Group Sparsity Residual Constraint for Image Denoising with External Nonlocal Self-Similarity Prior

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Abstract

Nonlocal image representation has been successfully used in many image-related inverse problems including denoising, deblurring and deblocking. However, most existing methods only consider the nonlocal self-similarity (NSS) prior of degraded observation image, and few methods use the NSS prior from natural images. In this paper we propose a novel method for image denoising via group sparsity residual constraint with external NSS prior (GSRC-ENSS). Different from the previous NSS prior-based denoising methods, two kinds of NSS prior (e.g., NSS priors of noisy image and natural images) are used for image denoising. In particular, to enhance the performance of image denoising, the group sparsity residual is proposed, and thus the problem of image denoising is translated into reducing the group sparsity residual. Because the groups contain a large amount of NSS information of natural images, to reduce the group sparsity residual, we obtain a good estimation of the group sparse coefficients of the original image by the external NSS prior based on Gaussian Mixture Model (GMM) learning, and the group sparse coefficients of noisy image are used to approximate the estimation. To combine these two NSS priors better, an effective iterative shrinkage algorithm is developed to solve the proposed GSRC-ENSS model. Experimental results demonstrate that the proposed GSRC-ENSS not only outperforms

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several state-of-the-art methods, but also delivers the best qualitative denoising results with finer details and less ringing artifacts.

Keywords: Image denoising, group sparsity residual constraint, nonlocal self-similarity, Gaussian Mixture Model, iterative shrinkage algorithm.

1. Introduction

Image denoising is not only an important problem in various image processing studies, but also an idea test bed for measuring the statistical image modeling techniques. It has attracted a lot of research interest in the past few decades [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Image denoising aims to estimate the latent clean image $\mathbf{X}$ from its noisy observation $\mathbf{Y} = \mathbf{X} + \mathbf{V}$, where $\mathbf{V}$ is usually assumed to be additive white Gaussian noise. Due to the ill-posed nature of image denoising, it is critical to exploit the prior knowledge that characterizes the statistical features of the images.

Previous models mainly employed the priors on level of pixel, such as Tikhonov regularization [19], total variation (TV) regularization [2]. These methods are effective in removing the noise artifacts but smear out details and tend to over-smooth the images.

Recently, patch-based prior has shown promising performance in image denoising. One representative example is sparse coding based scheme, which assumes that each patch of an image can be precisely modeled as a sparse linear combination of basic elements. These elements are called atoms and they compose a dictionary. As such, one key issue of sparse coding based scheme is to train a dictionary from natural images. Popular techniques for dictionary learning are KSVD [20], ODL [21] and SDL [22]. Gaussian mixture models (GMMs) have been successfully employed as prior for natural image patches [7, 8] and its connections with block sparse representation have been explored [8]. In [8], it is observed that the estimation of an image patch from its corresponding noisy observation under GMM model with K Gaussian components. However, patch-based model usually suffers from some limits, such as great computational

26 complexity, neglecting the relationship among similar patches [23, 24].

27 Inspired by the fact that natural images contain a large number of mutually
28 similar patches at different locations, this so-called nonlocal self-similarity (NSS)
29 prior was initially utilized in the work of nonlocal means denoising [1]. Due to
30 its effectiveness, a large amount of further developments [11, 12, 13, 14, 23,
31 24] have been proposed. For instance, a very popular method is BM3D [12],
32 which exploited nonlocal similar 2D image patches and 3D transform domain
33 collaborative filtering. In [13], Marial *et al.* advanced the idea of NSS by group
34 sparse coding. LPG-PCA [14] utilized nonlocal similar patches as data samples
35 to estimate statistical parameters for PCA training. However, all the methods
36 above only exploited the NSS prior of noisy image. In other words, for a given
37 patch, its similar patches are collected only from degraded input, which makes
38 the quality of image denoising largely depend on the input itself.

39 In fact, most existing methods only focused on the input but ignored the
40 NSS prior of clean natural images. As we know, the information from image
41 dataset with a large number of clean images is far richer than the noisy input
42 image. Mention that the NSS prior from natural images was firstly developed in
43 [11], and achieved state-of-the-art denoising performance, but how to explicitly
44 learn the priors from input and natural images is still an open issue. To the best
45 of our knowledge, few works have exploited the NSS priors from both input and
46 natural images.

47 With the above considerations, in this work we propose a novel method for
48 image denoising via group sparsity residual constraint with external nonlocal
49 NSS prior (GSRC-ENSS). The major contributions of this paper are summarized
50 as follows. First, to improve the performance of image denoising, we propose
51 the concept of group sparsity residual, and thus the problem of image denoising
52 is turned into reducing the group sparsity residual. Second, due to the fact that
53 the groups contain a large amount of NSS information of natural images, to
54 reduce the group sparsity residual, we obtain a good estimation of the group
55 sparse coefficients of the original image by the NSS prior of natural images based
56 on Gaussian Mixture Model (GMM) learning, and the group sparse coefficients

of noisy image are used to approximate the estimation. The flowchart of the proposed method is illustrated in Fig. 1. To fuse these two NSS priors better, we present an effective iterative shrinkage algorithm to solve the proposed GSRC-ENSS model. Our experimental results demonstrate that the proposed GSRC-ENSS model outperforms many state-of-the-art methods. What’s more, the proposed GSRC-ENSS delivers the best qualitative denoising results with finer details and less ringing artifacts.

The reminder of this paper is organized as follows. Section 2 provides a brief survey of related work. Section 3 introduces the modeling of group sparsity residual constraint (GSRC), learns the NSS prior from natural image by GMM and presents an effective iterative shrinkage algorithm to solve the GSRC-ENSS model. Section 4 discusses the main difference among the GSRC-ENSS method, the NCSR method [25], most existing NSS prior-based denoising methods and the popular methods combining external and internal priors. Section 5 presents the experimental results. Finally, several concluding remarks are given in Section 6.

2. Related Work

Image denoising is a typical ill-posed inverse problem. To solve such ill-posed problem, it is of great importance to exploit the prior knowledge we know about the original image so that we can regularize the solution and various image prior models have been developed [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

Early prior models mainly consider the local correlation among image pixels. One representative work is the total variation (TV) regularization method [2], which usually uses the local structure information and high effectiveness to recover the smooth regions. The TV regularization method actually assumes that natural image gradients have a heavy-tailed distribution, which can be fitted by Laplacian or hyper-Laplacian models [26, 27]. In addition, many statistical models of wavelet coefficients have also been proposed, such as generalized

Figure 1: Flowchart of image denoising by group sparsity residual constraint with external NSS prior.

86 Gaussian [28] and Gaussian scale mixture models [29], etc.

87 Besides the above prior models, another significant property of natural im-
88 ages is to model the prior on patches. One representative example is sparse
89 coding based scheme [6, 20], which assumes that each patch of an image can be
90 perfectly represented by a sparse coefficient vector whose entries are mostly zero
91 or close to zero based on a basis set called as dictionary. The dictionary is usual-
92 ly learned from a natural image dataset [6, 20]. Compared with the conventional
93 analytically designed dictionaries, such as those based on wavelet, curvelet and
94 DCT, dictionaries learned directly from images have an advantage of being bet-
95 ter adapted to image local structures [30], and thus could enhance the sparsity
96 which leads to better performance. For instance, the seminal of KSVD dic-
97 tionary learning [20] has not only achieved promising denoising performance,
98 but also extended and successfully exploited it in various image processing and
99 computer vision tasks [31, 32, 33, 34].

100 Considering that natural images are non-Gaussian and image patches are
101 regarded as samples of a multivariate vector, Gaussian mixture models (GMM-
102 s) have emerged as favored prior for natural image patches in various image
103 restoration studies [7, 8, 9]. For instance, Zoran *et al.* [7] exploited GMM mod-
104 el to learn image patches from clean natural image dataset, and proposed a
105 method which can reconstruct the latent image by maximizing the expected
106 patch log likelihood. Yu *et al.* [8] estimated the latent clean image from its
107 noisy observation under GMM model with K Gaussian components. Niknejad
108 *et al.* [9] used GMM model with spatially constraint patch clustering to recover
109 the latent image from noisy image.

