On linear independence measures of the values of Mahler functions

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Abstract

In this paper, we estimate the linear independence measures for the values of a class Mahler functions of degree one and two. For the purpose, we study the determinants of suitable Hermite-Padé approximation polynomials. Based on the non-vanishing of these determinants, we apply the functional equations to get an infinite sequence of approximations which is used to produce the linear independence measures.

1 Introduction and results

In the present work our aim is to obtain linear independence measures for the values of a class of Mahler functions $F(z), G(z) \in \mathbb{Q}[[z]]$ converging on some open disc $D_r := \{z : |z| < r \leq 1\}$ and satisfying a system of Mahler type functional equations

$$\begin{cases} F(z^d) = p_{11}(z)F(z) + p_{12}(z)G(z) + p_{10}(z), \\ G(z^d) = p_{21}(z)F(z) + p_{22}(z)G(z) + p_{20}(z) \end{cases} \tag{1}$$

with $p_{ij}(z) \in \mathbb{Q}(z)$ satisfying $p_{11}(z)p_{22}(z) - p_{12}(z)p_{21}(z) \neq 0$. Note that Mahler functions of degree one or two satisfy functional equations of the above type, if $F(z)$ and $G(z)$ are Mahler functions of degree one, then $p_{12}(z) = p_{21}(z) = 0$, and if $F(z)$ is of degree two, then we choose $G(z) = F(z^d)$. Our general result (Theorem 6 in Sec. 4) needs some technical notations to be presented later, and therefore to introduce our results we give here applications to some well-known functions.

The linear independence measures studied here are lower bounds for linear forms (in 1 and certain numbers γ_1 and γ_2) of the form

$$|h_0 + h_1\gamma_1 + h_2\gamma_2| > CH^{-\mu} \tag{2}$$

valid for any integers h_0, h_1, h_2, not all zero, where the exponent μ is given explicitly, $H = \max\{|h_1|, |h_2|, H_0\}$, and positive constants C and H_0 are independent of h_i. In our results γ_1 and γ_2 are the values of the functions under consideration at rational points $a/b \in D_r \backslash \{0\}$, where $\log|a| = \lambda \log b$ $(0 \leq \lambda < \log(rb)/\log b)$. We note that generally [12, Theorem 4.4.1] implies the existence of a μ (≥ 2) in our cases below, and here our aim is to obtain an explicit upper bound for the linear independence exponent

$$\mu(\gamma_1, \gamma_2) := \inf\{\mu : (2) \text{ holds for some } C > 0, H_0 > 0\}.$$

This work is a continuation to [14], where we studied simultaneous approximations of similar numbers γ_1 and γ_2. We also note that, after Bugeaud's remarkable work [4] on Thue-Morse numbers, there has appeared several works on the irrationality exponents of the values of degree one Mahler functions, see [2, 5, 9, 11, 13, 15] and the references in [5]. In particular, the irrationality exponents of the numbers in Theorem 1-3 below equal 2.

1.1 Thue-Morse number and its square

Our first result studies the product

$$T(z) = \prod_{j=0}^{\infty}(1 - z^{2^j}),$$

the generating function of the Thue-Morse sequence on $\{-1, 1\}$, satisfying

$$T(z) = (1 - z)T(z^2). \tag{3}$$

Theorem 1. *We have*

$$\mu\left(T\left(\frac{1}{b}\right), T^2\left(\frac{1}{b}\right)\right) \leq \frac{91}{32} \approx 2.843\ldots.$$

More generally, if $0 \leq \lambda < 7/29$*, then*

$$\mu\left(T\left(\frac{a}{b}\right), T^2\left(\frac{a}{b}\right)\right) \leq \frac{91}{32 - 104\lambda}.$$

1.2 Stern's sequence and its twisted version

Next, let $A(z)$ and $B(z)$ be generating functions of Stern's diatomic sequence and its twisted version. These functions satisfy functional equations

$$A(z) = (1 + z + z^2)A(z^2), \qquad B(z) = 2 - (1 + z + z^2)B(z^2), \tag{4}$$

of type (I), see e.g. [7].

Theorem 2. *We have*

$$\mu\left(A\left(\frac{1}{b}\right), B\left(\frac{1}{b}\right)\right) \leq \frac{26}{9} \approx 2.888\ldots.$$

More generally,

$$\mu\left(A\left(\frac{a}{b}\right), B\left(\frac{a}{b}\right)\right) \leq \begin{cases} \dfrac{130}{45 - 149\lambda}, & \text{if } \lambda < \dfrac{145}{1289}, \\ \dfrac{69}{25 - 89\lambda}, & \text{if } \dfrac{145}{1289} \leq \lambda < \dfrac{5}{29}. \end{cases}$$

1.3 Lambert series $G_3(z)$ and $F_3(z)$

The functions

$$G_3(z) = \sum_{j=0}^{\infty} \frac{z^{3^j}}{1 - z^{3^j}}, \qquad F_3(z) = \sum_{j=0}^{\infty} \frac{z^{3^j}}{1 + z^{3^j}} = -G_3(-z)$$

satisfy

$$(1 - z)G_3(z) - (1 - z)G_3(z^3) - z = 0, \qquad (1 + z)F_3(z) - (1 + z)F_3(z^3) - z = 0. \tag{5}$$

The following result studies the values of these typical examples of Mahler functions.

Theorem 3. *We have*

$$\mu\left(G_3\left(\frac{1}{b}\right), F_3\left(\frac{1}{b}\right)\right) \le \frac{129}{37} \approx 3.486\ldots.$$

More generally,

$$\mu\left(G_3\left(\frac{a}{b}\right), F_3\left(\frac{a}{b}\right)\right) \le \begin{cases} \dfrac{129}{37-119\lambda}, & \text{if } \lambda < \dfrac{25}{443}, \\ \dfrac{83}{24-80\lambda}, & \text{if } \dfrac{25}{443} \le \lambda < \dfrac{43}{337}, \\ \dfrac{57}{17-59\lambda}, & \text{if } \dfrac{43}{337} \le \lambda < \dfrac{7}{29}. \end{cases}$$

1.4 The Rudin-Shapiro sequence

Let $(r_n)_{n\ge 0}$ be the Rudin-Shapiro sequence defined by $r_0 = 1, r_{2n} = r_n, r_{2n+1} = (-1)^n r_n$. Its generating function $R(z) = \sum_{n\ge 0} r_n z^n$ satisfies

$$R(z) = R(z^2) + zR(-z^2). \tag{6}$$

We shall investigate the values of $R(z)$ and $R(-z)$ at some rational points.

Theorem 4. *We have*

$$\mu\left(R\left(\frac{1}{b}\right), R\left(-\frac{1}{b}\right)\right) \le \frac{13}{4} = 3.25.$$

More generally,

$$\mu\left(R\left(\frac{a}{b}\right), R\left(-\frac{a}{b}\right)\right) \le \begin{cases} \dfrac{39}{12-40\lambda}, & \text{if } \lambda < \dfrac{21}{187}, \\ \dfrac{47}{15-53\lambda}, & \text{if } \dfrac{21}{187} \le \lambda < \dfrac{3}{13}. \end{cases}$$

1.5 A degree 2 Mahler function

As an example of degree 2 Mahler functions we take the function $S(z)$ satisfying $S(0) = 1$ and

$$zS(z) - (1 + z + z^2)S(z^4) + S(z^{16}) = 0. \tag{7}$$

This function was introduced by Dilcher and Stolarsky [10], and it has been studied recently in several works , see e.g. [1], [3] and [8], in particular the algebraic independence of $S(\alpha)$, $S'(\alpha)$, $S(\alpha^4)$ and $S'(\alpha^4)$ is proved in [3] for all algebraic α, $0 < |\alpha| < 1$. Note also that in [8] an upper bounded 5 is obtained for the irrationality exponent of $S(1/b)$.

Theorem 5. *We have*

$$\mu\left(S\left(\frac{1}{b}\right), S\left(\frac{1}{b^4}\right)\right) \le \frac{167}{25} = 6.68.$$

More generally, if $0 \le \lambda < 1/5$*, then*

$$\mu\left(S\left(\frac{a}{b}\right), S\left(\left(\frac{a}{b}\right)^4\right)\right) \le \frac{167}{25-93\lambda}.$$

All results above are based on non-vanishing of the determinants of suitable Hermite-Padé approximation polynomials studied in Section 2. This non-vanishing is verified here by computing the determinants, but it would be of great interest to find a more general criterion for this. After having some non-zero determinants the functional equations can be used to produce a sufficiently dense infinite sequence of approximations with non-zero determinants. It is well-known that such approximations can be used to produce linear independence measures. Section 3 contains this consideration, and it is then applied to prove a general result in Section 4. The proofs of Theorems 1-5 are given in Section 5.

2 Important determinants

We first note that the above system (1) can be given in the form

$$P(z)F(z^d) = P_{11}(z)F(z) + P_{12}(z)G(z) + P_{10}(z), \tag{8}$$
$$P(z)G(z^d) = P_{21}(z)F(z) + P_{22}(z)G(z) + P_{20}(z), \tag{9}$$

where $P(z)$, the least common denominator of $p_{ij}(z)$, and $P_{ij}(z) = P(z)p_{ij}(z)$ belong to $\mathbb{Z}[z]$ and satisfy $P_{11}(z)P_{22}(z) - P_{12}(z)P_{21}(z) \neq 0$.

For an integer $k \geq 1$, let $A_k(z), B_k(z), C_k(z) \in \mathbb{Z}[z]$ denote $(d_1, d_2, d_3) = (d_1(k), d_2(k), d_3(k))$ Hermite-Padé approximation polynomials of $F(z), G(z)$ and 1, so

$$A_k(z)F(z) + B_k(z)G(z) + C_k(z) = R_k(z), \tag{10}$$

where $\deg A_k(z) \leq d_1$, $\deg B_k(z) \leq d_2$, $\deg C_k(z) \leq d_3$ and the order of zero at $z = 0$ of the remainder term $R_k(z)$ satisfies $\operatorname{ord} R_k(z) =: o(k) \geq d_1 + d_2 + d_3 + 2$. Clearly such polynomials, where at least one of $A_k(z), B_k(z)$ is not zero, exist. Substituting in (10) z^d for z and applying (8) and (9), we obtain

$$\begin{aligned}(P_{11}(z)A_k(z^d) + P_{21}(z)B_k(z^d))F(z) + (P_{12}(z)A_k(z^d) + P_{22}(z)B_k(z^d))G(z)\\ + P_{10}(z)A_k(z^d) + P_{20}(z)B_k(z^d) + P(z)C_k(z^d) = P(z)R_k(z^d).\end{aligned}$$

Repeating this procedure m times, we have

$$A_{k,m}(z)F(z) + B_{k,m}(z)G(z) + C_{k,m}(z) = R_{k,m}(z), \ m = 0, 1, \ldots, \tag{11}$$

where $A_{k,0}(z) = A_k(z)$, $B_{k,0}(z) = B_k(z)$, $C_{k,0}(z) = C_k(z)$, $R_{k,0}(z) = R_k(z)$ and, for $m = 1, 2, \ldots,$

$$\begin{cases} A_{k,m}(z) = P_{11}(z)A_{k,m-1}(z^d) + P_{21}(z)B_{k,m-1}(z^d), \\ B_{k,m}(z) = P_{12}(z)A_{k,m-1}(z^d) + P_{22}(z)B_{k,m-1}(z^d), \\ C_{k,m}(z) = P_{10}(z)A_{k,m-1}(z^d) + P_{20}(z)B_{k,m-1}(z^d) \\ \qquad\qquad + P(z)C_{k,m-1}(z^d), \\ R_{k,m}(z) = P(z)R_{k,m-1}(z^d). \end{cases} \tag{12}$$

We are interested in determinants

$$\Delta(\underline{k}, m, z) := \det \begin{pmatrix} A_{k_1,m}(z) & B_{k_1,m}(z) & C_{k_1,m}(z) \\ A_{k_2,m}(z) & B_{k_2,m}(z) & C_{k_2,m}(z) \\ A_{k_3,m}(z) & B_{k_3,m}(z) & C_{k_3,m}(z) \end{pmatrix},$$

where $1 \leq k_1 < k_2 < k_3$. By the above recursions (12)

$$\Delta(\underline{k}, m, z) = \Phi(z)\Delta(\underline{k}, m-1, z^d), \quad \Phi(z) := (P_{11}(z)P_{22}(z) - P_{12}(z)P_{21}(z))P(z),$$

and so

$$\Delta(\underline{k},m,z)=\Delta(\underline{k},0,z^{d^m})\prod_{j=0}^{m-1}\Phi(z^{d^j}). \tag{13}$$

In particular, for degree one functions we have $\Phi(z)=P_{11}(z)P_{22}(z)P(z)$, and for degree two function $F(z)$ with $G(z)=F(z^d)$ the function $\Phi(z)=P_{21}(z)P^2(z)$.

Let $\bar{d}(k):=\max\{d_1(k),d_2(k),d_3(k)\}$. By our assumption $k_1<k_2<k_3$ it is natural to assume that $\bar{d}(k_1)\le\bar{d}(k_2)\le\bar{d}(k_3)$ and $o(k_1)\le o(k_2)\le o(k_3)$. Since

$$\Delta(\underline{k},0,z)=\det\begin{pmatrix}A_{k_1}(z) & B_{k_1}(z) & R_{k_1}(z)\\ A_{k_2}(z) & B_{k_2}(z) & R_{k_2}(z)\\ A_{k_3}(z) & B_{k_3}(z) & R_{k_3}(z)\end{pmatrix},$$

it follows that $o(k_1)\le \operatorname{ord}\Delta(\underline{k},0,z)\le\deg\Delta(\underline{k},0,z)\le\bar{d}(k_1)+\bar{d}(k_2)+\bar{d}(k_3)$, if $\Delta(\underline{k},0,z)\neq0$. Thus in this case

$$\Delta(\underline{k},0,z)=:z^{o(k_1)}D(\underline{k},z) \tag{14}$$

with some polynomial $D(\underline{k},z)\neq0,\deg D(\underline{k},z)\le\bar{d}(k_1)+\bar{d}(k_2)+\bar{d}(k_3)-o(k_1)$. Further, if $o(k_1)>\bar{d}(k_1)+\bar{d}(k_2)+\bar{d}(k_3)$, then $\Delta(\underline{k},0,z)=0$.

