FIELD OF OPTIMAL QUOTIENTS AND HERMITIANITY

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Abstract. For an eigenvalue problem, be it generalized or not, the field of optimal quotients is an inclusion region containing the eigenvalues. Convexity properties, connectedness and continuity of this set are addressed. Since any notion of quotients is inherently basis dependent, varying the basis is shown to provide a lot of information. Then, due to its non-convexity, the field of optimal quotients allows recovering the spectrum exactly. In the case of a standard eigenvalue problem, the classical field of values of a matrix is recovered through a limit process. The notion leads to a serious claim how normal, Hermitian and unitary generalized eigenvalue problems should be formulated.

Key words. generalized eigenvalue problem, field of optimal quotients, field of values, non-convexity, convexity, Hermitianity

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1. Introduction. Let the matrices $M, N \in \mathbb{C}^{n \times n}$ be linearly independent such that there exist nonsingular linear combinations of them. The field of optimal quotients, denoted by $\mathcal{F}(M, N)$, for the eigenvalue problem

$$Mx = \lambda Nx \quad (1.1)$$

consist of the complex numbers in the polar form

$$\frac{q^* N^* M q}{|q^* N^* M q|} \frac{\|Mq\|}{\|Nq\|} \quad (1.2)$$

with nonzero $q \in \mathbb{C}^n$ whenever $q^* N^* M q \neq 0$. If $Mq = 0$ (resp. $Nq = 0$) for $q \neq 0$, define the optimal quotient to have the value 0 (resp. ∞). Optimal quotients (1.2) were introduced for a rapid iterative solution of very large eigenvalue problems [11].¹ Analogously to the connection between the field of values and Rayleigh quotients, the location of the field of optimal quotients determines where numerical approximations to the eigenvalues can then appear. (For the Rayleigh quotients, see, e.g., [14, 16, 3, 1] and references therein. See also [13, Section 8].) This paper is concerned with the properties of the field of optimal quotients. Its convexity, connectedness and continuity are addressed. The field of optimal quotients of linear combinations of M and N is shown to carry a lot of information, yielding classical notions such as the spectrum and the field of values of a matrix. A notion of normality for the generalized eigenvalue problem is suggested. Then vanishing of the planar dimension of the field of optimal quotients allows formulating Hermitian and unitary generalized eigenvalue problems in a natural way. These, moreover, are shown to be entirely indistinguishable.

Regarding properties of $\mathcal{F}(M, N)$, we first recall some basics from [11]. Then it is shown that the field of optimal quotients is radially convex, i.e., if two points on a ray

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¹It can be argued that the iteration introduced is faster than the classical Rayleigh quotient iteration in the case of the standard Hermitian eigenvalue problem. It hence resolves the open problem of whether the Rayleigh quotient iteration is “best”, a claim occasionally made without evidence.

away from the origin belong to $\mathcal{F}(M, N)$, then the corresponding line segment belongs to $\mathcal{F}(M, N)$. There is also a natural association of the field of optimal quotients with the Davis-Wielandt shell, a classical convex spectral set. Connectedness is shown to be a generic property; see Theorem 2.10. If $0 \notin \mathcal{F}(N^*M)$, the field of values of the matrix N^*M , then $\mathcal{F}(M, N)$ is shown to be closed and simply connected. Moreover, then the field of optimal quotients is a continuous set-valued function at $(M, N) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$.

Inspecting $\mathcal{F}(M, N)$ alone is neither sufficient nor arguable. Since any notion of a quotient is inherently basis dependent, we associate with the eigenvalue problem (1.1) the matrix subspace

$$\mathcal{V} = \text{span}\{M, N\}. \quad (1.3)$$

There arises the problem of choosing a basis which can be regarded as yielding the most informative field of optimal quotients. There is a tremendous amount of data available. To illustrate this, consider the set-valued map

$$\mu \mapsto \mathcal{F}(M - \mu N, N) + \mu \quad (1.4)$$

for $\mu \in \mathbb{C}$. For any μ , the image contains the eigenvalues of (1.1). Due to their non-convexity, it is shown that the intersection, while μ varies, allows recovering the spectrum exactly. However, the field of optimal quotients is convex at infinity in the sense that taking the limit at infinity yields the field of values of the matrix MN^{-1} . Consequently, in the case $N = I$ we are dealing with a far more nuanced notion than the field of values $\mathcal{F}(M)$ of the matrix M which hence is just one manifestation of the field of optimal quotients. Of course, $\mu \mapsto \mathcal{F}(M - \mu I) + \mu$ is a particularly modest set-valued map by being merely a constant, i.e., independent on μ . It seems plausible that numerical methods can be devised to benefit from this basis dependence.

The case of vanishing planar dimension of the field of optimal quotients is particularly intriguing; see Figure 3.1. Although formulated for matrices, the operator theoretic relevance of this is considerable by the fact that the polar form (1.2) hints a way to identify Hermitian and unitary generalized eigenvalue problems in a natural manner. This takes place when dealing with an appropriate notion of normality. Again, this requires inspecting the whole matrix subspace $\mathcal{V}$ and thereafter, in the positive case, taking a carefully chosen basis of it. It is shown that the Hermitian and unitary structures suggested are entirely indistinguishable, so it suffices to concentrate on Hermitianity. First, besides admitting accurate eigenvalue inclusion regions, the Hermitian structure suggested allows avoiding applying the inverse in a transformation of the eigenvalue problem (1.1) to the standard form. Second, there is a natural notion of orthogonality for eigenvectors with respect to the standard Euclidean inner product. Third, then the problem of finding eigenvalues can be formulated as an extremum problem for a ratio of two quadratic forms. This means, together with convincing numerical evidence given in [11], that all the relevant criteria classically associated with Hermitianity² are satisfied.

The paper is organized as follows. In Section 2 properties such as convexity and connectedness of the field of optimal quotients is studied. Section 3 is concerned with showing that varying the basis of $\mathcal{V}$ provides a large amount of spectral information.

²C. Hermite showed the reality of the eigenvalues of a Hermitian form [8], a statement expressible in terms of the field of values of the associated matrix. This observation resulted in these matrices named for him.

In Section 4 vanishing of the planar dimension of the field of optimal quotients is investigated, leading to a natural notion of Hermitianity and unitarity for generalized eigenvalue problems.

