =: H$  and  $f \in L^2(D)$  (see [21]).

There are several studies on how to extend an elliptic BVP to irregular loads or irregular boundary data. In [3] and references therein, BVPs are taken to be deterministic with no loads but highly irregular boundary values. Obviously, the above problem can be cast in such a form. For smooth boundaries, the several proposed extensions in [3] work nicely but for polygonal domains turn out to be problematic. A similar theme can be found in [5]. Rozanov [24] treats random fields as Hilbert space processes, and applies the theory of distributions in defining the boundary traces for  $C^2$ -smooth boundaries. The smoothness of the boundary facilitates the definition of distributions on the boundary. An attempt to solve the Neumann boundary value problem with the help of Lax–Milgram theorem is made in [17]. However, the paper does not take into account that some of the stochastically integrated functions are anticipating which suggests that correct formulation would involve multidimensional Skorohod integrals. Also the interpretation of the normal derivative is left vague. A correct formulation with more regular loads can be found e.g. in [4,27], but it is clear that the white noise loads do not fulfill the required conditions. The work in [17] can be appreciated from the point of view of a more pragmatic question, which asks

whether the white noise could be approximated by more regular stochastic fields (in the sense of an existing limit).

The approximations of white noise in Dirichlet problems are often carried out together with finite element methods [17,27,1,6,12,13,30]. Also, the convergence of approximated solutions has been verified [27,1,6,12,13,30].

In [11], the homogeneous Dirichlet solution is acquired by replacing the BVP with a Hammerstein integral equation. A similar integral equation could be written in the Neumann or Robin case (see [14]) by updating the Dirichlet Green’s function  $G(x, y)$  with a correct boundary value. In the linear case, the conjectured integral equation would be  $X(x) + \lambda \int_D G(x, y)X(y)dy = \int_D G(x, y)dW_y$  for a stochastic field  $X$  with a.s. continuous realizations, where  $dW_y$  represents multidimensional Itô integral. This is referred to as the mild form of the Neumann problem. However, it is not clear whether the realizations of  $X$  would fulfill the Neumann boundary condition  $\partial_n X|_{\partial D} = 0$  in any other than mild sense. For smooth domains, a partial answer can be found in [3] for the description of the BVP, where such a formulation is compared to another generalization of irregular boundary values.

We proceed in different direction than in [3,5]. Instead of trying to stretch the definition of the differentiability, we stretch the definition of the boundary trace with measure theoretic methods. Indeed, replacing  $f$  in (2) with an  $L^2$ -approximation of white noise hints that a rigorous definition of the normal derivative of  $X$  might not call for continuity the linear forms (2) on  $H^1(D)$  but only measurability. Similar phenomenon appears in the variational formulations of BVPs with different boundary conditions. For  $f \in L^2(D)$ , the variational form of the homogeneous Dirichlet BVP is to find  $u \in H_0^1(D)$  that satisfies

$$\int_D f(x)\psi(x)dx = \int_D \nabla u(x) \cdot \nabla \psi(x) + \lambda u(x)\psi(x)dx \tag{3}$$

for all  $\psi \in H_0^1(D)$ . Replacing  $f$  with a regular approximation of white noise hints again to the measurability of linear forms. Indeed, in the case of homogeneous Dirichlet problem, such approximative variational solutions are known to converge in  $L^2(\Omega, \Sigma, P; L^2(D))$ -norm to the correct solution [12]. The corresponding limit, when refining the white noise approximations, is then

$$\int_D \psi(x)dW_x = \lim_{n \rightarrow \infty} \int \nabla X_n \cdot \nabla \psi + \lambda X_n \psi dx \tag{4}$$

for every  $\psi \in H_0^1(D)$ , where  $dW_x$  represents a multidimensional Itô integral and  $X_n$  are the variational solutions of (3) with the approximated white noise.

The present paper contributes in this area by giving an explicit formula for the normal derivative of the solution of (1) as a measurable mapping (see Definition 3.1). Instead of tackling directly the variational formulations of general BVP or trying to interpret the normal derivative in distributional sense, we reformulate the irregular elliptic BVP so that existing results for more regular elliptic BVP can be easily utilized. The approach also avoids the need to provide new estimates for the corresponding Green’s functions, as is often the case in mild formulations. For example, the continuity of solutions of two and three dimensional Neumann and Robin problems with the white noise load follows from the regularity of a deterministic problem via well-known Gaussian arguments. Moreover, the unique solvability of high-dimensional problems is also guaranteed.

The present formulation of BVPs involves Cameron–Martin space techniques. The main tool is the method of extending a continuous linear mapping  $L$  on the Cameron–Martin space of a Gaussian field  $X \sim \mathcal{N}(0, C_X)$  into a measurable linear mapping  $\hat{L}$  on the sample space of  $X$  (see

e.g. [8]). Careful choice of the sample space is the key for allowing measurable linear extensions of the boundary operators outside their usual domain of definition.

In order to demonstrate the admissibility of the BVP, we show that the finite element approximations  $X_n$  of the solution  $X$  converge to the solution of the problem. The proof reduces essentially to a one-linear (23), even for high-dimensional problems.

The main approach to finite element methods (FEM) with irregular stochastic loads was introduced in [1], where the stochastic load  $f$  is first approximated by a spatially piecewise constant function, and then the ordinary FEM is applied (see also [12,17]). However, even in 1D the solutions of (1) with white noise load are not regular enough for standard pathwise error methods [1]. The convergence of FEM approximations is therefore recast as a question of convergence of random variables, where several other modes of convergence are available besides to pathwise convergence. From previous studies [12] it is known that the random fields  $X_n$  converge to  $X$  in the norm

$$\|X\| := \left( E \|X\|_{L^2(D)}^2 \right)^{\frac{1}{2}} \tag{5}$$

for 2D Dirichlet problems. In [13], a 3D case on a convex smooth domain is considered. Also, estimates for the speed of convergence are known [12]. Other similar works are [1,16,17,30]. More regular loads are considered in [4,27].

We improve the previous results by replacing  $L^2(D)$  with  $C(D)$  and giving generalization to the cases of Neumann and Robin boundary data. The cases of Neumann and Robin boundary conditions are new.

The contents of this paper is the following. In Section 2 we recall known results about Gaussian random variables and their linear transformations. In Section 3 we define the measurable boundary trace and measurable normal derivatives (see Definition 3.1). In Section 4 we formulate the BVP and study its unique solvability. In Section 5 the regularity of solutions is considered. In Section 6 the finite element approximations are studied.

## 2. Measure theoretic preliminaries

Let  $(\Omega, \Sigma, P)$  denote a complete probability space. We make a standing assumption that all random variables are defined on  $(\Omega, \Sigma, P)$ . Moreover, all appropriate function spaces appearing below are endowed with their Borel  $\sigma$ -algebras. We will denote with  $H^s(D)$ ,  $s \in \mathbb{R}$ , the usual Sobolev spaces on  $D$  and with  $H_0^s(D)$  the usual closure of compactly supported smooth functions on  $D$  (see e.g. [21]).

In this work, we extensively use the theory of Gaussian function-valued random variables and their linear functionals. As an introduction to present ideology, we recall the basic definitions in the case of white noise.

Let  $\mathcal{B}(D)$  denote the Borel sets of a bounded Lipschitz domain  $D \subset \mathbb{R}^d$ . Recall, that  $\dot{W}$  is the white noise on  $D$  if  $\{\dot{W}(A) : A \in \mathcal{B}(D)\}$  are Gaussian random variables with zero mean and covariance  $\mathbb{E}\dot{W}(A)\dot{W}(B) = |A \cap B|$ , where  $|A|$  denotes the Lebesgue’s measure of  $A$ , and  $\dot{W}(A \cup B) = \dot{W}(A) + \dot{W}(B)$  a.s. for disjoint  $A$  and  $B \in \mathcal{B}(D)$ . A common way to construct functionals  $\dot{W}(A)$  is through stochastic integrals

$$\dot{W}(A) = \int 1_A(x) dW_x,$$

with respect to  $d$ -dimensional Wiener field  $W_x$ , which is a Gaussian field with zero mean and covariance  $\mathbb{E}W_x W_y = \min(x_1, y_1) \cdots \min(x_d, y_d)$  for all  $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{R}^d$ .

The Itô isometry allows us to replace characteristic functions  $1_A$  of Borel sets  $A \in \mathcal{B}(D)$  by functions  $\phi \in L^2(D)$ , and hence define white noise functionals

$$\dot{W}(\phi) := \int_D \phi(x) dW_x \tag{6}$$

as Gaussian random variables with zero mean and variance  $\mathbb{E}\dot{W}(\phi)^2 = \|\phi\|_{L^2(D)}^2$ .

Instead of considering solutions for the elliptic boundary value problem as stochastic fields, we take the more general approach by considering solutions (and the white noise) as Banach space valued random variables. Let us recall some definitions (e.g. [8,9]).

Let  $\mathbb{B}$  be a separable Banach space. A mapping  $X : \Omega \rightarrow \mathbb{B}$  is a  $\mathbb{B}$ -valued random variable if  $X^{-1}(A) \in \Sigma$  for all Borel sets  $A \subset \mathbb{B}$ . Denote  $\mu_X = P \circ X^{-1}$  the image measure of  $X$  on  $\mathbb{B}$ . Let  $\mathbb{B}^*$  denote the topological dual of  $\mathbb{B}$  and  $\langle \cdot, \cdot \rangle_{\mathbb{B}, \mathbb{B}^*}$  denote the duality. A  $\mathbb{B}$ -valued random variable  $X$  is called Gaussian if  $\langle X, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}$  is Gaussian for all  $b^* \in \mathbb{B}^*$ .