We note that the condition $D(\underline{k},z)\neq0$ gives a strong restriction to $o(k_1)$. For example, if $d_j(k_1)=k,d_j(k_2)=k+1,d_j(k_3)=k+2$ $(j=1,2,3)$, then deg $\Delta(\underline{k},0,z)\le3k+3$ and $o(k_1)\ge3k+2$. Thus the condition $D(\underline{k},z)\neq0$ is possible only if $3k+2\le o(k_1)\le3k+3$.

The above means that one determinant $\Delta(\underline{k},0,z)\neq0$ gives an infinite sequence of determinants $\Delta(\underline{k},m,z)\neq0,m=0,1,\dots$. When considering the values of the functions at rational points $z=a/b$ we need to know that $\Delta(\underline{k},m,a/b)\neq0$ at least for all sufficiently large m. This condition can be verified in many concrete cases by using (13) and (14), since deg $D(\underline{k},z)$ is small.

3 Fundamental lemma

In this section γ_1 and γ_2 denote real numbers and $b\ge2$ is an integer. Let $\underline{k}=\underline{k}(\ell)=(k_{\ell,1},k_{\ell,2},k_{\ell,3})$ $(\ell=1,\dots,L)$ be vectors with positive integer components $k_{\ell,i}$ satisfying $k_{\ell,1}<k_{\ell,2}<k_{\ell,3}$ and $k_{\ell,3}\le k_{\ell+1,1}$ $(\ell=1,\dots,L-1),k_{L,3}\le dk_{1,1}$. Assume that for each $k=k_{\ell,i}$ there exists an integer $m_0(k)$ such that for all $m\ge m_0(k)$ we have linear forms

$$a_{k,m}\gamma_1+b_{k,m}\gamma_2+c_{k,m}=r_{k,m}$$

with the following properties (i) – (iii).

(i) The coefficients $a_{k,m},b_{k,m},c_{k,m}\in\mathbb{Z}$ and satisfy

$$\max\{|a_{k,m}|,|b_{k,m}|\}\le c_1(k)b^{E(k)d^m}, \tag{15}$$

where $E(k)$ and $c_1(k)$ (as also $c_2(k),\dots$ later) are positive constants independent of m.

(ii) We have

$$|r_{k,m}|\le c_2(k)b^{-V(k)d^m}, \tag{16}$$

where $V(k)>0$ is independent of m.

(iii) The determinant

$$\det\begin{pmatrix}a_{k_{\ell,1},m} & b_{k_{\ell,1},m} & c_{k_{\ell,1},m}\\ a_{k_{\ell,2},m} & b_{k_{\ell,2},m} & c_{k_{\ell,2},m}\\ a_{k_{\ell,3},m} & b_{k_{\ell,3},m} & c_{k_{\ell,3},m}\end{pmatrix}\neq0.$$

for all $\ell=1,\dots,L;m\ge m_0(\underline{k}(\ell))=\max_{1\le i\le3}\{m_0(k_{\ell,i})\}$.

For the following fundamental lemma, we finally define, for all $\ell = 1, \dots, L$, the notations

$$\theta(\ell) = \max_{1\le i<j\le 3}\{E(k_{\ell,i}) + E(k_{\ell,j})\},$$
$$\nu(\ell) = \min_{\substack{1\le i,j\le 3\\ i\neq j}}\{V(k_{\ell,i}) - E(k_{\ell,j})\},$$

and denote $K := (\underline{k}(1), \dots, \underline{k}(L))$.

Lemma 1. *Suppose that $0 < \nu(1) < \cdots < \nu(L) < d\nu(1)$. Then there exist positive constants $C = C(K)$ and $H_0 = H_0(K)$ such that for any integers h_0, h_1, h_2, not all zero,*

$$|h_0 + h_1\gamma_1 + h_2\gamma_2| > CH^{-\mu},$$

where $H = \max\{|h_1|, |h_2|, H_0\}$ and

$$\mu = \max_{1\le\ell\le L}\mu(\ell), \quad \mu(\ell) := \frac{\theta(\ell+1)}{\nu(\ell)}, \quad \theta(L+1) := d\theta(1).$$

Proof. Let

$$\Lambda = h_0 + h_1\gamma_1 + h_2\gamma_2.$$

By the condition (iii) above, for all $\ell = 1, \dots, L$ there exist $1 \le i < j \le 3$ such that

$$D(\underline{k}(\ell), \underline{h}) := \det\begin{pmatrix} h_1 & h_2 & h_0 \\ a_{k_{\ell,i},m} & b_{k_{\ell,i},m} & c_{k_{\ell,i},m} \\ a_{k_{\ell,j},m} & b_{k_{\ell,j},m} & c_{k_{\ell,j},m} \end{pmatrix} = \det\begin{pmatrix} h_1 & h_2 & \Lambda \\ a_{k_{\ell,i},m} & b_{k_{\ell,i},m} & r_{k_{\ell,i},m} \\ a_{k_{\ell,j},m} & b_{k_{\ell,j},m} & r_{k_{\ell,j},m} \end{pmatrix} \neq 0.$$

Since $D(\underline{k}(\ell), \underline{h})$ is an integer, we obtain, by (15) and (16),

$$1 \le 2|\Lambda|\, c_1(k_{\ell,i})c_1(k_{\ell,j})b^{(E(k_{\ell,i})+E(k_{\ell,j}))d^m} + 2hc_1(k_{\ell,j})c_2(k_{\ell,i})b^{-(V(k_{\ell,i})-E(k_{\ell,j}))d^m} + \qquad (17)$$
$$2hc_1(k_{\ell,i})c_2(k_{\ell,j})b^{-(V(k_{\ell,j})-E(k_{\ell,i}))d^m}$$

with $h = \max\{|h_1|, |h_2|\}$. The definitions of $\theta(\ell)$ and $\nu(\ell)$ then give

$$1 \le C_1(K)\,|\Lambda|\, b^{\theta(\ell)d^m} + C_2(K)hb^{-\nu(\ell)d^m} \qquad (18)$$

for all $m \ge M_0 := \max\{m_0(\underline{k}(1)), \dots, m_0(\underline{k}(L))\}$, and here $C_1(K)$ and $C_2(K)$ (and also $C_3(K)$ later) are positive constants depending on K. Note that $C_1(K)$ and $C_2(K)$ are the same for all ℓ.

We now choose H_0 in such a way that

$$2C_2(K)H_0 \ge b^{\nu(1)d^{M_0}},$$

and fix the pair (ℓ, m) from the sequence $(1, M_0)$, …, (L, M_0), $(1, M_0+1)$, …, (L, M_0+1), $(1, M_0+2)$, … to be the first one satisfying

$$2C_2(K)H < b^{\nu(\ell)d^m},$$

where $H = \max\{h, H_0\}$. Then $(\ell, m) \neq (1, M_0)$, and the pair just before it is $(\ell-1, m)$, if $\ell > 1$, and $(L, m-1)$, if $\ell = 1$. The above choice means that

$$2C_2(K)H \ge b^{\nu(\ell-1)d^m}, \quad \ell > 1,$$

$$2C_2(K)H \ge b^{\nu(L)d^{m-1}}, \quad \ell = 1.$$

In the first case, by (18),

$$\frac{1}{2} < C_1(K)\,|\Lambda|\,b^{\theta(\ell)d^m} = C_1(K)\,|\Lambda|\,(b^{\nu(\ell-1)d^m})^{\theta(\ell)/\nu(\ell-1)} \le C_3(K)\,|\Lambda|\,H^{\mu}.$$

In the case $\ell = 1$ we similarly have

$$\frac{1}{2} < C_1(K)\,|\Lambda|\,b^{\theta(1)d^m} = C_1(K)\,|\Lambda|\,(b^{\nu(L)d^{m-1}})^{d\theta(1)/\nu(L)} \le C_3(K)\,|\Lambda|\,H^{\mu}.$$

This proves our lemma. □

4 General theorem

We now assume that $F(z), G(z) \in \mathbb{Q}[[z]]$ converge in some disk D_r and satisfy (8) and (9). Our aim is to apply Lemma 1 to consider the function values $F(a/b)$ and $G(a/b)$ at non-zero rational points $a/b \in D_r$, where $\log|a| = \lambda \log b$, $0 \le \lambda < \log(rb)/\log b$. We also assume that

$$(P_{11}((a/b)^{d^j})P_{22}((a/b)^{d^j}) - P_{12}((a/b)^{d^j})P_{21}((a/b)^{d^j}))P((a/b)^{d^j}) \ne 0,\ j = 0, 1, \ldots. \tag{19}$$

The approximation forms we use are obtained from (11) at $z = a/b$. The recursions (12) imply, for all $m \ge 1$,

$$\deg A_{k,m}(z),\ \deg B_{k,m}(z),\ \deg C_{k,m}(z) \le \left(\bar{e}(k) + \frac{\tau}{d-1}\right)\cdot d^m - \frac{\tau}{d-1} \tag{20}$$

where $\bar{e}(k)$ and τ are non-negative integers satisfying $\bar{e}(k) \le \bar{d}(k) := \max\{d_1(k), d_2(k), d_3(k)\}$ and $\tau \le \nu$, the maximum of the degrees of $P_{ij}(z)$ and $P(z)$. Thus the multiplication of (11) at $z = a/b$ by

$$Q_{k,m} := b^{(\bar{e}(k)+\frac{\tau}{d-1})d^m - \frac{\tau}{d-1}}$$

leads to linear forms

$$a_{k,m}F(\frac{a}{b}) + b_{k,m}G(\frac{a}{b}) + c_{k,m} = r_{k,m}, \quad m = 0, 1, \ldots,$$

where all $a_{k,m}, b_{k,m}$ and $c_{k,m}$ are integers. To be able to apply Lemma 1 with $\gamma_1 = F(a/b), \gamma_2 = G(a/b)$ we need to estimate the coefficients $a_{k,m}$ and $b_{k,m}$ and the remainders $r_{k,m}$. For this we apply the recursions (12).

Let $\widetilde{P}(z)$ denote the polynomial, where the coefficient of z^j is the maximum of the absolute values of the corresponding coefficients in $P_{ij}(z), 1 \le i, j \le 2$. Then, for all $m = 1, 2, \ldots$,

$$|A_{k,m}(z)| \le \widetilde{P}(|z|)(\left|A_{k,m-1}(z^d)\right| + \delta\left|B_{k,m-1}(z^d)\right|),$$

$$|B_{k,m}(z)| \le \widetilde{P}(|z|)(\delta\left|A_{k,m-1}(z^d)\right| + \left|B_{k,m-1}(z^d)\right|),$$

where $\delta = 0$ for degree one functions $F(z)$ and $G(z)$, and $\delta = 1$ otherwise. Applying these inequalities we obtain

$$\max\{|A_{k,m}(z)|, |B_{k,m}(z)|\} \le (1+\delta)^m \max\{\left|A_k(z^{d^m})\right|, \left|B_k(z^{d^m})\right|\} \prod_{j=0}^{m-1} \widetilde{P}(|z|^{d^j}).$$

Therefore, for all $m \ge m_1(k)$,

$$\max\{|a_{k,m}|, |b_{k,m}|\} \le c_3(k) b^{(\bar{e}(k)+\frac{\tau}{d-1})d^m},$$

if the condition

$$(1+\delta)\left|\widetilde{P}(0)\right| \le 1 \tag{21}$$

holds. Generally, for any given $\delta_1 > 0$,

$$\max\{|a_{k,m}|, |b_{k,m}|\} \le c_3(k) b^{(\bar{e}(k)+\frac{\tau}{d-1}+\delta_1)d^m} \tag{22}$$

for all $m \ge m_2(k, \delta_1)$, and under the condition (21) we may choose here $\delta_1 = 0$.

Since

$$R_{k,m}(z) = R_k(z^{d^m}) \prod_{j=0}^{m-1} P(z^{d^j}),$$

we also have

$$|r_{k,m}| \le c_4(k) \max\{1, |P(0)|^m\} b^{-((1-\lambda)o(k)-\bar{e}(k)-\frac{\tau}{d-1})d^m}$$

for all $m \ge m_3(k)$. Thus, for any given $\delta_2 > 0$,

$$|r_{k,m}| \le c_4(k) b^{-((1-\lambda)o(k)-\bar{e}(k)-\frac{\tau}{d-1}-\delta_2)d^m} \tag{23}$$

for all $m \ge m_4(k, \delta_2)$, and we may use here the value $\delta_2 = 0$, if the condition

$$|P(0)| \le 1 \tag{24}$$

holds.

Thus we have the estimates (15) and (16) for all $m \ge m_5(k, \delta_1, \delta_2)$, where

$$E(k) = \bar{e}(k) + \frac{\tau}{d-1} + \delta_1, \; V(k) = (1-\lambda)o(k) - \bar{e}(k) - \frac{\tau}{d-1} - \delta_2. \tag{25}$$

By using these values with Lemma 1 we get the following theorem, we only need to note that the condition $D(\underline{k}, z) \ne 0$ implies $D(\underline{k}, (a/b)^{d^m}) \ne 0$ for all $m \ge m_6(\underline{k}, a/b)$.

Theorem 6. *Assume that the condition (19) holds and $D(\underline{k}, z) \ne 0$ for all $\ell = 1, \ldots, L$. Let $\theta(\ell)$ and $\nu(\ell)$ be defined as in Lemma 1 with $E(k)$ and $V(k)$ given in (25). If $0 < \nu(1) < \cdots < \nu(L) < d\nu(1)$, then there exist positive constants $\lambda_0 = \lambda_0(K, F, G), C = C(K, a/b, F, G)$ and $H_0 = H_0(K, a/b, F, G)$ such that for all $0 \le \lambda < \lambda_0$ and any integers h_0, h_1, h_2, not all zero,*

$$\left|h_0 + h_1 F(\frac{a}{b}) + h_2 G(\frac{a}{b})\right| > CH^{-\mu}$$

with H and μ as in Lemma 1.

5 Proof of Theorems 1-5

We are ready to prove Theorems 1-5. We start by giving the following formulae which follow from (25):

$$\begin{cases} \theta(\ell) = \max\limits_{1\le i<j\le 3} \{\bar{e}(k_{\ell,i}) + \bar{e}(k_{\ell,j})\} + \dfrac{2\tau}{d-1} + 2\delta_1, \\ \nu(\ell) = \min\limits_{\substack{1\le i,j\le 3 \\ i\ne j}} \{(1-\lambda)o(k_{\ell,i}) - \bar{e}(k_{\ell,i}) - \bar{e}(k_{\ell,j})\} - \dfrac{2\tau}{d-1} - \delta_1 - \delta_2. \end{cases} \tag{26}$$

Thus we should choose τ and δ_i as small as possible while applying Lemma 1.

