GENERALIZED CONTINUED FRACTION EXPANSIONS WITH CONSTANT PARTIAL DENOMINATORS

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Abstract

We study generalized continued fraction expansions of the form

$$\frac{a_1}{N+}\frac{a_2}{N+}\frac{a_3}{N+\cdots},$$

where N is a fixed positive integer and the partial numerators a_i are positive integers for all i. We call these expansions dn_N expansions and show that every positive real number has infinitely many dn_N expansions for each N. In particular we study the dn_N expansions of rational numbers and quadratic irrationals. Finally we show that every positive real number has for each N a dn_N expansion with bounded partial numerators.

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1. Introduction

In [1] Anselm and Weintraub introduced a generalization of simple continued fractions, the cf_N expansion

$$a_0+\frac{N}{a_1+}\frac{N}{a_2+}\frac{N}{a_3+\cdots},$$

where N is a fixed positive integer, a_0 is a non-negative integer and a_i is a positive integer for every i. They showed that every positive real number has infinitely many cf_N expansions for all $N>1$ and studied the properties of these expansions for rational numbers and quadratic irrationals. In particular, they focused on the so-called *best* cf_N *expansion*, where the partial denominators a_i are chosen as large as possible and which is unique for each real number.

In this paper we flip the roles of the partial numerators and denominators of the cf_N expansions and study generalized continued fraction expansions of the form

$$\frac{a_1}{N+}\frac{a_2}{N+}\frac{a_3}{N+\cdots}, \tag{1}$$

where N is a fixed positive integer and a_i are positive integers. We shall call these continued fractions dn_N expansions and denote them by $\langle a_1, a_2, \ldots\rangle_N$. While a general study of the dn_N expansions of real numbers doesn't seem to have been done, continued fractions of form (1) have been studied quite a lot. For example, Ramanujan presented many such continued fractions in his notebooks ([2],[3]). Among them were

$$1 = \frac{x+N}{N\quad+}\frac{(x+N)^2-N^2}{N\qquad+}\frac{(x+2N)^2-N^2}{N\qquad+}\frac{(x+3N)^2-N^2}{N\qquad+\cdots},$$

where $x \neq -kN$ for all positive integers k,

$$1 + 2N^2\sum_{k=1}^{\infty}\frac{(-1)^k}{(N+k)^2} = \frac{1}{N+}\frac{1^2}{N+}\frac{1\cdot 2}{N\ +}\frac{2^2}{N+}\frac{2\cdot 3}{N\ +}\frac{3^2}{N+\cdots} \tag{2}$$

and perhaps most famously Ramanujan's AGM continued fraction

$$\mathcal{R}_N(a,b) = \frac{a}{N+}\frac{b^2}{N+}\frac{(2a)^2}{N\ +}\frac{(3b)^2}{N\ +}\frac{(4a)^2}{N\ +}\frac{(5b)^2}{N\ +\cdots}$$

that satisfies the remarkable equation

$$\mathcal{R}_N\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_N(a,b)+\mathcal{R}_N(b,a)}{2}$$

connecting the arithmetic and geometric mean of numbers a and b ([4]). Some examples of well-known dn_N expansions for real numbers are Lord Brouncker's dn_2 expansion

$$\pi = \frac{8}{2+}\frac{2}{2+}\frac{3^2}{2\ +}\frac{5^2}{2\ +\cdots}$$

(see [6]), the dn_1 expansion

$$\ln 2 = \mathcal{R}_1(1,1) = \frac{1}{1+}\frac{1^2}{1\ +}\frac{2^2}{1\ +}\frac{3^2}{1\ +\cdots}$$

(see [4]), and the dn_1 expansion

$$\zeta(2) - 1 = \frac{\pi^2}{6} - 1 = \frac{1}{1+}\frac{1^2}{1+}\frac{1\cdot 2}{1\ +}\frac{2^2}{1+}\frac{2\cdot 3}{1\ +}\frac{3^2}{1+\cdots}$$

derived from (2).

We will begin with some preliminaries in Section 2, followed by the dn_N algorithm in Section 3. We will show that every positive real number has infinitely many dn_N expansions for every N and define a special dn_N expansion called the least dn_N expansion. In Sections 4 and 5 we will examine the dn_N expansions of positive rational numbers and positive real quadratic irrationals, respectively. We will prove that for any rational number there exist infinitely many finite, periodic and aperiodic dn_N expansions, and that for any quadratic irrational number there exist infinitely many periodic and aperiodic dn_N expansions. Special attention is paid to the least dn_N expansion of these numbers. In Section 6 we will show that every positive real number has a dn_N expansion with bounded partial numerators.

In this paper, we denote the set of positive integers with $\mathbb{Z}_+$ and the set of non-negative integers with $\mathbb{N}$.

2. On continued fractions

We begin with some preliminaries on (generalized) continued fractions

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}} = b_0 + \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n} = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots, \tag{3}$$

where the partial numerators a_n and the partial denominators b_n are positive integers for all $n \in \mathbb{Z}_+$ and $b_0 \in \mathbb{Z}$. If the limit of the n:th convergent

$$\frac{A_n}{B_n} = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}$$

at infinity exists, it is called the value of the continued fraction. The numerators A_n and denominators B_n of the convergents can be obtained from the recurrence relations

$$\begin{cases} A_{n+2} &= b_{n+2}A_{n+1} + a_{n+2}A_n, \\ B_{n+2} &= b_{n+2}B_{n+1} + a_{n+2}B_n \end{cases} \tag{4}$$

with initial values $A_0 = b_0$, $B_0 = 1$, $A_1 = b_0 b_1 + a_1$ and $B_1 = b_1$. These relations imply the formula

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{(-1)^n a_1 \ldots a_{n+1}}{B_n B_{n+1}},$$

valid for all $n \in \mathbb{N}$. If continued fraction (3) converges to $\tau \in \mathbb{R}$, then

$$\tau = b_0 + \sum_{k=0}^{\infty} \frac{(-1)^k a_1 \ldots a_{k+1}}{B_k B_{k+1}}, \tag{5}$$

as shown for example in [5]. Using recurrence relations (4) and the standard error estimates of alternating series we get

$$\frac{b_{n+2}a_1\dots a_{n+1}}{B_nB_{n+2}} < \left|\tau - \frac{A_n}{B_n}\right| < \frac{a_1\dots a_{n+1}}{B_nB_{n+1}}. \tag{6}$$

We can also determine the sign of $\tau - A_n/B_n$ since equations (4) and (5) imply

$$\frac{A_0}{B_0} < \frac{A_2}{B_2} < \dots < \frac{A_{2k}}{B_{2k}} < \tau < \frac{A_{2l+1}}{B_{2l+1}} < \dots < \frac{A_3}{B_3} < \frac{A_1}{B_1} \tag{7}$$

for all $k, l \in \mathbb{N}$.