2. Convexity and connectedness of the field of optimal quotients. A numerical approximation of an eigenpair of the eigenvalue problem (1.1) typically starts with a generation of an approximate unit eigenvector $q \in \mathbb{C}^n$. Once available, let us compute

$$w_1 = \frac{Mq}{\|Mq\|} \text{ and } w_2 = \frac{Nq}{\|Nq\|} \quad (2.1)$$

assuming the norms are nonzero. Regarding these vectors as elements of the Grassmannian $\text{Gr}_1(\mathbb{C}^n)$, we have an eigenvector exactly when they represent the same element. When they differ, there arises the problem of choosing an element z of $\text{Gr}_1(\mathbb{C}^n)$ best simultaneously approximating w_1 and w_2 . To this end we may consider

$$|z^*w_1| \text{ and } |z^*w_2|.$$

If we use the Euclidean norm in (2.1), then it is natural to take the squares and impose

$$\max_{z \in \mathbb{C}^n, \|z\|=1} (|z^*w_1|^2 + |z^*w_2|^2). \quad (2.2)$$

Whenever w_1 and w_2 are non-orthogonal, this is uniquely solved in terms of q as

$$z = \frac{1}{\sqrt{2 + 2|w_1^*w_2|}} \left(\frac{w_1^*w_2}{|w_1^*w_2|} w_1 + w_2 \right) \quad (2.3)$$

modulo multiplications by $e^{i\theta}$ with $\theta \in \mathbb{R}$. With respect to q and z (depending hence on q) the corresponding quotient is defined to be the scalar λ satisfying the identity

$$z^*Mq = \lambda z^*Nq. \quad (2.4)$$

This then yields (1.2) to approximate an eigenvalue of the eigenvalue problem (1.1); see [11, Section 2] for more details. To associate this with approximating eigenvectors, define

$$(w_1, w_2) \mapsto 1 + |w_1^*w_2| \quad (2.5)$$

for any unit vectors w_1 and w_2 .

PROPOSITION 2.1. *Assume $w_1, w_2 \in \mathbb{C}^n$ are non-orthogonal unit vectors. Then z given by (2.3) yields $|z^*w_1|^2 + |z^*w_2|^2 = 1 + |w_1^*w_2|$.*

Clearly, the function (2.5) is bounded above by two such that two is attained if and only if q is an eigenvector corresponding to an eigenvalue $\notin \{0, \infty\}$.³ Maximizing this function yields thereby a very natural route to improve the initial eigenvector approximation q . In general, unless the inverse can be applied (possibly only approximately) [11], the vector z does not appear to be of any use in this.

The matrix subspace (1.3) is said to be nonsingular if there exist nonsingular linear combinations of M and N . (See [10] for the concept of nonsingularity of matrix subspaces.) Then collecting all the possible quotients (1.2) yields us the following notion.

³The eigenvalues $\{0, \infty\}$ are in this sense exceptional

DEFINITION 2.2. Assume (1.3) is nonsingular. The set

$$\left\{ \frac{q^* N^* M q}{|q^* N^* M q|} \frac{\|Mq\|}{\|Nq\|} : q^* N^* M q \neq 0 \right\}$$

is said to be the field of optimal quotients of the eigenvalue problem (1.1) such that if $Mq = 0$ (resp. $Nq = 0$) for $q \neq 0$, define the optimal quotient to have the value 0 (resp. ∞).⁴ The field of optimal quotients is denoted by $\mathcal{F}(M, N)$.

It is readily seen that $\mathcal{F}(M, N)$ contains the eigenvalues of the eigenvalue problem (1.1).

Regarding changing the coordinates, an equivalence transformation of the eigenvalue problem (1.1) means forming the matrix subspace

$$X\mathcal{V}Y^{-1} = \{XVY^{-1} : V \in \mathcal{V}\}$$

with two invertible matrices $X, Y \in \mathbb{C}^{n \times n}$. The field of values of a matrix remains invariant only under a unitary similarity. As opposed to this, $\mathcal{F}(M, N)$ is more flexible and far more nonsymmetric notion with respect to performing equivalence transformations.

THEOREM 2.3. [11] If $U \in \mathbb{C}^{n \times n}$ is unitary and $Y \in \mathbb{C}^{n \times n}$ invertible, then

$$\mathcal{F}(UMY^{-1}, UNY^{-1}) = \mathcal{F}(M, N).$$

This result is sharp, i.e., the unitary matrix U cannot be replaced with an invertible matrix in general. This is sufficient, generically, to allow to reduce the problem to

$$\mathcal{F}(M, I) \tag{2.6}$$

involving a single matrix M which can, furthermore, taken to be upper triangular. (Recall that for the field of values of a matrix it suffices to study the upper triangular case. This follows from using the Schur decomposition.) Namely, by assuming N to be invertible, first take $Y = N$. Thereafter take the Schur decomposition of MN^{-1} to perform a unitary equivalence transformation.

This has other similar consequences.

COROLLARY 2.4. Suppose there exist a unitary $U \in \mathbb{C}^{n \times n}$ and an invertible $Y \in \mathbb{C}^{n \times n}$ such that the matrices UMY^{-1} and UNY^{-1} are real. Then $\mathcal{F}(M, N)$ is symmetric with respect to the real axis.

COROLLARY 2.5. Suppose $M \in \mathbb{C}^{n \times n}$ is invertible. Then $\mathcal{F}(M^{-1}, I) = \frac{1}{\mathcal{F}(M, I)}$.

The matrix N^*M is seemingly important for the generalized eigenvalue problem (1.1). This is due to the fact any optimal quotient (1.2) has its argument among the arguments of $\mathcal{F}(N^*M)$, the field of values of N^*M . (See [9, Chapter 1] for the field of values of matrices.)

THEOREM 2.6. [11] Assume $0 \notin \mathcal{F}(N^*M)$. Then $\mathcal{F}(M, N)$ is closed.

If $0 \in \mathcal{F}(N^*M)$, then $\mathcal{F}(M, N)$ need not be closed; see [11, Example 2.3].

Regarding the moduli of optimal quotients, $\mathcal{F}(M, N)$ belongs to the origin centered annulus with the inner and outer radii

$$r(M, N) = \min_{\|q\|=1} \frac{\|Mq\|}{\|Nq\|} \quad \text{and} \quad R(M, N) = \max_{\|q\|=1} \frac{\|Mq\|}{\|Nq\|}. \tag{2.7}$$

⁴If $Mq = 0$ (resp. $Nq = 0$), choose z to be a unit vector in the direction of Nq (resp. Mq).

This annulus can be very thin, of course.

EXAMPLE 1. Assume there exists an invertible $Y \in \mathbb{C}^{n \times n}$ such that both MY^{-1} and NY^{-1} are unitary. Then $\mathcal{F}(M, N)$ is a subset of the unit circle. In particular, it is not convex.