Proof of Theorem 1. To prove Theorem 1, we apply Theorem 6 with $F(z) = T(z)$, $G(z) = T^2(z)$. Now, by (3),

$$(1-z)^2F(z^2) = (1-z)F(z), \quad (1-z)^2G(z^2) = G(z).$$

Therefore $r = 1$, $\delta = 0$, $P(z) = (1-z)^2$, $\widetilde{P}(z) = 1+z$ and $\widetilde{P}(0) = P(0) = 1$ gives $\delta_1 = \delta_2 = 0$. We shall use $(k, k+1, k-1)$ approximations and we may take $\bar{e}(k) = k+1$, $\tau = 0$. Our $\underline{k}(\ell)$ are $(k_{\ell,1}, k_{\ell,1}+1, k_{\ell,1}+2)$ and the choices for $k = k_{\ell,1}$ are $29, 31, 34, 43$ and 49. For all these values $o(k) = 3k+2$. Since $\deg\Delta(\underline{k}(\ell), z) \le 3k+3$, we have $D(\underline{k}(\ell), z) = s_{\ell,0} + s_{\ell,1}z$, where

$$s_{\ell,0} = \det\begin{pmatrix} A_{k+1}(0) & B_{k+1}(0) \\ A_{k+2}(0) & B_{k+2}(0) \end{pmatrix} c \neq 0$$

where c is the coefficient of z^{3k+2} in $R_k(z)$ (see Appendix). In fact $s_{\ell,0}$ is nonzero in all of our cases, also in the proofs of Theorems 2-5. By using (26) we get

$$\theta(\ell) = 2k+5, \quad \nu(\ell) = k-2-\lambda(3k+2)$$

for all $\lambda < 2/3$. So we have the following table.

ℓ	1	2	3	4	5
k	29	31	34	43	49
$\theta(\ell)$	63	67	73	91	103
$\nu(\ell)$	$27-89\lambda$	$29-95\lambda$	$32-104\lambda$	$41-131\lambda$	$47-149\lambda$

For the condition $0 < \nu(1) < \cdots < \nu(5) < 2\nu(1)$ we need to assume $\lambda < \lambda_0 := 7/29 \approx 0.241\ldots$. When $\lambda < \lambda_0$, the comparison of $\mu(\ell)$ gives

$$\mu = \max_{\ell} \frac{\theta(\ell+1)}{\nu(\ell)} = \frac{\theta(4)}{\nu(3)} = \frac{91}{32-104\lambda}.$$

This prove Theorem 1. □

To prove Theorem 2-5, we need to modify the choices of parameters.

Proof of Theorem 2. Here we apply Theorem 6 with $F(z) = A(z)$, $G(z) = B(z)$, and the use of (4) gives $r = 1$, $\delta = 0$, $P(z) = 1+z+z^2$, $\widetilde{P}(z) = 1$ and $\delta_1 = \delta_2 = 0$. The $(k, k+1, k-1)$ approximations give $\bar{e}(k) = k+1$, $\tau = 1$. By choosing $\underline{k}(l)$ as above, where $k = k_{\ell,1}$ are 29, 31, 34, 38, 43 and 49, we get $o(k) = 3k+2$ and the determinants $D(\underline{k}(\ell), z) \neq 0$ (see Appendix). Further,

$$\theta(\ell) = 2k+7, \quad \nu(\ell) = k-4-\lambda(3k+2)$$

for all $\lambda < 2/3$, and this leads to the following table.

ℓ	1	2	3	4	5	6
k	29	31	34	38	43	49
$\theta(\ell)$	65	69	75	83	93	105
$\nu(\ell)$	$25-89\lambda$	$27-95\lambda$	$30-104\lambda$	$34-116\lambda$	$39-131\lambda$	$45-149\lambda$

To satisfy the condition $0 < \nu(1) < \cdots < \nu(6) < 2\nu(1)$, we need to assume $\lambda < \lambda_0 := 5/29 \approx 0.172\ldots$. After the comparison of $\mu(\ell) = \theta(\ell+1)/\nu(\ell)$ we see that

$$\mu = \max_{1\le\ell\le6} \mu(\ell) = \begin{cases} \mu(6) = \dfrac{130}{45-149\lambda}, & \text{if } \lambda < \dfrac{145}{1289}, \\ \mu(1) = \dfrac{69}{25-89\lambda}, & \text{if } \dfrac{145}{1289} \le \lambda < \dfrac{5}{29}. \end{cases}$$

This proves Theorem 2. □

Remark 1. *We note that here all determinants $D(\underline{k}(\ell), z) \neq 0$, $1 \leq k \leq 50$. In all other theorems most of these determinants equal zero.*

Proof of Theorem 3. In this case we apply Theorem 6 with $d = 3$, $F(z) = G_3(z)$ and $G(z) = F_3(z)$. Then (5) implies $r = 1, \delta = 0$, $P(z) = 1 - z^2$, $\widetilde{P}(z) = 1 + z^2$ and $\delta_1 = \delta_2 = 0$. The use of (k, k, k) approximations give $\bar{e}(k) = k$, $\tau = 2$. If $\underline{k}(\ell)$ is the same as above and $k = k_{\ell,1}$ are $19, 26$ and 39, then $o(k) = 3k + 2$ and $D(\underline{k}(\ell), z) \neq 0$ (see Appendix). By (26), if $\lambda < 2/3$, we get

$$\theta(\ell) = 2k + 5, \quad \nu(\ell) = k - 2 - \lambda(3k + 2).$$

Now the table is the following.

ℓ	1	2	3
k	19	26	39
$\theta(\ell)$	43	57	83
$\nu(\ell)$	$17 - 59\lambda$	$24 - 80\lambda$	$37 - 119\lambda$

The condition $0 < \nu(1) < \nu(2) < \nu(3) < 3\nu(1)$ holds, if $\lambda < \lambda_0 := 7/29 \approx 0.241\ldots$. Similarly to the above proofs we now get Theorem 3. □

Proof of Theorem 4. Here we may use Theorem 6 with $F(z) = R(z)$ and $G(z) = R(-z)$. By (6), we have

$$2zF(z^2) = zF(z) + zG(z), \quad 2zG(z^2) = F(z) + G(z).$$

Therefore, we can choose $r = 1, \delta = 1, P(z) = 2z$, and $\widetilde{P}(z) = 1 + z$. Since $P(0) = 0$, (23) holds and we may take $\delta_2 = 0$. We use the (k, k, k) approximations and we can take $\bar{e}(k) = k$ and $\tau = 1$. We also choose $\underline{k}(\ell) = (k_{\ell,1}, k_{\ell,1} + 1, k_{\ell,1} + 2)$ where $k = k_{\ell,1}$ are 17, 21 and 26. Then we get $o(k) = 3k + 2$ and the determinants $D(\underline{k}(\ell), z) \neq 0$ (see Appendix). Moreover,

$$\theta(\ell) = 2k + 5 + 2\delta_1, \quad \nu(\ell) = k - 2 - \delta_1 - \lambda(3k + 2)$$

for all $\lambda < 2/3$. This gives the following table.

ℓ	1	2	3
k	17	21	26
$\theta(\ell)$	$39 + 2\delta_1$	$47 + 2\delta_1$	$57 + 2\delta_1$
$\nu(\ell)$	$15 - \delta_1 - 53\lambda$	$19 - \delta_1 - 65\lambda$	$24 - \delta_1 - 80\lambda$

The condition $0 < \nu(1) < \nu(2) < \nu(3) < 2\nu(1)$ holds, if $\lambda < 3/13 \approx 0.230\ldots$ and δ_1 is sufficiently small. If $\lambda < 21/187 \approx 0.112\ldots$ and δ_1 is small enough, then

$$\mu = \frac{\theta(4)}{\nu(3)} = \frac{78 + 4\delta_1}{24 - \delta_1 - 80\lambda}.$$

If $21/187 \leq \lambda < 3/13$, then

$$\mu = \frac{\theta(2)}{\nu(1)} = \frac{47 + 2\delta_1}{15 - \delta_1 - 53\lambda}.$$

This proves Theorem 4, since we may choose δ_1 arbitrarily small. □

Proof of Theorem 5. We now apply Theorem 6 with $F(z) = S(z)$ and $G(z) = S(z^4)$. The use of (7) gives $d = 4$, $r = 1$ and

$$F(z^4) = G(z), \quad G(z^4) = -zF(z) + (1 + z + z^2)G(z).$$

Since $P(0) = 1$, we may choose $\delta_2 = 0$ in (23). We shall use of $(k, k-1, k)$ approximations and we may take $\bar{e}(k) = k$, $\tau = 1$. Again our $\underline{k}(\ell) = (k_{\ell,1}, k_{\ell,1}+1, k_{\ell,1}+2)$ and the choices for $k = k_{\ell,1}$ are 10 and 26. For both of these values $o(k) = 3k+1$ and the determinants $D(\underline{k}(\ell), z) \neq 0$ (see Appendix). By using (26), if $\lambda < 2/3$, we get,

$$\theta(\ell) = 2k + 3 + \frac{2}{3} + 2\delta_1, \quad \nu(\ell) = k - 1 - \frac{2}{3} - \delta_1 - \lambda(3k+1).$$

Thus we have the following table.

ℓ	1	2
k	10	26
$\theta(\ell)$	$23 + \frac{2}{3} + 2\delta_1$	$55 + \frac{2}{3} + 2\delta_1$
$\nu(\ell)$	$9 - \frac{2}{3} - \delta_1 - 31\lambda$	$25 - \frac{2}{3} - \delta_1 - 79\lambda$

If $\lambda < \lambda_0 := \frac{1}{5}$ and $\delta_1 > 0$ is sufficiently small, then $0 < \nu(1) < \nu(2) < 4\nu(1)$. Since

$$\frac{55 + \frac{2}{3} + 2\delta_1}{9 - \frac{2}{3} - \delta_1 - 31\lambda} > \frac{4(23 + \frac{2}{3} + 2\delta_1)}{25 - \frac{2}{3} - \delta_1 - 79\lambda}$$

for all $0 \leq \lambda < \lambda_0$, and $\delta_1 > 0$ can be arbitrarily small, Theorem 5 follows from Theorem 6. □

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Appendix

In the appendix we shall give the values of $\Delta(\underline{k},0,z)$, where $\underline{k}=(k,k+1,k+2)$. Tables in this appendix are organized in the following form.

k	$o(k)$	$A_k(z)$
		$B_k(z)$
		The first two non-zero terms of $R_k(z)$
$\underline{k}$		$\Delta(\underline{k},0,z)$

Approximation polynomials and related determinants in Theorem 1

29	89	$-5500\,z^{29}+264\,z^{28}-432\,z^{27}-432\,z^{26}-5908\,z^{25}-144\,z^{24}-816\,z^{23}-816\,z^{22}-6268\,z^{21}-504\,z^{20}-1152\,z^{19}-1152\,z^{18}-6580\,z^{17}-816\,z^{16}-1440\,z^{15}-1440\,z^{14}+4132\,z^{13}-1632\,z^{12}-864\,z^{11}-864\,z^{10}+4684\,z^{9}-1080\,z^{8}-336\,z^{7}-336\,z^{6}+25048\,z^{5}+1992\,z^{4}+1224\,z^{3}+1224\,z^{2}-18708\,z-1416$
		$8561\,z^{30}+9917\,z^{29}+18922\,z^{28}+20722\,z^{27}+24653\,z^{26}+27143\,z^{25}+31050\,z^{24}+33516\,z^{23}+36870\,z^{22}+40224\,z^{21}+43404\,z^{20}+46584\,z^{19}+42727\,z^{18}+46075\,z^{17}+41528\,z^{16}+44186\,z^{15}+43421\,z^{14}+46979\,z^{13}+45716\,z^{12}+48776\,z^{11}+40170\,z^{10}+43092\,z^{9}+33508\,z^{8}+35452\,z^{7}+27159\,z^{6}+28953\,z^{5}+19844\,z^{4}+20822\,z^{3}+10939\,z^{2}+11143\,z+528$
		$-57640\,z^{89}+61912\,z^{90}$
30	92	$744\,z^{30}-764\,z^{28}+844\,z^{26}-664\,z^{24}+944\,z^{22}-564\,z^{20}+1044\,z^{18}-464\,z^{16}-444\,z^{14}+1064\,z^{12}-544\,z^{10}+964\,z^{8}+444\,z^{6}-1064\,z^{4}-1388\,z^{2}+1628$

