NORM, ESSENTIAL NORM AND WEAK COMPACTNESS OF WEIGHTED COMPOSITION OPERATORS BETWEEN DUAL BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract. In this paper we estimate the norm and the essential norm of weighted composition operators from a large class of- non-necessarily reflexive- Banach spaces of analytic functions on the open unit disk into weighted type Banach spaces of analytic functions and Bloch type spaces. We also show the equivalence of compactness and weak compactness of weighted composition operators from these weighted type spaces into a class of Banach spaces of analytic functions, that includes a large family of conformally invariant spaces like BMOA and analytic Besov spaces.

1. Introduction

Very recently Colonna and Tjani [7] have obtained estimates of the essential norm of weighted composition operators acting on a family of reflexive Banach spaces of analytic functions on the open unit disk into weighted Banach spaces of analytic functions as well as into Bloch type spaces. The natural question is whether these essential norm formulas hold for non-reflexive Banach spaces of analytic functions as it was already hinted by them. In this paper we show that this is the case for a large class of dual Banach spaces of analytic functions, among them the Hardy space H^1 and the Bergman space A^1. Our approach is mainly based on a new estimate of the essential norm of a weighted composition operator (Theorem 4.2), which allows us to work with compact dual operators instead of just compact operators when calculating the essential norm. The result is based on the fact that for many spaces the weighted composition operator is a dual operator together with an extension of a result due to Axler, Jewell and Shields [3] concerning the relationship between the essential norm of an operator and the essential norm of its adjoint.

We also obtain (Section 3) formulas for the norm of a bounded operator acting from a general Banach space of analytic functions on the open unit disk into a weighted Banach space of analytic functions or a Bloch type space. From these formulas we then derive expressions of the norms of weighted composition operators.

Moreover we show (Theorem 5.1) that a weakly compact weighted composition operator from a Bloch type space into a wide class of Banach spaces of analytic functions on the open unit disk is always compact, and thereby much earlier work of many authors is unified into a single

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theorem. Motivation for such a result comes from the recent work of Contreras, Díaz-Madrigal and Vukotić [10]. Their main result states that every weakly compact composition operator from the Bloch space into any space of a large class of conformally invariant spaces is compact. In the literature there exist numerous results on the equivalence of weak compactness and compactness for composition operators, see for example [19, 24, 22]. The corresponding much harder problem whether every weakly compact composition self-operator of BMOA is compact was settled in the affirmative by Laitila, Nieminen, Tylli and Saksman [16].

Finally, we derive estimates of the essential norms of the weighted composition operator acting on the Hardy spaces and the weighted Bergman spaces in terms of the type n^{th} power of the symbol.

Some of our results in this paper are related to the work in [12] and [13].

Properties of composition operators have been extensively studied on various function spaces during recent decades. We refer to [11] and [23].

2. Preliminaries and notation

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc and $H(\mathbb{D})$ the space of all analytic functions $\mathbb{D} \to \mathbb{C}$. The weighted Banach spaces of analytic functions H_v^∞ and H_v^0 are defined by

$$\begin{aligned} H_v^\infty &= \Big\{ f \in H(\mathbb{D}) \,:\, \|f\|_{H_v^\infty} := \sup_{z\in\mathbb{D}} v(z)|f(z)| \,<\, +\infty \Big\} \\ H_v^0 &= \Big\{ f \in H_v^\infty \,:\, \lim_{|z|\to 1^-} v(z)|f(z)| = 0 \Big\}, \end{aligned}$$

where the weight $v : \mathbb{D} \to \mathbb{R}$ is a continuous and strictly positive function which is also assumed to be radial, that is $v(z) = v(|z|)$ for all $z \in \mathbb{D}$, and non-increasing with respect to $|z|$. We also want it to satisfy $\lim_{|z|\to 1} v(z) = 0$. Throughout the paper we will assume that the weight has these properties.