As the partial coefficients a_n and b_n of continued fraction (3) are positive integers for all $n \in \mathbb{Z}_+$, the following theorem gives us a convergence criterion.

Theorem 2.1. *[The Seidel-Stern Theorem] Let a_n and b_n be positive real numbers for all n. Then the continued fraction $\mathrm{K}_{n=1}^{\infty}\frac{a_n}{b_n}$ converges if and only if the* Stern-Stolz series

$$\sum_{n=1}^{\infty} b_n \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}} \tag{8}$$

diverges to ∞.

Proof. See [7], Chapter III, Theorem 3 and the subsequent Remark 2. □

Corollary 2.2. *Let a_n and b_n be positive integers for all n. If the sequence (a_n) has a bounded subsequence, then the continued fraction $\mathrm{K}_{n=1}^{\infty}\frac{a_n}{b_n}$ converges.*

Proof. Let us assume that (a_n) has a bounded subsequence (a_{k_i}) such that $a_{k_i} \leq M$ for all $i \in \mathbb{Z}_+$ and some $M \in \mathbb{Z}_+$. Without loss of generality, we may also assume that $k_{i+1} \geq k_i + 2$. By denoting

$$S_n = \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}}$$

the Stern-Stolz series of the continued fraction $\mathrm{K}_{n=1}^{\infty}\frac{a_n}{b_n}$ can be written as $\sum_{n=1}^{\infty} b_n S_n$, where $S_1 = 1/a_1$ and $S_{n+1} = 1/(S_n a_{n+1})$. Now either $S_{k_i} \geq 1$ or $S_{k_i} < 1$ and $S_{k_i+1} = 1/(S_{k_i}a_{k_i+1}) > 1/M$ so

$$\sum_{n=1}^{\infty} b_n S_n \geq \sum_{i=1}^{\infty}(S_{k_i} + S_{k_i+1}) \geq \sum_{i=1}^{\infty}\frac{1}{M} \to \infty.$$

Hence the Stern-Stolz series of $\mathrm{K}_{n=1}^{\infty}\frac{a_n}{b_n}$ diverges to infinity and by Theorem 2.1 the continued fraction $\mathrm{K}_{n=1}^{\infty}\frac{a_n}{b_n}$ converges. □

We say that the (infinite) expansion $\langle a_1, a_2, \ldots \rangle_N$ is (eventually) periodic if there exist positive integers k and m such that $a_i = a_{i+k}$ for every $i \geq m$. Then we denote

$$\langle a_1, a_2, \ldots \rangle_N = \langle a_1, \ldots, a_{m-1}, \overline{a_m, \ldots, a_{m+k-1}} \rangle_N.$$

Every periodic dn_N expansion converges by Corollary 2.2 since the partial numerators of periodic continued fractions are bounded. It is easy to see that every periodic dn_N expansion represents a rational number or a quadratic irrational.

Finally, we recall some useful results from the theory of simple continued fractions

$$c_0 + \frac{1}{c_1 +} \frac{1}{c_2 + \cdots} = [c_0; c_1, c_2, \ldots],$$

which are a special case of continued fractions (3) with $a_n = 1$ and $b_n = c_n$ for all n. We denote the convergents of the simple continued fraction expansion by C_n/D_n. As is well known, the simple continued fraction expansion of a real number τ is finite if and only if τ is rational and periodic if and only if τ is a quadratic irrational. Especially,

$$\sqrt{d} = [c_0; \overline{c_1, \ldots, c_{k-1}, 2c_0}],$$

where d is a positive non-square integer, $c_0 = \lfloor \sqrt{d} \rfloor$ and $c_i = c_{k-i}$ for all $1 \leq i \leq k-1$ (see [8]).

For simple continued fractions error estimates (6) take the form

$$\frac{1}{(d_{n+1}+2)D_n^2} < \frac{d_{n+2}}{D_n D_{n+2}} < \left| \tau - \frac{C_n}{D_n} \right| < \frac{1}{D_n D_{n+1}} < \frac{1}{d_{n+1} D_n^2}, \tag{9}$$

which suggests the convergents C_n/D_n are good approximants for τ. In a way they are the only very good approximants as the following Theorem shows:

THEOREM 2.3. *If τ is a real number, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ are coprime and*

$$\left| \tau - \frac{p}{q} \right| < \frac{1}{2q^2},$$

then p/q is a convergent of the simple continued fraction expansion of τ.

For the proof, see for example Lemma 2.33 in [5].

3. The dn_N expansion

Through the rest of this paper, N is a fixed positive integer and τ_0 is an arbitrary positive real number unless stated otherwise.

We now present *the* dn_N *algorithm* for obtaining a dn_N expansion for τ_0:

1) Let $i = 1$.

2) Choose a positive integer a_i such that $a_i/\tau_{i-1} \geq N$.

3) Let $\tau_i = \frac{a_i}{\tau_{i-1}} - N$. If $\tau_i = 0$, terminate. Otherwise let $i = i + 1$ and go to step 2.

As the only criterion for choosing each a_i is to keep τ_i non-negative, we can obtain uncountably many dn_N expansions for τ_0. However, we would like our continued fraction to converge to the number τ_0. Therefore the partial numerators a_i should be chosen so that the series

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}}$$

diverges to infinity, which in the case of dn_N expansions implies the divergence of the Stern-Stolz series (8).