		$755\,z^{31}+755\,z^{30}+380\,z^{29}+380\,z^{28}+674\,z^{27}+674\,z^{26}+129\,z^{25}+129\,z^{24}+625\,z^{23}+625\,z^{22}-100\,z^{21}-100\,z^{20}+155\,z^{19}+155\,z^{18}-495\,z^{17}-495\,z^{16}+318\,z^{15}+318\,z^{14}-422\,z^{13}-422\,z^{12}-107\,z^{11}-107\,z^{10}-622\,z^{9}-622\,z^{8}-268\,z^{7}-268\,z^{6}-553\,z^{5}-553\,z^{4}-663\,z^{3}-663\,z^{2}-598\,z-598$
		$-5784\,z^{92}+5784\,z^{93}$
31	95	$526640\,z^{31}-403916\,z^{30}-1128412\,z^{29}-282452\,z^{28}+377436\,z^{27}-130140\,z^{26}-1216884\,z^{25}+52056\,z^{24}+349696\,z^{23}+265100\,z^{22}-1183892\,z^{21}+508028\,z^{20}+443420\,z^{19}+781804\,z^{18}-1029436\,z^{17}+1085464\,z^{16}-1338644\,z^{15}+860852\,z^{14}+498604\,z^{13}+921584\,z^{12}-825048\,z^{11}+951468\,z^{10}+951468\,z^{9}+951468\,z^{8}+2249624\,z^{7}+50128\,z^{6}+412376\,z^{5}-10604\,z^{4}-2417124\,z^{3}-133032\,z^{2}+953712\,z-315228$
		$1708923\,z^{32}+1514436\,z^{31}+2757186\,z^{30}+2413761\,z^{29}+4598714\,z^{28}+3949701\,z^{27}+5376256\,z^{26}+4497570\,z^{25}+7296661\,z^{24}+6077442\,z^{23}+7720942\,z^{22}+6297837\,z^{21}+8631316\,z^{20}+6946485\,z^{19}+7931494\,z^{18}+6167133\,z^{17}+8925958\,z^{16}+6968556\,z^{15}+8175910\,z^{14}+6253212\,z^{13}+8038642\,z^{12}+6059550\,z^{11}+6430156\,z^{10}+4622415\,z^{9}+5576084\,z^{8}+3949575\,z^{7}+3907170\,z^{6}+2659272\,z^{5}+2203930\,z^{4}+1283310\,z^{3}+521312\,z^{2}+34251\,z+281488$
		$10659096\,z^{95}-8858344\,z^{96}$
$(29,30,31)$		$-8520627916960\,z^{90}-15548742756800\,z^{89}$
32	98	$736\,z^{32}+64\,z^{30}-68\,z^{28}+96\,z^{26}-36\,z^{24}+128\,z^{22}-4\,z^{20}+160\,z^{18}-148\,z^{16}+32\,z^{14}+164\,z^{12}+296\,z^{8}-32\,z^{6}+232\,z^{4}-96\,z^{2}+344$
		$-58\,z^{33}-58\,z^{32}-92\,z^{31}-92\,z^{30}-127\,z^{29}-127\,z^{28}-122\,z^{27}-122\,z^{26}-168\,z^{25}-168\,z^{24}-142\,z^{23}-142\,z^{22}-102\,z^{21}-102\,z^{20}+34\,z^{19}+34\,z^{18}-z^{17}-z^{16}+84\,z^{15}+84\,z^{14}+121\,z^{13}+121\,z^{12}+278\,z^{11}+278\,z^{10}+203\,z^{9}+203\,z^{8}+232\,z^{7}+232\,z^{6}+145\,z^{5}+145\,z^{4}+114\,z^{3}+114\,z^{2}+37\,z+37$
		$-64\,z^{98}+64\,z^{99}$
33	101	$-64192\,z^{33}-64192\,z^{32}-4800\,z^{31}-4800\,z^{30}+16828\,z^{29}-7936\,z^{28}-8704\,z^{27}-8704\,z^{26}+12732\,z^{25}-12032\,z^{24}-12992\,z^{23}-12992\,z^{22}+8252\,z^{21}-16512\,z^{20}-17664\,z^{19}-17664\,z^{18}-20660\,z^{17}+53632\,z^{16}-5760\,z^{15}-5760\,z^{14}-27964\,z^{13}-3200\,z^{12}-3008\,z^{11}-3008\,z^{10}-47224\,z^{9}+2304\,z^{8}+2880\,z^{7}+2880\,z^{6}-40952\,z^{5}+8576\,z^{4}+9536\,z^{3}+9536\,z^{2}-9864\,z-59392$
		$10958\,z^{34}+9534\,z^{33}+18828\,z^{32}+15740\,z^{31}+28681\,z^{30}+23049\,z^{29}+33158\,z^{28}+24694\,z^{27}+48192\,z^{26}+34544\,z^{25}+52282\,z^{24}+32874\,z^{23}+55378\,z^{22}+28354\,z^{21}+42570\,z^{20}+7258\,z^{19}+45071\,z^{18}+14783\,z^{17}+41812\,z^{16}+740\,z^{15}+32217\,z^{14}-4407\,z^{13}+15854\,z^{12}-31986\,z^{11}+8339\,z^{10}-19437\,z^{9}+10440\,z^{8}-27784\,z^{7}+1009\,z^{6}-13535\,z^{5}+8906\,z^{4}-11990\,z^{3}+8709\,z^{2}-1547\,z+14704$
		$49528\,z^{101}-44792\,z^{102}$
$(31,32,33)$		$-42632104148736\,z^{96}+77338989665280\,z^{95}$
34	104	$548\,z^{34}-448\,z^{32}+16\,z^{30}+16\,z^{28}+20\,z^{26}+20\,z^{24}+24\,z^{22}+24\,z^{20}-548\,z^{18}+448\,z^{16}-4\,z^{14}-4\,z^{12}-12\,z^{10}-12\,z^{8}-20\,z^{6}-20\,z^{4}+548\,z^{2}-448$
		$5\,z^{35}+5\,z^{34}+15\,z^{33}+15\,z^{32}+31\,z^{31}+31\,z^{30}+53\,z^{29}+53\,z^{28}+87\,z^{27}+87\,z^{26}+133\,z^{25}+133\,z^{24}+193\,z^{23}+193\,z^{22}+267\,z^{21}+267\,z^{20}+224\,z^{19}+224\,z^{18}+313\,z^{17}+313\,z^{16}+282\,z^{15}+282\,z^{14}+380\,z^{13}+380\,z^{12}+221\,z^{11}+221\,z^{10}+303\,z^{9}+303\,z^{8}+118\,z^{7}+118\,z^{6}+164\,z^{5}+164\,z^{4}+78\,z^{3}+78\,z^{2}+109\,z+109$
		$-8\,z^{104}+8\,z^{105}$
35	112	$4\,z^{35}-4\,z^{34}-4\,z^{33}+4\,z^{32}-4\,z^{19}+4\,z^{18}+4\,z^{17}-4\,z^{16}+4\,z^{3}-4\,z^{2}-4\,z+4$
		$-z^{20}-z^{16}-2\,z^{12}-2\,z^{8}-z^{4}-1$
		$8\,z^{112}-16\,z^{113}$
36	112	$4\,z^{35}+8\,z^{34}-4\,z^{33}-4\,z^{32}-4\,z^{20}-4\,z^{19}-8\,z^{18}+4\,z^{17}+8\,z^{16}+4\,z^{4}+4\,z^{3}+8\,z^{2}-4\,z-8$

		$z^{37}+z^{36}+2\,z^{35}+2\,z^{34}+3\,z^{33}+3\,z^{32}+4\,z^{31}+4\,z^{30}+6\,z^{29}+6\,z^{28}+8\,z^{27}+8\,z^{26}+10\,z^{25}+10\,z^{24}+12\,z^{23}+12\,z^{22}+15\,z^{21}+14\,z^{20}+15\,z^{19}+15\,z^{18}+17\,z^{17}+16\,z^{16}+16\,z^{15}+16\,z^{14}+18\,z^{13}+16\,z^{12}+14\,z^{11}+14\,z^{10}+14\,z^{9}+12\,z^{8}+8\,z^{7}+8\,z^{6}+8\,z^{5}+7\,z^{4}+5\,z^{3}+5\,z^{2}+4\,z+3$
		$-16\,z^{112}+8\,z^{113}$
$(34,35,36)$		$32\,z^{105}-32\,z^{104}$
43	131	$4\,z^{43}-8\,z^{42}+8\,z^{40}-8\,z^{39}+8\,z^{38}-8\,z^{36}+4\,z^{35}$
		$-z^{44}+z^{43}-z^{36}+z^{35}-2\,z^{28}+2\,z^{27}-2\,z^{20}+2\,z^{19}-z^{12}+z^{11}-z^{4}+z^{3}$
		$-8\,z^{131}+24\,z^{132}$
44	134	$64\,z^{44}-88\,z^{42}-44\,z^{40}+88\,z^{38}-4\,z^{36}-4\,z^{32}+4\,z^{20}+4\,z^{16}+4\,z^{12}+4\,z^{8}+4\,z^{4}+4$
		$-15\,z^{45}-15\,z^{44}-8\,z^{43}-8\,z^{42}-4\,z^{41}-4\,z^{40}-13\,z^{37}-13\,z^{36}-4\,z^{35}-4\,z^{34}+z^{33}+z^{32}+6\,z^{31}+6\,z^{30}-22\,z^{29}-22\,z^{28}-6\,z^{27}-6\,z^{26}+2\,z^{25}+2\,z^{24}+10\,z^{23}+10\,z^{22}-18\,z^{21}-18\,z^{20}-2\,z^{19}-2\,z^{18}+4\,z^{17}+4\,z^{16}+10\,z^{15}+10\,z^{14}-5\,z^{13}-5\,z^{12}+2\,z^{11}+2\,z^{10}+4\,z^{9}+4\,z^{8}+6\,z^{7}+6\,z^{6}-10\,z^{5}-10\,z^{4}-4\,z^{3}-4\,z^{2}-2\,z-2$
		$-176\,z^{134}+176\,z^{135}$
45	137	$304192\,z^{45}+262504\,z^{44}-72072\,z^{43}-72072\,z^{42}-40788\,z^{41}+35640\,z^{40}+146432\,z^{39}+146432\,z^{38}+84828\,z^{37}+77880\,z^{36}+128128\,z^{35}+128128\,z^{34}+69956\,z^{33}+63008\,z^{32}+116688\,z^{31}+116688\,z^{30}+106392\,z^{29}+106392\,z^{28}+96096\,z^{27}+96096\,z^{26}+85800\,z^{25}+85800\,z^{24}+75504\,z^{23}+75504\,z^{22}+109652\,z^{21}+116600\,z^{20}+38896\,z^{19}+38896\,z^{18}+69612\,z^{17}+76560\,z^{16}-4576\,z^{15}-4576\,z^{14}+53596\,z^{13}+60544\,z^{12}+6864\,z^{11}+6864\,z^{10}+61604\,z^{9}+68552\,z^{8}+11440\,z^{7}+11440\,z^{6}+62748\,z^{5}+69696\,z^{4}+9152\,z^{3}+9152\,z^{2}+57028\,z+63976$
		$-57215\,z^{46}-102271\,z^{45}-130028\,z^{44}-145626\,z^{43}-190986\,z^{42}-229398\,z^{41}-260744\,z^{40}-285142\,z^{39}-385343\,z^{38}-469911\,z^{37}-525496\,z^{36}-565448\,z^{35}-628583\,z^{34}-683033\,z^{33}-700980\,z^{32}-710242\,z^{31}-835054\,z^{30}-932074\,z^{29}-949350\,z^{28}-938834\,z^{27}-966634\,z^{26}-980538\,z^{25}-927114\,z^{24}-859794\,z^{23}-928550\,z^{22}-969514\,z^{21}-918722\,z^{20}-840138\,z^{19}-825524\,z^{18}-800488\,z^{17}-715518\,z^{16}-620126\,z^{15}-616995\,z^{14}-601705\,z^{13}-531364\,z^{12}-448864\,z^{11}-396040\,z^{10}-339742\,z^{9}-258032\,z^{8}-172848\,z^{7}-169960\,z^{6}-156650\,z^{5}-117726\,z^{4}-68380\,z^{3}-58456\,z^{2}-45058\,z-24266$
		$180648\,z^{137}+288392\,z^{138}$
$(43,44,45)$		$247104\,z^{132}-247104\,z^{131}$
49	149	$-30924\,z^{49}+7224\,z^{48}+8904\,z^{47}+8904\,z^{46}+74616\,z^{45}-1680\,z^{44}-1176\,z^{43}-1176\,z^{42}-64704\,z^{41}+11592\,z^{40}+13272\,z^{39}+13272\,z^{38}+45876\,z^{37}+7728\,z^{36}+7728\,z^{35}+7728\,z^{34}-89412\,z^{33}+25032\,z^{32}+25704\,z^{31}+25704\,z^{30}+24192\,z^{29}+24192\,z^{28}+22680\,z^{27}+22680\,z^{26}-109080\,z^{25}+43512\,z^{24}+42168\,z^{23}+42168\,z^{22}+5532\,z^{21}+43680\,z^{20}+39648\,z^{19}+39648\,z^{18}+172020\,z^{17}+32144\,z^{16}+10416\,z^{15}+10416\,z^{14}-22692\,z^{13}+15456\,z^{12}+14952\,z^{11}+14952\,z^{10}-18660\,z^{9}+19488\,z^{8}+18480\,z^{7}+18480\,z^{6}-15636\,z^{5}+22512\,z^{4}+21000\,z^{3}+21000\,z^{2}+116628\,z+2184$
		$17547\,z^{50}+19199\,z^{49}+33722\,z^{48}+32350\,z^{47}+37321\,z^{46}+39113\,z^{45}+42950\,z^{44}+43608\,z^{43}+98001\,z^{42}+98351\,z^{41}+143378\,z^{40}+134362\,z^{39}+190369\,z^{38}+179617\,z^{37}+226090\,z^{36}+205804\,z^{35}+279265\,z^{34}+254177\,z^{33}+314912\,z^{32}+277098\,z^{31}+305202\,z^{30}+269726\,z^{29}+292874\,z^{28}+252442\,z^{27}+306568\,z^{26}+265324\,z^{25}+310126\,z^{24}+259558\,z^{23}+266266\,z^{22}+222110\,z^{21}+227222\,z^{20}+181470\,z^{19}+195921\,z^{18}+156329\,z^{17}+167882\,z^{16}+125392\,z^{15}+88177\,z^{14}+60499\,z^{13}+29290\,z^{12}+7618\,z^{11}+16837\,z^{10}+3803\,z^{9}+11636\,z^{8}-2784\,z^{7}-1788\,z^{6}-7150\,z^{5}-6238\,z^{4}-11684\,z^{3}-1502\,z^{2}-4036\,z+4340$
		$228888\,z^{149}-195624\,z^{150}$
		$-120\,z^{50}-120\,z^{48}-36\,z^{46}-36\,z^{44}-120\,z^{42}-120\,z^{40}-48\,z^{34}-48\,z^{32}+108\,z^{30}+108\,z^{28}+96\,z^{26}+96\,z^{24}+288\,z^{22}+288\,z^{20}+932\,z^{18}-296\,z^{16}+36\,z^{14}+36\,z^{12}+72\,z^{10}+72\,z^{8}+108\,z^{6}+108\,z^{4}+156\,z^{2}+156$
50	152	

		$179\,z^{51}+179\,z^{50}+230\,z^{49}+230\,z^{48}+433\,z^{47}+433\,z^{46}+481\,z^{45}+481\,z^{44}+866\,z^{43}+866\,z^{42}+974\,z^{41}+974\,z^{40}+1329\,z^{39}+1329\,z^{38}+1317\,z^{37}+1317\,z^{36}+1561\,z^{35}+1561\,z^{34}+1447\,z^{33}+1447\,z^{32}+1544\,z^{31}+1544\,z^{30}+1238\,z^{29}+1238\,z^{28}+1230\,z^{27}+1230\,z^{26}+906\,z^{25}+906\,z^{24}+844\,z^{23}+844\,z^{22}+430\,z^{21}+430\,z^{20}+353\,z^{19}+353\,z^{18}-z^{17}-z^{16}-36\,z^{15}-36\,z^{14}-366\,z^{13}-366\,z^{12}-290\,z^{11}-290\,z^{10}-422\,z^{9}-422\,z^{8}-211\,z^{7}-211\,z^{6}-271\,z^{5}-271\,z^{4}-149\,z^{3}-149\,z^{2}-152\,z-152$
		$-96\,z^{152}+96\,z^{153}$
51	160	$4\,z^{19}-4\,z^{18}-4\,z^{17}+4\,z^{16}$
		$z^{52}+z^{48}+2\,z^{44}+2\,z^{40}+2\,z^{36}+2\,z^{32}+2\,z^{28}+2\,z^{24}+2\,z^{20}+2\,z^{16}+2\,z^{12}+2\,z^{8}+z^{4}+1$
		$8\,z^{160}-16\,z^{161}$
$(49,50,51)$		$-30517344\,z^{150}+35706528\,z^{149}$