Moreover, we will study the Bloch-type spaces $\mathcal{B}_v$ and $\mathcal{B}_v^0$ defined by

$$\begin{aligned} \mathcal{B}_v &= \Big\{ f \in H(\mathbb{D}) \,:\, \|f\|_{\mathcal{B}_v} := |f(0)| + \sup_{z\in\mathbb{D}} v(z)|f'(z)| \,<\, +\infty \Big\} \\ \mathcal{B}_v^0 &= \Big\{ f \in \mathcal{B}_v \,:\, \lim_{|z|\to 1^-} v(z)|f'(z)| = 0 \Big\}. \end{aligned}$$

as well as the spaces $\tilde{\mathcal{B}}_v := \{f \in \mathcal{B}_v :\ f(0) = 0\}$ and $\tilde{\mathcal{B}}_v^0 := \{f \in \mathcal{B}_v^0 :\ f(0) = 0\}$.

By $E \approx F$ we mean that the spaces E and F are isomorphic. We shall also use the notation $A \lesssim B$ if there is a positive constant c, not depending on properties of A and B, such that $A \le cB$. We write $A \asymp B$ whenever $A \lesssim B$ and $B \lesssim A$.

Lusky [18] has shown that $H_v^\infty \approx l^\infty$ and $H_v^0 \approx c_0$ for a large class of weights including the normal weights, i.e. weights v for which

$$\inf_k \limsup_n \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1 \quad \text{and} \quad \sup_n \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty.$$

Therefore H_v^∞ and H_v^0 have the λ-metric approximation property whenever v is a normal weight. By the differentiation operator $D : f \mapsto f'$, the spaces H_v^∞ and $\tilde{\mathcal{B}}_v$ as well as their "vanishing at the boundary" subspaces H_v^0 and $\tilde{\mathcal{B}}_v^0$ are isometrically isomorphic. The map $(f, \lambda) \mapsto f + \lambda$

gives that also $\tilde{\mathcal{B}}_v \oplus_1 \mathbb{C} \approx \mathcal{B}_v$ and $\tilde{\mathcal{B}}_v^0 \oplus_1 \mathbb{C} \approx \mathcal{B}_v^0$ are isometrically isomorphic. Hence also $\mathcal{B}_v$ and $\mathcal{B}_v^0$ have the λ-metric approximation property for normal weights.

Weights of the type $v_\alpha(z) := (1-|z|^2)^\alpha$ with $\alpha > 0$ are called standard weights, and they are clearly normal.

Recall that the Hardy space H^p with $1 \le p < \infty$ consists of all $f \in H(\mathbb{D})$ such that $\|f\|_{H^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} < \infty$. The standard weighted Bergman space A^p_α with $\alpha > -1$ and $1 \le p < \infty$ is the space

$$A^p_\alpha = \{f \in H(\mathbb{D}) : \|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty\},$$

where $dA(z)$ denotes the normalized area measure of $\mathbb{D}$.

Let $\mathcal{X}$ be a Banach space of analytic functions on the unit disc $\mathbb{D}$ and let $\|\cdot\|$ denote its norm. Recall that for each $z \in \mathbb{D}$, δ_z is the evaluation functional defined by $\delta_z(f) = f(z)$ for all $f \in \mathcal{X}$. If $\mathcal{X}$ contains the constant functions, then all δ_z are non-zero.

In [7] Colonna and Tjani introduced several conditions on $\mathcal{X}$ of which we will use the following ones:

(I) The closed unit ball $B_{\mathcal{X}}$ of $\mathcal{X}$ is compact with respect to the compact open topology co. In particular, the identity map $Id : (\mathcal{X}, \|\cdot\|) \to (\mathcal{X}, co)$ is continuous, hence $\delta_z \in X^*$. Moreover, since every δ_z is $co-$continuous, we have have that the norm $\|\delta_z\|$ is attained at some $f_z \in \mathcal{X}$ with $\|f_z\|_{\mathcal{X}} \le 1$ for every $z \in \mathbb{D}$, that is, $f_z(z) = \|\delta_z\|$.