LEMMA 3.1. *If the* dn_N *expansion obtained for* τ_0 *by the* dn_N *algorithm converges, then it converges to* τ_0.

PROOF. By induction,

$$\tau_0 = \frac{A_n + A_{n-1}\tau_n}{B_n + B_{n-1}\tau_n}.$$

Then

$$\begin{aligned}
\left|\tau_0 - \frac{A_n}{B_n}\right| &= \left|\frac{A_n + A_{n-1}\tau_n}{B_n + B_{n-1}\tau_n} - \frac{A_n}{B_n}\right| \\
&= \left|\frac{A_nB_n + A_{n-1}B_n\tau_n - A_nB_n - A_nB_{n-1}\tau_n}{B_n(B_n + B_{n-1}\tau_n)}\right| \\
&= \frac{\tau_n \prod_{i=1}^{n} a_i}{B_n(B_n + B_{n-1}\tau_n)} < \frac{\prod_{i=1}^{n} a_i}{B_nB_{n-1}}.
\end{aligned}$$

Since the continued fraction converges, we have

$$\lim_{n\to\infty} \frac{\prod_{i=1}^{n} a_i}{B_nB_{n-1}} = 0$$

by (5), and hence $\lim\limits_{n\to\infty} A_n/B_n = \tau_0$. □

In the multitude of possibilities for choosing the partial numerators there is a natural method for making the choices uniquely, and that is by choosing each a_i to be as small as possible. Since the smallest positive integer a_i such that $\tau_i = a_i/\tau_{i-1} - N \geq 0$ is $\lceil N\tau_{i-1} \rceil$, we give the following definition:

DEFINITION 1. The dn_N expansion of τ_0 obtained from the dn_N algorithm by choosing $a_i = \lceil N\tau_{i-1} \rceil$ for every i is the *least* dn_N *expansion of* τ_0.

THEOREM 3.2. *The least* dn_N *expansion of* τ_0 *converges.*

PROOF. If the least dn_N expansion of τ_0 is finite, we interpret it as converging. Let the least dn_N expansion $\langle a_1, a_2, \ldots \rangle_N$ of τ_0 be infinite. If $\tau_i \geq 1$, then

$$0 < \tau_{i+1} = \frac{\lceil N\tau_i \rceil}{\tau_i} - N < \frac{N\tau_i + 1}{\tau_i} - N = \frac{1}{\tau_i} \leq 1$$

so there are infinitely many i such that $\tau_i \leq 1$. Since $\tau_i \leq 1$ implies that $a_{i+1} = \lceil N\tau_i \rceil \leq N$, there are infinitely many i such that $a_i \leq N$. Therefore the sequence (a_i) has a bounded subsequence, so by Corollary 2.2 the continued fraction $\langle a_1, a_2, \ldots \rangle_N$ converges and by Lemma 3.1 it converges to τ_0. $\square$

We have now established that every positive real number has at least one converging dn_N expansion. In fact, there are uncountably many such expansions since we may choose $a_i = \lceil N\tau_{i-1} \rceil + 1$ instead of $a_i = \lceil N\tau_{i-1} \rceil$ and still get a converging dn_N expansion. From this point forward, when we talk about a dn_N expansion $\langle a_1, a_2, \ldots \rangle_N$ of a positive real number τ_0, we indicate that the expansion converges to τ_0, that is $\tau_0 = \langle a_1, a_2, \ldots \rangle_N$.

EXAMPLE 1. Here are some least dn_N expansions of different numbers.

τ_0	N	least dn_N expansion of τ_0
$5/17$	1	$\langle 1, 3, 1, 3 \rangle_1$
	10	$\langle 3, 2 \rangle_{10}$
$\sqrt{2}$	1	$\langle \overline{2, 1} \rangle_1$
	2	$\langle 3, 1, 13, 1, 21, 1, 24, 1, 27, 1, 136, 1, 140, 1, 7849, \ldots \rangle_2$
	7	$\langle 10, \overline{1, 50} \rangle_7$
π	1	$\langle 4, 1, 3, 1, 7, 1, 37, 1, 71, 1, 449, 1, 657, 1, 991, \ldots \rangle_1$
	2	$\langle 7, 1, 5, 1, 17, 1, 20, 1, 108, 1, 204, 1, 239, 1, 326, \ldots \rangle_2$
e	1	$\langle 3, 1, 9, 1, 24, 1, 65, 1, 67, 1, 335, 1, 881, 1, 1152, \ldots \rangle_1$
	7	$\langle 20, 3, 10, 2, 23, 2, 5, 6, 4, 5, 9, 2, 4, 2, 22, \ldots \rangle_7$

4. Rational numbers

Throughout this section $\tau_0 = p/q$ is a positive rational number with $p, q \in \mathbb{Z}_+$.

THEOREM 4.1. *Let (k_j) be a sequence of positive integers such that $k_j q > N$ for all j. Then*

$$\frac{p}{q} = \langle k_1 p,\ k_1^2 q^2 - N^2,\ k_1(p+Nq),\ k_2 p,\ k_2^2 q^2 - N^2,\ k_2(p+Nq), \ldots\rangle_N$$
$$= {}_{j=1}^{\infty}\langle\overline{k_j p,\ k_j^2 q^2 - N^2,\ k_j(p+Nq)}\rangle_N. \tag{10}$$

PROOF. With the choices

$$\begin{cases} a_{3j-2} &= k_j p, \\ a_{3j-1} &= k_j^2 q^2 - N^2, \\ a_{3j} &= k_j(p+Nq) \end{cases}$$

for all $j \in \mathbb{Z}_+$ we get inductively from the dn_N algorithm that

$$\tau_{3j-2} = \frac{k_j p}{p/q} - N = k_j q - N > 0,$$

$$\tau_{3j-1} = \frac{k_j^2 q^2 - N^2}{k_j q - N} - N = k_j q$$

and

$$\tau_{3j} = \frac{k_j(p+Nq)}{k_j q} - N = \frac{p}{q} = \tau_0.$$

Hence we obtain the dn_N expansion

$${}_{j=1}^{\infty}\langle\overline{k_j p,\ k_j^2 q^2 - N^2,\ k_j(p+Nq)}\rangle_N.$$