The field of optimal quotients is hence not convex. For radial convexity, first observe that

$$\mathcal{F}(e^{i\theta_1}M, e^{i\theta_2}N) = e^{i(\theta_1 - \theta_2)}\mathcal{F}(M, N) \quad (2.8)$$

for any $\theta_1, \theta_2 \in \mathbb{R}$.

THEOREM 2.7. Let $\theta \in \mathbb{R}$ and $0 < r < R$. If $re^{i\theta}$ and $Re^{i\theta}$ belong to $\mathcal{F}(M, N)$, then

$$tre^{i\theta} + (1-t)Re^{i\theta} \in \mathcal{F}(M, N)$$

for any $0 \leq t \leq 1$.

Proof. By (2.8) we may assume $\theta = 0$, otherwise consider $\mathcal{F}(e^{-i\theta}M, N)$. Let

$$N^*M = H + iK \quad (2.9)$$

with Hermitian $H, K \in \mathbb{C}^{n \times n}$. By assumption, there exist unit vectors $q_1, q_2 \in \mathbb{C}^n$ such that

$$r = \frac{\|Mq_1\|}{\|Nq_1\|}, \quad q_1^*Hq_1 > 0 \quad \text{and} \quad q_1^*Kq_1 = 0$$

and

$$R = \frac{\|Mq_2\|}{\|Nq_2\|}, \quad q_2^*Hq_2 > 0 \quad \text{and} \quad q_2^*Kq_2 = 0.$$

For any $t_1, t_2 \in \mathbb{R}$ also the vectors $e^{it_1}q_1$ and $e^{it_2}q_2$ satisfy these conditions. Replacing q_2 with $e^{it_2}q_2$, with appropriately chosen t_2 , we may assume $\text{Re } q_1^*Kq_2 = 0$. Set $q(t) = tq_1 + (1-t)q_2$ with $0 \leq t \leq T$, where $0 < T \leq 1$ is the least positive value of t yielding $\frac{\|Mq(T)\|}{\|Nq(T)\|} = r$. Then we have

$$q(t)^*Hq(t) = t^2q_1^*Hq_1 + (1-t)^2q_2^*Hq_2 + 2t(1-t)\text{Re } q_1^*Hq_2 > 0$$

and

$$q(t)^*Kq(t) = t^2q_1^*Kq_1 + (1-t)^2q_2^*Kq_2 + 2t(1-t)\text{Re } q_1^*Kq_2 = 0$$

for any $0 \leq t \leq T$. Then, by continuity, the path

$$t \longmapsto \frac{q(t)^*N^*Mq(t)}{|q(t)^*N^*Mq(t)|} \frac{\|Mq(t)\|}{\|Nq(t)\|}$$

for $0 \leq t \leq T$ covers the interval claimed. $\square$

To associate the field of optimal quotient with a convex set, recall that the Davis-Wielandt shell of a matrix $M \in \mathbb{C}^{n \times n}$ is a subset of $\mathbb{C} \times \mathbb{R}$ defined as

$$\{(q^*Mq, \|Mq\|^2) : \|q\| = 1\}; \quad (2.10)$$

see [9, Chapter 1.8] and [12]. Consider (2.6). Then the function

$$f(z, r) = \frac{z}{|z|} \sqrt{r}, \quad z \neq 0 \quad (2.11)$$

maps the Davis-Wielandt shell of M into $\mathcal{F}(M, I)$. If M is nonsingular, then the map is onto. This can be used, for example, in the 2-by-2 case since then the Davis-Wielandt shell can be characterized [12, Theorem 2.2]. Moreover, in dimensions larger than two, the Davis-Wielandt shell is always convex. This convex set is related with $\mathcal{F}(M, I)$ in terms of (2.11).

We have angular convexity as follows.

PROPOSITION 2.8. *Let $N \in \mathbb{C}^{n \times n}$ be invertible and suppose nonzero $\lambda_1, \lambda_2 \in \mathcal{F}(M, N)$ satisfy $|\arg(\lambda_1) - \arg(\lambda_2)| \neq \pi$. If $|\lambda_1| = |\lambda_2|$, then $\mathcal{F}(M, N)$ contains the origin centered circular arc connecting λ_1 and λ_2 .*

Proof. We may consider $\mathcal{F}(A, I)$ with $A = MN^{-1}$.

If $n = 1$ then $\lambda_1 = \lambda_2$. If $n = 2$, then the Davis-Wielandt (2.10) is an ellipsoid without the interior. Take the path on this ellipsoid, where the second co-ordinate attains the constant value $|\lambda_1|^2$. When this path is projected to the first co-ordinate, we have an ellipse which does not reduce to a point. This ellipse thus may or may not contain the origin. The map (2.11) connects λ_1 and λ_2 through a circular arc when we traverse on this ellipsoid such that the first co-ordinate avoids the origin. Finally, if $n \geq 3$, then by the convexity of the Davis-Wielandt shell, there are unit vectors $q(t)$ such that

$$q(t)^* A q(t) = (1-t)q_1^* A q_1 + tq_2^* A q_2$$

and $\|Aq(t)\|^2 = (1-t)|\lambda_1|^2 + t|\lambda_2|^2 = |\lambda_1|^2$ for $0 \leq t \leq 1$. Here $\lambda_j = \frac{q_j^* A q_j}{|q_j^* A q_j|} \|Aq_j\|$ for $j = 1, 2$. The map (2.11) then connects λ_1 and λ_2 through a circular arc when t varies from 0 to 1. $\square$

These results pave way to study connectedness of the field of optimal quotients. First, consider the annulus with the radii (2.7) containing $\mathcal{F}(M, N)$. If N is invertible, then $r(M, N)$ and $R(M, N)$ equal the smallest and the largest singular value of the matrix MN^{-1} . If N is singular, then the field of optimal quotients is unbounded under the following assumptions.