110 Motivated by the fact that image patches that have similar structures can
111 be spatially far from each other and thus can be collected in the whole image,
112 this so-called nonlocal self-similarity (NSS) prior is among the most remarkable
113 priors for image restoration. The seminal work of nonlocal means (NLM) [1]
114 utilized the NSS prior to implement a form of the weighted filtering for image
115 denoising. After this, inspired by the success of the NLM denoising filtering,
116 a flurry of NSS based methods [11, 12, 13, 14, 23, 24, 25, 27, 30, 35] were

117 proposed to solve various image inverse problems. Among these NSS based
 118 schemes, the famous BM3D method [12] exploited nonlocal similar patches and
 119 3D transform domain collaborative filtering, which can achieve state-of-the-
 120 art performance in image denoising. Dong *et al.* [25] proposed the nonlocally
 121 centralized sparse representation (NCSR) model for image restoration, which
 122 obtained sparse coding coefficients estimate of the original image by the principle
 123 of NLM [1], and then according to those estimates, NCSR, centralized the sparse
 124 coding coefficients of the observed image to improve the denoising performance.
 125 Xu *et al.* [11] proposed a patch group based on NSS scheme to learn NSS models
 126 from natural images for high performance denosing.

127 Motivated by the observation that the nonlocal similar patches (forming a
 128 matrix) in a natural image are linearly correlated with each other, in [24, 36, 37],
 129 the authors transformed the image denoising into the problem of low rank matrix
 130 approximation, which achieved highly competitive recovery results.

131 Different image priors describe different and complementary aspects of nat-
 132 ural image statistics, and it is possible to combine multiple priors to improve
 133 the denoising performance. For instance, Zhang *et al.* [38] proposed joint statis-
 134 tical modeling for various image inverse problems. Mosseri *et al.* [39] exploited
 135 a patch-SNR as a metric to decide whether a noisy patch is denoised using in-
 136 ternal priors or external priors. Yue *et al.* [40] used a frequency domain method
 137 to fuse the internal and external denoising results. Lu *et al.* [41] proposed a
 138 unified method that learns a generic prior from external images and adapts the
 139 prior to a specific images. Luo *et al.* [42] proposed a prior learning algorithm
 140 that combines the internal and external priors from a Bayesian hyper-prior
 141 perspective.

142 Very recently, deep learning based methods have been proposed for image
 143 denoising tasks [43, 44, 45, 46, 47, 48]. For example, Jain *et al.* [43] proposed
 144 to use convolutional neural network (CNN) for natural image denoising and
 145 claimed that CNN has similar or even better representation power than Markov
 146 random field (MRF) model [49]. Burger *et al.* [45] proposed a patch-based de-
 147 noising algorithm that is learned on a large dataset with a plain neural network.

148 Liu *et al.* [46] considered the denoising problem as recursive image filtering via a
 149 hybrid neural network. Mao *et al.* [48] proposed a very deep fully convolutional
 150 encoding-decoding framework for image denoising.

151 **3. Group Sparsity Residual Constraint for Image Denoising with Ex-** 152 **ternal Nonlocal Self-Similarity Prior**

153 As we know, a majority of existing methods only consider the nonlocal self-
 154 similarity (NSS) prior of noisy input image, and few methods use the NSS prior
 155 from natural images. In this section, two kinds of NSS prior (i.e., NSS priors of
 156 noisy image and natural images) are exploited for image denoising. To improve
 157 the performance of image denoising, we propose the concept of group sparsity
 158 residual, and thus the problem of image denoising is turned into reducing the
 159 group sparsity residual. To this end, a good estimation of the group sparse
 160 coefficients of the original image are obtained by the external NSS prior based
 161 on GMM learning, and the group sparse coefficients of noisy image are used to
 162 approximate the estimation. To fuse these two NSS priors better, we present
 163 an iterative shrinkage algorithm to solve the proposed scheme.

164 *3.1. Group-based sparse coding*

Recent advances [12, 13, 23, 50, 51, 52] have suggested that structured or
 group sparsity can offer powerful reconstruction performance for image denois-
 ing. More specifically, given a clean image $\mathbf{X}$, for each image patch $\mathbf{x}_i$ of size
 $d \times d$ in $\mathbf{X}$, $i = 1, 2, \dots, n$, its m best matched patches are selected from a $W \times W$
 sized search window to form a group $\mathbf{X}_i$, denoted by $\mathbf{X}_i = \{\mathbf{x}_{i,1}, \mathbf{x}_{i,2}, \dots, \mathbf{x}_{i,m}\}$,
 where $\mathbf{x}_{i,m}$ denotes the m -th similar patch (column vector) of the i -th group.
 Similar to patch-based sparse coding [6, 20], given a dictionary $\mathbf{D}_i$, each group
 $\mathbf{X}_i$ can be sparsely represented as $\mathbf{B}_i = \mathbf{D}_i^{-1} \mathbf{X}_i$ and solved by the following
 ℓ_p -norm minimization problem,

$$\mathbf{B}_i = \arg \min_{\mathbf{B}_i} \{ \|\mathbf{X}_i - \mathbf{D}_i \mathbf{B}_i\|_F^2 + \lambda_i \|\mathbf{B}_i\|_p \} \quad (1)$$

165 where $\|\bullet\|_F^2$ denotes the Frobenious norm, λ_i is the regularization parame-
 166 ter, and p characterizes the sparsity of $\mathbf{B}_i$. Then the whole image $\mathbf{X}$ can be
 167 represented by the set of group sparse codes $\{\mathbf{B}_i\}$.

In image denoising, each noise patch $\mathbf{y}_i$ is extracted from the noisy image $\mathbf{Y}$. Like in the above paragraph, we search for its similar patches to generate a group $\mathbf{Y}_i$, i.e., $\mathbf{Y}_i = \{\mathbf{y}_{i,1}, \mathbf{y}_{i,2}, \dots, \mathbf{y}_{i,m}\}$. Thus, image denoising is turned into how to reconstruct $\mathbf{X}_i$ from $\mathbf{Y}_i$ by using group sparse coding,

$$\mathbf{A}_i = \arg \min_{\mathbf{A}_i} \{\|\mathbf{Y}_i - \mathbf{D}_i \mathbf{A}_i\|_F^2 + \lambda_i \|\mathbf{A}_i\|_p\} \quad (2)$$

168 Once all group sparse codes $\{\mathbf{A}_i\}$ are obtained, the latent clean image $\mathbf{X}$
 169 can be reconstructed as $\hat{\mathbf{X}} = \mathbf{D}\mathbf{A}$, where the group sparse code $\mathbf{A}$ includes the
 170 set of $\{\mathbf{A}_i\}$.

171 3.2. Group Sparsity Residual Constraint

Although group sparsity has demonstrated its effectiveness in image denois-
 ing, due to the influence of noise, it is very difficult to estimate the true group
 sparse code $\mathbf{B}$ from noisy image $\mathbf{Y}$. That is, the group sparse code $\mathbf{A}$ achieved
 by solving Eq. (2) is expected to be close enough to the true group sparse code
 $\mathbf{B}$ of the original image $\mathbf{X}$ in Eq. (1). Therefore, the quality of image denoising
 largely depends on the level of the group sparsity residual, which is defined as
 the difference between group sparse code $\mathbf{A}$ and true group sparse code $\mathbf{B}$,

$$\mathbf{R} = \mathbf{A} - \mathbf{B} \quad (3)$$

Therefore, to reduce the group sparsity residual $\mathbf{R}$ and enhance the accuracy
 of $\mathbf{A}$, we propose the group sparsity residual constraint (GSRC) [53] to image
 denoising, and thus Eq. (2) can be rewritten as

$$\mathbf{A}_i = \arg \min_{\mathbf{A}_i} \{\|\mathbf{Y}_i - \mathbf{D}_i \mathbf{A}_i\|_F^2 + \lambda_i \|\mathbf{A}_i - \mathbf{B}_i\|_p\} \quad (4)$$

172 However, it can be seen that the true group sparse code $\mathbf{B}$ and p are unknown
 173 because the original image $\mathbf{X}$ is not available. We now discuss how to obtain
 174 $\mathbf{B}$ and p .