For notational simplicity, we focus on reflexive  $\mathbb{B}$ . In the case of reflexive  $\mathbb{B}$ , we denote with  $m \in \mathbb{B}$  the mean of  $X$ , i.e.

$$\langle m, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} = \mathbb{E}\langle X, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}$$

for all  $b^* \in \mathbb{B}^*$ , and with  $C_X : \mathbb{B}^* \rightarrow \mathbb{B}$  the covariance operator of  $X$  i.e.

$$\langle C_X b^*, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} = \mathbb{E}\langle X - m, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} \langle X - m, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}$$

for all  $b^* \in \mathbb{B}^*$ .

Next, we recall that the white noise  $\dot{W}$  is  $H^{-d/2-\epsilon}(D)$ -valued Gaussian random variable for any  $\epsilon > 0$ . Indeed, properties of realizations of the  $d$ -dimensional white noise can be derived by using random functionals  $\dot{W}(\phi)$ . The definition of the stochastic integral helps in identifying realizations of the white noise as weak derivatives of realizations of the Wiener field. Then the random functional  $\dot{W}(\phi)$  can be identified with the linear functional  $\langle \dot{W}, \phi \rangle$  between a distribution  $\dot{W}$  and a test function  $\phi \in C_0^\infty(D)$ . It is an easy task to apply duality to study Sobolev norms  $\|\dot{W}\|_{H^{-s}(D)} = (\sum_{k=1}^\infty |\langle \dot{W}, \phi_k \rangle_{H^{-s}(D), H^s(D)}|^2)^{\frac{1}{2}}$  where  $\phi_k \in C_0^\infty(D)$  form an orthonormal basis in  $H_0^s(D)$  and  $s \geq 0$ . Recall, that orthonormal basis can be chosen from a dense set, and the dual of  $H^{-s}(D)$  can be identified with  $H_0^s(D)$  (see Theorem 3.30 in [21]). In particular,  $\dot{W}$  belongs a.s. to the Sobolev space  $H^{-d/2-\epsilon}(D)$  for any  $\epsilon > 0$ , since the series of variances  $\sum_{k=1}^\infty \|\phi_k\|_{L_2(D)}^2$  converges (see [19], Theorem 2 in Chapter. 3) by Maurin’s theorem (e.g. [15]). Similarly,  $\dot{W}$  belongs to  $H^{-d/2}(D)$  with probability zero.

The measurability of white noise can be checked by the well-known Pettis’ measurability theorem, which says that a  $\mathbb{B}$ -valued mapping is a  $\mathbb{B}$ -valued random variable, if it is weakly measurable i.e. mappings  $\langle X, b^* \rangle$  are random variables for all  $b^* \in \mathbb{B}^*$ . Hence,  $\dot{W}$  is  $H^{-d/2-\epsilon}(D)$ -valued Gaussian random variable for any  $\epsilon > 0$ .

White noise  $\dot{W}$  has mean zero and identity as the covariance operator.

**Definition 2.1.** Let  $\mathbb{B}$  be a separable reflexive Banach space and let  $X$  be a Gaussian  $\mathbb{B}$ -valued zero mean random variable whose covariance operator  $C_X$  is nontrivial. Set

$$\|b^*\|_{\mu_X} := \sqrt{\langle C_X b^*, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}}$$

for all  $b^* \in \mathbb{B}^*$  and denote with  $\mathbb{B}_{\mu_X}^*$  the closure of  $\mathbb{B}^*$  in the norm  $\|\cdot\|_{\mu_X}$ .

It is well-known that the elements of  $\mathbb{B}_{\mu_X}^*$  can be identified with  $\mu_X$ -measurable linear functionals on  $\mathbb{B}$ . More precisely, a functional on  $\mathbb{B}$  is a  $\mu$ -measurable linear functional if it is  $\mu$ -measurable and it has a version that is linear on a linear subspace of full  $\mu$ -measure. The

measurability of  $\widehat{h} \in \mathbb{B}_{\mu_X}^*$  can be seen as follows. For every  $\widehat{h} \in \mathbb{B}_{\mu_X}^*$  there exists a sequence  $(b_k^*) \subset \mathbb{B}^*$  so that  $\lim_{k \rightarrow \infty} b_k^* = \widehat{h}$  in  $\mathbb{B}_{\mu_X}$ . But then the linear functionals

$$b \mapsto \langle b, b_k^* \rangle_{\mathbb{B}, \mathbb{B}^*}, k \in \mathbb{N},$$

form a Cauchy sequence in  $L^2(\mu_X)$ . By taking a suitable subsequence, we obtain  $\mu_X$ -a.s. limit

$$b \mapsto \lim_{j \rightarrow \infty} \langle b, b_{k_j}^* \rangle_{\mathbb{B}, \mathbb{B}^*} =: \widehat{h}(b) \tag{7}$$

and

$$\|\widehat{h} - b_{k_j}\|_{\mu_X}^2 = \mathbb{E} \left( \widehat{h}(X) - \langle X, b_{k_j}^* \rangle_{\mathbb{B}, \mathbb{B}^*} \right)^2.$$

Each  $\widehat{h} \in \mathbb{B}_{\mu_X}^*$  defines a measurable functional (7) which is linear on a full measure linear subspace (for details, see Theorem 2.10.9 and Theorem 3.2.3 in [8]). We summarize the above facts in the next lemma.

**Lemma 1.** *The elements  $\widehat{h}$  of  $\mathbb{B}_{\mu_X}^*$  can be identified with the  $\mu_X$ -measurable linear functionals  $b \mapsto \widehat{h}(b)$  that are Gaussian zero mean random variables on Lebesgue’s completion of the probability space  $(\mathbb{B}, \mathcal{B}(\mathbb{B}), \mu_X)$ . Moreover, the covariance*

$$(\widehat{h}, \widehat{g})_{\mu_X} := \left( \int \widehat{h}(b)\widehat{g}(b)\mu_X(db) \right),$$

where  $\widehat{h}, \widehat{g} \in \mathbb{B}_{\mu_X}^*$ , defines an inner product on  $\mathbb{B}_{\mu_X}^*$  and  $(\widehat{h}, \widehat{h})_{\mu_X} = \|\widehat{h}\|_{\mu_X}^2$ .

The difference between just measurable and a measurable linear functional is that the versions of measurable linear functionals are not allowed to be modified on arbitrary null sets but only those null sets that will not destroy the linearity.

Especially, the mapping

$$B_{\mu_X}^* \times B^* \ni (\widehat{h}, b_2^*) \mapsto \mathbb{E}\widehat{h}(X)\langle X, b_2^* \rangle_{\mathbb{B}, \mathbb{B}^*}$$

is bilinear, and by Fernique’s theorem (see e.g. [8]) bounded in the sense that

$$|\mathbb{E}\widehat{h}(X)\langle X, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}| \leq (\mathbb{E}\widehat{h}(X)^2)^{\frac{1}{2}} (\mathbb{E}\|X\|_{\mathbb{B}}^2 \|b^*\|_{\mathbb{B}^*}^2)^{\frac{1}{2}} \leq C \|\widehat{h}\|_{\mu_X} \|b^*\|_{\mathbb{B}^*}. \tag{8}$$

**Remark 1.** By (8) and the reflexivity of  $\mathbb{B}$ , we may extend the covariance operator  $C_X : \mathbb{B}^* \rightarrow \mathbb{B}$  to a continuous mapping from  $\mathbb{B}_{\mu_X}^*$  to  $\mathbb{B}$ , and we continue to denote the extension with  $C_X$  i.e.

$$\langle C_X \widehat{h}, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} = \mathbb{E}\widehat{h}(X)\langle X, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} \tag{9}$$

for all  $\widehat{h} \in \mathbb{B}_{\mu_X}^*$  and  $b^* \in B^*$ .

**Definition 2.2.** Let  $X, C_X, \mathbb{B}$ , and  $\mathbb{B}_{\mu_X}^*$  be as in Definition 2.1 and extend  $C_X$  as in Remark 1. The Cameron–Martin space of  $X$  is the set

$$H_{\mu_X} = C_X(\mathbb{B}_{\mu_X}^*)$$

equipped with the inner product

$$(h, g)_{H_{\mu_X}} = \int_B \widehat{h}(b)\widehat{g}(b)\mu_X(db), \tag{10}$$

where for all  $h \in H_{\mu_X}$  the notation  $\widehat{h}$  means such a vector in  $\mathbb{B}_{\mu_X}^*$  that  $C_X \widehat{h} = h$ . The corresponding inner product norm is denoted with  $\|h\|_{H_{\mu_X}}$ .

**Remark 2.** The Cameron–Martin space  $H_{\mu_X}$  and the space of measurable linear functionals  $\mathbb{B}_{\mu_X}^*$  are separable Hilbert spaces for all Gaussian random variables  $X$  that have values in separable Banach spaces (see Theorem 3.2.7 in [8]).