PROPOSITION 2.9. *Assume N is singular. If $\mathcal{N}(N) \cap \mathcal{N}(N^*M) = \{0\}$, then $\mathcal{F}(M, N) \setminus \{\infty\}$ is unbounded.*

Proof. The field of optimal quotients is readily seen to be invariant under unitary equivalence transformations. After performing the generalized Schur transformation, we may assume M and N are upper triangular such that the $(1, 1)$ -entry of N is zero. Since (1.3) is nonsingular, the $(1, 1)$ -entry of M is $\mu_1 \neq 0$. Take $q = \sqrt{1-\epsilon}e_1 + \sqrt{\epsilon}v$, where e_1 is the first standard basis vector and v^* is the first row of N . Since $\mathcal{N}(N) \cap \mathcal{N}(N^*M) = \{0\}$, necessarily $v \neq 0$. Then

$$q^* N^* M q = \sqrt{\epsilon(1-\epsilon)} \|v\|^2 \mu_1 + \epsilon(v^* N^*) M v$$

which is non-zero for small enough $\epsilon > 0$. Also $\frac{\|Mq\|}{\|Nq\|} = \frac{\|\sqrt{1-\epsilon}\mu_1 + \sqrt{\epsilon}Mv\|}{\sqrt{\epsilon}\|Nv\|}$ which is unbounded as ϵ approaches zero. $\square$

EXAMPLE 2. Even if N is singular, $\mathcal{F}(M, N) \setminus \{\infty\}$ can be bounded. As an extreme case, suppose $N^*M = 0$. Then, since the matrix subspace (1.3) is assumed to be nonsingular for $\mathcal{F}(M, N)$ to be defined, necessarily $\mathcal{N}(N) \cap \mathcal{N}(N^*M) \neq \{0\}$.

Now M and N must be singular both, so that $\mathcal{F}(M, N) = \{0, \infty\}$. In particular, $\mathcal{F}(M, N)$ is disconnected.

This illustrates that the field of optimal quotients need not be connected, although this is exceptional in the following sense.

THEOREM 2.10. *Assume $M, N \in \mathbb{C}^{n \times n}$ are nonsingular. Then $\mathcal{F}(M, N)$ is disconnected if and only if $e^{i\theta} N^* M$ is Hermitian indefinite for some $\theta \in \mathbb{R}$.*

Proof. Let us first deal with the case $e^{i\theta} N^* M$ is Hermitian for some $\theta \in \mathbb{R}$. Assume first that $e^{i\theta} N^* M$ is positive definite. Then $\mathcal{F}(M, N)$ is located on the ray $\{re^{i\theta} : r \geq 0\}$. Moreover, by Theorem 2.7 we know that then any two finite nonzero points of $\mathcal{F}(M, N)$ are connected by a path. Next assume $e^{i\theta} N^* M$ is indefinite. Since M is invertible, then by the bounds (2.7) the origin is not included in $\mathcal{F}(M, N)$. Moreover, since $e^{i\theta} N^* M$ is indefinite, there are both negative and positive eigenvalues in $\mathcal{F}(M, N)$.

So we are left with assuming that $e^{i\theta} N^* M$ is not Hermitian for any $\theta \in \mathbb{R}$. Consider any two points

$$\lambda_1 = \frac{q_1^* N^* M q_1}{|q_1^* N^* M q_1|} \frac{\|M q_1\|}{\|N q_1\|} \quad \text{and} \quad \lambda_2 = \frac{q_2^* N^* M q_2}{|q_2^* N^* M q_2|} \frac{\|M q_2\|}{\|N q_2\|}$$

of $\mathcal{F}(M, N)$ satisfying $|\text{Arg}(\lambda_1) - \text{Arg}(\lambda_2)| \neq \pi$. By (2.8), possibly multiplying M by an appropriate $e^{i\theta_1}$, we may assume that we have $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) > 0$. Set $q(t) = \sqrt{1-t}q_1 + \sqrt{t}q_2$ for $0 \leq t \leq 1$. Then

$$q(t)^* N^* M q(t) = (1-t)q_1^* N^* M q_1 + tq_2^* N^* M q_2 + \sqrt{1-t}\sqrt{t}(q_1^* N^* M q_2 + \overline{q_1^* (N^* M)^* q_2}).$$

In terms of (2.9) we have $q_1^* N^* M q_2 + \overline{q_1^* (N^* M)^* q_2} = 2(\text{Re}(q_1^* H q_2) - \text{Im}(q_1^* K q_2))$. So choose $\hat{q}_2 = \pm e^{-\alpha i} q_2$ with $\alpha = \text{Arg}(q_1^* K q_2)$ and the sign $\pm$ such that $\text{Re}(q_1^* H \hat{q}_2) \geq 0$. Then with $q(t) = \sqrt{1-t}q_1 + \sqrt{t}\hat{q}_2$ instead we have

$$q(t)^* N^* M q(t) = (1-t)q_1^* N^* M q_1 + t\hat{q}_2^* N^* M \hat{q}_2 + 2\sqrt{1-t}\sqrt{t}\text{Re}(q_1^* H \hat{q}_2).$$

This yields us a smooth path $t \mapsto \frac{q(t)^* N^* M q(t)}{|q(t)^* N^* M q(t)|} \frac{\|M q(t)\|}{\|N q(t)\|}$ connecting λ_1 and λ_2 in $\mathcal{F}(M, N)$ while $0 \leq t \leq 1$.

If $|\text{Arg}(\lambda_1) - \text{Arg}(\lambda_2)| = \pi$, then we can do the same twice with three points satisfying the preceding angular assumption. Such three points can be found since $e^{i\theta} N^* M$ is not Hermitian for any $\theta \in \mathbb{R}$. $\square$

COROLLARY 2.11. *If $e^{i\theta} N^* M$ is Hermitian indefinite and invertible for some $\theta \in \mathbb{R}$, then $\mathcal{F}(M, N)$ consists of two components.*

COROLLARY 2.12. *If $0 \notin \mathcal{F}(N^* M)$, then $\mathcal{F}(M, N)$ is simply connected.*

Proof. Suppose two points λ_1 and λ_2 of $\mathcal{F}(M, N)$ and 0 are collinear. Since $0 \notin \mathcal{F}(N^* M)$, the line segment connecting λ_1 and λ_2 does not include 0. This line segment belongs to $\mathcal{F}(M, N)$ by Theorem 2.7. From this it follows that $\mathcal{F}(M, N)$ cannot contain any holes. $\square$

3. Recovering the spectrum and the field of values. To associate the field of optimal quotients with classical spectral sets besides the Davis-Wiedlant shell (2.10), consider linearly independent linear combinations

$$A = aM + bN \quad \text{and} \quad B = cM + dN \tag{3.1}$$

of M and N with $a, b, c, d \in \mathbb{C}$.

PROPOSITION 3.1. *Let $M, N \in \mathbb{C}^{n \times n}$ and suppose (3.1) with $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \neq 0$. Then the matrix $\alpha A + \beta B$ is singular if and only if $\delta M + \gamma N$ is singular, where $\begin{bmatrix} \delta \\ \gamma \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.*

This means that the original eigenvalue problem (1.1) can be solved by solving the eigenvalue problem involving linear combinations A and B . Such an invariance property cannot be expected to hold for any reasonable notion of quotients.