Using the same notation as in the proof of Corollary 2.2, if $S_{3j-3} \le 1$, then since $k_j q \ge N+1$, we have

$$S_{3j} = \frac{a_{3j-1}}{a_{3j-2} a_{3j} S_{3j-3}} > \frac{k_j^2 q^2 - N^2}{k_j^2 p(p+Nq)} \ge \frac{q^2(2N+1)}{p(p+Nq)(N+1)^2}.$$

Therefore the sequence (S_{3j}) is bounded below by a positive constant and the Stern-Stolz series of continued fraction (10) diverges to infinity. Then the convergence of continued fraction (10) to p/q follows from Theorem 2.1 and Lemma 3.1. □

COROLLARY 4.2. *The rational number p/q has infinitely many periodic dn_N expansions and uncountably many aperiodic dn_N expansions.*

PROOF. By Theorem 4.1 the rational number p/q has the dn_N expansion (10) for any sequence of positive integers (k_j) that satisfies $k_j q > N$ for all j. If (k_j) is periodic of period length m, then the dn_N expansion

$$\frac{p}{q} = {}_{j=1}^{\infty}\langle\overline{k_j p,\ k_j^2 q^2 - N^2,\ k_j(p+Nq)}\rangle_N$$

is periodic of period length $3m$ at most. As there are infinitely many periodic sequences (k_j), it follows that there are infinitely many periodic dn_N expansions for p/q.

On the other hand, if we choose the sequence (k_j) to be such that it has a strictly increasing subsequence, then the dn_N exapansion (10) is aperiodic because it contains arbitrarily large partial numerators. As there are uncountably many such sequences (k_j), there are uncountably many aperiodic dn_N expansions for p/q. $\square$

EXAMPLE 2. Let $\tau_0 = 22/7$ and $N = 13$. It is then that Theorem 4.1 gives the dn_{13} expansion

$$\frac{22}{7} = {}_{j=1}^{\infty}\langle\overline{22k_j,\ 49k_j^2 - 169,\ 127k_j}\rangle_{13},$$

where the sequence (k_j) satisfies $k_j \geq 2$ for all j. Using different sequences (k_j) we get the following dn_{13} expansions:

(k_j)	dn_{13} expansion of $22/7$
$k_j = 2$ for all j	$\langle\overline{44,\ 27,\ 254}\rangle_{13}$
$k_{2i-1} = 2,\ k_{2i} = 3$	$\langle\overline{44,\ 27,\ 254,\ 66,\ 272,\ 381}\rangle_{13}$
$k_j = j+1$ for all j	$\langle 44,\ 27,\ 254,\ 66,\ 272,\ 381,\ 88,\ 615,\ldots\rangle_{13}$

In Example 1 both of the least dn_N expansions calculated for $5/17$ were finite. It turns out this is the case for every least dn_N expansion of a positive rational number.

THEOREM 4.3. *The least* dn_N *expansion of* $\tau_0 = p/q$ *is finite.*

PROOF. Let us denote $P_0 = p$, $Q_0 = q$ and $S_0 = P_0 + Q_0$. By the division algorithm there exist unique $q_1, r_1 \in \mathbb{N}$ such that $NP_0 = q_1 Q_0 + r_1$, where $0 < r_1 \leq Q_0$. Then $\lceil NP_0/Q_0 \rceil = q_1 + 1$. Using the dn_N algorithm we have

$$\tau_1 = \frac{\lceil NP_0/Q_0 \rceil}{P_0/Q_0} - N = \frac{Q_0(q_1+1) - NP_0}{P_0} = \frac{Q_0 - r_1}{P_0}.$$

If $r_1 = Q_0$, then $\tau_1 = 0$ and the algorithm terminates. If $0 < r_1 < Q_0$, we put $P_1 = Q_0 - r_1$ and $Q_1 = P_0$. Note that now

$$S_0 = P_0 + Q_0 > P_0 + Q_0 - r_1 = P_1 + Q_1 = S_1.$$

Suppose we have reached $\tau_i = P_i/Q_i$, $P_i, Q_i \in \mathbb{Z}_+$ and $S_i = P_i + Q_i$. By the division algorithm there exist unique $q_{i+1}, r_{i+1} \in \mathbb{N}$ such that

$$NP_i = q_{i+1}Q_i + r_{i+1},$$

where $0 < r_{i+1} \leq Q_i$. Then $\lceil NP_i/Q_i \rceil = q_{i+1} + 1$ and

$$\tau_{i+1} = \frac{\lceil NP_i/Q_i \rceil}{P_i/Q_i} - N = \frac{Q_i(q_{i+1}+1) - NP_i}{P_i} = \frac{Q_i - r_{i+1}}{P_i}.$$

If $r_{i+1} = Q_i$, then $\tau_{i+1} = 0$ and the algorithm terminates. If $0 < r_{i+1} < Q_i$, we put $P_{i+1} = Q_i - r_{i+1}$ and $Q_{i+1} = P_i$. Then

$$S_i = P_i + Q_i > P_i + Q_i - r_{i+1} = P_{i+1} + Q_{i+1} = S_{i+1}.$$

Because the sequence (S_i) is a strictly decreasing sequence of positive integers and $Q_i > 0$, it follows there must exist an $n \in \mathbb{Z}_+$ such that $P_n = 0$. Then $\tau_n = 0$ and the algorithm terminates. Thus the least dn_N expansion of τ_0 is finite. □

We get infinitely many finite dn_N expansions for p/q by choosing the first finitely many a_i as we please and then making the least choice from there on.

5. Quadratic irrationals

Let us start by noting that because there are uncountably many infinite dn_N expansions for every positive real number but there exist only countably many periodic dn_N expansions, it follows that every positive quadratic irrational number has uncountably many aperiodic dn_N expansions.