**Remark 3.** The covariance operator  $C_X : \mathbb{B}_{\mu_X}^* \rightarrow H_{\mu_X}$  is an isometric isomorphism. From the inner product (10), we derive the bilinear form

$$\langle h, \widehat{g} \rangle_{H_{\mu_X}, \mathbb{B}_{\mu_X}^*} = \langle C_X \widehat{h}, \widehat{g} \rangle_{H_{\mu_X}, \mathbb{B}_{\mu_X}^*} := \int_B \widehat{h}(b) \widehat{g}(b) \mu_X(db),$$

and thus identify  $\mathbb{B}_{\mu_X}^*$  as the dual space of the Cameron–Martin space. By (7) and (9),

$$\langle h, \widehat{g} \rangle_{H_{\mu_X}, \mathbb{B}_{\mu_X}^*} = \widehat{g}(C_X \widehat{h}) = \widehat{g}(h), \tag{11}$$

for proper linear versions of  $b \mapsto \widehat{g}(b)$  since the Cameron–Martin space is contained in every linear subspace of full measure (see [8], Theorem 2.4.7). By density, we may always choose an orthonormal basis  $(\widehat{e}_k)$  of  $\mathbb{B}_{\mu_X}^*$  that consists of functions in  $\mathbb{B}^*$  and the corresponding image  $C_X \widehat{e}_k \subset \mathbb{B}$  is an orthonormal basis of the Cameron–Martin space.

We recall that the Cameron–Martin space of  $\dot{W}$  is  $L^2(D)$ . In general, the Cameron–Martin space of a  $\mathbb{B}$ -valued random variable is separable and the Cameron–Martin space does not depend on the sample space of the  $\mathbb{B}$ -valued random variable  $X$  (see Theorem 3.2.7 and Lemma 3.2.2 in [8]). Moreover, since the zero-mean Gaussian  $X$  has values in  $\mathbb{B}$ , then the inclusion mapping of the Cameron–Martin space into  $\mathbb{B}$  is Hilbert–Schmidt (see [8], Corollary 3.5.11).

Let us recall the definition of measurable linear operator in our setting (see Definition 3.7.1 in [8] for a more general formulation).

**Definition 2.3.** Let  $\mathbb{B}_1, \mathbb{B}_2$  be separable Banach spaces equipped with their Borel  $\sigma$ -algebras and let  $\mu$  be a Borel probability measure on  $\mathbb{B}_1$ . A mapping  $T : \mathbb{B}_1 \rightarrow \mathbb{B}_2$  is a  $\mu$ -measurable linear operator if there exists a linear mapping  $S : \mathbb{B}_1 \rightarrow \mathbb{B}_2$  such that  $S$  is  $\mu$ -measurable and  $S = T$   $\mu$ -a.e. The linear mapping  $S$  is called a proper linear version of  $T$ .

In the case of Gaussian measures, there is a close relationship between measurable linear operators and Gaussian random series (see [9], Theorem 1.4.5 and Corollary 1.4.6-7). In the next theorem, we explicitly state the form of the measurable linear operators (the result is a minor modification of [9], Corollary 1.4.6).

**Theorem 2.** Let  $X$  be a zero mean Gaussian random variable on a separable reflexive Banach space  $\mathbb{B}_1$ , let  $H_{\mu_X}$  denote the Cameron–Martin space of  $X$  and let  $(\widehat{e}_k)_{k=1}^\infty \subset \mathbb{B}_{\mu_X}^*$  denote an orthonormal basis of  $\mathbb{B}_{\mu_X}^*$ . If  $T$  is a continuous linear mapping from  $H_{\mu_X}$  into a separable Hilbert space  $\mathbb{H}$ , then

$$\widehat{T}(b) = \sum_{k=1}^\infty \widehat{e}_k(b) T C_X \widehat{e}_k \tag{12}$$

defines a  $\mu_X$ -measurable linear operator  $\widehat{T} : \mathbb{B}_1 \rightarrow \mathbb{B}_2$  for any separable Hilbert space  $\mathbb{B}_2$  such that the inclusion mapping  $\mathbb{H} \hookrightarrow \mathbb{B}_2$  is Hilbert–Schmidt.

Moreover, if  $R : \mathbb{B}_1 \rightarrow \mathbb{B}_2$  is a  $\mu_X$ -measurable linear operator whose proper linear version  $R_0$  coincides with  $T$  on  $H_{\mu_X}$ , then  $R = \widehat{T}$   $\mu_X$ -a.e.

**Proof.** The proof follows the well-known lines (e.g. [8], Theorem 3.7.6). For completeness, we provide a sketch of the proof.

First, we verify that any two proper linear measurable mappings  $S_1, S_2 : \mathbb{B}_1 \rightarrow \mathbb{B}_1$  that coincide on the Cameron–Martin space of  $X$  coincide  $\mu_X$ -a.s. on  $\mathbb{B}_1$ .

The space  $\mathbb{B}_2$  is separable and reflexive. Therefore, there exists a countable subset  $G$  of  $\mathbb{B}_2^*$  that separates the points of  $\mathbb{B}_2$ . We only need to verify that

$$\langle S_1x, b^* \rangle = \langle S_2x, b^* \rangle$$

$\mu_X$ -almost surely for every  $b^* \in G$ .

The both mappings  $x \mapsto \langle S_i x, b^* \rangle, i = 1, 2$ , are linear and measurable functionals, and they coincide on the Cameron–Martin space. Therefore, they coincide  $\mu_X$ -almost surely (see Theorem 2.10.7 in [8]). Hence the two proper linear measurable mappings coincide  $\mu_X$ -almost surely.

Next, we verify that the series is  $\mu_X$ -almost surely convergent. Since the inclusion mapping of  $\mathbb{H}$  into  $\mathbb{B}_2$  is Hilbert–Schmidt, also the mapping  $T : H_{\mu_X} \rightarrow \mathbb{B}_2$  is Hilbert–Schmidt. By definition,

$$\sum_{k=1}^{\infty} \|T(C_X \widehat{e}_k)\|_{\mathbb{B}_2}^2 < \infty$$

for any orthonormal basis  $(C_X \widehat{e}_k)$  of  $H_{\mu_X}$ .

Then the random series (12) is  $\mu_X$ -a.s. convergent, since the sum of variances  $\sum_{k=1}^{\infty} \|TC_X e_k\|_{\mathbb{B}_2}^2$  converges (e.g. [19], Theorem 2 in Chapter 3). Indeed, the coefficients  $b \mapsto \widehat{e}_k(b)$  of  $TC_X e_k$  in the series (12) are normal random variables on the Lebesgue’s completion of the probability space  $(\mathbb{B}_1, \mathcal{B}(\mathbb{B}_1), \mu_X)$  by Lemma 1. Moreover, they are independent since  $(\widehat{e}_k)$  is an orthonormal basis.

The set  $L \subset \mathbb{B}_1$ , where the series (12) converges is  $\mu_X$ -measurable linear subspace of full measure. Moreover,  $\widehat{T} : L \rightarrow \mathbb{B}_2$  is linear since  $T$  is linear. We extend  $\widehat{T}$  linearly onto  $\mathbb{B}_1$  by taking such a linear subspace  $M$  of  $\mathbb{B}_1$  that  $\mathbb{B}_1$  is a direct algebraic sum of  $L$  and  $M$  and defining  $\widehat{T}(b + b') := \widehat{T}b$  for  $b \in L$  and  $b' \in M$ . Since the convergence holds a.s., the mapping  $\widehat{T} : \mathbb{B}_1 \rightarrow \mathbb{B}_2$  is measurable with respect to the Lebesgue’s completion of the Borel  $\sigma$ -algebra of  $\mathbb{B}_1$ .  $\square$

**Definition 2.4.** Let  $T$  and  $\widehat{T}$  be as in Theorem 2. The mapping  $\widehat{T}$  is called *measurable linear extension of  $T$* .

**Corollary 3.** Let the assumptions in Theorem 2 hold. The following claims hold for a measurable linear extension  $\widehat{T}$  of  $T : H_{\mu_X} \rightarrow \mathbb{H} \hookrightarrow \mathbb{B}_2$ .

- (i) The set  $T(H_{\mu_X})$  coincides with the Cameron–Martin space of  $\widehat{T}X$  and the mapping  $T : H_{\mu_X} \rightarrow H_{\mu_{\widehat{T}X}}$  is an isometry.
- (ii) (Measurable transpose) Let  $T^* : \mathbb{B}_2^* \rightarrow (\mathbb{B}_1^*)_{\mu_X}$  denote the transpose of  $T$  and let  $b^* \in \mathbb{B}_2^*$ . Then

$$\langle \widehat{T}b, b^* \rangle_{\mathbb{B}_2, \mathbb{B}_2^*} = T^*b^*(b)$$

for  $\mu_X$ -a.e.  $b \in \mathbb{B}_1$ .

- (iii) (Associativity of compositions) Let  $\mathbb{H}_2$  be a separable Hilbert space, whose inclusion into separable Hilbert space  $\mathbb{B}_3$  is Hilbert–Schmidt. When  $\widehat{S} : \mathbb{B}_2 \rightarrow \mathbb{B}_3$  is a measurable linear



extension of the continuous linear mapping  $S : H_{\mu_{\widehat{T}X}} \rightarrow \mathbb{H}_2$ , then

$$\widehat{ST}X = \widehat{S}(\widehat{T}X)$$

almost surely.

(iv) When  $T$  is the identity mapping, we have

$$b = \widehat{T}b$$

$\mu_X$ -a.e.

**Proof.** (i) The characterization of the elements of the Cameron–Martin space follows as in Theorem 3.7.3 in [8], which also shows that the mapping  $T$  is an isometry.