175 3.3. Learning the NSS Prior from Natural Images by GMM

176 Since the original image $\mathbf{X}$ is not available, it seems to be difficult to obtain
 177 the true group sparse code $\mathbf{B}$. Nonetheless, we can obtain some good estimation
 178 of $\mathbf{B}$. Generally speaking, there are various methods to estimate the true group
 179 sparse code $\mathbf{B}$, which depends on prior knowledge of the original image $\mathbf{X}$ we
 180 have. In recent years, patch-based or group-based priors referring to denoising
 181 operators learned from natural images achieved the state-of-the-art denoising
 182 results [6, 7, 11, 20]. For instance, in [6], a dictionary learning-based method is
 183 introduced for compact patch representation, whereas in [11], a GMM model is
 184 learned from natural image groups based on NSS scheme and used as a prior
 185 for denoising. Due to the fact that the groups contain a rich amount of NSS
 186 information of natural images, we can achieve the group sparse code $\mathbf{B}$ by the
 187 NSS prior of natural images based on GMM learning.

Like in subsection 3.1, we extract N groups from a given clean natural image dataset, and we denote one group as

$$\overline{\mathbf{Z}}_i = \{\bar{\mathbf{z}}_{i,j}\}_{j=1}^m, \quad i = 1, 2, \dots, N \quad (5)$$

where $\overline{\mathbf{Z}}_i$ ¹ is the group mean subtraction of each group $\mathbf{Z}_i$ and $\bar{\mathbf{z}}_{i,j}$ denotes the j -th similar patch (column vector) of the i -th group. Since GMM has been successfully used to model the image patch or group priors such as EPLL [7], PLE [8] and PGPD [11], we adopt the strategy in [11] and learn a finite GMM over natural image groups $\{\overline{\mathbf{Z}}_i\}$ as group priors. By using the GMM model, the likelihood of a given group $\{\overline{\mathbf{Z}}_i\}$ is:

$$P(\overline{\mathbf{Z}}_i) = \sum_{k=1}^K \pi_k \prod_{j=1}^m \mathcal{N}(\bar{\mathbf{z}}_{i,j} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \quad (6)$$

where K is the total number of mixture components selected from the GMM, and the GMM model is parameterized by mean vectors $\{\boldsymbol{\mu}_k\}$, covariance matrices $\{\boldsymbol{\Sigma}_k\}$, and mixture weights of mixture components $\{\pi_k\}$. By assuming

¹The advantage of group mean subtraction is that it can further facilitate the NSS prior learning because the possible number of patterns is reduced, while the training samples of each pattern are increased.

that all the groups are independent, the overall objective likelihood function is $\mathcal{L} = \prod_{i=1}^N P(\bar{\mathbf{Z}}_i)$. Then by applying log to it, we maximize the objective function as:

$$\ln \mathcal{L} = \sum_{i=1}^N \ln \left(\sum_{k=1}^K \pi_k \prod_{j=1}^m \mathcal{N}(\bar{\mathbf{z}}_{i,j} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \right) \quad (7)$$

188 We collectively represent three parameters $\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k$ and π_k by $\boldsymbol{\Theta} = \{\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k, \pi_k\}_{k=1}^K$,
 189 and $\boldsymbol{\Theta}$ is learned using Expectation Maximization algorithm (EM) [54].

In the **E-step**, we calculate the posterior probability for the component k as:

$$P(k | \bar{\mathbf{z}}_{i,j}, \boldsymbol{\Theta}) = \frac{\pi_k \prod_{j=1}^m \mathcal{N}(\bar{\mathbf{z}}_{i,j} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_{l=1}^K \pi_l \prod_{j=1}^m \mathcal{N}(\bar{\mathbf{z}}_{i,j} | \boldsymbol{\mu}_l, \boldsymbol{\Sigma}_l)} \quad (8)$$

$$N_k = \sum_{i=1}^N P(k | \bar{\mathbf{z}}_{i,j}, \boldsymbol{\Theta}) \quad (9)$$

In the **M-step**, for each group $\bar{\mathbf{Z}}_i$, we update the model parameters as follows:

$$\pi_k = N_k / N \quad (10)$$

$$\boldsymbol{\mu}_k = \frac{\sum_{i=1}^N \pi_k \sum_{j=1}^m \bar{\mathbf{z}}_{i,j}}{\sum_{i=1}^N \pi_k} \quad (11)$$

$$\boldsymbol{\Sigma}_k = \frac{\sum_{i=1}^N P(k | \bar{\mathbf{z}}_{i,j}, \boldsymbol{\Theta}) \bar{\mathbf{z}}_{i,j} \bar{\mathbf{z}}_{i,j}^T}{N_k} \quad (12)$$

190 We iterate over the E-Step and M-Step until convergence. For more details
 191 about EM algorithm, please refer to [54].

Thus, for each noisy group $\mathbf{Y}_i$ ² of noisy image $\mathbf{Y}$, the best suitable Gaussian component is selected from this group-based GMM learning stage. To be concrete, assume that the image is corrupted by the Gaussain white noise with variance σ^2 , then the covariance matrix of the k -th Gaussian component will

²In this work, all noisy groups are preprocessed by mean subtraction. The mean $\boldsymbol{\mu}_i$ of each noisy group $\mathbf{Y}_i$ is very close to the mean of the original group $\mathbf{X}_i$ because the mean of noise $\mathbf{V}$ is nearly zero. Thus, the mean $\boldsymbol{\mu}_i$ can be added back to the denoised group $\hat{\mathbf{X}}_i$ to achieve the latent clean image $\hat{\mathbf{X}}$.

turn into $\Sigma_k + \sigma^2 \mathbf{I}$, where $\mathbf{I}$ represents the identity matrix. The selection that $\mathbf{Y}_i$ belongs to the k -th Gaussian component can be accomplished by computing the following posterior probability,

$$P(k|\mathbf{Y}_i) = \frac{\prod_{j=1}^m \mathcal{N}(\mathbf{y}_{i,j}|\mathbf{0}, \Sigma_k + \sigma^2 \mathbf{I})}{\sum_{l=1}^K \prod_{j=1}^m \mathcal{N}(\mathbf{y}_{i,j}|\mathbf{0}, \Sigma_l + \sigma^2 \mathbf{I})} \quad (13)$$

192 We maximize it, and finally, the Gaussian component with the highest prob-
193 ability is selected to operate each group $\mathbf{Y}_i$.

Then, we assume that the k -th Gaussian component is selected for the group $\mathbf{Y}_i$. Actually, here GMM model is equivalent to the block sparse estimation with a block dictionary having K blocks wherein each block corresponds to the principle component analysis (PCA) [30] basis of one of the Gaussian components in the mixture [8, 55]. Thus, the covariance matrix of the k -th Gaussian component is denoted by Σ_k . By using singular value decomposition to Σ_k , we have

$$\Sigma_k = \mathbf{U}_k \mathbf{\Lambda}_k \mathbf{U}_k^T \quad (14)$$

194 where $\mathbf{U}_k$ is an orthonormal matrix formed by the eigenvector of Σ_k and $\mathbf{\Lambda}_k$
195 is the diagonal matrix of eigenvalues. With the group-based GMM learning, as
196 the statistical structures of NSS variations in natural image are captured by the
197 eigenvectors in $\mathbf{U}_k$, thus $\mathbf{U}_k$ can be used to represent the structural variations
198 of the groups in that component. Finally, for each group $\mathbf{Y}_i$, each true group
199 sparse code $\mathbf{B}_i$ can be estimated by $\mathbf{B}_i = \mathbf{U}_k^{-1} \mathbf{Y}_i$.