Throughout this section τ_0 is a positive real quadratic irrational. Now there exist $P, Q, d \in \mathbb{Z}$ such that $\tau_0 = (\sqrt{d} + P)/Q$, $d \geq 2$ is not a perfect square and $Q \,\big|\, (d - P^2)$ (see for example [8], Lemma 10.5). Then we denote $Q' = |(d - P^2)/Q| = |\sqrt{d} - P|\tau_0$.

LEMMA 5.1. *If $|P| < \sqrt{d}$ and $k \in \mathbb{Z}_+$ is such that $k(\sqrt{d} - P) > N$, then*

$$\tau_0 = \langle kQ', \overline{D - 2kPN - N^2, D} \rangle_N, \tag{11}$$

where $D = k^2(d - P^2)$.

PROOF. Since $|P| < \sqrt{d}$ and τ_0 is positive, it follows that Q and D are positive and $Q' = (d - P^2)/Q$. If we choose $a_1 = kQ'$, we get from the dn_N algorithm

$$\tau_1 = \frac{kQ'}{\tau_0} - N = \frac{k(d - P^2)}{\sqrt{d} + P} - N = k\sqrt{d} - (kP + N) > 0.$$

As $k(\sqrt{d} + P) + N > 0$, we may continue by choosing

$$a_2 = D - 2kPN - N^2 = k^2 d - (kP + N)^2 = (k(\sqrt{d} + P) + N)\tau_1 > 0$$

and get

$$\tau_2 = \frac{(k(\sqrt{d} + P) + N)\tau_1}{\tau_1} - N = k\sqrt{d} + kP > 0.$$

Finally with $a_3 = D$ we have

$$\tau_3 = \frac{k^2(d - P^2)}{k\sqrt{d} + kP} - N = k\sqrt{d} - (kP + N) = \tau_1,$$

and thus we get the periodic expansion $\tau_0 = \langle kQ', \overline{D - 2kPN - N^2, D}\rangle_N$. $\square$

THEOREM 5.2. *There exists a periodic* dn_N *expansion of the positive real quadratic irrational* τ_0.

PROOF. We begin by constructing the desired dn_N expansion for τ_0. Let us denote $P_0 = P$, $Q_0 = Q$ and $R_0 = 1$. Let k_1 be the smallest positive integer such that $k_1|\sqrt{d} - P_0| > N$ and $a_1 = k_1 Q'$. Then we get from the dn_N algorithm

$$\tau_1 = \frac{a_1}{\tau_0} - N = \frac{k_1|\sqrt{d} - P_0|\tau_0}{\tau_0} - N = k_1|\sqrt{d} - P_0| - N > 0.$$

Now we denote $\tau_1 = R_1\sqrt{d} + P_1$, where

$$\begin{cases} R_1 = k_1 \text{ and } P_1 = -k_1 P_0 - N, & \text{when } \sqrt{d} > P_0, \\ R_1 = -k_1 \text{ and } P_1 = k_1 P_0 - N, & \text{when } \sqrt{d} < P_0. \end{cases}$$

If $\tau_i = R_i\sqrt{d} + P_i$, we choose $a_{i+1} = k_{i+1}|R_i^2 d - P_i^2|$, where k_{i+1} is the smallest positive integer such that $k_{i+1}|R_i\sqrt{d} - P_i| > N$ and get

$$\tau_{i+1} = \frac{k_{i+1}|R_i^2 d - P_i^2|}{R_i\sqrt{d} + P_i} - N = k_{i+1}|R_i\sqrt{d} - P_i| - N > 0.$$

Then we denote $\tau_{i+1} = R_{i+1}\sqrt{d} + P_{i+1}$, where

$$\begin{cases} R_{i+1} = k_{i+1}R_i \text{ and } P_{i+1} = -k_{i+1}P_i - N, & \text{when } R_i\sqrt{d} > P_i, \\ R_{i+1} = -k_{i+1}R_i \text{ and } P_{i+1} = k_{i+1}P_i - N, & \text{when } R_i\sqrt{d} < P_i. \end{cases}$$

It remains to be shown that the dn_N expansion $\langle a_1, a_2, \ldots\rangle_N$ constructed above is periodic. Note that if we choose k in Lemma 5.1 to be as small as possible, then the periodic dn_N expansion (11) is a special case of the dn_N expansion under study. Therefore it suffices to show that there exists a $j \geq 1$ such that R_j is positive and

$$|P_j| < |R_j\sqrt{d}| = R_j\sqrt{d} = \sqrt{R_j^2 d}$$

in which case Lemma 5.1 gives us the periodicity.

Suppose on the contrary that

$$|P_i| > |R_i\sqrt{d}| \text{ for all } i \geq 1. \tag{12}$$

Then P_i is positive for all i because $\tau_i = R_i\sqrt{d} + P_i$ is positive for all i. If $k_i = 1$ for every large i, then $P_{i+1} = P_i - N$ and $R_{i+1} = -R_i$ for every large i. In this case the sequence (P_i) is a strictly decreasing sequence of integers and so there exists a j such that $P_j < 0$, which we can't have. Hence there exist infinitely many j such that $k_j > 1$. This implies that the sequence $(|R_i|)$ is tending to infinity so there exists an n such that $|R_i\sqrt{d}| > N$ for all $i \geq n$.