(ii) Note that the mappings  $b \mapsto \langle \widehat{T}b, b^* \rangle_{\mathbb{B}_2, \mathbb{B}_3^*}$  and  $b \mapsto T^*b^*(b)$  are measurable and have proper linear versions that coincide on  $H_{\mu_X}$  for  $b^* \in \mathbb{B}^*$  by (11). By Theorem 2.10.7 in [8], these measurable functionals coincide  $\mu_X$ -a.s.

(iii) Both mappings are measurable linear operators. Indeed,  $\widehat{ST}X$  is well-defined  $P$ -measurable mapping, since  $\widehat{S}$  is  $\mu_{\widehat{T}X}$ -measurable and the set  $\{\widehat{T}X \in B\}$  has zero measure whenever  $\mu_{\widehat{T}X}(B) = 0$ . The linearity on a full measure linear subspace follows then from the definition of extension. Considering approximating sequences of measurable linear functionals and (ii), we obtain

$$\langle \widehat{ST}b, \widehat{h} \rangle_{\mathbb{B}_3, \mathbb{B}_3^*} = S^*\widehat{h}(\widehat{T}b) = T^*S^*\widehat{h}(b) = \langle \widehat{ST}b, \widehat{h} \rangle_{\mathbb{B}_3, \mathbb{B}_3^*}$$

$\mu_X$ -a.e. for each  $\widehat{h} \in \mathbb{B}_3^*$ . Taking  $\widehat{h}$  from some countable dense subset of  $\mathbb{B}_3^*$  proves the claim.

(iv) See Theorem 3.5.1 in [8].  $\square$

### 3. Measurable boundary operators

Theorem 2 allows us to define the measurable linear extensions of the boundary operators

$$Bu = u|_{\partial D}$$

and

$$Bu = \partial_n u|_{\partial D} + \lambda u|_{\partial D}$$

for  $u \in H^1(D)$ . Here we omit writing out the inclusion mappings.

For simplicity, the sample space of the boundary mapping is taken to be a scale space (for Banach scale spaces, see [20]). In particular, let us denote with  $H_{sc}(\partial D)$ , the closure of  $H^{-\frac{1}{2}}(\partial D)$  with respect to the norm

$$\|u\|_{sc} := \left( \sum_{k=1}^{\infty} k^{-2} (u, f_k)_{H^{-\frac{1}{2}}(\partial D)}^2 \right)^{\frac{1}{2}},$$

where  $(f_k)$  is a fixed orthonormal basis of  $H^{-\frac{1}{2}}(\partial D)$ .

The choice of sample space has little effect for the stochastic analysis.

**Corollary 4.** Let  $X$  be an  $H^{-r}(D)$ -valued Gaussian zero mean random variable for some  $r \geq 0$ . Let  $B : H_{\mu_X} \rightarrow H^{-\frac{1}{2}}(\partial D)$  be a continuous linear mapping and let  $(e_k)$  be an orthonormal basis of  $H_{\mu_X}$ . Then

$$\widehat{B}b = \sum_{k=1}^{\infty} \widehat{e}_k(b) B e_k$$

belongs to  $H_{sc}(\partial D)$  for  $\mu_X$ -a.e.  $b$ , the mapping  $\widehat{B} : H^{-r}(D) \rightarrow H_{sc}(\partial D)$  is a  $\mu_X$ -measurable linear operator, and  $\widehat{B}X$  is  $H_{sc}(\partial D)$ -valued Gaussian random variable that has zero mean and covariance operator  $C_{\widehat{B}X}$  satisfying  $C_{\widehat{B}X}u = BC_XB^*u$  for all  $u \in H^{\frac{1}{2}}(\partial D)$ .

**Proof.** The claim follows from Theorem 2 after we verify that the inclusion of  $H^{-\frac{1}{2}}(\partial D)$  into  $H_{sc}(\partial D)$  is Hilbert–Schmidt. By definition, this follows from

$$\sum_{k=1}^{\infty} \|f_k\|_{sc}^2 = \sum_{k=1}^{\infty} k^{-2} < \infty. \quad \square$$

Corollary 4 allows us to define the measurable linear extensions of boundary operators.

**Definition 3.1.** Let  $X$  be  $H^{-r}(D)$ -valued Gaussian zero mean random variable for some  $r \geq 0$  whose Cameron–Martin space  $H_{\mu_X}$  can be continuously included in  $H^1(D)$ , and let  $(e_k)$  be an orthonormal basis of  $H_{\mu_X}$ .

The measurable trace of  $X$  on  $\partial D$  is the  $H_{sc}(\partial D)$ -valued Gaussian zero mean random variable

$$\widehat{Tr}X = \sum_{k=1}^{\infty} \widehat{e}_k(X)e_k|_{\partial D}.$$

A proper linear version of the corresponding mapping  $\widehat{Tr}$  is called the  $\mu_X$ -measurable trace.

Assuming additionally that  $\Delta u \in L^2(D)$  for all  $u \in H_{\mu_X}$ , the measurable normal derivative of  $X$  on  $\partial D$  is the  $H_{sc}(\partial D)$ -valued Gaussian zero mean random variable

$$\widehat{\partial}_n X = \sum_{k=1}^{\infty} \widehat{e}_k(X)\partial_n e_k,$$

where  $\partial_n e_k$  denotes the usual conormal derivative of  $e_k$ . A proper linear version of the corresponding mapping  $\widehat{\partial}_n$  is called the  $\mu_X$ -measurable normal derivative.

Let us now verify that the  $\mu_X$ -measurable trace and the  $\mu_X$ -measurable normal derivative are extensions of the usual operations.

**Lemma 5.** Let  $\widehat{B}$  be a  $\mu_X$ -measurable trace or a  $\mu_X$ -measurable normal derivative.

- (a) If  $X$  has values a.s. in  $H^1(D)$ , then  $\widehat{B}X = BX$  almost surely.
- (b) If  $u$  belongs to the Cameron–Martin space of  $X$  and  $\widehat{B}u = 0$  in  $H_{sc}$ , then  $Bu = 0$  in  $H^{-\frac{1}{2}}(\partial D)$ .

**Proof.** (a) The random series

$$X = \sum_{k=1}^{\infty} \widehat{e}_k(X)e_k$$

converges in  $H^1(D)$ , and we have

$$\widehat{Tr}X = \sum_{k=1}^{\infty} \widehat{e}_k(X)e_k|_{\partial D} = X|_{\partial D}$$

by continuity of the trace operator. Similar result holds for the normal derivative.

(b) Since we are dealing with proper linear versions, we have that  $\widehat{B} = B$  on the Cameron–Martin space, and hence  $Bu = 0$  in  $H_{sc}$ . Moreover,  $H^{-\frac{1}{2}}(\partial D)$  is dense in  $H_{sc}$ .  $\square$

**Remark 4.** Take  $X$  to be  $H^{-r}(D)$ -valued Gaussian zero mean random variable. Since the dual  $H^r(D)$  of the sample space  $H^{-r}(D)$  of  $X$  is dense in  $(H^r(D))_{\mu_X}$  and  $C_X : (H^r(D))_{\mu_X} \rightarrow H_{\mu_X}$  is an isometry by Remark 3, the orthonormal basis  $(e_k)$  of  $H_{\mu_X}$  can be always chosen so that  $\widehat{e}_k$  is from the dual  $H^r(D)$  of the sample space  $H^{-r}(D)$  of  $X$ . Then

$$\widehat{e}_k(X) = \langle X, \widehat{e}_k \rangle_{H^{-r}(D), H^r(D)},$$

by (7) which is a notationally simple choice of the proper linear basis.

#### 4. Existence and uniqueness of the solution

A rough description of the Cameron–Martin space of a Gaussian random variable  $X$  leads to a crude idea of the regularity of  $X$ .

**Lemma 6.** *Let  $X$  be a zero mean Gaussian  $H^{-r}(D)$ -valued random variable, whose Cameron–Martin space  $H(\mu_X)$  can be continuously embedded into  $H^1(D)$ . Then  $X$  has realizations in  $H^{1-d/2-\delta}(D)$  a.s. for each  $\delta > 0$ .*

**Proof.** A Lipschitz domain is an extension domain (see Theorem A.4 in [21]). Hence, we may apply Maurin’s theorem, which tells that the inclusion of  $H^1(D)$  into  $H^{1-d/2-\delta}(D)$  is Hilbert–Schmidt [15]. Hence, also the Cameron–Martin space can be embedded into  $H^{1-d/2-\delta}(D)$  by a Hilbert–Schmidt mapping. Hence,  $X$  belongs a.s. in  $H^{1-d/2-\delta}$ .  $\square$

We are now ready to formulate the measurable form of the BVP.