(II) $\lim_{|z|\to 1} \|\delta_z\| = \infty$.

(III) For $0 < r < 1$, the linear operator $T_r : \mathcal{X} \to \mathcal{X}, f \mapsto f_r$ is compact, where $f_r(z) = f(rz)$.

(IV) There is a constant $C > 0$ such that for all $f \in \mathcal{X}$ and all $z \in \mathbb{D}$ we have

$$(1-|z|^2)|f'(z)| \le C\|f\|_{\mathcal{X}} \cdot \|\delta_z\|.$$

If (I) holds, then the functional evaluation $\delta'_z : \mathcal{X} \to \mathbb{C}$ defined as $\delta'_z(f) = f'(z)$ is bounded, and from (IV) we have

$$\|\delta'_z\| = \sup_{\|f\|_{\mathcal{X}} \le 1} |f'(z)| \lesssim \frac{\|\delta_z\|}{1-|z|^2}.$$

(V) We have that $\sup_{0<r<1} \|T_r\|_{\mathcal{X}\to\mathcal{X}} < \infty$ for the operators T_r in (III).

Moreover, we need an extra condition:

(VI) For all $u \in H^\infty$ the pointwise multiplication operator $M_u : \mathcal{X} \to \mathcal{X}$ satisfies $\|M_u\|_{\mathcal{X}\to\mathcal{X}} \le \|u\|_\infty$. In particular, $H^\infty \subset \mathcal{X}$.

From condition (I) we obtain by using the Dixmier-Ng theorem [21] that the space

$${}^*\mathcal{X} := \{l \in \mathcal{X}^* : l_{|B_{\mathcal{X}}} \text{ is } co\text{-continuous}\},$$

endowed with the norm induced by $\mathcal{X}^*$ is a Banach space and the evaluation map $\mathcal{X} \to ({}^*\mathcal{X})^*, f \mapsto [l \mapsto l(f)]$ is an onto isometric isomorphism. In particular, ${}^*\mathcal{X}$ is a predual of $\mathcal{X}$. Moreover, it follows from the Hahn-Banach theorem that the linear span of the set $\{\delta_z : z \in \mathbb{D}\}$ is contained and norm dense in ${}^*\mathcal{X}$. See [6] for more details.

The spaces H_v^∞, H^p and A^p, $1 \le p < \infty$, fulfill all the assumptions (I)-(VI), when v is normal and equivalent with its associated weight $\tilde{v}$ (see [4]). But e.g. H^∞ does not satisfy (II) and in

$\mathcal{B}_v$ condition (VI) fails. That condition (IV) holds for H_v^∞ follows from a result of Lusky [17] stating that $\mathcal{B}_{v_1\cdot v} = H_v^\infty$ when v is normal. Condition (I) is valid for all these spaces and any weight v. This can be proved by using Montel's theorem and for H^p and A^p combining it with Fatou's lemma. However (I) does not hold for $\mathcal{B}_v^0$ and H_v^0. Furthermore, it can be shown (see e.g. [7]) that $\|\delta_z\|_{H^p} = (1-|z|^2)^{\frac{-1}{p}}$, $\|\delta_z\|_{H^\infty} = 1$, $\|\delta_z\|_{A^p_\alpha} = (1-|z|^2)^{\frac{-2-\alpha}{p}}$, $\|\delta_z\|_{H_v^\infty} = 1/\tilde{v}(z)$, $\|\delta_z\|_{\mathcal{B}} = \max\left\{1, \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)\right\}$.