This, however, turns out to be something of interest. That is, in what follows it is shown that the field of optimal quotients of linear combinations of M and N carries an impressive amount of spectral information. To this end it suffices to consider (1.4). Clearly, for any μ , the image contains the eigenvalues of (1.1).

EXAMPLE 3. The map (1.4) is a set-valued function of μ . The corresponding maps for the spectrum and the field of values of a matrix are constant, i.e., independent on μ . There is a fundamental difference in this regard.

Consider (1.4). The spectrum of (1.1) can be recovered in terms of the optimal field of values in the following way.

THEOREM 3.2. *Assume (1.3) is nonsingular. Then the intersection*

$$\bigcap_{\mu \in \mathbb{C}} \{\mathcal{F}(M - \mu N, N) + \mu\} \quad (3.2)$$

equals the spectrum of (1.1).

Proof. Since the spectrum of (1.1) is included in every set intersected, it remains to show that the intersection does not contain points outside the spectrum.

Take a point $\mu \in \mathbb{C}$ that is not an eigenvalue of (1.1). Form linear combinations $A = M - \mu N$ and $B = N$. Then A is invertible. Then also $\frac{\|Aq\|}{\|Bq\|} > 0$ whenever $q^* B^* A q \neq 0$. Therefore the origin is not included in $\mathcal{F}(A, B)$ and, consequently, μ is not included in (3.2). $\square$

Observe that the lack of convexity of the field of optimal quotients was instrumental to have this result. In particular, in this manner non-convexity can be very useful in approximating interior eigenvalues.

Also the field of values of a matrix is directly related with the field optimal quotients as follows.

THEOREM 3.3. *Let $N \in \mathbb{C}^{n \times n}$ be invertible. Then*

$$\lim_{\mu \rightarrow \infty} \mathcal{F}(M - \mu N, N) + \mu = \mathcal{F}(MN^{-1}).$$

Proof. Denote by A the matrix MN^{-1} . By Theorem 2.3 there holds

$$\mathcal{F}(M - \mu N, N) = \mathcal{F}(A - \mu I, I).$$

Now for any unit vector $q \in \mathbb{C}^n$ we have by the Pythagorean theorem

$$\|(A - \mu I)q\|^2 = |q^* A q - \mu|^2 + \|(A - q^* A q I)q\|^2 = |q^* A q - \mu|^2 + \|Aq\|^2 - |q^* A q|^2.$$

Therefore

$$\begin{aligned} & \frac{q^*(A - \mu I)q}{|q^*(A - \mu I)q|} \|(A - \mu I)q\| = \\ & \frac{q^* A q - \mu}{|q^* A q - \mu|} \sqrt{|q^* A q - \mu|^2 + \|Aq\|^2 - |q^* A q|^2} = (q^* A q - \mu) \sqrt{1 + \frac{\|Aq\|^2 - |q^* A q|^2}{|q^* A q - \mu|^2}}, \end{aligned}$$

FIG. 3.1. $\mathcal{F}(M - \mu I, I) + \mu$ for values $\mu = 0$, $\mu = 1$ and $\mu = 100$. The eigenvalues are depicted with o . See Example 4.

so that

$$\frac{q^*(A - \mu I)q}{|q^*(A - \mu I)q|} \|(A - \mu I)q\| + \mu = q^* A q + O\left(\frac{1}{|q^* A q - \mu|}\right). \quad (3.3)$$

The speed of convergence can be bounded uniformly in q when q is restricted to be of unit length. $\square$

This can be interpreted, conversely, as follows. The more nearly linearly dependent M and N are, the better $\mathcal{F}(M, N)$ can be approximated with a convex set. Hence this is also a convexity result, i.e., the image of (1.4) is convex at infinity. (Recall that the field of values of a matrix is convex [6].)

Suppose $N = I$. Being recoverable through a limit process, the field of values of a matrix $M \in \mathbb{C}^{n \times n}$ is just one manifestation of the optimal field of values.

COROLLARY 3.4. $\lim_{\mu \rightarrow \infty} \mathcal{F}(M - \mu I, I) + \mu = \mathcal{F}(M)$.

Because of all this versatility, it is tempting to identify linear combinations yielding as informative field of optimal quotients as possible. It is clear that vanishing of the planar dimension of the field of optimal quotients can be expected to correspond to intriguing classes of eigenvalue problems; see Section 4. For another example, it is possible to locate the spectrum exactly with a finite number of intersections instead of (3.2).

EXAMPLE 4. Let M be unitary with the eigenvalues on a half-circle; see Figure 3.1. Then $\mathcal{F}(M, I)$ is the path on the origin centered unit circle with the angles varying from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. That is, the planar dimension of $\mathcal{F}(M, I)$ is zero. We have depicted $\mathcal{F}(M - \mu I, I) + \mu$ for the values $\mu = 0$, $\mu = 1$ and $\mu = 100$. Now Corollary 3.4 yields that we attain $\mathcal{F}(M)$, the field of values of M , at infinity. (It is of interest, regarding approximating the spectrum, that the information at infinity with $\mathcal{F}(M)$ is the least accurate.) This takes place already for $\mu = 100$ within the resolution of the figure since the error is of order $O(10^{-2})$; see (3.3). Now intersecting $\mathcal{F}(M, I)$ and $\mathcal{F}(M - \mu I, I) + \mu$ for μ large enough yields the eigenvalues exactly.

In varying parameters, let us make some remarks on continuity.

THEOREM 3.5. Suppose $0 \notin \mathcal{F}(N^* M)$. Then the map

$$(A, B) \mapsto \mathcal{F}(A, B)$$

from $\mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ to compact subsets of $\mathbb{C}$ is continuous at (M, N) .

Proof. Since $0 \notin \mathcal{F}(N^*M)$, both A and B are necessarily invertible when sufficiently close to M and N . Moreover, the field of values of a matrix is a continuous function of the matrix, so that $\mathcal{F}(B^*A)$ does not include the origin when A and B are sufficiently close to M and N . Therefore $\mathcal{F}(A, B)$ equals the image of the map

$$q \mapsto \frac{q^* B^* A q}{|q^* B^* A q|} \frac{\|Aq\|}{\|Bq\|}$$

for unit vectors $q \in \mathbb{C}^n$. Because of continuity of this function on the compact set of unit vectors, the claim follows. $\square$

Somewhat surprisingly, if the assumptions of this theorem are not satisfied, continuity cannot be assured.