Let $m \geq n$ be such that $k_{m+1} \geq 2$. As k_{m+1} is the least positive integer such that

$$k_{m+1}P_m - k_{m+1}R_m\sqrt{d} = k_{m+1}|P_m - R_m\sqrt{d}| > N,$$

then

$$(k_{m+1} - 1)P_m - (k_{m+1} - 1)R_m\sqrt{d} < N.$$

Combining the above inequalities yields

$$0 < k_{m+1}P_m - k_{m+1}R_m\sqrt{d} - N < P_m - R_m\sqrt{d} < N, \tag{13}$$

where the last inequality holds because $k_{m+1} \geq 2$. Since $P_m > |R_m\sqrt{d}| > N$ and $P_m - R_m\sqrt{d} < N$, then $R_m > 0$ and

$$P_{m+1} - R_{m+1}\sqrt{d} = k_{m+1}P_m - N + k_{m+1}R_m\sqrt{d} > 3N.$$

Thus $k_{m+2} = 1$ and by (13) we have

$$P_{m+2} = P_{m+1} - N = k_{m+1}P_m - 2N < k_{m+1}R_m\sqrt{d} = R_{m+2}\sqrt{d},$$

where $R_{m+2} = k_{m+1}R_m$ is positive. This is a contradiction with assumption (12). Thus there exists a $j \geq 1$ such that R_j is positive and $|P_j| < R_j\sqrt{d}$ and by Lemma 5.1

$$\tau_0 = \langle a_1, a_2, \ldots, a_j, D/k_{j+1}, \overline{D - 2k_{j+1}P_jN - N^2, D}\rangle_N,$$

where $D = k_{j+1}^2(R_j^2 d - P_j^2)$. □

Since we may choose the first finitely many a_i as we please and then continue as described in Lemma 5.1 and Theorem 5.2, every positive quadratic irrational has infinitely many different periodic dn_N expansions.

Example 3. Let $\tau_0 = (7 + \sqrt{10})/13$ and $N = 4$. Constructing the dn_4 expansion described in Theorem 5.2 we get

$$\begin{aligned}
&k_1 = 2, && a_1 = 6, && \tau_1 = 10 - 2\sqrt{10},\\
&k_2 = 1, && a_2 = 60, && \tau_2 = 6 + 2\sqrt{10},\\
&k_3 = 13, && a_3 = 52, && \tau_3 = 26\sqrt{10} - 82,\\
&k_4 = 1, && a_4 = 36, && \tau_4 = 26\sqrt{10} + 78,\\
&k_5 = 1, && a_5 = 676, && \tau_5 = 26\sqrt{10} - 82 = \tau_3
\end{aligned}$$

and so

$$\frac{7+\sqrt{10}}{13} = \langle 6, 60, 52, \overline{36, 676}\rangle_4.$$

We now turn our attention to the least dn_N expansions of positive real quadratic irrationals. In [1] it is conjectured that the best cf_N expansion of a positive quadratic irrational is not periodic for every N. It seems likely that this is the case for the least dn_N expansion as well. For example the dn_1 expansion of $\sqrt{3}$ is

$$\begin{aligned}\sqrt{3} = \langle & 2, 1, 6, 1, 10, 1, 11, 1, 18, 1, 50, 1, 65, 1, 750, 1, 8399, 1, 11727, 1, 12855,\\ & 1, 66368, 1, 281130, 1, 437015, 1, 482182, 1, 643701, 1, 743770, 1,\\ & 2808107, 1, 11306550, 1, 12268089, 1, 24304646, 1, 98323268, 1, \ldots\rangle_1,\end{aligned}$$

where the partial numerators seem to alternate between 1 and a rapidly increasing sequence of positive integers.

However, there are some cases when we can find a periodic least dn_N expansion. Recall that if $|P| < \sqrt{d}$ and $k \in \mathbb{Z}_+$ is such that $k(\sqrt{d}-P) > N$, then by Lemma 5.1

$$\tau_0 = \langle kQ', \overline{D - 2kPN - N^2, D}\rangle_N, \tag{14}$$

where $D = k^2(d-P^2)$.

Theorem 5.3. *Let $|P| < \sqrt{d}$ and $k \in \mathbb{Z}_+$ be such that $k(\sqrt{d}-P) > N$. Then expansion* (14) *is the least* dn_N *expansion of τ_0 if and only if*

$$0 < (\sqrt{d}-P) - \frac{N}{k} < \frac{1}{(\sqrt{d}+P)k^2}. \tag{15}$$

Proof. As noted in the proof of 5.1, in this case the numbers Q, Q' and D are positive integers. Expansion (14) is the least dn_N expansion if and only if $a_i = \lceil N\tau_{i-1}\rceil$ for every i. Since expansion (14) is periodic, it suffices to check that $kQ' = \lceil N\tau_0\rceil$, $D-2kPN-N^2 = \lceil N\tau_1\rceil$ and $D = \lceil N\tau_2\rceil$. From the proof of 5.1 we have that $\tau_1 = k\sqrt{d} - (kP+N)$ and $\tau_2 = k\sqrt{d} + kP$. If $D = \lceil N\tau_2\rceil$ then

$$k^2(d-P^2) = \lceil Nk(\sqrt{d}+P)\rceil = Nk(\sqrt{d}+P) + c, \tag{16}$$

where $0 < c < 1$. Now

$$kQ' = N\frac{\sqrt{d}+P}{Q} + \frac{c}{kQ} = N\tau_0 + \frac{c}{kQ},$$

where $0 < c/kQ < 1$ and

$$\begin{aligned} D - 2kPN - N^2 &= Nk(\sqrt{d}+P) - 2kPN - N^2 + c\\ &= N(k\sqrt{d} - (kP+N)) + c = N\tau_1 + c,\end{aligned}$$

so $kQ' = \lceil N\tau_0 \rceil$ and $D - 2kPN - N^2 = \lceil N\tau_1 \rceil$. It is therefore enough to study when $D = \lceil N\tau_2 \rceil$.

By (16) $D = \lceil N\tau_2 \rceil$ if and only if

$$0 < c = k^2(d - P^2) - Nk(\sqrt{d} + P) = k(\sqrt{d} + P)(k(\sqrt{d} - P) - N) < 1,$$

which is equivalent to (15). $\square$

REMARK. If $\sqrt{d} + P > 2$, then by Theorem 2.3 and inequalities (7) condition (15) can hold only if N/k is an even convergent of the simple continued fraction expansion of $\sqrt{d} - P$. If

$$\sqrt{d} - P = [c_0; \overline{c_1, \dots, c_{m-1}, c_m}],$$

then by (9) we have

$$\frac{1}{(c_{2n+1} + 2)D_{2n}^2} < (\sqrt{d} - P) - \frac{C_{2n}}{D_{2n}} < \frac{1}{c_{2n+1} D_{2n}^2} \tag{17}$$

for all $n \in \mathbb{N}$. Thus, if there exists a $c_{2n+1} > \sqrt{d} + P$, then by Theorem 5.3 expansion (14) is the least dn_N expansion of τ_0 when $N = C_{2n+lm}$ and $k = D_{2n+lm}$ for any $l \in \mathbb{N}$. By contrast, if $c_{2n+1} + 2 < \sqrt{d} + P$ for all n, then (14) is never the least dn_N expansion of τ_0.