**Theorem 7.** *Let  $D \subset \mathbb{R}^d$  be a bounded Lipschitz domain,  $\dot{W}$  be the Gaussian white noise on  $D$ ,  $r > d/2 - 1$ , and  $\lambda, \beta > 0$ .*

*There exists a pathwise unique Gaussian zero mean  $H^{-r}(D)$ -valued random field  $X$  that solves Dirichlet’s (or Neumann’s or Robin’s) BVP in the following sense:*

1. *the Cameron–Martin space  $H(\mu_X)$  of  $X$  can be continuously embedded into  $H^1(D)$  and all  $h \in H(\mu_X)$  satisfy  $\Delta h \in L^2(D)$ ,*
2. *the field  $X$  satisfies*

$$-\Delta X + \lambda X = \dot{W} \tag{13}$$

*in the sense of generalized functions, and*

3. *the field  $X$  satisfies the  $\mu_X$ -measurable boundary condition*

$$\widehat{Tr} X = 0 \text{ in } H_{sc}(\partial D) \tag{14}$$

*in the Dirichlet case (or*

$$\widehat{\partial}_n X = 0 \text{ in } H_{sc}(\partial D) \tag{15}$$

*in the Neumann case, or*

$$\widehat{\partial}_n X + \beta \widehat{Tr} X = 0 \text{ in } H_{sc}(\partial D) \tag{16}$$

*in the Robin case, correspondingly).*

At a glance, the above formulation may seem eccentric. However, it involves typical elements of elliptic BVPs. Namely, the regularity of the desired solution  $X$  is explicitly specified. This is done by requiring that (a) the sample space of  $X$  is at least in the Sobolev space  $H^{-r}(D)$ , (b)  $X$  has Gaussian distribution, and (c) the inclusion of the Cameron–Martin space of  $X$  into  $H^1(D)$  is continuous. The space  $H^{-r}(D)$  may seem unnecessary irregular, but this is not a hindrance, since the local and global regularity of the solution can be further studied and refined. On the other hand, such a weak condition is easy to verify.

The Gaussianity of the solution is explicitly required in order to apply Cameron–Martin space techniques. In particular,  $\widehat{B} : H^{-r}(D) \rightarrow (H^{-\frac{1}{2}}(\partial D))_{sc}$  is the measurable linear extension of the continuous linear operator  $B \circ I : H(\mu_X) \rightarrow H^{-\frac{1}{2}}(D)$  (see e.g. [8]). The restriction (i) on the Cameron–Martin space is needed for the definition of the normal derivative. In ordinary elliptic BVPs, the boundary trace of  $H^1(D)$ -functions is defined as a continuous linear extension of the trace operator defined originally on continuous functions. In the same spirit, the boundary operator is extended from  $H^1(D)$  onto aspired solutions. However, the extended boundary operator  $\widehat{B}$  appearing in (14), (15), and (16) is no longer required to be continuous but only measurable, which makes  $\widehat{B}X$  well-defined generalized random field on the boundary (see Definition 3.1).

Another significant difference is that  $\widehat{B}$  depends on the solution  $X$  through its Cameron–Martin space. However, it can be shown that the extension  $\widehat{B}$  coincide with the ordinary continuous boundary operator  $B$  for  $L^2$ -loads, which are dense in negatively indexed Sobolev spaces and  $X$  contributes to assigning probabilities to sets.

**Theorem 8.** *Let  $X$  be as in Theorem 7. Then*

$$X = \widehat{T}\dot{W},$$

where  $T : L^2(D) \rightarrow H^1(D)$  is defined by setting  $Tf := u$ , where

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } D \\ Bu = 0 & \text{on } \partial D, \end{cases} \tag{17}$$

and the boundary operator  $Bu = u|_{\partial D}$  in the Dirichlet case or  $Bu = \partial_n u|_{\partial D}$  in the Neumann case or  $Bu = \partial_n u|_{\partial D} + \beta u|_{\partial D}$  in the Robin case.

**Proof of Theorems 7 and 8.** We consider only Robin boundary values. Other boundary conditions are handled similarly (for the Neumann case, choose e.g.  $\beta = 0$ ). For the existence of the Gaussian field, we represent white noise  $\dot{W}$  as

$$\dot{W} = \sum_{k=1}^{\infty} \widehat{f}_k(\dot{W})f_k,$$

where  $(f_k)$  is an orthonormal basis of  $L^2(D)$  (see Corollary 3(iv)).

Now define  $T$  as in Theorem 8. Such  $T$  exists according to the well-known theory of elliptic BVPs (e.g. Theorem 4.11 in [21]).

Then the random series

$$X := \sum_{k=1}^{\infty} \widehat{f}_k(\dot{W})Tf_k$$

is convergent in  $H^{-s}(D)$  for any  $s > d/2$  by Theorem 2 and defines a zero mean Gaussian random field, whose Cameron–Martin space is  $T(L^2)$  equipped with the norm

$$\|Tf\|_{H\mu_X} = \|f\|_{L^2(D)}$$

by Corollary 3. Application of the well-known stability estimate

$$\|Tf\|_{H^1(D)} \leq C\|f\|_{L^2(D)}$$

shows that the Cameron–Martin space  $H_{\mu_X}$  can be continuously included in  $H^1(D)$ . This shows also that the realizations of  $X$  belong to  $H^{-r}(D)$  by Lemma 6. Furthermore, the operator  $-\Delta + \lambda$  is continuous on distributions, and we have

$$(-\Delta + \lambda)X = \sum_{k=1}^{\infty} \widehat{f}_k(\dot{W})(-\Delta + \lambda)Tf_k = \dot{W}$$

almost surely, since  $(-\Delta + \lambda)Tf_k = f_k$  for all  $k$ . Moreover, by Corollary 3

$$\begin{aligned} \widehat{\partial}_n X + \beta \widehat{Tr} X &= \widehat{\partial}_n \widehat{T} \dot{W} + \beta \widehat{Tr} \widehat{T} \dot{W} = \widehat{\partial}_n \widehat{T} \dot{W} + \beta \widehat{Tr} \widehat{T} \dot{W} \\ &= \sum_{k=1}^{\infty} \widehat{f}_k(\dot{W})(\partial_n + \beta Tr)Tf_k = 0 \end{aligned}$$

almost surely.

To prove the uniqueness, assume that there are two solutions  $X$  and  $\widetilde{X}$  with measurable boundary operators  $\widehat{B}$  and  $\widetilde{B}$ , respectively. We show that the Cameron–Martin space of  $X - \widetilde{X}$  is then the trivial space  $\{0\}$ . Since the Cameron–Martin spaces of  $X$  and  $\widetilde{X}$  are included continuously in  $H^1(D)$ , also the Cameron–Martin space of  $X - \widetilde{X}$  is included continuously in  $H^1(D)$ . (Indeed, by Theorem 3.3.4 in [8] it suffices to consider continuous linear forms and apply Cauchy–Schwarz inequality.)

By assumption,  $(-\Delta + \lambda)(X - \widetilde{X}) = 0$  almost surely. Hence, the same holds by linearity of  $-\Delta + \lambda$  for all functions  $u \in H^1(D)$  that belong to the Cameron–Martin space of  $X - \widetilde{X}$ . Indeed, we may consider countably many continuous linear functionals that separate the points of  $H^{-r-2}(D)$  and apply Theorem 2.10.7 in [8]. Moreover,  $\widehat{B}u = Bu = \widetilde{B}u$  in  $H^{-\frac{1}{2}}(\partial D)$  by Lemma 5. By uniqueness of the deterministic problem, its solution  $u \equiv 0$ , which means that Cameron–Martin space reduces to  $\{0\}$ . This is possible only if  $X = \widetilde{X}$  a.s.  $\square$

### 5. Improvements in global regularity

We first show that the solution of the BVP with white noise load is square integrable. According to the following theorem, the dimension and regularity of the domain  $D$  have key roles.

**Theorem 9.** *Let  $X$  and  $T$  be as in Theorem 8. Then  $X \in L^2(D)$  almost surely if and only if  $T : L^2(D) \rightarrow L^2(D)$  is a Hilbert–Schmidt operator.*

**Proof.** Consider an orthonormal basis  $(\phi_k)$  of  $L^2(D)$  consisting of smooth functions. Then  $\langle X, \phi_k \rangle_{H^{-r}, H^r}$  are well-defined random variables and we can study finiteness of the expectation

$$\mathbb{E} \sum_{k=1}^{\infty} \langle X, \phi_k \rangle^2 = \sum_{k=1}^{\infty} \mathbb{E} \langle \dot{W}, T^* \phi_k \rangle^2 = \sum_{k=1}^{\infty} \|T^* \phi_k\|_{L^2(D)}^2.$$

This shows the claim, since  $T$  is a Hilbert–Schmidt operator if and only if  $T^*$  is a Hilbert–Schmidt operator.  $\square$

In low dimensional cases it is often more natural to show that the realizations are actually continuous. Let us equip the Hölder space  $C^{0,\alpha}(D)$ ,  $0 < \alpha < 1$  with the usual norm

$$\|f\|_{C^{0,\alpha}} = \sup_{x \in D} |f(x)| + \sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

**Theorem 10.** *Let  $X$  and  $T$  be as in Theorem 8. If the regular solution operator  $T : L^2(D) \rightarrow C^{0,\alpha}(\bar{D})$  continuously, then  $X$  has a.s. continuous realizations.*

**Proof.** Let  $(f_k)$  be an orthonormal basis of  $L^2(D)$ . As a linear combination of continuous functions, the random fields

$$\widehat{T}_N \dot{W} := \sum_{k=1}^N \widehat{f}_k(\dot{W}) T f_k$$

have a.s. continuous realizations for all  $N \in \mathbb{N}$ . Moreover, the a.s. limit  $\lim_{N \rightarrow \infty} \widehat{T}_N \dot{W}(x)$  exists for each  $x$ . Indeed, by the assumptions, the composition of  $T$  with the pointwise evaluation  $\langle \cdot, \delta_x \rangle$  is a continuous linear functional on the Cameron–Martin space of  $\dot{W}$ . Hence, it has a measurable linear extension

$$b \mapsto \sum_{k=1}^\infty \widehat{f}_k(b) T f_k(x)$$

on the sample space  $H^{-r}(D)$  of the white noise. Furthermore, the distributions of the sequences  $\widehat{T}_N \dot{W}(x)$  are tight on  $H^{-r}(D)$  for fixed  $x$ . Moreover,

$$\mathbb{E}|\widehat{T}_N \dot{W}(x) - \widehat{T}_N \dot{W}(y)|^2 = \sup_{\|f\|_{L^2(D)} \leq 1} |T_N f(x) - T_N f(y)|^2 \leq \|T\|_{L^2, C^{0,\alpha}} |x - y|^\alpha,$$

since  $T_N f = T \sum_{k=1}^N \langle f, e_k \rangle e_k$  and

$$\|T_N f\|_{C^{0,\alpha}} \leq \|T\|_{L^2, C^{0,\alpha}} \left\| \sum_{k=1}^N \langle f, e_k \rangle e_k \right\|_{L^2} \leq \|T\|_{L^2, C^{0,\alpha}} \|f\|_{L^2}.$$

By Kolmogorov tightness criterion (see [26] for a nice Besov space proof), the limit  $\widehat{T} \dot{W} = \sum_{k=1}^\infty \widehat{f}_k(\dot{W}) T f_k$  has a.s. continuous realizations.  $\square$

The following familiar examples demonstrate application of the above theorem.