Similar to $\mathbf{B}_i$, the covariance matrix of each group $\mathbf{Y}_i$ is defined as Σ_i and we have

$$\Sigma_i = \mathbf{D}_i \mathbf{\Lambda}_i \mathbf{D}_i^T \quad (15)$$

200 where $\mathbf{D}_i$ is an orthonormal matrix formed by the eigenvector of Σ_i and $\mathbf{\Lambda}_i$ is
201 the diagonal matrix of eigenvalues. Thus, $\mathbf{A}_i$ can be solved by $\mathbf{A}_i = \mathbf{D}_i^{-1} \mathbf{Y}_i$.

202 3.4. How to determine p

Besides estimating $\mathbf{B}$, we also need to determine the value of p . Here we perform some experiments to investigate the statistical property of $\mathbf{R}$, where $\mathbf{R}$

Figure 2: The distribution of $\mathbf{R}$, fitting Gaussian, Laplacian and hyper-Laplacian distribution for image *Monarch* with $\sigma = 30$ in (a) and *foreman* with $\sigma = 100$ in (b).

denotes the set of $\mathbf{R}_i = \mathbf{A}_i - \mathbf{B}_i$. In these experiments, image *Monarch* and *foreman* are used as examples, where Gaussian white noise is added to the images *Monarch* and *foreman* with standard deviation $\sigma = 30$ and $\sigma = 100$, respectively. We plot the histogram of $\mathbf{R}$ as well as the fitting Gaussian, Laplacian and hyper-Laplacian distribution in the log domain in Fig. 2(a) and Fig. 2(b), respectively. It can be seen that the histogram of $\mathbf{R}$ can be well characterized by the Laplacian distribution. Thus, the ℓ_1 -norm is adopted to regularize $\mathbf{R}_i$, and Eq. (4) can be rewritten as

$$\begin{aligned} \mathbf{A}_i &= \arg \min_{\mathbf{A}_i} \{ \|\mathbf{Y}_i - \mathbf{D}_i \mathbf{A}_i\|_F^2 + \lambda_i \|\mathbf{A}_i - \mathbf{B}_i\|_1 \} \\ &= \arg \min_{\tilde{\boldsymbol{\alpha}}_i} \{ \|\tilde{\mathbf{y}}_i - \tilde{\mathbf{D}}_i \tilde{\boldsymbol{\alpha}}_i\|_2^2 + \lambda_i \|\tilde{\boldsymbol{\alpha}}_i - \tilde{\boldsymbol{\beta}}_i\|_1 \} \end{aligned} \quad (16)$$

where $\tilde{\mathbf{y}}_i$, $\tilde{\boldsymbol{\alpha}}_i$, and $\tilde{\boldsymbol{\beta}}_i$ denote the vectorization of the matrix $\mathbf{Y}_i$, $\mathbf{A}_i$ and $\mathbf{B}_i$, respectively. Each column $\tilde{\mathbf{d}}_j$ of the matrix $\tilde{\mathbf{D}}_i = [\tilde{\mathbf{d}}_1, \tilde{\mathbf{d}}_2, \dots, \tilde{\mathbf{d}}_J]$ denotes the vectorization of the rank-one matrix, where J denotes the number of dictionary atoms.

207 3.5. Algorithm of GSRC-ENSS

For fixed $\tilde{\beta}_i$ and λ_i , it can be seen that Eq. (16) is convex and can be solved efficiently by using some iterative thresholding algorithms. We adopt the surrogate algorithm in [56] to solve Eq. (16). In the $t+1$ -iteration, the proposed shrinkage operator can be calculated as

$$\tilde{\alpha}_i^{t+1} = \mathbf{S}_{\lambda_i}(\tilde{\mathbf{D}}_i^{-1} \hat{\mathbf{x}}_i^t - \tilde{\beta}_i) + \tilde{\beta}_i \quad (17)$$

208 where $\mathbf{S}_{\lambda_i}(\cdot)$ is the soft-thresholding operator, $\hat{\mathbf{x}}_i$ represents the vectorization of
 209 the i -th reconstructed group $\hat{\mathbf{X}}_i$. In fact, according to Eq. (17), it can be seen
 210 that these two NSS priors can be better integrated into this surrogate algorithm.
 211 The above shrinkage operator follows the standard surrogate algorithm, and
 212 more details can be seen in [56].

The parameter λ_i for each group that balances the fidelity term and the regularization term should be adaptively determined for better denoising performance. Inspired by [57], the regularization parameter λ_i of each group $\mathbf{Y}_i$ is set as

$$\lambda_i = \frac{c * 2\sqrt{2}\sigma^2}{\sigma_i} \quad (18)$$

213 where σ_i denotes the estimated variance of each group sparsity residual $\mathbf{R}_i$, and
 214 c is a small constant.

215 After obtaining the solution $\mathbf{A}_i$ in Eq. (17), the clean group $\mathbf{X}_i$ can be
 216 reconstructed as $\hat{\mathbf{X}}_i = \mathbf{D}_i \mathbf{A}_i$. Then the clean image $\hat{\mathbf{X}}$ can be reconstructed
 217 by aggregating all the groups $\{\hat{\mathbf{X}}_i\}$.

Besides, we could execute the above denoising procedures for better results after several iterations. In the t -th iteration, the iterative regularization strategy [58] is used to update the estimation of noise variance. Then the standard deviation of noise in t -th iteration is adjusted as

$$\sigma^{t+1} = \gamma * \sqrt{(\sigma^2 - \|\mathbf{Y} - \hat{\mathbf{X}}^t\|_2^2)} \quad (19)$$

218 where γ is a constant.

219 According to the above analysis, it can be seen that the proposed GSRC-
 220 ENSS employs the group sparsity residual with external NSS prior for image
 221 denoising. The proposed denoising procedure is summarized in Table 1.

Table 1: Image denoising via GSRC-ENSS

Input:	Noisy image $\mathbf{Y}$ and Group-based GMM learning model.
Initialization:	$\hat{\mathbf{X}} = \mathbf{Y}, c, m, d, W, K, \sigma, \gamma, \rho;$
For	$t = 1, 2, \dots, Iter$ do
	Iterative regularization $\mathbf{Y}^{t+1} = \hat{\mathbf{X}}^t + \rho(\mathbf{Y} - \hat{\mathbf{X}}^t);$
	Re-estimate σ^{t+1} computing by Eq. (19);
For	each patch $\mathbf{y}_i$ in $\mathbf{Y}$ do
	Find a group $\mathbf{Y}_i^{t+1}$ for each patch $\mathbf{y}_i$.
	The best Gaussian component is selected by Eq. (13).
	Constructing dictionary $\mathbf{U}_k$ by Eq. (14).
	Update $\mathbf{B}_i^{t+1}$ computing by $\mathbf{B}_i = \mathbf{U}_k^{-1} \mathbf{Y}_i$.
	Constructing dictionary $\mathbf{D}_i^{t+1}$ by Eq. (15).
	Update λ_i^{t+1} computing by Eq. (18).
	Update $\mathbf{A}_i^{t+1}$ computing by Eq. (17).
	Get the estimation $\mathbf{X}_i^{t+1} = \mathbf{D}_i^{t+1} \mathbf{A}_i^{t+1}.$
End for	
	Aggregate $\mathbf{X}_i^{t+1}$ to form the recovered image $\hat{\mathbf{X}}^{t+1}.$
End for	
Output:	$\hat{\mathbf{X}}^{t+1}.$

222 4. Discussion

223 In this section, we provide detailed discussion about the main difference
 224 among the proposed GSRC-ENSS method, the NCSR method [25], a majori-
 225 ty of NSS prior-based denoising methods and the popular methods combining
 226 external and internal priors.