Approximation polynomials and related determinants in Theorem 2

29	89	$8838\,z^{29}-103526\,z^{28}+48074\,z^{27}+249006\,z^{26}-36267\,z^{25}-222096\,z^{24}-98200\,z^{23}-42694\,z^{22}+236901\,z^{21}+813160\,z^{20}+82188\,z^{19}+105172\,z^{18}-76623\,z^{17}-341910\,z^{16}-29528\,z^{15}+101076\,z^{14}+42482\,z^{13}+214028\,z^{12}-7828\,z^{11}-293424\,z^{10}-61460\,z^{9}+84522\,z^{8}+90968\,z^{7}-67100\,z^{6}-221774\,z^{5}-749016\,z^{4}-82002\,z^{3}+59754\,z^{2}+157247\,z+372960$
		$249196\,z^{30}+197392\,z^{29}+504078\,z^{28}-11390\,z^{27}+8214\,z^{26}-17793\,z^{25}-222796\,z^{24}-40936\,z^{23}-71854\,z^{22}-31047\,z^{21}+355216\,z^{20}+158208\,z^{19}-251360\,z^{18}-245645\,z^{17}-403198\,z^{16}-44440\,z^{15}-75264\,z^{14}+35828\,z^{13}-391672\,z^{12}-126276\,z^{11}+392512\,z^{10}+300274\,z^{9}+635406\,z^{8}+79336\,z^{7}+155388\,z^{6}-30926\,z^{5}+185780\,z^{4}+8222\,z^{3}-417854\,z^{2}-180793\,z-348900$
		$-341396\,z^{89}-904548\,z^{90}$
30	92	$284151\,z^{30}-303211\,z^{29}+34473\,z^{28}+230361\,z^{27}-272325\,z^{26}+122016\,z^{25}+180112\,z^{24}-334273\,z^{23}+109371\,z^{22}+36208\,z^{21}-196837\,z^{20}+184636\,z^{19}+92877\,z^{18}-222239\,z^{17}+170339\,z^{16}+15686\,z^{15}-212872\,z^{14}+178126\,z^{13}+20616\,z^{12}-143468\,z^{11}+163829\,z^{10}-56575\,z^{9}-134101\,z^{8}+230702\,z^{7}-66928\,z^{6}+9672\,z^{5}+97210\,z^{4}-157573\,z^{3}-40095\,z^{2}+125240\,z-109296$
		$-59086\,z^{31}+15283\,z^{30}-74369\,z^{29}+60109\,z^{28}+28737\,z^{27}-15235\,z^{26}+39866\,z^{25}-14610\,z^{24}-15469\,z^{23}+16375\,z^{22}-86676\,z^{21}+36259\,z^{20}+116870\,z^{19}-82905\,z^{18}+67053\,z^{17}+31655\,z^{16}-103912\,z^{15}+26480\,z^{14}+172732\,z^{13}-187648\,z^{12}-74152\,z^{11}+153851\,z^{10}-249333\,z^{9}+87897\,z^{8}+154690\,z^{7}-163750\,z^{6}-19530\,z^{5}+191628\,z^{4}-94209\,z^{3}-18717\,z^{2}+204414\,z-206502$
		$379224\,z^{92}+172268\,z^{93}$
31	95	$-4225266\,z^{31}+1316319\,z^{30}+2983440\,z^{29}-3968815\,z^{28}+1611693\,z^{27}+1846233\,z^{26}-4256835\,z^{25}+2557108\,z^{24}+1911015\,z^{23}-3531513\,z^{22}+3405999\,z^{21}+155295\,z^{20}-3334914\,z^{19}+2859543\,z^{18}-181131\,z^{17}-2388117\,z^{16}+2999370\,z^{15}-581472\,z^{14}-2191518\,z^{13}+2452914\,z^{12}-917898\,z^{11}-1244721\,z^{10}+2592741\,z^{9}-1622013\,z^{8}-1446126\,z^{7}+2251806\,z^{6}-2410077\,z^{5}+236958\,z^{4}+2263671\,z^{3}-2142621\,z^{2}+587307\,z+1506461$
		$303774\,z^{32}+398004\,z^{31}-205521\,z^{30}+755412\,z^{29}-338283\,z^{28}-77559\,z^{27}+87615\,z^{26}-492033\,z^{25}+285082\,z^{24}-79797\,z^{23}+281151\,z^{22}+665469\,z^{21}-1280281\,z^{20}-3960\,z^{19}+863013\,z^{18}-1755093\,z^{17}+770027\,z^{16}+705864\,z^{15}-1921572\,z^{14}+387582\,z^{13}+1832330\,z^{12}-1503042\,z^{11}+429153\,z^{10}+2481123\,z^{9}-2516947\,z^{8}+797970\,z^{7}+1745100\,z^{6}-2355387\,z^{5}+325532\,z^{4}+1275255\,z^{3}-1951479\,z^{2}+45261\,z+1580655$
		$5421516\,z^{95}-3588972\,z^{96}$
$(29,30,31)$		$-127293061031881800\,z^{90}-47224605589276632\,z^{89}$

32	98	$-22588299\,z^{32} + 13357080\,z^{31} - 660150\,z^{30} - 11245880\,z^{29} + 13273201\,z^{28} - 4213080\,z^{27} - 10169442\,z^{26} + 14802080\,z^{25} - 4524244\,z^{24} - 4588680\,z^{23} + 13870419\,z^{22} - 8577000\,z^{21} - 3890070\,z^{20} + 10892280\,z^{19} - 8706669\,z^{18} - 1315560\,z^{17} + 10323339\,z^{16} - 8303040\,z^{15} - 616950\,z^{14} + 7345200\,z^{13} - 8432709\,z^{12} + 1957560\,z^{11} + 6776259\,z^{10} - 9789480\,z^{9} + 2206179\,z^{8} + 2264400\,z^{7} - 9081531\,z^{6} + 6965040\,z^{5} + 2002044\,z^{4} - 6780360\,z^{3} + 6752958\,z^{2} + 61720\,z + 4831822$
		$1760400\,z^{33} + 449991\,z^{32} + 1533720\,z^{31} + 922782\,z^{30} - 1734360\,z^{29} + 1227135\,z^{28} - 974760\,z^{27} - 823668\,z^{26} + 1069520\,z^{25} - 1197466\,z^{24} + 2592120\,z^{23} + 473565\,z^{22} - 4485800\,z^{21} + 2729386\,z^{20} - 1450200\,z^{19} - 4290933\,z^{18} + 5043160\,z^{17} - 1619519\,z^{16} - 6351840\,z^{15} + 5718510\,z^{14} + 2455600\,z^{13} - 5795075\,z^{12} + 8621640\,z^{11} + 1149129\,z^{10} - 7986200\,z^{9} + 8595727\,z^{8} + 484800\,z^{7} - 7667985\,z^{6} + 5206240\,z^{5} - 338900\,z^{4} - 7833240\,z^{3} + 6179112\,z^{2} + 663720\,z + 4080948$
		$-37889148\,z^{98} + 9889572\,z^{99}$
33	101	$-176064633\,z^{33} - 85665591\,z^{32} + 74750757\,z^{31} - 47272970\,z^{30} + 21499188\,z^{29} + 14694666\,z^{28} - 95780913\,z^{27} + 43755587\,z^{26} + 68776500\,z^{25} - 4037217\,z^{24} + 64432656\,z^{23} + 27953142\,z^{22} - 75272931\,z^{21} + 19707900\,z^{20} + 4403736\,z^{19} - 33320676\,z^{18} + 62965368\,z^{17} + 7176237\,z^{16} - 53028738\,z^{15} + 18240612\,z^{14} - 16373169\,z^{13} - 11076483\,z^{12} + 61498080\,z^{11} - 43239204\,z^{10} - 55583784\,z^{9} - 12887979\,z^{8} - 46376811\,z^{7} - 4446624\,z^{6} + 58106862\,z^{5} - 9556881\,z^{4} + 3266673\,z^{3} + 35390365\,z^{2} + 47644845\,z + 47235177$
		$29638536\,z^{34} + 24799239\,z^{33} + 29661303\,z^{32} + 9226737\,z^{31} + 15614334\,z^{30} + 1923930\,z^{29} - 24820692\,z^{28} - 11807025\,z^{27} - 5265061\,z^{26} - 5139888\,z^{25} + 43840173\,z^{24} + 25801494\,z^{23} - 53349362\,z^{22} - 20883027\,z^{21} - 5941698\,z^{20} - 35963706\,z^{19} - 3232646\,z^{18} + 17530458\,z^{17} - 72115791\,z^{16} - 3288378\,z^{15} + 68504836\,z^{14} - 9686523\,z^{13} + 50244111\,z^{12} + 55708260\,z^{11} - 11129384\,z^{10} + 14958006\,z^{9} + 44808585\,z^{8} - 49038303\,z^{7} - 28482668\,z^{6} + 19260636\,z^{5} - 58513881\,z^{4} + 205749\,z^{3} + 9770433\,z^{2} + 39546873\,z + 55820127$
		$-332338020\,z^{101} - 144335068\,z^{102}$
$(31, 32, 33)$		$-445634331745438289520\,z^{96} + 417178155931626734568\,z^{95}$
34	104	$-399703431\,z^{34} + 1087916661\,z^{33} - 1630720744\,z^{32} - 123158126\,z^{31} - 21796076\,z^{30} + 189852624\,z^{29} - 422979590\,z^{28} + 46370805\,z^{27} + 286887150\,z^{26} - 644304791\,z^{25} + 995887336\,z^{24} + 26439608\,z^{23} - 84276316\,z^{22} + 67661308\,z^{21} + 23651188\,z^{20} - 47593026\,z^{19} + 55252839\,z^{18} - 61203883\,z^{17} - 32396137\,z^{16} + 103424620\,z^{15} - 63992303\,z^{14} + 4287153\,z^{13} + 91016151\,z^{12} - 54334722\,z^{11} - 247162617\,z^{10} + 629737225\,z^{9} - 1008730988\,z^{8} - 52474214\,z^{7} + 164818441\,z^{6} - 223558699\,z^{5} + 266169085\,z^{4} + 73506901\,z^{3} + 179099632\,z^{2} - 630460649\,z + 1041984688$
		$-94512652\,z^{35} + 266646659\,z^{34} - 490549293\,z^{33} + 857095194\,z^{32} + 108641546\,z^{31} - 38038978\,z^{30} - 12932166\,z^{29} - 194804864\,z^{28} + 54188017\,z^{27} - 85203872\,z^{26} - 50724817\,z^{25} + 364466056\,z^{24} - 113721868\,z^{23} - 151523710\,z^{22} + 505358802\,z^{21} - 756027622\,z^{20} + 2616640\,z^{19} + 24294241\,z^{18} + 176360093\,z^{17} - 629232539\,z^{16} + 29058172\,z^{15} + 224824483\,z^{14} - 663863213\,z^{13} + 1293732477\,z^{12} - 45638242\,z^{11} + 33431723\,z^{10} - 139780043\,z^{9} + 284577000\,z^{8} + 108864730\,z^{7} - 175863023\,z^{6} + 375097749\,z^{5} - 839815275\,z^{4} - 44561283\,z^{3} + 172085310\,z^{2} - 459202523\,z + 910810930$
		$-1702388928\,z^{104} - 837017980\,z^{105}$
35	107	$-1046716196\,z^{35} - 1705139966\,z^{34} + 1328534848\,z^{33} + 209186103\,z^{32} - 579850958\,z^{31} + 218235941\,z^{30} - 105588449\,z^{29} + 39130547\,z^{28} + 948112085\,z^{27} + 821966732\,z^{26} - 715563620\,z^{25} - 86195686\,z^{24} - 108468120\,z^{23} - 221335144\,z^{22} + 349140044\,z^{21} - 205764255\,z^{20} - 57327118\,z^{19} + 296572370\,z^{18} - 344630703\,z^{17} + 92536691\,z^{16} + 271430792\,z^{15} - 441926838\,z^{14} + 256544717\,z^{13} + 73948784\,z^{12} - 877888846\,z^{11} - 762881152\,z^{10} + 620094166\,z^{9} + 185407898\,z^{8} + 208876518\,z^{7} + 575933089\,z^{6} - 448129406\,z^{5} - 97527619\,z^{4} + 781030829\,z^{3} + 678076638\,z^{2} - 536082374\,z - 98863393$