**Example 5.1.**

1. When  $D \subset \mathbb{R}^d$ ,  $d = 2, 3$ , is a bounded convex Lipschitz domain then

$$\|Tf\|_{H^2(D)} \leq C \|f\|_{L^2(D)}$$

for the Dirichlet problem (e.g. [18], Theorem 3.2.1.2). The embedding  $H^2(D) \hookrightarrow C^{0,\alpha}(\bar{D})$  is continuous for  $0 < \alpha < 1/2$ . Hence, the assumptions are satisfied and  $X = \widehat{T} \dot{W}$  has a.s. continuous realizations.

2. When  $D \subset \mathbb{R}^2$  is a bounded Lipschitz domain, then there exists such  $0 < \alpha < \frac{1}{2}$  that

$$\|Tf\|_{H^{1+\alpha}(D)} \leq C \|f\|_{L^2(D)}$$

in Dirichlet boundary value problems (see [25]). The embedding  $H^{1+\alpha}(D) \hookrightarrow C^{0,\alpha}(\bar{D})$  is continuous.

3. When  $D \subset \mathbb{R}^d$ ,  $d = 2, 3$ , then

$$\|Tf\|_{C^{0,\alpha}} \leq C\|f\|_{L^2(D)}$$

in the case of Robin or Neumann boundary conditions (see [22], Theorem 3.14).

### 6. Approximations

At this point, it is yet unclear if the generalizations of the Neumann and Robin boundary values have any more value than mathematical eccentricity. However, we show now that when white noise is replaced with its regular approximations, the corresponding approximative solutions converge to the solution of the generalized problem. This clarifies the generalizations from a practical point of view. In the same spirit, we study convergence of Galerkin approximations of high-dimensional problems in Theorem 13. We emphasize that such approximations are interesting, for example, as priors in numerical Bayesian estimation of unknown multivariable functions [23], and they pave way for practical uncertainty quantification in high-dimensional problems. Finally, we consider in this section some low-dimensional problems as standard examples.

Our first convergence theorem covers only certain approximations of the white noise. From various possible approximations of the white noise, we choose here the truncated sums in the measurable linear extension of the identity mapping. However, the proof only requires  $L^2$ -convergence, and transfers therefore to a wider class of approximations.

We use the generic notation  $B$  for any boundary operator appearing in Theorem 7.

**Theorem 11.** *Let  $D \subset \mathbb{R}^d$  be a bounded Lipschitz domain, let  $\dot{W}$  be the Gaussian white noise on  $D$ , let  $\{f_k\}$  be an orthonormal basis of  $L^2(D)$ ,  $r > d/2 - 1$  and let  $X$  be as in Theorem 7.*

*If  $H^1(D)$ -valued random fields  $X^{(m)}$ ,  $m \in \mathbb{N}$ , satisfy*

$$\begin{cases} -\Delta X^{(m)} + \lambda X^{(m)} = \dot{W}^{(m)} & \text{in } D \\ BX^{(m)} = 0 & \text{on } \partial D, \end{cases}$$

*where  $\dot{W}^{(m)} = \sum_{k=1}^m f_k(\dot{W})f_k$ , then  $X^{(m)}$  converges to  $X$  in  $L^2(\Omega, \Sigma, P; H^{-r}(D))$  as  $m \rightarrow \infty$ .*

**Proof.** The realizations of the load  $\dot{W}^{(m)}$  are almost surely in  $L^2(D)$ . Hence, it is easy to see that the unique field  $X^{(m)}$  exists for every  $m \in \mathbb{N}$ .

Denote with  $Tf := u$  the solution of

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } D \\ Bu = 0 & \text{on } \partial D. \end{cases} \tag{18}$$

Then  $X = \widehat{T}\dot{W}$  almost surely by Theorem 8. Moreover,  $T : L^2(D) \rightarrow H^1(D)$  continuously and the mapping  $T : L^2(D) \rightarrow H^{-r}(D)$  is Hilbert–Schmidt (see [15]).

Denote with  $P_m$  the orthogonal projection operator on  $L^2(D)$  that projects onto the subspace spanned by  $\{f_1, \dots, f_m\}$ . Then  $X^{(m)} = T\dot{W}^{(m)} = T\widehat{P}_m\dot{W} = \widehat{T}P_m\dot{W}$  by Corollary 3. Denote with  $\{g_\ell\}$  the orthonormal basis of  $H_0^r(D)$ , which we identify with the dual space of  $H^{-r}(D)$ . By Corollary 3,

$$\mathbb{E}\|X^{(m)} - X\|_{H^{-r}(D)}^2 = \mathbb{E} \sum_{\ell=1}^{\infty} \langle X^{(m)} - X, g_\ell \rangle_{H^{-r}(D), H_0^r(D)}^2$$

$$\begin{aligned}
 &= \sum_{\ell=1}^{\infty} \mathbb{E}(P_m - I)T^*g_{\ell}(\dot{W})^2 \\
 &= \sum_{\ell=1}^{\infty} \|(P_m - I)T^*g_{\ell}\|_{L^2}^2,
 \end{aligned}$$

which converges to zero as  $m \rightarrow \infty$  by Lebesgue’s dominated convergence, since  $T : L^2(D) \rightarrow H^{-r}(D)$  is a Hilbert–Schmidt operator and

$$\|(P_m - I)T^*g_{\ell}\|_{L^2(D)} \leq \|T^*g_{\ell}\|_{L^2(D)}. \quad \square$$

Next, we study different approximations of the BVP arising from Ritz–Galerkin methods. As a preliminary step, we clarify connections between certain measurable linear forms and  $L^2$ -regular approximations of the white noise. We anticipate FEM by indexing the Galerkin subspaces with  $h > 0$ , which is typically connected to the size of elements in a finite element mesh.

**Lemma 12.** *Let  $Q_h : L^2(D) \rightarrow L^2(D)$  be the orthogonal projection onto a finite-dimensional linear subspace  $V_h$  of  $L^2(D)$ . Then  $\widehat{Q}_h \dot{W} \in V_h$  almost surely.*

**Proof.** Let  $(e_k)$  be an orthonormal basis of  $L^2(D)$  whose  $K$  first components span  $V_h$ . Then

$$\widehat{Q}_h \dot{W} = \sum_{k=1}^K \widehat{e}_k(\dot{W})e_k \in V_h$$

almost surely.  $\square$

The following theorem demonstrates how convergence of Ritz–Galerkin approximations reduces to study of regular cases. Here the Ritz–Galerkin approximations include also the approximation of white noise by its orthogonal projection that was introduced in Lemma 12. We focus on Neumann and Robin problems, which have not been studied before. The results are based on the careful choice of the sample space  $H^{-r}(D)$ ,  $r > d/2 - 1$ , which is significantly more irregular than the spaces appearing in the variational approach for regular load terms.

**Theorem 13.** *Let  $D \subset \mathbb{R}^d$  be a bounded Lipschitz domain, let  $\dot{W}$  be the Gaussian white noise on  $D$ , let  $\lambda, \beta > 0$ ,  $r > d/2 - 1$ , and let  $X$  be as in Theorem 7 in the case of Robin boundary value.*

*Let  $V_h$  be a finite-dimensional linear subspace of  $H^1(D)$ , and let  $X_h$  be a  $V_h$ -valued random variable that satisfies*

$$\int_D \nabla X_h \cdot \nabla \phi dx + \lambda \int_D X_h \phi dx + \beta \int_{\partial D} X_h \phi d\sigma = \int_D (\widehat{Q}_h \dot{W}) \phi dx \tag{19}$$

for all  $\phi \in V_h$ , where  $Q_h$  is the orthogonal projection onto  $V_h$  in  $L^2(D)$ . Then

$$\mathbb{E}\|X - X_h\|_{H^{-r+s}(D)}^2 \leq C \sup_{\|f\|_{L^2(D)} \leq 1} \|u^f - u_h^f\|_{H^{1+s}(D)}^2, \tag{20}$$

where  $s \in \mathbb{R}$  is such that  $u^f, u_h^f \in H^{1+s}(D)$ ,  $u^f$  is the solution of

$$\begin{cases} -\Delta u^f + \lambda u^f = f & \text{in } D \\ \partial_n u^f|_{\partial D} + \beta u^f|_{\partial D} = 0 & \text{on } \partial D, \end{cases} \tag{21}$$



and  $u_h^f \in V_h$  is the Ritz–Galerkin approximation defined by

$$\int_D \nabla u_h^f \cdot \nabla \phi dx + \lambda \int_D u_h^f \phi dx + \beta \int_{\partial D} u_h^f \phi d\sigma = \int_D \phi f dx \tag{22}$$

for all  $\phi \in V_h$ .