It is very important in our study that $(B_{\mathcal{B}_v}, co)$ is compact, i.e. condition (I) holds for $\mathcal{B}_v$. Indeed, let $z \in \mathbb{D}$ and $f \in \mathcal{B}_v$. Then

$$|f(z) - f(0)| \leq \int_0^1 |z| \cdot |f'(tz)| dt \leq ||f||_{\mathcal{B}_v} \int_0^1 \frac{|z|}{v(t|z|)} dt,$$

so

$$\sup_{||f||_{\mathcal{B}_v} \leq 1} \sup_{z\in K} |f(z)| \leq 1 + \sup_{z\in K} \int_0^1 \frac{|z|}{v(t|z|)} dt \leq 1 + \sup_{z\in K} \frac{1}{v(z)} < \infty$$

for every compact set $K \subset \mathbb{D}$. Therefore $(B_{\mathcal{B}_v}, co)$ is co-relatively compact by Montel's theorem. Moreover, it is co-closed by Weierstrass theorem. Hence $(B_{\mathcal{B}_v}, co)$ is compact.

For any weight v, the closed unit ball of H_v^0 is co-dense in the closed unit ball $B_{H_v^\infty}$, so by Theorem 1.1 (b) in [5], the restriction map $l \mapsto l_{|H_v^0}$ gives rise to an isometric isomorphism ${}^*H_v^\infty \approx (H_v^0)^*$. Since for normal weights v, we have $H_v^0 \approx c_0$, we conclude that ${}^*H_v^\infty \approx l^1$ if the weight v is normal.

Let $Aut(\mathbb{D})$ denote the group of all disk automorphisms. For a positive Borel measure μ on $\mathbb{D}$ and $1 \leq p < \infty$, the weighted Dirichlet type space $\mathcal{D}^p_\mu$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{\mathcal{D}^p_\mu} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p d\mu(z) < \infty.$$

Moreover, let $\mathcal{M}(\mathcal{D}^p_\mu)$ be the Möbius invariant subspace consisting of all $f \in H(\mathbb{D})$ for which

$$\|f\|^p_{\mathcal{M}(\mathcal{D}^p_\mu)} = |f(0)|^p + \sup_{\sigma\in Aut(\mathbb{D})} \int_{\mathbb{D}} |(f\circ\sigma)'(z)|^p d\mu(z) < \infty.$$

Clearly, $\mathcal{M}(\mathcal{D}^p_\mu) \subset \mathcal{D}^p_\mu$. Likewise, let $\mathcal{M}_0(\mathcal{D}^p_\mu)$ be the subspace of $\mathcal{M}(\mathcal{D}^p_\mu)$ consisting of all f for which

$$\lim_{|\sigma(0)|\to 1, \sigma\in Aut(\mathbb{D})} \int_{\mathbb{D}} |(f\circ\sigma)'(z)|^p d\mu(z) = 0.$$

In order that the spaces $\mathcal{D}^p_\mu$ become Banach spaces the authors of [10] assumed that the following condition holds:

$$\sup_{\|f\|_{\mathcal{D}^p_\mu} \leq 1} \sup_{z\in K} |f(z)| < \infty \tag{2.1}$$

for all compact sets $K \subset \mathbb{D}$, i.e. that the closed unit ball $B_{\mathcal{D}^p_\mu}$ is bounded with respect to the topology co. Throughout the paper we also make this assumption. Notice that $\mathcal{D}^p_\mu$ is isomorphic to a closed subspace of $L^p(\mathbb{D}, \mu)$, hence it is weakly sequentially complete (see for $p = 1$ [25, 14 Corollary, p. 140]). This fact will be used later on. For suitable choices of the measure μ the above three spaces give rise to many classical spaces of analytic functions on the open unit disk, such as e.g. Besov spaces B^p, $1 < p < \infty$, and Q_β spaces, $0 < \beta < \infty$. In particular, we have $BMOA = Q_1$ and $\mathcal{B} = Q_\beta$ for $\beta > 1$. See [10] for a thorough discussion. See also [1].