THEOREM 5.4. *Let $\tau_0 = \sqrt{d}$ where d is a positive integer and not a perfect square. If m is a positive integer, then*

$$\sqrt{d} = \langle 2kd, 2(k^2 d - m^2), \overline{k^2 d - m^2} \rangle_{2m}, \tag{18}$$

where k is a positive integer such that $k\sqrt{d} > m$. Expansion (18) *is the least dn_{2m} expansion of $\sqrt{d}$ if and only if*

$$0 < \sqrt{d} - \frac{m}{k} < \frac{1}{2k\sqrt{d}}. \tag{19}$$

PROOF. Let k be a positive integer such that $k\sqrt{d} > m$. By choosing $a_1 = 2kd$ we get from the dn_N algorithm

$$\tau_1 = \frac{2kd}{\sqrt{d}} - 2m = 2(k\sqrt{d} - m) > 0.$$

We continue by choosing $a_2 = 2(k^2 d - m^2) > 0$ and get

$$\tau_2 = \frac{2(k^2 d - m^2)}{2(k\sqrt{d} - m)} - 2m = k\sqrt{d} - m > 0.$$

Finally, with $a_3 = k^2 d - m^2$ we have

$$\tau_3 = \frac{k^2 d - m^2}{k\sqrt{d} - m} - 2m = k\sqrt{d} - m = \tau_2$$

and hence we get periodic expansion (18).

Now $a_1 = 2kd = \lceil 2m\sqrt{d}\rceil$ if and only if $0 < 2kd - 2m\sqrt{d} < 1$ which is equivalent to (19). If inequality (19) holds, then

$$0 < a_2 - 2m\tau_1 = 2(k\sqrt{d} - m)(k\sqrt{d} + m - 2m) < \frac{1}{2d}$$

and

$$0 < a_3 - 2m\tau_2 = (k\sqrt{d} - m)(k\sqrt{d} + m - 2m) < \frac{1}{4d}$$

so $a_2 = \lceil 2m\tau_1\rceil$ and $a_3 = \lceil 2m\tau_2\rceil$. Hence expansion (18) is the least dn_{2m} expansion of $\sqrt{d}$ if and only if inequality (19) holds. □

Remark. By (17) inequality (19) has infinitely many solutions in m/k for every $\sqrt{d}$, as we may choose $m = C_{2n}$ and $k = D_{2n}$ when n is large enough. Consequently every irrational $\sqrt{d}$ has infinitely many periodic least dn_N expansions.

Example 4. Periodic least dn_N expansions given by Theorem 5.3:

τ_0	N	least dn_N expansion of τ_0
$\sqrt{K^2+1}$	K	$\langle \overline{K^2+1, 1}\rangle_K$
$\sqrt{2}$	7	$\langle 10, \overline{1, 50}\rangle_7$
$\frac{1+\sqrt{5}}{2}$	1	$\langle 2, \overline{1, 4}\rangle_1$
$\frac{-2+\sqrt{13}}{3}$	5	$\langle 3, \overline{4, 9}\rangle_5$

Periodic least dn_N expansions given by Theorem 5.4:

τ_0	N	least dn_N expansion of τ_0
$\sqrt{2}$	14	$\langle 20, 2, \overline{1}\rangle_{14}$
$\sqrt{3}$	10	$\langle 18, 4, \overline{2}\rangle_{10}$
$\sqrt{6}$	44	$\langle 108, 4, \overline{2}\rangle_{44}$

Other periodic least dn_N expansions:

τ_0	N	least dn_N expansion of τ_0
$\sqrt{7}$	13	$\langle 35, 3, \overline{2, 59, 2}\rangle_{13}$
$3+\sqrt{2}$	1	$\langle 5, 1, 7, \overline{1, 14}\rangle_1$
$\frac{6+\sqrt{3}}{2}$	5	$\langle 20, 1, 4, \overline{1, 2, 2, 1, 5, 5}\rangle_5$

6. Bounded partial numerators

One of the major open questions of Diophantine approximation is if the simple continued fraction expansions of algebraic numbers of degree greater than 2 have bounded partial denominators. In the case of dn_N expansions the analogue is quickly solved. In fact, we show below that for every positive real number there exists a dn_N expansion that has partial numerators from a set of two digits only.

LEMMA 6.1. *Let α_1 and α_2 be positive integers such that $\alpha_1 < \alpha_2$ and*

$$\alpha_1\alpha_2/(\alpha_1+\alpha_2) \geq N^2, \tag{20}$$

and denote

$$\begin{aligned}
\tau_m &= \frac{-(N^2+\alpha_2-\alpha_1)+\sqrt{(N^2+\alpha_2-\alpha_1)^2+4\alpha_1N^2}}{2N}, \\
\tau_M &= \frac{-(N^2-\alpha_2+\alpha_1)+\sqrt{(N^2+\alpha_2-\alpha_1)^2+4\alpha_1N^2}}{2N} = \tau_m + \frac{\alpha_2-\alpha_1}{N}
\end{aligned}$$

and $I = [\tau_m, \tau_M]$. If $\tau_0 \in I$, there exists a dn_N expansion $\tau_0 = \langle a_1, a_2, \ldots\rangle_N$ such that $a_i \in \{\alpha_1, \alpha_2\}$ for all i.