**Proof.** By Lemma 12, the measurable linear forms  $\widehat{Q}_h \dot{W} \in V_h$  a.s. By the regularity of the load  $\widehat{Q}_h \dot{W}$ , the standard proof of existence and uniqueness of Ritz–Galerkin approximations holds also for  $X_h$ . Moreover, we can write  $X_h = T_h(\widehat{Q}_h \dot{W})$ , where  $T_h$  takes regular loads from  $L^2(D)$  to corresponding Ritz–Galerkin approximations in  $V_h$ .

By Corollary 3,  $T_h \widehat{Q}_h \dot{W} = \widehat{T}_h \widehat{Q}_h \dot{W}$ , which almost surely coincides with  $\widehat{T}_h \dot{W}$  since  $\langle Q_h f, \phi \rangle = \langle f, \phi \rangle$  for all  $f \in L^2(D)$  and  $\phi \in V_h$  and, by uniqueness of the solution,  $T_h Q_h = T_h$  on  $L^2(D)$ .

Denote with  $(\phi_k)$  an orthonormal basis in  $H^r(D)$ , and with  $\|L\|_{HS:V_1 \rightarrow V_2}$  the Hilbert–Schmidt norm of a linear mapping  $L$  from a separable Hilbert space  $V_1$  into a separable Hilbert space  $V_2$ . Then by Corollary 3(ii),

$$\begin{aligned} \mathbb{E} \|X - X_h\|_{H^{-r+s}(D)}^2 &= \sum_{k=1}^{\infty} \mathbb{E} \langle X - X_h, \phi_k \rangle_{H^{-r+s}(D), H^{r-s}(D)}^2 \\ &= \sum_{k=1}^{\infty} \|(T^* - T_h^*)\phi_k\|_{L^2(D)}^2 \\ &= \|T^* - T_h^*\|_{HS:H^{r-s}(D) \rightarrow L^2(D)}^2 \\ &\leq \|T^* - T_h^*\|_{H^{-1-s}(D), L^2(D)} \|I\|_{HS:H^{r-s}(D) \rightarrow H^{-1-s}(D)}, \end{aligned} \tag{23}$$

where the embedding  $I$  of  $H^{r-s}(D)$  into  $H^{-1-s}(D)$  is Hilbert–Schmidt by Maurin’s theorem. Moreover,

$$\|T^* - T_h^*\|_{H^{-1-s}(D), L^2(D)} = \|T - T_h\|_{L^2(D), H^{1+s}(D)} = \sup_{\|f\|_{L^2(D)} \leq 1} \|u^f - u_h^f\|_{H^{1+s}(D)}. \quad \square$$

The main difference to regular deterministic cases, as seen in (20), is uniform convergence with respect to the  $L^2$ -loads. However, the following corollary shows that Ritz–Galerkin approximations  $X_h$  automatically converge in our carefully chosen sample spaces.

**Corollary 14.** *Let the assumptions of Theorem 13 hold. If the finite-dimensional subspaces  $V_h$  fulfill the condition*

$$\lim_{h \rightarrow 0} \min_{v \in V_h} \|u - v\|_{H^1(D)} = 0$$

for all  $u \in H^1(D)$ , then  $X_h$  converges to  $X$  in  $L^2(\Omega, \Sigma, P; H^{-r}(D))$ , when  $h \rightarrow 0$  and  $r > d/2 - 1$ .

**Proof.** Let  $u^f$  and  $u_h^f$  satisfy (21) and (22), respectively. By Cea’s lemma

$$\begin{aligned} \sup_{\|f\|_{L^2(D)} \leq 1} \|u^f - u_h^f\|_{H^1(D)} &\leq \sup_{\|f\|_{L^2(D)} \leq 1} \min_{v \in V_h} \|u^f - v\|_{H^1(D)}^2 \\ &= \sup_{\|f\|_{L^2(D)} \leq 1} \|(I - Q_h)u^f\|_{H^1(D)}, \end{aligned}$$

where  $Q_h$  is the orthogonal projection in  $H^1(D)$  onto  $V_h$ .

By the  $H^1$ -regularity of the solution  $u^f$  of the Robin BVP, the corresponding inhomogeneous Neumann BVP has boundary data  $-\beta u^f|_{\partial D} = u^f|_{\partial D} \in H^{\frac{1}{2}}(\partial D)$ . By Theorem 4 in [25], the unique solution  $u^f$  of this Neumann problem belongs to  $H^{1+s}(D)$  for some  $0 < s < \frac{1}{2}$ .

The embedding of  $H^{1+s}(D) \rightarrow H^1(D)$  is compact. Under the mapping  $A := f \mapsto u^f$ , the image of the unit ball of  $L^2(D)$  is relatively compact in  $H^1(D)$ . For fixed  $\varepsilon > 0$ , all open balls  $B(\phi, \varepsilon)$  of  $H^1(D)$  whose center points  $\phi$  belong to the image of the unit ball under the mapping  $A$ , form a cover of the image and we may choose a finite subcover consisting of sets  $B(\phi_k, \varepsilon)$ ,  $k = 1, \dots, K$ . For any  $f$  satisfying  $\|f\|_{L^2(D)} \leq 1$ , we have

$$\begin{aligned} \|(I - Q_h)u^f\|_{H^1(D)} &\leq \|u^f - \phi_k\|_{H^1(D)} + \sum_{k=1}^K \|(I - Q_h)\phi_k\|_{H^1(D)} \\ &\leq \varepsilon + \sum_{k=1}^K \|(I - Q_h)\phi_k\|_{H^1(D)} \end{aligned}$$

for some  $k = 1, \dots, K$ .  $\square$

**Remark 5.** Replacing  $H^1(D)$  with  $H_0^1(D)$  in Theorem 13 gives the same results for the homogeneous Dirichlet problem (see [25] for the required regularity).

Many practical applications of boundary value problems involve finite element methods (e.g. [10]). Below, we give some examples of the speed of the convergence.

**Example 6.1.**

1. Consider the Dirichlet problem in a bounded Lipschitz domain  $D \subset \mathbb{R}^d$  with piecewise  $C^1$ -boundary. Let  $D$  be divided into curved elements as in [7] with maximal distance  $h$  between nodes. Then

$$\mathbb{E}\|X - X_h\|_{H^{-r}(D)}^2 \leq Ch^{2s}.$$

for  $0 < s < \frac{1}{2}$ . Indeed (see Equation 6.21 in [7]),

$$\|u^f - u_h^f\|_{H^1(D)}^2 \leq Ch^{2s} \|u\|_{1+s}^2$$

for  $0 < s < \frac{1}{2}$ . The solution  $u$  of the Dirichlet problem belongs to  $H^{1+s}(D)$  by Theorem 3 in [25].

2. Consider the Neumann problem in polygonal domain  $D = G \times (0, a) \subset \mathbb{R}^3$ , where  $G \subset \mathbb{R}^2$  is a bounded polygonal domain and  $a > 0$ . When the domain is divided into tetrahedral elements with maximum diameter of  $h$ , and the finite element space  $V_h \subset H^1(D)$  consists of piecewise linear functions on  $\mathcal{T}_h$  (see [2] for details), it holds that

$$\mathbb{E}\|X - X_h\|_{H^{-r}(D)}^2 \leq Ch^2.$$

Indeed, then  $\int_D |\nabla(u^f - u_h^f)|^2 dx \leq Ch^2 \|f\|_{L^2(D)}^2$  by Theorem 5.4 in [2]. The seminorm is comparable to  $H^1$ -norm since the constant in the generalized Poincaré inequality vanishes by choosing a constant test function in the variational form of (21) and in (22).

Next, we improve the previous convergence results by replacing  $H^{-r}(D)$  with the space of continuous functions  $C(\bar{D})$  equipped with the usual supremum norm. Instead of  $L^2$ -convergence, we study weak convergence of probability distributions. We apply the notations appearing in Theorem 13.

**Theorem 15.** Let  $D \subset \mathbb{R}^2$  be a bounded Lipschitz domain. Let  $X$  be as in Theorem 7 and let  $X_h$  be the corresponding finite element approximations of  $X$ . Assume that the solution  $u^f$  of (18) and its finite element approximations  $u_h^f$  are such that

$$\lim_{h \rightarrow 0} \sup_{\|f\|_{L^2(D)} \leq 1} |u_h^f(x) - u^f(x)| = 0$$

for any  $x \in \bar{D}$  and there exists  $\alpha, C > 0$  such that

$$\sup_{\|f\|_{L^2(D)} \leq 1} |u_h^f(x) - u_h^f(y)| \leq C|x - y|^\alpha \tag{24}$$

for all  $0 < h < 1$  and  $x, y \in \bar{D}$ .

Then the probability distribution of  $X_h$  converges to the probability distribution of  $X$  weakly in the sense of measures on  $C(\bar{D})$ .

**Proof.** It is clear from (24) that  $X_h$  has continuous realizations.

We show that the finite-dimensional probability distributions of  $X_h$  converge to the finite-dimensional probability distributions of  $X$ , and that the family of probability distributions of  $X_h$  on  $C(\bar{D})$  is tight.