		$144914334\,z^{36}+297095560\,z^{35}+1169289678\,z^{34}-426372626\,z^{33}-863497319\,z^{32}+361540034\,z^{31}-434779223\,z^{30}-368826463\,z^{29}+890265147\,z^{28}+6887337\,z^{27}-155113820\,z^{26}+542917024\,z^{25}-518948038\,z^{24}-879518516\,z^{23}-316730730\,z^{22}+557695048\,z^{21}-296928951\,z^{20}+74727036\,z^{19}-279021182\,z^{18}-506772977\,z^{17}+1040291399\,z^{16}+712099152\,z^{15}+918083242\,z^{14}-214816021\,z^{13}-804575800\,z^{12}-106173246\,z^{11}+406912124\,z^{10}-44306314\,z^{9}-497884300\,z^{8}+1563142\,z^{7}-1043442955\,z^{6}-18436418\,z^{5}+1246202671\,z^{4}+261494749\,z^{3}+905060170\,z^{2}-152646960\,z-822091223$
		$-550045468\,z^{107}-1597702416\,z^{108}$
36	110	$-72679207\,z^{36}-60042465\,z^{35}-27328323\,z^{34}+28175850\,z^{33}+38242280\,z^{32}-1018170\,z^{31}-19881498\,z^{30}-75330\,z^{29}+64690622\,z^{28}+35278740\,z^{27}+28066812\,z^{26}-14776020\,z^{25}-50238100\,z^{24}-8127540\,z^{23}+8674680\,z^{22}-1006830\,z^{21}+551847\,z^{20}+7718085\,z^{19}-12637842\,z^{18}-1883655\,z^{17}+10669710\,z^{16}-9245745\,z^{15}+6795732\,z^{14}+6166935\,z^{13}-45339792\,z^{12}-32359095\,z^{11}-28120197\,z^{10}+15981300\,z^{9}+56558841\,z^{8}+19362645\,z^{7}+6299418\,z^{6}-10006335\,z^{5}+27751201\,z^{4}+27999270\,z^{3}+23655654\,z^{2}-12371130\,z-34995848$
		$3546180\,z^{37}+24463035\,z^{36}+36719325\,z^{35}+40784937\,z^{34}-33465960\,z^{33}-43519224\,z^{32}-7654500\,z^{31}-35415090\,z^{30}+22383540\,z^{29}+35479690\,z^{28}-5341950\,z^{27}+29684478\,z^{26}-8292780\,z^{25}-67702526\,z^{24}-19643310\,z^{23}-9206988\,z^{22}+257580\,z^{21}+18201351\,z^{20}-5600745\,z^{19}-45297798\,z^{18}+24472935\,z^{17}+84518202\,z^{16}+34033365\,z^{15}+57933060\,z^{14}-29645595\,z^{13}-68674026\,z^{12}+9503325\,z^{11}+11405619\,z^{10}-15313860\,z^{9}-26194131\,z^{8}-28855035\,z^{7}-53742390\,z^{6}+38598525\,z^{5}+88270743\,z^{4}+28094040\,z^{3}+43862046\,z^{2}-28627020\,z-56682498$
		$-31540680\,z^{110}-98505000\,z^{111}$
$(34,35,36)$		$-10749530318906122061682776\,z^{105}+39437465990135596375969920\,z^{104}$
38	116	$-328392625\,z^{38}+290993025\,z^{37}-106716958\,z^{36}-80099400\,z^{35}+146334448\,z^{34}-82673700\,z^{33}-10372071\,z^{32}+109928850\,z^{31}+160680958\,z^{30}-236200200\,z^{29}+178868184\,z^{28}-25575900\,z^{27}-133420172\,z^{26}+154610400\,z^{25}-68735318\,z^{24}-68040750\,z^{23}+90264865\,z^{22}-26269275\,z^{21}-64890392\,z^{20}+97028775\,z^{19}-58008848\,z^{18}-40843875\,z^{17}+125617990\,z^{16}-79415775\,z^{15}-137346362\,z^{14}+191098125\,z^{13}-157753650\,z^{12}+30459150\,z^{11}+147917207\,z^{10}-161086425\,z^{9}+76506261\,z^{8}+42485775\,z^{7}+119836337\,z^{6}-146144100\,z^{5}+121742572\,z^{4}-26396550\,z^{3}-74539424\,z^{2}+109114050\,z-112978051$
		$-9403500\,z^{39}+99652845\,z^{38}-58716225\,z^{37}+37194802\,z^{36}-44106150\,z^{35}+13053908\,z^{34}+9717150\,z^{33}-87002903\,z^{32}+120547500\,z^{31}-88922450\,z^{30}-15160650\,z^{29}+146537548\,z^{28}-160122600\,z^{27}-75053458\,z^{26}+189482250\,z^{25}-103836420\,z^{24}-938400\,z^{23}+87715453\,z^{22}-87401325\,z^{21}-85839656\,z^{20}+176370225\,z^{19}+52126768\,z^{18}-167935725\,z^{17}+205536492\,z^{16}-158374125\,z^{15}+13195820\,z^{14}+116660925\,z^{13}-71477426\,z^{12}+1417950\,z^{11}-43444389\,z^{10}+16305675\,z^{9}-84638221\,z^{8}+124726875\,z^{7}+67343275\,z^{6}-135781050\,z^{5}+126170316\,z^{4}-97316700\,z^{3}-8126386\,z^{2}+76755450\,z-33574889$
		$-130668712\,z^{116}+529192388\,z^{117}$
39	119	$-149119706349\,z^{39}+63707949546\,z^{38}-26395317696\,z^{37}-36247180542\,z^{36}+59443318176\,z^{35}-31786065503\,z^{34}-9258370371\,z^{33}+37607627380\,z^{32}+80661854706\,z^{31}-57012122504\,z^{30}+62282372040\,z^{29}-7098995408\,z^{28}-61235049540\,z^{27}+50983686882\,z^{26}-20606379006\,z^{25}-28591419162\,z^{24}+35781687273\,z^{23}-9322212819\,z^{22}-30043911930\,z^{21}+39553662819\,z^{20}-20042522118\,z^{19}-18860076597\,z^{18}+53019807420\,z^{17}-38017902921\,z^{16}-67954416972\,z^{15}+44264940214\,z^{14}-55991529852\,z^{13}+15502477726\,z^{12}+68947673487\,z^{11}-41919255682\,z^{10}+22755960363\,z^{9}+17094442241\,z^{8}+57342356871\,z^{7}-33076245296\,z^{6}+43263248676\,z^{5}-12548417012\,z^{4}-34528437732\,z^{3}+38395278251\,z^{2}-43071249867\,z+10796816534$

		$3019569924\,z^{40} + 46348505073\,z^{39} + 12526842752\,z^{38} + 10272887556\,z^{37} - 25924554292\,z^{36} + 1310254080\,z^{35} - 18755197493\,z^{34} - 35923872231\,z^{33} + 58445245834\,z^{32} - 31003013634\,z^{31} + 5501615364\,z^{30} + 62566901664\,z^{29} - 71660413140\,z^{28} - 45616511226\,z^{27} + 49880906186\,z^{26} - 32766551160\,z^{25} - 23248226440\,z^{24} + 38184226665\,z^{23} - 46360350849\,z^{22} - 41824933494\,z^{21} + 97154733705\,z^{20} + 38120227062\,z^{19} - 17090569745\,z^{18} + 76837413774\,z^{17} - 60618822959\,z^{16} - 4884457938\,z^{15} + 41319863530\,z^{14} - 24244285020\,z^{13} - 24745519454\,z^{12} - 21728806557\,z^{11} - 28922373896\,z^{10} - 33575376855\,z^{9} + 69559600537\,z^{8} + 41361916065\,z^{7} - 9475846394\,z^{6} + 44214361152\,z^{5} - 34357388522\,z^{4} - 10977837426\,z^{3} + 18941010169\,z^{2} - 9408704265\,z - 13383501602$
		$-484474868376\,z^{119} - 461974967404\,z^{120}$
		$91864726291\,z^{40} + 319280781460\,z^{39} - 224030950487\,z^{38} + 27374191880\,z^{37} + 292533708140\,z^{36} - 292674605820\,z^{35} - 1510481652\,z^{34} + 44669145100\,z^{33} - 334085225207\,z^{32} - 134799607080\,z^{31} + 317018563878\,z^{30} - 254112973440\,z^{29} - 145256148136\,z^{28} + 252753057880\,z^{27} - 182489746860\,z^{26} + 36049279040\,z^{25} + 258332973123\,z^{24} - 158528733880\,z^{23} - 48928115708\,z^{22} + 193350332700\,z^{21} - 140746785575\,z^{20} + 38800707260\,z^{19} + 159559557623\,z^{18} - 333737842620\,z^{17} + 34029855687\,z^{16} + 103226012960\,z^{15} - 240864919868\,z^{14} + 290346874040\,z^{13} + 113028048961\,z^{12} - 180497459260\,z^{11} + 172178300702\,z^{10} - 72252586560\,z^{9} - 256768685827\,z^{8} - 94832110460\,z^{7} + 161086577681\,z^{6} - 220632804400\,z^{5} - 82608922550\,z^{4} + 163568737100\,z^{3} - 139940524770\,z^{2} + 286924169860\,z + 198899712593$
40	122	$58079840240\,z^{41} + 20779678345\,z^{40} + 103515938500\,z^{39} - 17011134207\,z^{38} - 147808517040\,z^{37} + 59923127422\,z^{36} - 212289811620\,z^{35} - 13631116930\,z^{34} + 321426076900\,z^{33} - 153779160721\,z^{32} + 233225695720\,z^{31} + 128766125602\,z^{30} - 434954929680\,z^{29} + 207808787358\,z^{28} + 64712750720\,z^{27} - 349503491512\,z^{26} - 73802320960\,z^{25} + 81918338683\,z^{24} - 322745890720\,z^{23} + 126471217758\,z^{22} + 533202716180\,z^{21} - 303556927977\,z^{20} + 206915469740\,z^{19} + 290555338185\,z^{18} - 398457916780\,z^{17} + 132114286577\,z^{16} + 39320376200\,z^{15} - 306305865594\,z^{14} - 132039615160\,z^{13} + 68052119533\,z^{12} - 229173884420\,z^{11} + 57726516830\,z^{10} + 402305026120\,z^{9} - 200625842251\,z^{8} + 162582080740\,z^{7} + 207397909869\,z^{6} - 204794211720\,z^{5} + 97428157174\,z^{4} - 17980049220\,z^{3} - 177803934800\,z^{2} - 84951726820\,z + 25430355927$
		$-537183155528\,z^{122} - 703138211928\,z^{123}$
$(38, 39, 40)$		$107066318523279421968072480l040\,z^{117} - 383714096776182783428173226848\,z^{116}$
		$654719760512\,z^{43} - 2821346833612\,z^{42} - 94694672644\,z^{41} + 2479488890145\,z^{40} - 512912959904\,z^{39} + 2552059180057\,z^{38} + 90318251584\,z^{37} - 540036131793\,z^{36} - 332932250408\,z^{35} + 689404484714\,z^{34} - 233019909196\,z^{33} - 1816808882738\,z^{32} + 350569737040\,z^{31} - 298843735826\,z^{30} + 80388796672\,z^{29} - 759331507046\,z^{28} - 193863522676\,z^{27} + 98917002890\,z^{26} + 239024931320\,z^{25} + 507105169032\,z^{24} + 55088360924\,z^{23} + 45426928350\,z^{22} - 450655798776\,z^{21} - 92389707567\,z^{20} + 127506668620\,z^{19} - 104589492891\,z^{18} + 423189766836\,z^{17} - 553853895119\,z^{16} - 211431381744\,z^{15} - 129812189536\,z^{14} - 130698692896\,z^{13} + 643148743452\,z^{12} - 340716989224\,z^{11} + 2775000173564\,z^{10} - 160997824948\,z^{9} - 2755147607899\,z^{8} + 301462445928\,z^{7} - 2157483293747\,z^{6} + 366504279088\,z^{5} + 596005489469\,z^{4} + 138760078784\,z^{3} - 725373895320\,z^{2} - 128931956988\,z + 2197279449742$
43	131	

		$1600354283224\,z^{44} + 229103252712\,z^{43} - 1515378673264\,z^{42} - 223609710724\,z^{41} + 626899969097\,z^{40} - 411335257848\,z^{39} + 1782268460381\,z^{38} + 514927759856\,z^{37} - 1635050608811\,z^{36} + 471980486624\,z^{35} - 2021350873784\,z^{34} - 698337174500\,z^{33} + 2025446830184\,z^{32} - 67928635024\,z^{31} + 2924131042610\,z^{30} - 13100256640\,z^{29} - 1559320311274\,z^{28} - 450365666348\,z^{27} - 1826792092706\,z^{26} + 681361714048\,z^{25} + 1581036201434\,z^{24} + 469409501316\,z^{23} - 2128757496492\,z^{22} - 625900977864\,z^{21} + 1072970184197\,z^{20} - 161130674252\,z^{19} + 4404057533735\,z^{18} + 2130376028\,z^{17} - 3832527512063\,z^{16} - 342823093552\,z^{15} - 1230015018030\,z^{14} + 496838785344\,z^{13} + 1676061259314\,z^{12} + 420162704880\,z^{11} - 2827316268964\,z^{10} - 350037673132\,z^{9} + 1078126371345\,z^{8} - 203066232000\,z^{7} + 4111288791141\,z^{6} - 26920017216\,z^{5} - 2150284069005\,z^{4} + 30602791784\,z^{3} - 1803737859838\,z^{2} - 50999731780\,z + 1712645539000$
		$1922418947808\,z^{131} + 6043431728416\,z^{132}$
		$-75947033832\,z^{44} - 5579107092\,z^{43} - 104238012156\,z^{42} + 4739827911\,z^{41} + 105950111578\,z^{40} + 4937188815\,z^{39} + 95161460838\,z^{38} - 1864833711\,z^{37} + 82640354966\,z^{36} + 1664230542\,z^{35} + 19169079328\,z^{34} - 1874055534\,z^{33} - 110419687504\,z^{32} - 2242997406\,z^{31} + 2455649268\,z^{30} - 41909634\,z^{29} - 71433888516\,z^{28} + 614444454\,z^{27} + 4379526516\,z^{26} - 773138496\,z^{25} + 38156981404\,z^{24} + 947504922\,z^{23} - 11826042596\,z^{22} + 1292556447\,z^{21} - 355903374\,z^{20} - 2056683525\,z^{19} + 16466899058\,z^{18} - 1263948465\,z^{17} - 53825333390\,z^{16} + 1616137536\,z^{15} - 20478872416\,z^{14} + 74222772\,z^{13} + 120456198304\,z^{12} + 4671106596\,z^{11} + 104403966276\,z^{10} - 3272846733\,z^{9} - 123793016934\,z^{8} - 5059153749\,z^{7} - 66892041266\,z^{6} + 393363555\,z^{5} - 64852742270\,z^{4} + 298156560\,z^{3} - 41110831292\,z^{2} + 2510715714\,z + 140992632208$
44	134	$2216770296\,z^{45} + 57129877336\,z^{44} - 2728300848\,z^{43} - 60713752852\,z^{42} + 2166681231\,z^{41} + 40970060514\,z^{40} + 2967191643\,z^{39} + 77208074222\,z^{38} - 4780052277\,z^{37} - 97090594078\,z^{36} - 2387742864\,z^{35} - 96526339780\,z^{34} + 5426834712\,z^{33} + 90857543716\,z^{32} + 2175781374\,z^{31} + 132551744076\,z^{30} - 167355630\,z^{29} - 46431083436\,z^{28} - 2857116366\,z^{27} - 61093058028\,z^{26} - 1027769106\,z^{25} + 6007230904\,z^{24} - 935469756\,z^{23} - 111414475152\,z^{22} + 2497628523\,z^{21} + 96742259546\,z^{20} + 5218568745\,z^{19} + 185829175462\,z^{18} - 3930963153\,z^{17} - 131339449078\,z^{16} - 2434706274\,z^{15} - 36798869300\,z^{14} + 377423982\,z^{13} + 10056749292\,z^{12} - 2820430860\,z^{11} - 128752306132\,z^{10} + 1823523351\,z^{9} + 71211869794\,z^{8} + 6026206563\,z^{7} + 164593057134\,z^{6} - 2461736979\,z^{5} - 56421410130\,z^{4} - 3939916266\,z^{3} - 63767607912\,z^{2} + 3454515480\,z + 25963043060$
		$-512186985504\,z^{134} - 522891150768\,z^{135}$
45	137	$541293377520\,z^{45} - 968893495884\,z^{44} - 1198079333952\,z^{43} + 447173795391\,z^{42} + 777341459536\,z^{41} - 88091856171\,z^{40} + 1040857866600\,z^{39} + 15100602735\,z^{38} - 1111190897800\,z^{37} + 1538190196584\,z^{36} + 371502990856\,z^{35} - 964937789790\,z^{34} + 168304420664\,z^{33} - 170359771302\,z^{32} - 770970946800\,z^{31} + 147538176606\,z^{30} + 383820068304\,z^{29} - 1091557612278\,z^{28} + 246926658864\,z^{27} + 587156627544\,z^{26} - 516091336808\,z^{25} + 390068663154\,z^{24} + 288431248168\,z^{23} - 1057237233597\,z^{22} + 510827218944\,z^{21} + 346537536069\,z^{20} - 713212429168\,z^{19} + 937402306665\,z^{18} - 174655591352\,z^{17} - 1070279230902\,z^{16} + 671183463200\,z^{15} - 291468004068\,z^{14} - 735491160272\,z^{13} + 1861353055764\,z^{12} + 812032969152\,z^{11} - 981007879581\,z^{10} - 176504582496\,z^{9} - 240996892647\,z^{8} - 1161587791976\,z^{7} + 1040368757721\,z^{6} + 435701866840\,z^{5} - 1674922778490\,z^{4} + 365855876968\,z^{3} + 103783298334\,z^{2} - 50206939352\,z + 1047984167376$