- 227 • The main difference between NCSR [25] and the proposed GSRC-ENSS
228 method is that NCSR first obtained some good estimates of the sparse
229 coding coefficients of the original image by the principle of NLM [1], and
230 then centralized the sparse coding coefficients of the observed image to
231 those estimates to improve the performance of denoising. However, s-
232 ince NLM depends on the weighted graph [59], it is unavoidable that
233 the weighted manner leads to disturbance and inaccuracy [60]. It is worth
234 mentioning that the proposed GSRC-ENSS method does not involve in the
235 weighted graph. In addition, NCSR is actually a patch-based sparse cod-
236 ing method, which usually ignores the relationship among similar patches
237 [13, 23, 50, 53].
- 238 • NSS prior has shown great success in image denoising. A majority of
239 existing denoising methods only exploit the NSS prior of noisy image [12,
240 13, 14, 23, 24, 25, 27, 30, 37], and few methods use the NSS prior from
241 natural images [11]. Different from the most existing NSS prior-based
242 denoising methods, in this work we consider two kinds of NSS prior, i.e.,
243 NSS priors of noisy input image and natural images.
- 244 • In recent years, several works about combining internal and external priors
245 for image denoising were proposed [39, 40, 41, 42, 61, 62, 63]. For example,
246 Mosseri *et al.* [39] developed a Patch-SNR measure to decide whether a
247 noisy patch is denoised patch using internal and external priors. Luo *et al.*
248 [42] proposed a prior learning algorithm that combines the internal and
249 external priors from a Bayesian hyper-prior perspective. However, differ-
250 ent from the most existing methods, which are only combining external
251 patch and internal patch priors to a new hybrid prior [39, 40, 41, 42, 61]
252 or using external patch prior guided internal prior [62, 63], the proposed
253 GSRC-ENSS method exploits external NSS prior and internal NSS prior
254 to image denoising.

Figure 3: All test images. Top: Airplane, J.Bean, Bridge, elaine, F.print, foreman, House; bottom: Lake, Leaves, straw, Miss, Monarch, Parthenon, starfish.

255 5. Experimental Results

256 In this section, we validate the performance of the proposed GSRC-ENSS
 257 and compare it with several state-of-the-art denoising methods, including BM3D
 258 [12], EPLL [7], NCSR [25], GID [64], LINC [9], PGPD [11] and aGMM [42],
 259 where EPLL, LINC, PGPD and aGMM are based on GMM learning model
 260 and aGMM exploits external and internal priors. We evaluate the competing
 261 methods on 14 typical natural images, whose scenes are displayed in Fig. 3. Zero
 262 mean additive white Gaussian noises are added to those test images to generate
 263 the noisy observations. To evaluate the quality of denoised image, both PSNR
 264 and SSIM [65] metrics are used. The training groups used in our experiments
 265 were sampled based on NSS scheme from the Kodak photoCD dataset³, which
 266 includes 24 natural images. The detailed setting of parameters is shown in
 267 Table 2, including the number of Gaussian components K ⁴, search window W ,
 268 the number of similar patches in a group m , patch size d and involved parameters
 269 c , γ , ρ . **The source code of the proposed method can be download at**

³<http://r0k.us/graphics/kodak/>.

⁴ Due to the operation of mean subtraction, the possible patterns and variables of each group are reduced, and thus, the number of Gaussian component is not necessarily large. For the parameter K , the number of GMM components, we referred to a most recent work [11], and experimental results also shown that $K = 64$ is the best choice under the limitation of testing environment.

<https://drive.google.com/open?id=0B0wKhHwcknCjWGx2LUd0NnZibmM>.

Table 2: Parameter settings

	GMM Learning Stage				Denoising Stage		
Noise level	K	W	d	m	c	ρ	γ
$\sigma \leq 10$	64	50	6	80	0.14	0.19	1.08
$10 < \sigma \leq 20$	64	50	6	80	0.13	0.20	1.05
$20 < \sigma \leq 30$	64	50	7	90	0.12	0.21	1.05
$30 < \sigma \leq 40$	64	50	8	100	0.11	0.22	1.05
$40 < \sigma \leq 50$	64	50	8	100	0.10	0.23	1.05
$50 < \sigma \leq 75$	64	50	9	120	0.09	0.24	1.00
$75 < \sigma \leq 100$	64	50	9	120	0.08	0.25	1.00

270

271 We present the denoising results on six noise levels, i.e., $\sigma=20, 30, 40, 50, 75$
 272 and 100. The PSNR and SSIM results under these noise levels for all competing
 273 denoising methods are reported in Table 3 and Table 4, respectively (the highest
 274 PSNR and SSIM values are marked in bold). In a majority of cases, it can be
 275 easily found that the proposed GSRC-ENSS achieves better PSNR and SSIM
 276 results than other competing methods. In terms of PSNR, the proposed GSRC-
 277 ENSS achieves 0.21dB, 0.56dB, 0.29dB, 1.29dB, 0.32dB, 0.13dB and 0.39dB
 278 improvement on average over the BM3D, EPLL, NCSR, GID, LINC, PGPD
 279 and aGMM, respectively. This clearly demonstrates the effectiveness of the
 280 proposed GSRC-ENSS.

281 Considering that human subjects are judging of the image quality at last,
 282 the visual quality of denoised images is also critical to evaluate a denoising al-
 283 gorithm. The visual comparisons of competing denoising methods at noise level
 284 40, 50, 75 and 100 are shown in Fig. 4, Fig. 5, Fig. 6 and Fig. 7, respectively.
 285 It can be seen that BM3D and LINC are resulting in over-smooth phenomena,
 286 while EPLL, NCSR, GID, PGPD and aGMM are likely to generate some un-
 287 desirable ringing artifacts. In contrast, the proposed GSRC-ENSS can not only
 288 recover sharp edges, fine details and suppress undesirable ringing artifacts, but

Figure 4: Denoising results on image *starfish* by different methods (noise level $\sigma = 40$). (a) Original image; (b) Noisy image; (c) BM3D [12] (PSNR=26.06dB, SSIM=0.7828); (d) EPLL [7] (PSNR=26.12dB, SSIM=0.7802); (e) NCSR [25] (PSNR=26.17dB, SSIM=0.7812); (f) GID [64] (PSNR=25.39dB, SSIM=0.7565); (g) LINC [9] (PSNR=26.07dB, SSIM=0.7758); (h) PGPD [11] (PSNR=26.21dB, SSIM= 0.7855); (i) aGMM [42] (PSNR= 26.16dB, SSIM=0.7824); (j) GSRC-ENSS (PSNR=**26.57dB**, SSIM= **0.7976**).

Figure 5: Denoising results on image *Monarch* by different methods (noise level $\sigma = 50$). (a) Original image; (b) Noisy image; (c) BM3D [12] (PSNR=25.82dB, SSIM=0.8200); (d) EPLL [7] (PSNR=25.78dB, SSIM=0.8124); (e) NCSR [25] (PSNR=25.73dB, SSIM=0.8252); (f) GID [64] (PSNR=25.28dB, SSIM=0.8022); (g) LINC [9] (PSNR=25.88dB, SSIM=**0.8314**); (h) PGPD [11] (PSNR=26.00dB, SSIM= 0.8269); (i) aGMM [42] (PSNR= 25.82dB, SSIM=0.8164); (j) GSRC-ENSS (PSNR=**26.18dB**, SSIM= 0.8269).

Table 3: PSNR (dB) Comparison of BM3D [12], EPLL [7], NCSR [25], GID [64], LINC [9], PGPD [11], aGMM [42] and GSRC-ENSS.