EXAMPLE 5. Let $M \in \mathbb{C}^{3 \times 3}$ be unitary with the eigenvalues $-i, i$ and 1 . Then $\mathcal{F}(M, I)$ is the path on the origin centered unit circle with the angles varying from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Now 0 is on the boundary of $\mathcal{F}(M)$. Perturb M such that $M + E$ is still unitary with the eigenvalues $-i$ and i slightly moved to the left half plane. Then $\mathcal{F}(M + E, I)$ is the origin centered unit circle.

4. Normal, Hermitian and unitary generalized eigenvalue problems.

In studying the eigenvalue problem and $\mathcal{F}(M, N)$, we may consider the generically attainable case (2.6). In terms of this, the field of optimal quotients strongly suggests a candidate for the notion of normality for two dimensional matrix subspaces. (For normal matrices, see [4].)

DEFINITION 4.1. A nonsingular matrix subspace (1.3) is said to be left normal if there exist a unitary $U \in \mathbb{C}^{n \times n}$ and an invertible $Y \in \mathbb{C}^{n \times n}$ such that UMY^{-1} and UNY^{-1} are diagonal.

If $\mathcal{V}$ is left normal, then the respective eigenvalue problem (1.1) is said to be left normal. Thus, the notion generalizes the case (2.6) with M being diagonal (or, equivalently, normal).

Correspondingly, $\mathcal{V}$ is said to be right normal if its Hermitian transposed matrix subspace

$$\mathcal{V}^* = \text{span}\{M^*, N^*\} \quad (4.1)$$

is left normal. (Observe that, in general, $\mathcal{F}(M^*, N^*)$ does not equal $\overline{\mathcal{F}(M, N)}$.)

Algebraically, and hence purely operator theoretically, the condition can be formulated as follows.

PROPOSITION 4.2. A nonsingular matrix subspace (1.3) is left normal if and only if AB^{-1} is a normal matrix for some linearly independent linear combinations (3.1) of M and N .

Suppose a matrix $M \in \mathbb{C}^{n \times n}$ is normal with exactly two separate eigenvalues λ_1 and λ_2 . Then, as is well-known, $\mathcal{F}(M)$ is the line segment parametrized as

$$\mathbf{r}(t) = t\lambda_1 + (1-t)\lambda_2, \quad \text{for } t \in [0, 1]. \quad (4.2)$$

In a similar vein, the planar dimension of $\mathcal{F}(M, I)$ vanishes. Compared with $\mathcal{F}(M)$, it “bulges outwards” as follows.

PROPOSITION 4.3. Suppose $M \in \mathbb{C}^{n \times n}$ is normal with exactly two separate eigenvalues λ_1 and λ_2 . Then $\mathcal{F}(M, I)$ is parametrized as

$$\mathbf{R}(t) = \mathbf{r}(t) \frac{\sqrt{|\mathbf{r}(t)|^2 + t(1-t)|\lambda_1 - \lambda_2|^2}}{|\mathbf{r}(t)|}, \quad \text{for } t \in [0, 1],$$

where $\mathbf{r}(t)$ is defined in (4.2).

Proof. We may assume $n = 2$. Use the family of unit vectors $q(t) = \sqrt{t}q_1 + \sqrt{1-t}q_2$, with $t \in [0, 1]$, where q_1 and q_2 are orthonormal eigenvectors of M . We have $q(t)^*Mq(t) = \mathbf{r}(t)$ and $\|Mq(t)\|^2 = t|\lambda_1|^2 + (1-t)|\lambda_2|^2$. Then

$$t|\lambda_1|^2 + (1-t)|\lambda_2|^2 - |\mathbf{r}(t)|^2 = t(1-t)|\lambda_1 - \lambda_2|^2,$$

so that $\|Mq(t)\| = \sqrt{|\mathbf{r}(t)|^2 + t(1-t)|\lambda_1 - \lambda_2|^2}$. $\square$

In applications, Hermitian and unitary problems are common. To have analogous notions for the generalized eigenvalue problem, we require the planar dimension of $\mathcal{F}(A, B)$ to vanish for some linear combinations (3.1) of M and N . This is instrumental in showing that any distinction between Hermitian and unitary problems can be regarded as artificial.

4.1. Hermitian generalized eigenvalue problem. The notion of Hermitian matrix arose in connection with studying the quadratic form

$$q \longmapsto q^*Mq$$

associated with a matrix $M \in \mathbb{C}^{n \times n}$ [8]. (See also [7].) As is well-known, M is said to be Hermitian if and only if the field of values of M is a subset of the real line. That is to say, the location of the field of values exactly determines whether or not a matrix is called Hermitian.

For a matrix subspace (1.3), an analogous notion can be identified in terms of the location of the field of optimal quotients. That is, in deciding whether the generalized eigenvalue problem (1.1) can be regarded as Hermitian, it is necessary to inspect the entire matrix subspace $\mathcal{V}$. Inspecting the pair of matrices M and N alone is not sufficient. To this end, let us proceed algebraically as follows.⁵

DEFINITION 4.4. *A nonsingular matrix subspace (1.3) is said to be left Hermitian if B^*A is a Hermitian matrix for some linearly independent linear combinations (3.1) of M and N .*

Correspondingly, a nonsingular matrix subspace (1.3) is said to be right Hermitian if (4.1) is left Hermitian.

If (1.3) is left Hermitian, then the eigenvalues of the original problem (1.1) are contained in a planar curve as follows.

PROPOSITION 4.5. *If $\mathcal{V}$ is left Hermitian with respect to the linearly independent combinations (3.1), then the eigenvalues of (1.1) belong to $\left\{ \frac{-b+\beta d}{a-\beta c} : \beta \in \mathbb{R} \right\}$.*

Proof. Because of the polar form (1.2), we may conclude that the field of optimal quotients for the eigenvalue problem $Ax = \lambda Bx$ is a subset of $\mathbb{R}$. Thereby using the corresponding linear fractional transformation yields the claim. $\square$

The notion introduced is such that the matrices M and N appearing in the original eigenvalue problem formulation (1.1) can obviously be very “non-Hermitian”. Still, it can be argued that this extends the standard Hermitian eigenvalue problem in a natural manner. First, if B is invertible, then B^*A being Hermitian is equivalent to having $A = HB$ for a Hermitian matrix H . The Hermitian factor $H = AB^{-1}$ is not explicitly available unless the inverse is applied. However, it is implicitly available since matrix-vector products with H in the form

$$Bx \longmapsto HBx$$

⁵Algebraic notions have the advantage that they extend to operators on infinite dimensional spaces.

are computable by separately applying B and A on a vector x . The eigenvalue problem

$$Ax = \lambda Bx \iff (H - \lambda I)Bx = 0 \quad (4.3)$$

is thus equivalent to a standard Hermitian eigenvalue problem such that $\mathcal{F}(H, I) = \mathcal{F}(A, B)$ holds by Theorem 2.3.