The Banach spaces $\mathcal{D}^p_\mu$ and $\mathcal{M}(\mathcal{D}^p_\mu)$ satisfy condition (I). For completeness we give the argument for $\mathcal{M}(\mathcal{D}^p_\mu)$. Indeed, by condition (2.1) also $B_{\mathcal{M}(\mathcal{D}^p_\mu)}$ is uniformly bounded on compact subsets of $\mathbb{D}$ and hence by Montel's theorem $B_{\mathcal{M}(\mathcal{D}^p_\mu)}$ is *co*-relatively compact. If $\{f_n\}$ is a *co*-convergent sequence in $B_{\mathcal{M}(\mathcal{D}^p_\mu)}$, then we apply Fatou's lemma to get

$$\lim_{n\to\infty} |f_n(0)|^p + \sup_{\sigma\in Aut(\mathbb{D})} \int_{\mathbb{D}} \lim_{n\to\infty} |(f_n \circ \sigma)'(z)|^p d\mu(z) \le \liminf_{n\to\infty} \|f_n\|^p_{\mathcal{M}(\mathcal{D}^p_\mu)} \le 1.$$

Therefore the limit function is in $B_{\mathcal{M}(\mathcal{D}^p_\mu)}$, hence it is a *co*-compact set.

For any $w \in \mathbb{D}$, let $\sigma_w(z) = \frac{w-z}{1-\overline{w}z},\ z \in \mathbb{D}$, which maps the disc conformally onto itself and exchanges w with 0.

The essential norm of a bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is defined to be the distance to the compact operators, i.e.,

$$\|T\|_{e,\mathcal{X}\to\mathcal{Y}} = \inf\{\|T - K\| : K : \mathcal{X} \to \mathcal{Y} \text{ is compact}\}.$$

Notice that $\|T\|_{e,\mathcal{X}\to\mathcal{Y}} = 0$ if and only T is compact.

Throughout the entire paper, φ will denote an analytic self-map of $\mathbb{D}$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, and $u \in H(\mathbb{D})$. These maps induce via composition and multiplication a linear weighted composition operator uC_φ which is defined on $H(\mathbb{D})$ by $(uC_\varphi f)(z) = (M_u C_\varphi f)(z) = u(z)f(\varphi(z)),\ z \in \mathbb{D}$.

For many spaces, we have the following for us crucial result that the weighted composition operator is a dual operator.

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces of analytic functions on the unit disc and both satisfying condition (I). Suppose that $uC_\varphi : \mathcal{X} \to \mathcal{Y}$ is well-defined. Then uC_φ is bounded and there exists a bounded operator ${}^*(uC_\varphi) : {}^*\mathcal{Y} \to {}^*\mathcal{X}$ such that $({}^*(uC_\varphi))^* = uC_\varphi$.*

Proof. If $uC_\varphi : \mathcal{X} \to \mathcal{Y}$ is well-defined, then by (I) and the Closed Graph Theorem, uC_φ is also bounded. Now define ${}^*(uC_\varphi) := (uC_\varphi)^*{}_{|{}^*\mathcal{Y}}$. Since $(uC_\varphi)^*(\delta_z) = u(z)\delta_{\varphi(z)}$, and as a consequence of condition (I) as we pointed out above,

$$\overline{span}^{\mathcal{Y}^*}\{\delta_z\colon z \in \mathbb{D}\} = {}^*\mathcal{Y} \quad \text{and} \quad \overline{span}^{\mathcal{X}^*}\{\delta_z\colon z \in \mathbb{D}\} = {}^*\mathcal{X},$$

it follows that the bounded operator ${}^*(uC_\varphi)$ maps ${}^*\mathcal{Y}$ into ${}^*\mathcal{X}$. Finally, we conclude from the definition of ${}^*(uC_\varphi)$ that $({}^*(uC_\varphi))^* = uC_\varphi$. □

3. Norm Estimates

In [7] Colonna and Tjani have obtained norm formulas for weighted composition operators, that also follow from the next results that are valid for all operators. The first norm formula below has been discussed both in [12] and [13], and can be proved straightforward.