The convergence of finite-dimensional distributions of zero mean Gaussian random variables  $(X_h(x_1), \dots, X_h(x_k))$  depends only on the convergence of the covariances

$$\mathbb{E}X_h(x)X_h(y).$$

Note that  $X_h(x) = \langle X_h, \delta_x \rangle$ . Moreover,

$$X_h = \widehat{T}_h \dot{W}, \tag{25}$$

where  $T_h f$  solves the variational boundary value problem for  $f \in L^2(D)$ . Hence

$$\mathbb{E}X_h(x)X_h(y) = \mathbb{E}\langle X_h, \delta_x \rangle \langle X_h, \delta_y \rangle = \mathbb{E}\langle \dot{W}, T_h^* \delta_x \rangle \langle \dot{W}, T_h^* \delta_y \rangle = \langle T_h^* \delta_x, T_h^* \delta_y \rangle.$$

We show that  $T_h^* \delta_x$  converge to  $T^* \delta_x$  in  $L^2(D)$ . Indeed,

$$\|T_h^* \delta_x - T^* \delta_x\|_{L^2(D)} = \sup_{\|f\|_{L^2(D)} \leq 1} |u_h^f(x) - u^f(x)|$$

which converges to zero by the assumptions. Therefore, the finite dimensional distributions converge.

Next, we verify tightness by Kolmogorov’s theorem (see [26]). By linearity

$$X_h(x) - X_h(y) = \langle X_h, \delta_x - \delta_y \rangle. \tag{26}$$

We insert (25) and (26) into

$$\begin{aligned} \mathbb{E}|X_h(x) - X_h(y)|^2 &= \mathbb{E}\langle X_h, \delta_x - \delta_y \rangle^2 = \mathbb{E}\langle \dot{W}, T_h^*(\delta_x - \delta_y) \rangle^2 \\ &= \|T_h^*(\delta_x - \delta_y)\|_{L^2(D)}^2. \end{aligned}$$

By the assumptions,

$$\mathbb{E}|X_h(x) - X_h(y)|^2 = \sup_{\|f\|_{L^2(D)} \leq 1} |u_h^f(x) - u_h^f(y)|^2 \leq C|x - y|^{2\alpha}.$$

Similarly  $\mathbb{E}|X_h(x)|^2 \leq C$ . Identification of the limit with the probability distribution of  $X$  is carried out with the help of characteristic functions.  $\square$

## 7. Conclusions

We have shown the surprising fact that the high dimensionality of domain does not prevent the solvability of Dirichlet, Neumann or Robin BVPs with Gaussian white noise loads.

The admissibility of the new formulation has been demonstrated by showing that the finite element approximations of BVP with regular discretized white noise loads converge to the solution of the new BVP when both the finite element approximations and the white noise discretizations are refined. For Neumann and Robin BVPs, such finite-dimensional approximations have been utilized before e.g. in Bayesian inverse problems. However, a proof of existence of the limit appears here for the first time. Moreover, we have derived estimates of the speed of convergence in carefully chosen sample spaces, although the regularity of the white noise load reduces as a function of dimension.

The present study carries relevance not only for Bayesian inverse problems but also for uncertainty quantification problems for stochastic partial differential equations. In particular, the new approach gives a unified framework for asymptotical studies by clarifying the meaning of the pathwise solution of Neumann and Robin BVPs.

The presented methodology is an effective tool tailored for Gaussian problems and does not directly generalize to nonlinear elliptic problems. However, the principle idea of replacing the normal derivative on the boundary with a measurable mapping may carry over to more general problems.

## Acknowledgments

This work has been funded by Academy of Finland (grant number 250215, Finnish Programme for Centre of Excellence in Research 2012–2017) and European Research Council (ERC advanced grant 267700—Inverse problems).

## References

- [1] E.J. Allen, S.J. Novosel, Z. Zhang, Finite element and difference approximation of some linear stochastic partial differential equations, *Stoch. Stoch. Rep.* 64 (1–2) (1998) 117–142.
- [2] T. Apel, D. Sirch,  $L^2$ -error estimates for Dirichlet and Neumann problems on anisotropic finite element meshes, *Appl. Math.* 56 (2) (2011) 177–206. URL <http://dx.doi.org/10.1007/s10492-011-0002-7>.
- [3] I. Babuška, V. Nistor, Boundary value problems in spaces of distributions on smooth and polygonal domains, *J. Comput. Appl. Math.* 218 (1) (2008) 137–148.
- [4] I. Babuška, R. Tempone, G.E. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations, *SIAM J. Numer. Anal.* 42 (2) (2004) 800–825.
- [5] G. Benfatto, G. Gallavotti, F. Nicolò, Elliptic equations and Gaussian processes, *J. Funct. Anal.* 36 (3) (1980) 343–400.
- [6] F.E. Benth, J. Gjerde, Convergence rates for finite element approximations of stochastic partial differential equations, *Stoch. Stoch. Rep.* 63 (3–4) (1998) 313–326.
- [7] C. Bernardi, Optimal finite-element interpolation on curved domains, *SIAM J. Numer. Anal.* 26 (5) (1989) 1212–1240. URL <http://dx.doi.org/10.1137/0726068>.
- [8] V.I. Bogachev, Gaussian Measures, in: *Mathematical Surveys and Monographs*, vol. 62, American Mathematical Society, Providence, RI, 1998, p. xii+433.
- [9] V.I. Bogachev, Differentiable Measures and the Malliavin Calculus, in: *Mathematical Surveys and Monographs*, vol. 164, American Mathematical Society, Providence, RI, 2010, p. xvi+488.
- [10] S.C. Brenner, L.R. Scott, *The Mathematical Theory of Finite Element Methods*, in: *Texts in Applied Mathematics*, vol. 15, Springer-Verlag, New York, 1994, p. xii+294.
- [11] R. Buckdahn, É. Pardoux, Monotonicity methods for white noise driven quasi-linear SPDEs, in: *Diffusion Processes and Related Problems in Analysis*, Vol. I (Evanston, IL, 1989), in: *Progr. Probab.*, vol. 22, Birkhäuser Boston, Boston, MA, 1990, pp. 219–233.

- [12] Y. Cao, H. Yang, L. Yin, Finite element methods for semilinear elliptic stochastic partial differential equations, *Numer. Math.* 106 (2) (2007) 181–198.
- [13] Y. Cao, R. Zhang, K. Zhang, Finite element and discontinuous Galerkin method for stochastic Helmholtz equation in two- and three-dimensions, *J. Comput. Math.* 26 (5) (2008) 702–715.
- [14] Y. Cao, R. Zhang, K. Zhang, Finite element method and discontinuous Galerkin method for stochastic scattering problem of Helmholtz type in  $\mathbb{R}^d$  ( $d = 2, 3$ ), *Potential Anal.* 28 (4) (2008) 301–319.
- [15] W.F. Donoghue Jr., On a theorem of K. Maurin, *Studia Math.* 24 (1964) 1–5.
- [16] Q. Du, T. Zhang, Numerical approximation of some linear stochastic partial differential equations driven by special additive noises, *SIAM J. Numer. Anal.* 40 (4) (2002) 1421–1445 (electronic).
- [17] S.R. Franklin, P. Seshaiyer, P.W. Smith, A three-field finite element method for elliptic partial differential equations driven by stochastic loads, *Stoch. Anal. Appl.* 23 (4) (2005) 757–783.
- [18] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, in: *Monographs and Studies in Mathematics*, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985, p. xiv+410.
- [19] J.-P. Kahane, *Some Random Series of Functions*, second ed, in: *Cambridge Studies in Advanced Mathematics*, vol. 5, Cambridge University Press, Cambridge, 1985, p. xiv+305.
- [20] S.G. Krein, Y.I. Petunin, Scales of banach spaces, *Russian Math. Surveys* 21 (2) (1966) 85. URL <http://stacks.iop.org/0036-0279/21/i=2/a=R03>.
- [21] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000, p. xiv+357.
- [22] R. Nittka, Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains, *J. Differential Equations* 251 (4–5) (2011) 860–880.
- [23] L. Roininen, J.M.J. Huttunen, S. Lasanen, Whittle-Matérn priors for Bayesian statistical inversion with applications in electrical impedance tomography, *Inverse Probl. Imaging* 8 (2) (2014) 561–586.
- [24] Y.A. Rozanov, Stochastic Sobolev spaces and their boundary trace, *Theory Probab. Appl.* 40 (1) (1996) 104–115.
- [25] G. Savaré, Regularity results for elliptic equations in Lipschitz domains, *J. Funct. Anal.* 152 (1) (1998) 176–201.
- [26] R.L. Schilling, Sobolev embedding for stochastic processes, *Expo. Math.* 18 (3) (2000) 239–242.
- [27] C. Schwab, R.-A. Todor, Sparse finite elements for elliptic problems with stochastic loading, *Numer. Math.* 95 (4) (2003) 707–734.
- [28] J.B. Walsh, A stochastic model of neural response, *Adv. Appl. Probab.* 13 (2) (1981) 231–281.
- [29] J.B. Walsh, An introduction to stochastic partial differential equations, in: *École e’été de Probabilités de Saint-Flour, XIV—1984*, in: *Lecture Notes in Math.*, vol. 1180, Springer, Berlin, 1986, pp. 265–439.
- [30] K. Zhang, R. Zhang, Y. Yin, S. Yu, Domain decomposition methods for linear and semilinear elliptic stochastic partial differential equations, *Appl. Math. Comput.* 195 (2) (2008) 630–640.