Classically Hermitianity is associated with having an eigenvalue problem involving a Hermitian and a positive definite matrix. Definition 4.4 is associated with such problems as follows.

EXAMPLE 6. Suppose $\mathcal{V}$ is left Hermitian with B invertible. Then there exists the option to transform the problem into the equivalent eigenvalue problem

$$B^*Ax = \lambda B^*Bx.$$

This is a classical structure since B^*A is Hermitian and B^*B positive definite. (For the origins of such a structure, see [7, p. 565]. Regarding its computational appearance, in a standard formulation of FEM, the positive definite matrix is the “mass” matrix resulting from the inner products of the finite elements used. See also [16, Chapter 15] and [15].) However, in this transformation we cannot guarantee that $\mathcal{F}(A, B)$ is preserved. In fact, we doubt this transformation is a good idea.

As opposed to having an eigenvalue problem involving a Hermitian and a positive definite matrix, there are significantly more degrees of freedom in terms of the notion of Definition 4.4. That is, by using (4.3), the number is roughly $\frac{3}{2}n^2$ as opposed to n^2 . It is preserved in an equivalence as follows.

PROPOSITION 4.6. *If $U \in \mathbb{C}^{n \times n}$ is unitary and $Y \in \mathbb{C}^{n \times n}$ invertible, then $\mathcal{V}$ is left Hermitian if and only if UVY^{-1} is left Hermitian.*

Definition 4.4 is algebraically (and hence purely operator theoretically) formulated. Its geometrical content is as follows.

THEOREM 4.7. *If $\mathcal{V}$ is left Hermitian, then $\mathcal{V}$ is left normal.*

Proof. If B is invertible, then $A = B^{-*}H$ for a Hermitian matrix H . Therefore applying $X = I$ and $Y = B$ yields an equivalence $XVY^{-1} = \text{span}\{B^{-*}HB^{-1}, I\}$. Since $B^{-*}HB^{-1}$ is Hermitian, it is unitarily similar to a Hermitian matrix which, in turn, is unitarily similar to a diagonal matrix.

If B is singular, then let $B = U\Sigma V^*$ be the singular value decomposition of B . Since $B^*A = H$ is Hermitian, it follows that

$$\Sigma^* \tilde{A} = V^*HV \quad (4.4)$$

is Hermitian, where $\tilde{A} = U^*AV$. Now $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$ with Σ_1 carrying the nonzero singular values of Σ . From (4.4) it follows that $V^*HV = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ \tilde{A}_2 & \tilde{A}_3 \end{bmatrix}$ with invertible $\tilde{A}_1 = \Sigma_1^{-*}H_1$. Since $\mathcal{V}$ is nonsingular, from the block structure of V^*HV it follows that $\tilde{A}_3$ is necessarily invertible. Take $W_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ with $C = -\tilde{A}_3^{-1}\tilde{A}_2$. Then $V^*H VW_1 = V^*HV$ and $\begin{bmatrix} \tilde{A}_1 & 0 \\ \tilde{A}_2 & \tilde{A}_3 \end{bmatrix} W_1 = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_3 \end{bmatrix}$. Then set $W_2 = \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & \tilde{A}_3^{-1} \end{bmatrix}$. We take $X = U^*$ and $Y = W_2^{-1}W_1^{-1}V^*$ to a the required similarity transformation to have $XY^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. One more block unitary similarity transformation is needed, exactly like in the case of invertible B , to diagonalize XAY^{-1} . $\square$

The notion of Hermitianity of matrices fits into this formulation as follows.

COROLLARY 4.8. *A matrix $M \in \mathbb{C}^{n \times n}$ is Hermitian if and only if $\text{span}\{M, I\}$ is a left Hermitian matrix subspace.*

The left and right Hermitianity are connected in terms of a very particular equivalence. To construct this, linearize the inversion operation on $\mathcal{V}$ at an invertible matrix $V \in \mathcal{V}$; see [10, p. 273]. This yields the equivalence

$$V^{-1}\mathcal{V}V^{-1}. \quad (4.5)$$

COROLLARY 4.9. *If $\mathcal{V}$ is left Hermitian and $V \in \mathcal{V}$ is invertible, then (4.5) is right Hermitian.*

Proof. Use the fact that $\mathcal{V}$ is left normal. $\square$

Orthogonality of eigenvectors is certainly a classical manifestation of Hermitianity for matrices. This property is completely preserved.

COROLLARY 4.10. *Assume (1.3) is right Hermitian. If q and $\hat{q}$ are eigenvectors associated with different eigenvalues of the eigenvalue problem (1.1), then q and $\hat{q}$ are orthogonal.*

Proof. If (1.3) is right Hermitian, then $M = Y^{-1}D_1U$ and $N = Y^{-1}D_2U$ for a nonsingular Y and a unitary matrix U . The claim follows by the unitarity of U . $\square$

Correspondingly, if (1.3) is left Hermitian, then we have to consider (4.1), i.e., eigenvectors associated with different eigenvalues of the eigenvalue problem

$$M^*x = \lambda N^*x, \quad (4.6)$$

are orthogonal. Alternatively, we can state this fact as follows.

COROLLARY 4.11. *Assume (1.3) is left Hermitian with respect to the linearly independent combinations (3.1). If q and $\hat{q}$ are eigenvectors associated with different eigenvalues of the eigenvalue problem (1.1), then*

$$(A + iB)q \text{ and } (A + iB)\hat{q}$$

are orthogonal.

Proof. Now $(A + iB)(A - iB)^{-1}$ is a unitary matrix. Moreover, it is immediate that y is an eigenvector of this matrix if and only if $(A - iB)^{-1}y$ is an eigenvector of the original eigenvalue problem (1.1). $\square$

For a further support that the notion of Hermitianity is arguable, let us show that then the eigenvalue problem can be solved by minimizing an appropriate quotient of norms. (For the classical PDE formulation concerning the principal eigenvalue in terms of the Rayleigh quotient, see, e.g., [2, p. 336]. See also [1, pp. 78–79].)

COROLLARY 4.12. *Assume (1.3) is left Hermitian with respect to the linearly independent combinations (3.1). If there is a unique eigenvalue of the least modulus for the eigenvalue problem $Ax = \lambda Bx$, then $q \neq 0$ solving*

$$\min_{q \neq 0} \frac{\|Aq\|}{\|Bq\|} \quad (4.7)$$

yields an eigenvector corresponding to that eigenvalue.