Theorem 3.1. *Assume that $\mathcal{X}$ is a Banach space of analytic functions on the unit disc satisfying condition (I). Then the norm of any operator $T : \mathcal{X} \to H^\infty_v$ is given by*

$$\|T\|_{\mathcal{X}\to H^\infty_v} = \sup_{z\in\mathbb{D}} \|T^*(\delta_z)\|v(z). \tag{3.1}$$

Furthermore, for the norm of any operator $T : \mathcal{X} \to \mathcal{B}_v$ we have

$$\max\left\{\|T^*(\delta_0)\|, \sup_{z\in\mathbb{D}} \|T^*(\delta'_z)\|v(z)\right\} \le \|T\|_{\mathcal{X}\to\mathcal{B}_v} \le \|T^*(\delta_0)\| + \sup_{z\in\mathbb{D}} \|T^*(\delta'_z)\|v(z). \tag{3.2}$$

If moreover $T(X) \subset \tilde{\mathcal{B}}_v$, then

$$\|T\|_{\mathcal{X}\to\mathcal{B}_v} = \sup_{z\in\mathbb{D}} \|T^*(\delta'_z)\|v(z). \tag{3.3}$$

Proof. We only prove (3.2). For $f \in \mathcal{X}, \|f\|_{\mathcal{X}} \le 1$

$$\|Tf\|_{\mathcal{B}_v} = |\delta_0(Tf)| + \sup_{z\in\mathbb{D}} v(z)|\delta'_z(Tf)| = |T^*(\delta_0)(f)| + \sup_{z\in\mathbb{D}} v(z)|T^*(\delta'_z)(f)|$$

$$\le \|T^*(\delta_0)\| + \sup_{z\in\mathbb{D}} v(z)\|T^*(\delta'_z)\|.$$

On the other hand, $\|T^*(\delta_0)\| = \sup_{\|f\|_{\mathcal{X}}\le 1} |Tf(0)| \le \sup_{\|f\|_{\mathcal{X}}\le 1} \|Tf\|_{\mathcal{B}_v} \le \|T\|$, and

$$v(z)\|T^*(\delta'_z)\| = \sup_{\|f\|_{\mathcal{X}}\le 1} v(z)|\delta'_z(Tf)| = \sup_{\|f\|_{\mathcal{X}}\le 1} v(z)|(Tf)'(z)| \le \sup_{\|f\|_{\mathcal{X}}\le 1} \|Tf\|_{\mathcal{B}_v} \le \|T\|.$$

Equation (3.3) follows in the same way bearing in mind that in this case $T^*(\delta_0) = 0$. □

Now we apply the above results to weighted composition operators.

Corollary 3.2. *Assume that $\mathcal{X}$ is a Banach space of analytic functions on the open unit disc satisfying condition (I). Then for the weighted composition operator $uC_\varphi : \mathcal{X} \to H^\infty_v$,*

$$\|uC_\varphi\|_{\mathcal{X}\to H^\infty_v} = \sup_{z\in\mathbb{D}} v(z)|u(z)| \cdot \|\delta_{\varphi(z)}\|. \tag{3.4}$$

Proof. Since $(uC_\varphi)^*(\delta_z) = u(z)\delta_{\varphi(z)}$, we get the statement by Theorem 3.1. □

Corollary 3.3. *Assume that $\mathcal{X}$ is a Banach space of analytic functions on the open unit disc satisfying conditions (I), (IV) and (VI). Then the weighted composition operator $uC_\varphi : \mathcal{X} \to \mathcal{B}_v$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} v(z)|u'(z)| \cdot \|\delta_{\varphi(z)}\| < \infty \text{ and } \sup_{z\in\mathbb{D}} \frac{v(z)|u(z)\varphi'(z)| \cdot \|\delta_{\varphi(z)}\|}{1-|\varphi(z)|^2} < \infty.$$

In that case

$$\|uC_\varphi\|_{\mathcal{X}\to\mathcal{B}_v} \asymp |u(0)| \cdot ||\delta_{\varphi(0)}|| + \sup_{z\in\mathbb{D}} v(z)|u'(z)| \cdot \|\delta_{\varphi(z)}\| + \sup_{z\in\mathbb{D}} \frac{v(z)|u(z)\varphi'(z)| \cdot \|\delta_{\varphi(z)}\|}{1-|\varphi(z)|^2}.$$