	$\sigma = 20$								$\sigma = 30$							
	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS
airplane	30.59	30.60	30.50	29.62	30.57	30.79	30.54	30.81	28.49	28.54	28.34	27.54	28.53	28.62	28.42	28.66
bridge	27.27	27.50	27.28	26.39	27.31	27.31	27.20	27.27	25.47	25.67	25.44	24.80	25.43	25.48	25.35	25.40
elaine	32.51	32.16	32.39	31.23	32.58	32.43	32.62	32.41	30.52	30.15	30.25	28.98	30.39	30.47	30.59	30.42
F.print	26.75	26.18	26.83	26.10	26.83	26.84	26.15	27.03	24.75	24.08	24.78	24.25	24.82	24.73	24.10	24.92
foreman	34.54	33.67	34.42	33.08	34.76	34.44	34.20	34.45	32.75	31.70	32.61	30.92	32.93	32.83	32.31	32.96
House	33.77	32.99	33.81	32.68	33.82	33.85	33.52	33.82	32.08	31.24	32.01	30.50	32.26	32.24	31.79	32.34
J.Bean	34.18	33.79	34.37	33.38	34.14	34.28	34.80	34.68	31.97	31.55	31.99	30.74	31.82	31.99	32.50	32.39
Lake	28.85	29.06	28.85	28.17	28.97	29.07	28.79	29.11	26.74	26.97	26.69	26.03	26.95	26.90	26.70	26.98
Leave	30.09	29.40	30.34	29.91	30.24	30.46	30.05	30.54	27.81	27.19	28.04	27.59	27.99	27.99	27.53	28.15
Miss	33.71	33.33	33.58	32.17	33.64	33.66	33.84	33.58	31.89	31.40	31.62	30.10	31.75	31.88	32.04	31.78
Monarch	30.35	30.49	30.52	29.75	30.64	30.68	30.31	30.81	28.36	28.36	28.38	27.68	28.74	28.49	28.27	28.69
Parthenon	29.45	29.23	29.41	28.15	29.38	29.33	29.42	29.26	27.54	27.32	27.48	26.25	27.49	27.39	27.43	27.37
starfish	29.67	29.58	29.85	29.05	29.58	29.84	29.74	30.05	27.65	27.52	27.69	26.90	27.52	27.67	27.61	28.02
straw	27.08	26.93	27.38	26.60	27.42	27.18	26.66	27.48	24.91	24.76	25.08	24.66	25.51	24.83	24.44	25.27
Average	30.63	30.35	30.68	29.73	30.71	30.72	30.56	30.81	28.64	28.32	28.60	27.64	28.72	28.68	28.50	28.81
	$\sigma = 40$								$\sigma = 50$							
	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS
airplane	26.88	27.08	26.78	26.17	27.08	27.12	26.95	27.18	25.76	25.96	25.63	25.10	26.04	25.98	25.83	25.99
bridge	24.33	24.51	24.21	23.73	24.10	24.38	24.25	24.31	23.58	23.69	23.41	22.78	23.18	23.60	23.50	23.55
elaine	28.95	28.73	28.91	27.67	28.96	29.08	29.10	28.95	27.96	27.63	27.68	26.54	27.73	27.90	27.93	27.78
F.print	22.88	22.66	23.29	23.00	23.42	23.33	22.77	23.50	22.59	21.55	22.29	21.89	22.19	22.36	21.74	22.50
foreman	31.29	30.28	31.52	29.61	31.31	31.55	30.95	31.77	30.56	29.20	30.41	28.64	30.33	30.45	29.80	30.57
House	30.65	29.89	30.79	29.02	31.00	31.02	30.40	31.12	29.69	28.79	29.61	27.76	29.87	29.93	29.28	30.14
J.Bean	30.22	29.96	30.49	29.53	30.24	30.39	30.79	30.74	29.26	28.73	29.24	28.43	29.01	29.20	29.46	29.47
Lake	25.21	25.58	25.21	24.61	25.51	25.51	25.39	25.56	24.29	24.56	24.15	23.68	24.30	24.49	24.39	24.51
Leave	25.69	25.62	26.20	25.82	26.31	26.29	25.76	26.43	24.68	24.39	24.94	24.41	25.11	25.03	24.42	25.23
Miss	30.50	29.93	30.46	28.89	30.29	30.54	30.65	30.49	29.48	28.76	29.31	27.80	29.22	29.45	29.42	29.31
Monarch	26.72	26.89	26.81	26.32	27.14	27.02	26.87	27.22	25.82	25.78	25.73	25.28	25.88	26.00	25.82	26.18
Parthenon	26.15	25.95	26.03	24.88	26.08	26.13	26.07	26.14	25.26	25.02	25.12	24.06	25.15	25.22	25.06	25.26
starfish	26.06	26.12	26.17	25.39	26.07	26.21	26.16	26.57	25.04	25.05	25.06	24.07	24.81	25.11	25.09	25.45
straw	23.16	23.19	23.62	23.25	23.91	23.39	22.94	23.82	22.41	22.02	22.48	22.02	22.34	22.30	21.82	22.64
Average	27.05	26.89	27.18	26.28	27.25	27.28	27.07	27.41	26.16	25.79	26.07	25.18	26.08	26.22	25.97	26.33
	$\sigma = 75$								$\sigma = 100$							
	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS
airplane	23.99	24.03	23.76	22.91	23.71	24.15	23.95	24.13	22.89	22.78	22.60	21.82	22.42	23.03	22.67	22.97
bridge	22.33	22.30	22.00	21.30	21.71	22.32	22.26	22.34	21.44	21.36	21.06	20.46	20.61	21.42	21.35	21.33
elaine	25.93	25.60	25.34	24.54	25.42	25.78	25.77	25.72	24.48	24.16	23.77	23.21	23.92	24.34	24.30	24.24
F.print	21.09	19.34	20.59	19.47	20.14	20.70	19.56	20.88	20.03	17.65	19.51	17.66	18.87	19.57	17.56	19.68
foreman	28.07	27.24	28.18	26.71	28.11	28.39	27.67	28.76	26.51	25.91	26.55	25.33	26.55	26.81	26.20	27.27
House	27.51	26.70	27.16	25.23	27.56	27.81	27.11	28.21	25.87	25.21	25.49	22.38	26.11	26.17	25.55	26.44
J.Bean	27.22	26.57	27.15	26.26	26.62	27.07	27.09	27.43	25.80	25.16	25.61	24.37	24.88	25.66	25.58	26.07
Lake	22.63	22.77	22.48	21.82	22.45	22.76	22.68	22.72	21.56	21.60	21.37	20.69	20.99	21.66	21.55	21.63
Leave	22.49	22.03	22.60	21.28	22.45	22.61	21.96	22.89	20.90	20.26	20.84	19.13	20.49	20.95	20.29	21.44
Miss	27.34	26.68	27.03	25.47	26.70	27.51	27.25	27.48	25.67	25.23	25.24	23.73	25.28	25.94	25.71	26.13
Monarch	23.91	23.73	23.67	22.77	23.91	24.00	23.85	24.21	22.52	22.24	22.10	20.73	22.13	22.56	22.42	22.83
Parthenon	23.61	23.45	23.35	22.52	23.37	23.68	23.38	23.85	22.57	22.39	22.39	21.50	22.21	22.70	22.42	22.86
starfish	23.28	23.17	23.18	21.99	22.74	23.23	23.22	23.45	22.10	21.92	21.91	20.84	21.10	22.08	21.95	22.10
straw	20.72	20.09	20.62	19.70	19.85	20.58	19.90	20.81	19.59	19.03	19.39	18.38	18.75	19.57	18.81	19.49
Average	24.29	23.84	24.08	23.00	23.92	24.33	23.97	24.49	22.99	22.49	22.70	21.45	22.45	23.03	22.60	23.18

Table 4: SSIM Comparison of BM3D [12], EPLL [7], NCSR [25], GID [64], LINC [9], PGPD [11], aGMM [42] and GSRC-ENSS.