Proof. We have $A = UD_1Y^{-1}$ and $B = UD_2Y^{-1}$ for a unitary matrix U and an invertible matrix Y . The matrices $D_1 = \text{diag}(d_1, \dots, d_n)$ and $D_2 = \text{diag}(c_1, \dots, c_n)$ are Hermitian and diagonal. We may choose Y in such a way that D_2 has ones and possibly zeros on its diagonal. (Then D_1 has the eigenvalues on its diagonal.)

Moreover, we have $\frac{\|Aq\|}{\|Bq\|} = \frac{\|D_1w\|}{\|D_2w\|} \geq \frac{\|D_1w\|}{\|w\|}$ which is bounded below by the smallest eigenvalue in modulus. This lower bound is attained with the corresponding eigenvector q , so that $\frac{q^*B^*Aq}{|q^*B^*Aq|} \frac{\|Aq\|}{\|Bq\|}$ is the eigenvalue nearest to the origin. $\square$

Taking the squares of (4.7) yields $\min_{q \neq 0} \frac{(q, A^* A q)}{(q, B^* B q)}$ which, by being a ratio of two quadratic forms, bears more resemblance to the Rayleigh quotient. Of course, now taking the square root (with appropriate sign) is required to have the eigenvalue.

Unlike with Rayleigh quotients, we can find gaps in the spectrum. That is, to locate “interior” eigenvalues, one can proceed entirely similarly by solving extremum problems.

EXAMPLE 7. To minimize for other eigenvalues, one can translate by $\mu \in \mathbb{R}$ and consider the eigenvalue problem

$$(A - \mu B)x = \lambda Bx.$$

Since $B^*(A - \mu B)$ is Hermitian if B^*A is, Corollary 4.12 can be applied accordingly. In doing this, one can also interchange the roles of A and B and still preserve left Hermitianity. Then use the fact that

$$\mathcal{F}(B, A) = \frac{1}{\mathcal{F}(A, B)};$$

see [11, Theorem 2.2]

Most notably, “max-min” or “min-max” subspace constructions are not necessary to find eigenvalues “inside” the spectrum.

4.2. Unitary generalized eigenvalue problem. Hermitian and unitary matrices are intimately connected.⁶ This can be seen also in terms of the field of optimal quotients since the polar form (1.2) hints at another instance of vanishing planar dimension. We also argue that these two instances are indistinguishable.

DEFINITION 4.13. *The matrix subspace (1.3) is said to be left unitary if, for some linearly independent linear combinations (3.1) of M and N ,*

$$\frac{\|Aq\|}{\|Bq\|} = 1$$

for any $q \neq 0$.

This means that both A and B are necessarily nonsingular. The condition implies that $\mathcal{F}(A, B)$ is a subset of the unit circle. Observe that, by the fact that any matrix is a linear combination of two unitary matrices, M and N can be very “non-unitary”. Therefore the structure cannot be immediately identified.

PROPOSITION 4.14. *If the matrix subspace (1.3) is left unitary, then it is left normal.*

Proof. Take the polar decompositions $A = Q_1 P_1$ and $B = Q_2 P_2$ of A and B . Thus, we have $\|P_1 q\| = \|P_2 q\|$ for every $q \in \mathbb{C}^n$. This condition implies $P_1 = P_2$. By the generalized Schur decomposition, there exist unitary U and V such that $UQ_1 V^*$ and $UQ_2 V^*$ are upper triangular. Since both of these matrices are unitary, and thus normal, they must actually be diagonal. Therefore $UAP_1^{-1}V^*$ and $UBP_1^{-1}V^*$ are diagonal, so that $\mathcal{V}$ is left normal. $\square$

The Cayley transformation can be formulated in terms of taking linear combinations (3.1) of M and N . This allows interpreting the left Hermitian and the left unitary structures as entirely indistinguishable.

⁶The exponential function or the Cayley transformation are typically used to provide a connection.

THEOREM 4.15. *Assume (1.3) is nonsingular. The eigenvalue problem (1.1) is left Hermitian if and only if it is left unitary.*

Proof. In both cases we know that (1.3) is normal. Therefore it suffices to consider the location of the eigenvalues.

Assume (1.1) is left Hermitian. We may, without loss of generality, assume N^*M is Hermitian. Then, by using the linear fractional transformation $f(z) = \frac{z+i}{z-i}$, choose $a = 1$, $b = i$, $c = 1$ and $d = -i$ to have linear combinations (3.1) for which the eigenvalues are contained in the unit circle. From this it follows that (1.3) is left unitary.

For the converse, if (1.3) is left unitary, then we may, without loss of generality, assume $\frac{\|Mq\|}{\|Nq\|} = 1$ for any $q \neq 0$. Then choose $a = 1$, $b = i$, $c = i$ and $d = 1$ to have a left Hermitian eigenvalue problem. $\square$

This is undoubtedly useful, e.g., for devising numerical methods since no distinction should be made between the left Hermitian and the left unitary eigenvalue problems. For the standard eigenvalue problem this means the following.

EXAMPLE 8. Suppose the matrix subspace (1.3) is such that M is a Hermitian matrix and $N = I$. Then (1.1) is the usual Hermitian eigenvalue problem formulation. However, if we take the basis of $\mathcal{V}$ to be

$$A = M + iI \text{ and } B = M - iI$$

instead, then $Ax = \lambda Bx$ corresponds to the unitary eigenvalue problem formulation. (To associate this with a classical concept, one may then perform an equivalence transformation of $\mathcal{V}$ with $X = I$ and $Y = B$. This gives rise to the standard unitary eigenvalue problem and the notion of Cayley transformation.) Hence there is absolutely no difference between these problems, the distinction is just a matter of the choice of a basis.

In view of this and Example 7, the “basis problem” for a given eigenvalue problem consists of choosing a basis of $\mathcal{V}$ which can, in some sense, be regarded as best. It is not a trivial matter.

Conclusions. The field of optimal quotients is a set containing the spectrum of a given eigenvalue problem. Its non-convexity admits recovering the spectrum exactly, by varying the basis of $\mathcal{V}$. Although non-convex in general, this set is shown to possess convexity properties which allow drawing conclusions about its structure. The field of values of a matrix is recoverable from the field of optimal quotients in a limit process at infinity. Based on the vanishing of the planar dimension of the field of optimal quotients, a serious claim can be made how Hermitian (equiv. unitary) eigenvalue problems should be formulated.

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