		$375826069824\,z^{46} + 389037204976\,z^{45} - 51783796104\,z^{44} - 564345721792\,z^{43} - 316470592089\,z^{42} + 407337506496\,z^{41} + 65331297489\,z^{40} + 607967063384\,z^{39} + 718385445921\,z^{38} - 941731846600\,z^{37} - 402580388634\,z^{36} - 318233580952\,z^{35} - 1105125087036\,z^{34} + 1260902030248\,z^{33} + 928573409322\,z^{32} - 214019408016\,z^{31} - 139378083822\,z^{30} + 240560920848\,z^{29} - 848300707458\,z^{28} - 355077245616\,z^{27} + 1794280909542\,z^{26} - 467081765816\,z^{25} - 922373507592\,z^{24} + 312051187944\,z^{23} - 1321527938829\,z^{22} + 256058214800\,z^{21} + 1870245875859\,z^{20} + 455245169152\,z^{19} - 685952191995\,z^{18} - 410889238072\,z^{17} - 463142659512\,z^{16} - 436655545808\,z^{15} + 1478403246906\,z^{14} - 1591217568\,z^{13} - 1100965032744\,z^{12} - 137175061936\,z^{11} - 672625854345\,z^{10} + 130115107936\,z^{9} + 1183740872769\,z^{8} + 880615121736\,z^{7} - 470011423005\,z^{6} - 356422892616\,z^{5} + 207556342200\,z^{4} - 913159201752\,z^{3} + 369523810776\,z^{2} + 926067028376\,z - 751138064892$
		$-4505097105248\,z^{137} - 544781681936\,z^{138}$
$(43, 44, 45)$		$-5615357513311320786518398192359682562\,z^{132} - 2559004740032102166224682984522055568\,z^{131}$
49	149	$15572671419924\,z^{49} - 3772543101346\,z^{48} - 16300018328191\,z^{47} + 4319661615519\,z^{46} - 4488346337109\,z^{45} + 5940727404948\,z^{44} + 17964463312288\,z^{43} - 9518113636788\,z^{42} - 23465636694092\,z^{41} - 9034214618364\,z^{40} + 13310977691102\,z^{39} - 3432119444066\,z^{38} + 4541780918154\,z^{37} - 4658180508108\,z^{36} - 15817903132432\,z^{35} + 10273041871757\,z^{34} + 16891806144813\,z^{33} + 11029947581076\,z^{32} - 8975515997286\,z^{31} + 3990991996645\,z^{30} + 1843162338871\,z^{29} + 2819751726454\,z^{28} + 5712528940514\,z^{27} - 3123325650527\,z^{26} - 4568857516124\,z^{25} + 3079446451107\,z^{24} - 1561054956031\,z^{23} + 2253742987675\,z^{22} + 8070075436642\,z^{21} - 5285683683185\,z^{20} - 6018866523615\,z^{19} + 2242220741691\,z^{18} - 16850715359180\,z^{17} + 10042194978521\,z^{16} + 21478221788849\,z^{15} - 7030756589503\,z^{14} + 2147040772453\,z^{13} - 6644126415292\,z^{12} - 20281107186208\,z^{11} + 10615688297782\,z^{10} + 23604136877588\,z^{9} + 4747990415974\,z^{8} - 9464569013826\,z^{7} + 554641056330\,z^{6} - 10417936355944\,z^{5} + 7521510211014\,z^{4} + 18118150311174\,z^{3} - 9798756899644\,z^{2} - 10300388851658\,z - 14410340200754$
		$-4599090143590\,z^{50} + 47240971614\,z^{49} - 7371501379626\,z^{48} + 3684666222245\,z^{47} - 2239828388971\,z^{46} - 12147728070459\,z^{45} + 1592748784080\,z^{44} + 8564296897524\,z^{43} - 965625646126\,z^{42} + 10131713521668\,z^{41} - 11133490379610\,z^{40} - 15177903407224\,z^{39} + 3577268952172\,z^{38} - 2784273097486\,z^{37} + 11204317248478\,z^{36} + 14206959307062\,z^{35} - 6344191607115\,z^{34} - 2797061583877\,z^{33} - 6009296669398\,z^{32} - 9068120053960\,z^{31} + 2840232781361\,z^{30} + 11928343993591\,z^{29} - 7364408409198\,z^{28} - 1772349624642\,z^{27} + 496840832203\,z^{26} - 19016054531576\,z^{25} + 9858371330193\,z^{24} + 19295872493353\,z^{23} - 4521827637741\,z^{22} + 1208342237300\,z^{21} - 4374640110055\,z^{20} - 18502749486377\,z^{19} + 6104435716535\,z^{18} + 19146661474462\,z^{17} + 488274966683\,z^{16} - 3316326725319\,z^{15} + 2182282334815\,z^{14} - 14824597559769\,z^{13} + 7532277927728\,z^{12} + 16314787492980\,z^{11} - 6917870993576\,z^{10} - 723482062748\,z^{9} - 8442497375404\,z^{8} - 10800247171176\,z^{7} + 1817577563136\,z^{6} + 7697242742116\,z^{5} - 521297676820\,z^{4} + 2233470317936\,z^{3} - 420000649522\,z^{2} - 2445166706486\,z - 554792510556$
		$-44864565312812\,z^{149} - 68768563833944\,z^{150}$

		$70701440915972\,z^{50}+2485750409310\,z^{49}-48433419862414\,z^{48}-4768774365096\,z^{47}-43697106743944\,z^{46}-53486277061320\,z^{45}+52228629883448\,z^{44}+113082234644520\,z^{43}-54801050915832\,z^{42}-64681454517156\,z^{41}+173187609378876\,z^{40}+4321810781880\,z^{39}+38273938178088\,z^{38}+41634873091296\,z^{37}-48143733242184\,z^{36}-109412908047690\,z^{35}+9716456542658\,z^{34}+80250859917927\,z^{33}-168465758454760\,z^{32}-47271090077022\,z^{31}-10584097269228\,z^{30}+41168723771976\,z^{29}-21017776669584\,z^{28}+5162721672354\,z^{27}+7283096893976\,z^{26}-41509416153159\,z^{25}-13740013133908\,z^{24}+24267526979784\,z^{23}+2091821742234\,z^{22}+29437466574372\,z^{21}+9032398055592\,z^{20}-60386478472440\,z^{19}-73936925599220\,z^{18}+18728137584669\,z^{17}+3482309709856\,z^{16}+37500696189300\,z^{15}+47510069761656\,z^{14}+12055597311288\,z^{13}-35159704302024\,z^{12}-96663979023996\,z^{11}+40073524325696\,z^{10}+86053467392322\,z^{9}-128842256441152\,z^{8}-18762432745968\,z^{7}-29965966283212\,z^{6}-63882704291400\,z^{5}+39994318420184\,z^{4}+139982181869676\,z^{3}-668440393388\,z^{2}-78045602999628\,z+162004856572564$
50	152	$36933555661356\,z^{51}+47920471863576\,z^{50}-55361681802540\,z^{49}+88101855745410\,z^{48}+75160967295972\,z^{47}-58185505771584\,z^{46}-89151046727544\,z^{45}+78638578340376\,z^{44}+28944982607844\,z^{43}+32365270828792\,z^{42}+70767237608154\,z^{41}-2897837630372\,z^{40}-101584076501208\,z^{39}+3986832279908\,z^{38}+909499929144\,z^{37}-47865768754168\,z^{36}+108831322045818\,z^{35}+6264960278182\,z^{34}-90486535189641\,z^{33}+62189759907232\,z^{32}+10409686675518\,z^{31}+31099124171420\,z^{30}+22971196235304\,z^{29}+35718305011712\,z^{28}-18763618774458\,z^{27}-83115522399416\,z^{26}-26374290250401\,z^{25}+22490348502472\,z^{24}+35347663844856\,z^{23}+15415934152918\,z^{22}+26457758570172\,z^{21}-45284574830120\,z^{20}-88493800774524\,z^{19}+72588911514944\,z^{18}+56207093663109\,z^{17}-66424921001812\,z^{16}+8638528221384\,z^{15}-92292496625888\,z^{14}-26271602491416\,z^{13}+34816523179000\,z^{12}+34907926208424\,z^{11}+43851949251232\,z^{10}+8179745438964\,z^{9}+16540887976792\,z^{8}-22321126797840\,z^{7}+39555650921560\,z^{6}-14655642670176\,z^{5}+6978532909192\,z^{4}+48835302448260\,z^{3}-26285180837460\,z^{2}-39059008911378\,z+32278938483072$
		$10869781914776\,z^{152}+87172975855064\,z^{153}$
51	155	$13210080195340\,z^{51}+4522603679090\,z^{50}-9427546683654\,z^{49}-2774004593602\,z^{48}-7400333366216\,z^{47}-18389042628384\,z^{46}+12548156717848\,z^{45}+27341468684832\,z^{44}-15441422259600\,z^{43}-12543292885956\,z^{42}+36178289170772\,z^{41}+3532845840508\,z^{40}+6486818170568\,z^{39}+14736874206568\,z^{38}-11142318969776\,z^{37}-27268258119784\,z^{36}+6499641451100\,z^{35}+15552688098917\,z^{34}-35741550364067\,z^{33}-14297220262025\,z^{32}+320638622102\,z^{31}+6579237422834\,z^{30}-6364978145088\,z^{29}+4708521934772\,z^{28}+1245708648396\,z^{27}-10366430267285\,z^{26}-265982221073\,z^{25}+1459777281547\,z^{24}-1127570531968\,z^{23}+10634813564780\,z^{22}+287337494344\,z^{21}-12986467293314\,z^{20}-10770234249082\,z^{19}-3992766908299\,z^{18}-370829803149\,z^{17}+11556474774739\,z^{16}+6599951121868\,z^{15}+10041200526700\,z^{14}-6971238164168\,z^{13}-26523100115088\,z^{12}+11857021456152\,z^{11}+18519668298230\,z^{10}-29057767792850\,z^{9}-2806926416618\,z^{8}-3971521872128\,z^{7}-21968919390312\,z^{6}+10644380502624\,z^{5}+33267432960812\,z^{4}-6611034284340\,z^{3}-11149342064328\,z^{2}+34702276800324\,z+981145960816$

		$9471847058336\,z^{52}+8059856303860\,z^{51}-12177152418460\,z^{50}+18749291515692\,z^{49}+21599773201460\,z^{48}-14444819794628\,z^{47}-20952257010196\,z^{46}+20136224631640\,z^{45}+1321024846664\,z^{44}+4077738215064\,z^{43}+24044002669190\,z^{42}-4263021234426\,z^{41}-20107393058282\,z^{40}+6339907974024\,z^{39}-9225054027280\,z^{38}-9725987385536\,z^{37}+28020194595724\,z^{36}-4139834380024\,z^{35}-14970556979527\,z^{34}+17056674980517\,z^{33}-4174676625749\,z^{32}+5502941089898\,z^{31}+7629557222846\,z^{30}+5037844502128\,z^{29}+681762605644\,z^{28}-14551651872212\,z^{27}-12917474596739\,z^{26}+5950916078361\,z^{25}+6715836234525\,z^{24}+390569499776\,z^{23}+12966422525876\,z^{22}-9308517476872\,z^{21}-24871693623182\,z^{20}+18287268771674\,z^{19}+10514655058465\,z^{18}-16446979062877\,z^{17}+7325484577311\,z^{16}-17826103849112\,z^{15}-10601837806464\,z^{14}+8519029342488\,z^{13}+6309601105656\,z^{12}+6465454287952\,z^{11}+7127843952492\,z^{10}+2525682365292\,z^{9}-3622009476772\,z^{8}+9053500583600\,z^{7}-5281247431456\,z^{6}+1470684239088\,z^{5}+13831158561592\,z^{4}-7428167087168\,z^{3}-7434850976830\,z^{2}+8602076610826\,z-3467479638186$
		$-62721998839472\,z^{155}-92277981264624\,z^{156}$
$(49,50,51)$		$1468080338609682609694211349074754070816\,0\,z^{150}+2662348061846113222191673773793536608246\,72\,z^{149}$