	$\sigma = 20$								$\sigma = 30$							
	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS
airplane	0.9006	0.9017	0.9016	0.8837	0.8988	0.8986	0.9018	0.9022	0.8631	0.8628	0.8660	0.8449	0.8632	0.8641	0.8647	0.8673
bridge	0.7938	0.8188	0.7933	0.7489	0.8064	0.7913	0.7928	0.7917	0.7026	0.7295	0.7033	0.6636	0.6972	0.6989	0.6917	0.6897
elaine	0.8900	0.8813	0.8880	0.8645	0.8927	0.8867	0.8914	0.8865	0.8530	0.8399	0.8498	0.8076	0.8517	0.8522	0.8542	0.8518
F.print	0.9252	0.9212	0.9226	0.9034	0.9318	0.9219	0.9158	0.9255	0.8786	0.8680	0.8746	0.8520	0.8789	0.8699	0.8577	0.8760
foreman	0.9076	0.8955	0.9065	0.8900	0.9113	0.9023	0.9055	0.9052	0.8823	0.8617	0.8846	0.8551	0.8894	0.8818	0.8766	0.8862
House	0.8726	0.8609	0.8735	0.8563	0.8685	0.8693	0.8646	0.8687	0.8480	0.8338	0.8479	0.8243	0.8485	0.8471	0.8435	0.8500
J.Bean	0.9582	0.9523	0.9632	0.9577	0.9615	0.9508	0.9633	0.9597	0.9357	0.9240	0.9435	0.9338	0.9449	0.9317	0.9413	0.9425
Lake	0.8798	0.8826	0.8779	0.8543	0.8798	0.8803	0.8768	0.8809	0.8287	0.8318	0.8271	0.8013	0.8276	0.8294	0.8242	0.8325
Leave	0.9534	0.9480	0.95550	0.9493	0.9561	0.9562	0.9559	0.9576	0.9378	0.9197	0.9311	0.9248	0.9339	0.9300	0.9273	0.9344
Miss	0.9015	0.8980	0.9001	0.8777	0.9028	0.8969	0.9038	0.8999	0.8715	0.8648	0.8715	0.8408	0.8747	0.8706	0.8762	0.8748
Monarch	0.9179	0.9166	0.9192	0.9027	0.9262	0.9187	0.9202	0.9198	0.8822	0.8789	0.8829	0.8628	0.8970	0.8853	0.8831	0.8857
Parthenon	0.8664	0.8571	0.8625	0.8179	0.8642	0.8493	0.8586	0.8430	0.8048	0.7913	0.8015	0.7447	0.7963	0.7876	0.7914	0.7777
starfish	0.8748	0.8756	0.8748	0.8546	0.8691	0.8756	0.8756	0.8769	0.8289	0.8248	0.8283	0.8028	0.8195	0.8277	0.8263	0.8355
straw	0.8973	0.8963	0.9038	0.8783	0.9108	0.8951	0.8822	0.9050	0.8290	0.8227	0.8336	0.8151	0.8559	0.8156	0.7959	0.8409
Average	0.8956	0.8933	0.8959	0.8742	0.8986	0.8924	0.8935	0.8945	0.8526	0.8467	0.8533	0.8267	0.8556	0.8494	0.8467	0.8532
	$\sigma = 40$								$\sigma = 50$							
	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS
airplane	0.8277	0.8264	0.8330	0.8120	0.8313	0.8338	0.8305	0.8384	0.8044	0.7922	0.8066	0.7795	0.8021	0.8053	0.7990	0.8094
bridge	0.6365	0.6582	0.6187	0.5994	0.6027	0.6325	0.6225	0.6242	0.5834	0.6034	0.569	0.5353	0.5366	0.5824	0.5745	0.5748
elaine	0.8180	0.8047	0.8258	0.7838	0.8244	0.8223	0.8209	0.8219	0.7971	0.7741	0.7976	0.7494	0.7945	0.7926	0.7890	0.7925
F.print	0.8188	0.8117	0.8123	0.8072	0.8243	0.8212	0.8015	0.8286	0.7933	0.7528	0.7598	0.7581	0.7698	0.7760	0.7472	0.7831
foreman	0.8565	0.8315	0.8723	0.8262	0.8675	0.8621	0.8515	0.8701	0.8445	0.8051	0.8559	0.8080	0.8534	0.8410	0.8270	0.8482
House	0.8256	0.8089	0.8323	0.7979	0.8340	0.8302	0.8221	0.8355	0.8122	0.7845	0.8160	0.7718	0.8180	0.8125	0.8002	0.8202
J.Bean	0.9122	0.8956	0.9296	0.9139	0.9248	0.9133	0.9170	0.9262	0.9006	0.8677	0.9134	0.8956	0.9085	0.8934	0.8911	0.9047
Lake	0.7793	0.7871	0.7788	0.7541	0.7834	0.7859	0.7812	0.7907	0.7433	0.7477	0.7420	0.7121	0.7423	0.7483	0.7428	0.7517
Leave	0.8961	0.8916	0.9028	0.8994	0.9093	0.9039	0.8979	0.9105	0.8680	0.8638	0.8787	0.8693	0.8925	0.8794	0.8673	0.8887
Miss	0.8443	0.8334	0.8555	0.8185	0.8520	0.8469	0.8504	0.8539	0.8287	0.8042	0.8359	0.7935	0.8326	0.8235	0.8235	0.8302
Monarch	0.8446	0.8441	0.8522	0.8266	0.8621	0.8549	0.8478	0.8553	0.8200	0.8124	0.8252	0.8022	0.8314	0.8269	0.8164	0.8269
Parthenon	0.7468	0.7272	0.7372	0.6798	0.7349	0.7384	0.7294	0.7276	0.7106	0.6840	0.6962	0.6253	0.6939	0.6951	0.6839	0.6847
starfish	0.7828	0.7802	0.7812	0.7565	0.7758	0.7855	0.7824	0.7976	0.7433	0.7392	0.7440	0.7011	0.7326	0.7457	0.7419	0.7609
straw	0.7452	0.7361	0.7631	0.7475	0.7811	0.7425	0.7076	0.7772	0.6881	0.6489	0.6924	0.6706	0.6684	0.6694	0.6187	0.7046
Average	0.8096	0.8026	0.8139	0.7874	0.8148	0.8124	0.7989	0.8184	0.7813	0.7628	0.7809	0.7480	0.7769	0.7780	0.7659	0.7843
	$\sigma = 75$								$\sigma = 100$							
	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS
airplane	0.7488	0.7168	0.7547	0.6754	0.7470	0.7484	0.7248	0.7553	0.7036	0.6523	0.7107	0.6393	0.6931	0.6940	0.6571	0.7038
bridge	0.5086	0.5089	0.4820	0.4413	0.4391	0.5013	0.4939	0.5007	0.4585	0.4483	0.4259	0.3883	0.3797	0.4481	0.4361	0.4661
elaine	0.7359	0.7062	0.7342	0.6762	0.7280	0.7303	0.7172	0.7353	0.6817	0.6459	0.6824	0.6139	0.6812	0.6721	0.6548	0.6801
F.print	0.7102	0.5814	0.6574	0.5952	0.6224	0.6755	0.5705	0.6944	0.6381	0.3837	0.5814	0.4179	0.5165	0.5875	0.3460	0.6074
foreman	0.7933	0.7467	0.8171	0.7556	0.8162	0.7965	0.7676	0.8096	0.7489	0.6949	0.7833	0.7050	0.7826	0.7452	0.7129	0.7656
House	0.7645	0.7251	0.7749	0.7052	0.7850	0.7709	0.7419	0.7877	0.7203	0.6695	0.7397	0.4735	0.7550	0.7195	0.6854	0.7401
J.Bean	0.8573	0.8019	0.8792	0.8574	0.8669	0.8503	0.8243	0.8658	0.8181	0.7429	0.8472	0.7862	0.8339	0.7999	0.7628	0.8218
Lake	0.6716	0.6644	0.6743	0.6366	0.6590	0.6760	0.6612	0.6787	0.6178	0.5977	0.6235	0.5728	0.5940	0.6171	0.5965	0.6224
Leave	0.8072	0.7921	0.8234	0.7751	0.8247	0.8121	0.7867	0.8337	0.7482	0.7163	0.7622	0.6857	0.7499	0.7469	0.7106	0.7874
Miss	0.7722	0.7422	0.7936	0.7353	0.7819	0.7757	0.7626	0.7867	0.7240	0.6901	0.7576	0.6607	0.7438	0.7236	0.7090	0.7465
Monarch	0.8200	0.8124	0.8252	0.8022	0.8314	0.8269	0.8164	0.8269	0.7021	0.6771	0.7109	0.6361	0.7076	0.7029	0.6823	0.7874
Parthenon	0.6202	0.5968	0.5980	0.5550	0.5929	0.6182	0.5888	0.6183	0.557	0.5397	0.5535	0.5052	0.5425	0.5638	0.5340	0.5635
starfish	0.6670	0.6502	0.6685	0.6111	0.6416	0.6638	0.6525	0.6808	0.6053	0.5799	0.6062	0.5415	0.5635	0.6018	0.5813	0.6111
straw	0.5462	0.4567	0.5291	0.4337	0.4143	0.5177	0.4150	0.5568	0.4224	0.3295	0.3722	0.3013	0.2687	0.4033	0.2819	0.3916
Average	0.7113	0.6735	0.7108	0.6543	0.6922	0.7072	0.6752	0.7195	0.6533	0.5977	0.6541	0.5662	0.6294	0.6447	0.5965	0.6597