PROOF. As the positive solutions x to equations

$$x = \frac{\alpha_1}{N+\frac{\alpha_2}{N+x}} \quad \text{and} \quad x = \frac{\alpha_2}{N+\frac{\alpha_1}{N+x}}$$

are $x = \tau_m$ and $x = \tau_M$, respectively, it follows that

$$\tau_m = \langle\overline{\alpha_1, \alpha_2}\rangle_N \quad \text{and} \quad \tau_M = \langle\overline{\alpha_2, \alpha_1}\rangle_N.$$

Let us denote $T_1(x) = \alpha_1/(N+x)$ and $T_2(x) = \alpha_2/(N+x)$ for $x \in I$. Then

$$T_1(\tau_M) = \tau_m, \quad T_2(\tau_m) = \tau_M, \quad T_1(\tau_m) = \frac{\alpha_1}{\alpha_2}\tau_M \quad \text{and} \quad T_2(\tau_M) = \frac{\alpha_2}{\alpha_1}\tau_m.$$

Because

$$\begin{aligned}
& \frac{\alpha_1}{\alpha_2}\tau_M = \frac{\alpha_1}{\alpha_2}\left(\tau_m + \frac{\alpha_2-\alpha_1}{N}\right) \geq \frac{\alpha_2}{\alpha_1}\tau_m \\
\Leftrightarrow \quad & \frac{\alpha_1(\alpha_2-\alpha_1)}{\alpha_2 N}\cdot\frac{\alpha_1\alpha_2}{\alpha_2^2-\alpha_1^2} = \frac{\alpha_1^2}{N(\alpha_2+\alpha_1)} \geq \tau_m \\
\Leftrightarrow \quad & \frac{2\alpha_1^2}{(\alpha_2+\alpha_1)} + N^2+\alpha_2-\alpha_1 \geq \sqrt{(N^2+\alpha_2-\alpha_1)^2+4\alpha_1N^2} \\
\Leftrightarrow \quad & \frac{\alpha_1^3}{\alpha_2+\alpha_1} + \alpha_1(N^2+\alpha_2-\alpha_1) \geq (\alpha_2+\alpha_1)N^2 \\
\Leftrightarrow \quad & \frac{\alpha_1\alpha_2}{\alpha_2+\alpha_1} \geq N^2,
\end{aligned}$$

then inequality (20) implies that $T_1(\tau_m) \geq T_2(\tau_M)$. Therefore

$$T_1(I)\cup T_2(I) = [\tau_m, T_1(\tau_m)] \cup [T_2(\tau_M), \tau_M] = [\tau_m, \tau_M] = I. \tag{21}$$

Let $\tau_0 \in I$. As the functions T_1 and T_2 are injective on I, then by (21) there exists a $\tau_1 \in I$ such that

$$\tau_0 = \frac{a_1}{N + \tau_1} \quad \Leftrightarrow \quad \tau_1 = \frac{a_1}{\tau_0} - N,$$

where $a_1 \in \{\alpha_1, \alpha_2\}$. Similarily, if $\tau_i \in I$, then there exists a $\tau_{i+1} \in I$ such that

$$\tau_i = \frac{a_{i+1}}{N + \tau_{i+1}} \quad \Leftrightarrow \quad \tau_{i+1} = \frac{a_{i+1}}{\tau_i} - N,$$

where $a_{i+1} \in \{\alpha_1, \alpha_2\}$. It follows by induction that $\tau_0 = \langle a_1, a_2, a_3, \ldots \rangle_N$, where $a_i \in \{\alpha_1, \alpha_2\}$ for all i. □

THEOREM 6.2. *Let τ_0 be a positive real number. Then there exist positive integers α_1 and α_2 such that $\tau_0 = \langle a_1, a_2, \ldots \rangle_N$, where $a_i \in \{\alpha_1, \alpha_2\}$ for all i.*

PROOF. Due to Lemma 6.1 it is sufficient to show that there exist positive integers α_1 and α_2 such that $\alpha_1 < \alpha_2$, $\alpha_1\alpha_2/(\alpha_1 + \alpha_2) \geq N^2$ and $\tau_0 \in [\tau_m, \tau_M]$, where τ_m and τ_M are as in Lemma 6.1. Now

$$\begin{aligned} \tau_m &= \frac{-(N^2 + \alpha_2 - \alpha_1) + \sqrt{(N^2 + \alpha_2 - \alpha_1)^2 + 4\alpha_1 N^2}}{2N} \\ &= \frac{N^2 + \alpha_2 - \alpha_1}{2N} \left(-1 + \sqrt{1 + \frac{4\alpha_1 N^2}{(N^2 + \alpha_2 - \alpha_1)^2}} \right). \end{aligned}$$

Since the function

$$f(x) = x \left(-1 + \sqrt{1 + \frac{\alpha_1}{x^2}} \right)$$

is strictly decreasing and tends to 0 as x tends to infinity for all positive α_1, then

$$\tau_m < \tau_0 < \tau_m + \frac{\alpha_2 - \alpha_1}{N} = \tau_M \tag{22}$$

when $\alpha_2 - \alpha_1$ is large enough. Because

$$\frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2} = \frac{1}{1/\alpha_1 + 1/\alpha_2} \geq \frac{\alpha_1}{2},$$

then we may choose $\alpha_1 \geq 2N^2$ and α_2 such that $\alpha_2 - \alpha_1$ is large enough for (22) to hold true. □

EXAMPLE 5. Here are the first 20 digits of some dn_1 expansions with bounded numerators.

τ_0	$\{a, b\}$	bounded dn_N expansion of τ_0
$\sqrt[3]{2}$	$\{2, 4\}$	$\langle 2, 2, 4, 2, 4, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 2, 2, 4, \ldots \rangle_1$
π	$\{2, 5\}$	$\langle 5, 2, 5, 2, 2, 5, 5, 2, 2, 2, 2, 5, 5, 2, 2, 5, 2, 5, 5, 5, \ldots \rangle_1$
e	$\{3, 7\}$	$\langle 7, 3, 3, 7, 7, 7, 3, 3, 7, 3, 7, 3, 3, 7, 3, 3, 3, 7, 3, 3, \ldots \rangle_1$
$\ln 2$	$\{2, 4\}$	$\langle 2, 4, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4, 2, 2, 4, 2, 2, 4, 4, 4, \ldots \rangle_1$

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