Approximation polynomials and related determinants in Theorem 3

19	59	$-153\,z^{19}-4\,z^{18}+z^{17}+15\,z^{13}+z^{11}+56\,z^{10}+12\,z^{9}+15\,z^{7}+z^{5}+56\,z-4$
		$153\,z^{19}-4\,z^{18}-z^{17}-15\,z^{13}-z^{11}+56\,z^{10}-12\,z^{9}-15\,z^{7}-z^{5}-56\,z-4$
		$2\,z^{59}+30\,z^{61}$
20	62	$41\,z^{20}+z^{19}-3\,z^{18}-4\,z^{14}-15\,z^{11}-3\,z^{10}+z^{9}-4\,z^{8}-15\,z^{2}+z+1$
		$-41\,z^{20}+z^{19}+3\,z^{18}+4\,z^{14}-15\,z^{11}+3\,z^{10}+z^{9}+4\,z^{8}+15\,z^{2}+z-1$
		$-8\,z^{62}-8\,z^{64}$
21	65	$76\,z^{21}+2\,z^{20}-6\,z^{19}-2\,z^{18}-8\,z^{15}-28\,z^{12}-6\,z^{11}+2\,z^{10}-2\,z^{9}-28\,z^{3}+2\,z^{2}+2\,z-2$
		$-76\,z^{21}+2\,z^{20}+6\,z^{19}-2\,z^{18}+8\,z^{15}-28\,z^{12}+6\,z^{11}+2\,z^{10}+2\,z^{9}+28\,z^{3}+2\,z^{2}-2\,z-2$
		$-16\,z^{65}-16\,z^{69}$
$(19,20,21)$		$-8\,z^{59}$
26	80	$-379\,z^{26}+z^{25}+z^{24}+11\,z^{23}+z^{22}+z^{21}+11\,z^{20}+z^{19}+z^{18}+141\,z^{17}+z^{16}+z^{15}+11\,z^{14}+z^{13}+z^{12}+11\,z^{11}+z^{10}+z^{9}+141\,z^{8}+z^{7}+z^{6}+11\,z^{5}+z^{4}+z^{3}+11\,z^{2}+z+1$
		$379\,z^{26}+z^{25}-z^{24}+11\,z^{23}-z^{22}+z^{21}-11\,z^{20}+z^{19}-z^{18}+141\,z^{17}-z^{16}+z^{15}-11\,z^{14}+z^{13}-z^{12}+11\,z^{11}-z^{10}+z^{9}-141\,z^{8}+z^{7}-z^{6}+11\,z^{5}-z^{4}+z^{3}-11\,z^{2}+z-1$
		$-1040\,z^{80}+2\,z^{82}$
27	83	$1798\,z^{27}-z^{26}-z^{25}-11\,z^{24}-z^{23}-z^{22}-11\,z^{21}-z^{20}-z^{19}-141\,z^{18}-z^{17}-z^{16}-11\,z^{15}-z^{14}-z^{13}-11\,z^{12}-z^{11}-z^{10}-141\,z^{9}-z^{8}-z^{7}-11\,z^{6}-z^{5}-z^{4}-11\,z^{3}-z^{2}-z-1419$
		$-z^{26}+z^{25}-11\,z^{24}+z^{23}-z^{22}+11\,z^{21}-z^{20}+z^{19}-141\,z^{18}+z^{17}-z^{16}+11\,z^{15}-z^{14}+z^{13}-11\,z^{12}+z^{11}-z^{10}+141\,z^{9}-z^{8}+z^{7}-11\,z^{6}+z^{5}-z^{4}+11\,z^{3}-z^{2}+z+379$
		$-2\,z^{83}-2\,z^{85}$
28	108	$z^{28}+z^{27}-z-1$
		$z^{28}+z^{27}+z+1$
		$2\,z^{108}+2\,z^{109}$
$(26,27,28)$		$1081600\,z^{81}+1081600\,z^{80}$
39	119	$z^{39}+z^{38}-z^{12}-z^{11}$
		$z^{39}+z^{38}+z^{12}+z^{11}$
		$2\,z^{119}+2\,z^{120}$
40	136	$-226980\,z^{40}+16380\,z^{38}+16380\,z^{36}+18720\,z^{34}+18720\,z^{28}+35100\,z^{22}-2340\,z^{20}-2340\,z^{18}+131040\,z^{13}-9360\,z^{11}-9360\,z^{9}-9360\,z^{7}+35100\,z^{4}-2340\,z^{2}-9360\,z-2340$
		$226980\,z^{40}-16380\,z^{38}-16380\,z^{36}-18720\,z^{34}-18720\,z^{28}-35100\,z^{22}+2340\,z^{20}+2340\,z^{18}+131040\,z^{13}-9360\,z^{11}-9360\,z^{9}-9360\,z^{7}-35100\,z^{4}+2340\,z^{2}-9360\,z+2340$

		$18720\,z^{136}+18720\,z^{142}$
41	135	$-2340\,z^{41}-226980\,z^{39}+17836\,z^{37}+1552\,z^{35}+18832\,z^{33}+1552\,z^{29}+18832\,z^{27}+556\,z^{23}+35116\,z^{21}-2548\,z^{19}+1448\,z^{14}+131048\,z^{12}-10192\,z^{10}-776\,z^{8}-9416\,z^{6}+556\,z^{5}+35116\,z^{3}-776\,z^{2}-2548\,z-9416$
		$2340\,z^{41}+226980\,z^{39}-17836\,z^{37}-1552\,z^{35}-18832\,z^{33}-1552\,z^{29}-18832\,z^{27}-556\,z^{23}-35116\,z^{21}+2548\,z^{19}+1448\,z^{14}+131048\,z^{12}-10192\,z^{10}-776\,z^{8}-9416\,z^{6}-556\,z^{5}-35116\,z^{3}-776\,z^{2}+2548\,z-9416$
		$18832\,z^{135}+1552\,z^{137}$
$(39,40,41)$		$88133760\,z^{119}\,(1+z)$

Approximation polynomials and related determinants in Theorem 4

17	53	$z^{17}+z^{16}-z^{15}+z^{14}-z^{5}-z^{4}-z^{3}+z^{2}$
		$z^{17}-z^{16}-z^{15}-z^{14}-z^{5}+z^{4}-z^{3}-z^{2}$
		$16\,z^{53}-16\,z^{61}$
18	56	$2\,z^{18}-6\,z^{17}-5\,z^{16}-z^{15}+3\,z^{14}+z^{13}-z^{12}+3\,z^{11}-3\,z^{10}-z^{9}-z^{8}+3\,z^{7}+3\,z^{6}+z^{5}+z^{4}+5\,z^{3}-3\,z^{2}-5\,z-8$
		$2\,z^{18}+6\,z^{17}-5\,z^{16}+z^{15}+3\,z^{14}-z^{13}-z^{12}-3\,z^{11}-3\,z^{10}+z^{9}-z^{8}-3\,z^{7}+3\,z^{6}-z^{5}+z^{4}-5\,z^{3}-3\,z^{2}+5\,z-8$
		$-64\,z^{56}-64\,z^{58}$
19	59	$z^{19}+z^{18}+z^{17}-z^{16}-z^{15}-z^{14}+z^{13}-z^{12}+z^{11}+z^{10}+z^{9}-z^{8}-z^{7}-z^{6}+z^{5}-z^{4}+2\,z^{3}+2\,z^{2}+2\,z-2$
		$z^{19}-z^{18}+z^{17}+z^{16}-z^{15}+z^{14}+z^{13}+z^{12}+z^{11}-z^{10}+z^{9}+z^{8}-z^{7}+z^{6}+z^{5}+z^{4}+2\,z^{3}-2\,z^{2}+2\,z+2$
		$32\,z^{59}+32\,z^{67}$
$(17,18,19)$		$-512\,z^{53}$
21	65	$2\,z^{21}+2\,z^{20}-2\,z^{19}+2\,z^{18}-z^{17}-z^{16}-z^{15}+z^{14}-z^{13}-z^{12}-3\,z^{11}+3\,z^{10}-z^{9}-z^{8}-5\,z^{7}+5\,z^{6}-z^{5}-z^{4}-3\,z^{3}+3\,z^{2}-4\,z-4$
		$2\,z^{21}-2\,z^{20}-2\,z^{19}-2\,z^{18}-z^{17}+z^{16}-z^{15}-z^{14}-z^{13}+z^{12}-3\,z^{11}-3\,z^{10}-z^{9}+z^{8}-5\,z^{7}-5\,z^{6}-z^{5}+z^{4}-3\,z^{3}-3\,z^{2}-4\,z+4$
		$-64\,z^{65}-64\,z^{69}$
22	70	$4\,z^{22}-4\,z^{21}+z^{20}+z^{19}+z^{18}-z^{17}-2\,z^{16}-2\,z^{15}+4\,z^{14}+4\,z^{13}+6\,z^{12}-2\,z^{11}-4\,z^{10}-4\,z^{9}+4\,z^{8}-4\,z^{7}+2\,z^{6}+6\,z^{5}+3\,z^{4}-5\,z^{3}+5\,z^{2}+3\,z-8$
		$4\,z^{22}+4\,z^{21}+z^{20}-z^{19}+z^{18}+z^{17}-2\,z^{16}+2\,z^{15}+4\,z^{14}-4\,z^{13}+6\,z^{12}+2\,z^{11}-4\,z^{10}+4\,z^{9}+4\,z^{8}+4\,z^{7}+2\,z^{6}-6\,z^{5}+3\,z^{4}+5\,z^{3}+5\,z^{2}-3\,z-8$
		$128\,z^{70}-64\,z^{72}$
23	71	$4\,z^{23}-4\,z^{22}+z^{21}+z^{20}+z^{19}-z^{18}-2\,z^{17}-2\,z^{16}+z^{15}+z^{14}+3\,z^{13}+z^{12}-z^{11}-z^{10}+z^{9}-z^{8}-z^{7}+3\,z^{6}-2\,z^{4}+2\,z^{3}-5\,z-3$
		$4\,z^{23}+4\,z^{22}+z^{21}-z^{20}+z^{19}+z^{18}-2\,z^{17}+2\,z^{16}+z^{15}-z^{14}+3\,z^{13}-z^{12}-z^{11}+z^{10}+z^{9}+z^{8}-z^{7}-3\,z^{6}+2\,z^{4}+2\,z^{3}-5\,z+3$
		$32\,z^{71}-64\,z^{73}$
$(21,22,23)$		$3072\,z^{65}$
26	80	$8\,z^{26}+5\,z^{24}-3\,z^{23}+5\,z^{22}-5\,z^{21}-z^{20}-z^{19}+3\,z^{18}+z^{17}+4\,z^{16}-4\,z^{13}+2\,z^{12}-2\,z^{11}+2\,z^{10}+2\,z^{9}+3\,z^{8}-z^{7}+3\,z^{6}+z^{5}-5\,z^{4}-z^{3}-5\,z^{2}-3\,z-8$
		$8\,z^{26}+5\,z^{24}+3\,z^{23}+5\,z^{22}+5\,z^{21}-z^{20}+z^{19}+3\,z^{18}-z^{17}+4\,z^{16}+4\,z^{13}+2\,z^{12}+2\,z^{11}+2\,z^{10}-2\,z^{9}+3\,z^{8}+z^{7}+3\,z^{6}-z^{5}-5\,z^{4}+z^{3}-5\,z^{2}+3\,z-8$
		$64\,z^{80}+64\,z^{88}$
27	91	$4\,z^{27}+4\,z^{26}+4\,z^{25}-4\,z^{24}+2\,z^{19}+2\,z^{18}+2\,z^{17}-2\,z^{16}-2\,z^{15}-2\,z^{14}+2\,z^{13}-2\,z^{12}+2\,z^{11}+2\,z^{10}+2\,z^{9}-2\,z^{8}+2\,z^{7}+2\,z^{6}-2\,z^{5}+2\,z^{4}-4\,z^{3}-4\,z^{2}-4\,z+4$

		$4z^{27}-4z^{26}+4z^{25}+4z^{24}+2z^{19}-2z^{18}+2z^{17}+2z^{16}-2z^{15}+2z^{14}+2z^{13}+2z^{12}+2z^{11}-2z^{10}+2z^{9}+2z^{8}+2z^{7}-2z^{6}-2z^{5}-2z^{4}-4z^{3}+4z^{2}-4z-4$
		$128z^{91}+64z^{99}$
28	88	$2z^{28}+4z^{27}+8z^{26}+4z^{25}+2z^{24}-4z^{23}+4z^{22}-4z^{21}+z^{19}+3z^{18}+3z^{17}+2z^{16}-2z^{15}-2z^{14}-2z^{13}-2z^{11}+4z^{10}+2z^{9}+2z^{8}+4z^{6}-2z^{4}-z^{3}-9z^{2}-3z-6$
		$2z^{28}+4z^{26}+6z^{24}+4z^{23}+4z^{22}+4z^{21}+z^{19}+z^{18}-z^{17}+4z^{16}+4z^{13}+2z^{12}+4z^{11}+2z^{10}+4z^{8}+2z^{7}+2z^{6}-2z^{5}-4z^{4}-3z^{3}-5z^{2}-z-10$
		$64z^{88}+128z^{90}$
(26, 27, 28)		$-4096z^{80}$

Approximation polynomials and related determinants in Theorem 5

10	31	$z^{10}-z^{9}-z^{8}+z^{6}-z^{5}-2z^{4}+z^{2}-z$
		$z^{9}+2z^{8}-z^{7}+3z^{5}+2z^{4}+1$
		$z^{31}-z^{47}$
11	43	$z^{9}+z^{5}+z^{4}+z+1$
		$-z^{10}-z^{9}-z^{8}-z^{6}-2z^{5}-z^{4}-z^{3}-z^{2}-2z-2$
		$z^{43}-z^{64}$
12	43	$z^{10}+3z^{9}+z^{6}+z^{5}+z^{4}+z^{2}+z+1$
		$-z^{11}-4z^{10}-4z^{9}-3z^{8}-z^{7}-2z^{6}-2z^{5}+z^{4}-2z^{3}-2z^{2}-3z-4$
		$z^{43}-2z^{44}$
(10, 11, 12)		$-3z^{32}-2z^{31}$
26	79	$z^{24}+z^{20}+z^{16}$
		$-z^{25}-z^{24}-z^{23}-z^{21}-z^{20}-z^{17}-z^{16}-z^{15}$
		$-z^{79}-z^{207}$
27	83	$-z^{24}-z^{20}+z^{4}+1$
		$z^{25}+z^{24}+z^{21}+z^{20}+z^{19}-z^{5}-z^{4}-z^{3}-z-1$
		$z^{83}-z^{91}$
28	85	$z^{28}-z^{26}-2z^{22}-z^{21}-2z^{10}-2z^{9}-2z^{6}-3z^{5}+z^{4}-z^{2}-2z+1$
		$z^{26}+2z^{23}+3z^{22}+2z^{21}+2z^{11}+4z^{10}+4z^{9}+2z^{8}+2z^{7}+5z^{6}+2z^{5}+3z^{2}+2z$
		$z^{85}-z^{88}$
(26, 27, 28)		$-z^{80}-z^{79}$