289 also yield visually pleasant results. For example, GSRC-ENSS achieves the most
 290 visually pleasant restoration of the images *starfish* in Fig. 4 and *House* in Fig. 6
 291 especially when one inspects the zoomed portions. Moreover, it can be seen
 292 that although PSNR results are slightly lower than BM3D and PGPD methods
 293 in Fig. 8 and Fig. 9, BM3D and PGPD methods generate obvious ringing ar-
 294 tifacts. Note that BM3D only uses the NSS prior of noisy image and the NSS
 295 prior of natural images is used in PGPD. However, the proposed GSRC-ENSS
 296 can generate more details and less ringing artifacts. The reason for this is that
 297 two NSS priors (i.e., NSS priors of noisy image and natural images) are used
 298 in our GSRC model and the iterative shrinkage algorithm (Eq. (17)) can better
 299 combine these two NSS priors.

300 We also evaluate different denoising methods on two real images, including
 301 BM3D, PGPD and the proposed GSRC-ENSS methods. The visual comparisons
 302 of competing denoising methods are shown in Fig. 10 and Fig. 11. One can
 303 observe that the proposed GSRC-ENSS is able to preserve the sharp edges and
 304 small fine details more effective than BM3D and PGPD methods. The results
 305 show the feasibility of deploying our GSRC-ENSS method for some practical
 306 image denoising applications.

307 Moreover, to evaluate the computational cost of eight denoising methods,
 308 we compare the running time on all test images with different noise levels. All
 309 experiments are run under the Matlab2012b environment on a machine with
 310 Intel(R) Core(TM) i3-4150 with a 3.50GHz CPU and 4GB RAM, and Windows
 311 10 operating system. The average run time(s) of computing methods on all
 312 test images is shown in Table 5. It's noted that BM3D and GID methods use
 313 parallel computing. Meanwhile, BM3D method is implemented with compiled
 314 C++ mex-function, while the other methods are implemented in Matlab. It
 315 can be seen that the proposed GSRC-ENSS method often needs more time than
 316 other competing methods. Nonetheless, the proposed GSRC-ENSS delivers the
 317 best qualitative denoising results with finer details and less ringing artifacts,
 318 such as Figs. 4- 11.

319 To sump up, such experimental findings clearly demonstrate that the GSRC-

Table 5: Average run time (s) on all test images with different methods.

Methods	BM3D	EPLL	NCSR	GID	LINC	PGPD	aGMM	GSRC-ENSS
Average Time (s)	3	112	355	346	254	10	249	699

ENSS model is a stronger prior for the class of photographic images containing large variations in edges/textures.

Figure 6: Denoising results on image *House* by different methods (noise level $\sigma = 75$). (a) Original image; (b) Noisy image; (c) BM3D [12] (PSNR=27.51dB, SSIM=0.7645); (d) EPLL [7] (PSNR=26.70dB, SSIM=0.7251); (e) NCSR [25] (PSNR=27.16dB, SSIM=0.7749); (f) GID [64] (PSNR=25.23dB, SSIM=0.7052); (g) LINC [9] (PSNR=27.56dB, SSIM=0.7850); (h) PGPD [11] (PSNR=27.81dB, SSIM=0.7709); (i) aGMM [42] (PSNR=27.11dB, SSIM=0.7419); (j) GSRC-ENSS (PSNR=**28.21dB**, SSIM= **0.7877**).

6. Conclusion

In this paper, a novel denoising method based on group sparsity residual constraint with external NSS prior (GSRC-ENSS) was presented. Our approach differed from typical NSS based denoising methods, which only exploited the nonlocal self-similarity (NSS) prior of noisy image or natural images, two NSS priors (i.e., NSS priors of noisy image and natural images) were used for image denoising. To boost the performance of image denoising, we proposed the concept of group sparsity residual, and thus the problem of image denoising was

Figure 7: Denoising results on image *Leaves* by different methods (noise level $\sigma = 100$). (a) Original image; (b) Noisy image; (c) BM3D [12] (PSNR=20.90dB, SSIM=0.7482); (d) EPLL [7] (PSNR=20.26dB, SSIM=0.7163); (e) NCSR [25] (PSNR=20.84dB, SSIM=0.7622); (f) GID [64] (PSNR=19.13dB, SSIM=0.6857); (g) LINC [9] (PSNR=20.49dB, SSIM=0.7499); (h) PGPD [11] (PSNR=20.95dB, SSIM= 0.7469); (i) aGMM [42] (PSNR= 20.29dB, SSIM=0.7106); (j) GSRC-ENSS (PSNR=**21.44dB**, SSIM= **0.7874**).

Figure 8: Denoising results on image *Lake* by different methods (noise level $\sigma = 75$). (a) Original image; (b) Noisy image; (c) BM3D [12] (PSNR=22.63dB, SSIM=0.6716); (d) PGPD [11] (PSNR=**22.76dB**, SSIM=0.6760); (e) GSRC-ENSS (PSNR=22.72dB, SSIM=**0.6787**).

Figure 9: Denoising results on image *elaine* by different methods (noise level $\sigma = 100$). (a) Original image; (b) Noisy image; (c) BM3D [12] (PSNR=**24.48dB**, SSIM=**0.6817**); (d) PGPD [11] (PSNR=24.34dB, SSIM=0.6721); (e) GSRC-ENSS (PSNR=24.24dB, SSIM=0.6801).

transformed into reducing the group sparsity residual. To this end, we achieved a good estimation of the group sparse coefficients of the original image by the NSS prior of natural images based GMM learning, and the group sparse coefficients of noisy image were exploited to approximate the estimation. To combine these two NSS priors better, we presented an iterative shrinkage algorithm to solve the GSRC-ENSS model. Experimental results have demonstrated that GSRC-ENSS can not only leads to visible improvements over many state-of-the-art methods, but also preserves much better the image local structures and suppresses the ringing artifacts.

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(a) Noisy image

(b) BM3D

(c) PGPD

(d) GSRC-ENSS

Figure 10: Denoising results on real image by different methods. (a) Noisy image; (b) BM3D [12]; (c) PGPD [11]; (d) GSRC-ENSS.

(a) Noisy image

(b) BM3D

(c) PGPD

(d) GSRC-ENSS

Figure 11: Denoising results on real image by different methods. (a) Noisy image; (b) BM3D [12]; (c) PGPD [11]; (d) GSRC-ENSS.

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