Proof. For any $f \in \mathcal{X}$ and $z \in \mathbb{D}$ we have that

$$\begin{aligned}(uC_\varphi)^*(\delta'_z)f &= \delta'_z(uC_\varphi f) = \delta'_z(u \cdot f \circ \varphi) = u'(z)f(\varphi(z)) + u(z)f'(\varphi(z))\varphi'(z) \\ &= u'(z)\delta_{\varphi(z)}f + u(z)\varphi'(z)\delta'_{\varphi(z)}f,\end{aligned}$$

and thus

$$(uC_\varphi)^*(\delta'_z) = u'(z)\delta_{\varphi(z)} + u(z)\varphi'(z)\delta'_{\varphi(z)}. \tag{3.5}$$

Inserting (3.5) in (3.2) gives

$$\begin{aligned}
\|uC_\varphi\|_{\mathcal{X}\to\mathcal{B}_v} - |u(0)|\cdot||\delta_{\varphi(0)}|| &\asymp \sup_{z\in\mathbb{D}} \|(uC_\varphi)^*(\delta'_z)\|v(z)\\
&= \sup_{z\in\mathbb{D}} \|u'(z)|\delta_{\varphi(z)} + u(z)\varphi'(z)\delta'_{\varphi(z)}\|v(z)\\
&\le \sup_{z\in\mathbb{D}} v(z)|u'(z)|\cdot\|\delta_{\varphi(z)}\| + \sup_{z\in\mathbb{D}} v(z)|u(z)\varphi'(z)|\cdot\|\delta'_{\varphi(z)}\|\\
&\lesssim \sup_{z\in\mathbb{D}} v(z)|u'(z)|\cdot\|\delta_{\varphi(z)}\| + \sup_{z\in\mathbb{D}} \frac{v(z)|u(z)\varphi'(z)|\cdot\|\delta_{\varphi(z)}\|}{1-|\varphi(z)|^2}\\
&:= N_1 + N_2,
\end{aligned}$$

where the estimate

$$\|\delta'_{\varphi(z)}\| \lesssim \frac{\|\delta_{\varphi(z)}\|}{1-|\varphi(z)|^2}$$

follows from condition (IV). Thus if N_1 and N_2 are finite then the operator $uC_\varphi : \mathcal{X} \to \mathcal{B}_v$ is bounded.

Now we prove the reverse implication. By condition (I), for each $z \in \mathbb{D}$ there is $f_z \in \mathcal{X}$, $\|f_z\|_{\mathcal{X}} \le 1$, with $f_z(z) = \|\delta_z\|$. For $w \in \mathbb{D}$, we can use (VI) to define $Q_w(z) := \sigma_{\varphi(w)}(z)f_{\varphi(w)}(z) \in \mathcal{X}$ with $\|Q_w\|_{\mathcal{X}} \le 1$. Now we obtain

$$\|uC_\varphi(Q_w)\|_{\mathcal{B}_v} \ge \frac{v(z)|u(w)\varphi'(w)|\cdot\|\delta_{\varphi(w)}\|}{1-|\varphi(w)|^2},$$

and $N_2 \lesssim \|uC_\varphi\|_{\mathcal{X}\to\mathcal{B}_v}$. Further, using the test functions $f_{\varphi(w)}$ and condition (IV), we obtain

$$|f'_{\varphi(w)}(\varphi(w))| \le C\|f_{\varphi(w)}\|_{\mathcal{X}}\cdot\|\delta_{\varphi(w)}\|(1-|\varphi(w)|^2)^{-1}.$$

Hence

$$\|uC_\varphi(f_{\varphi(w)})\|_{\mathcal{B}_v} \ge v(w)|u'(w)|\cdot\|\delta_{\varphi(w)}\| - C\frac{v(w)|u(w)\varphi'(w)|\cdot\|\delta_{\varphi(w)}\|}{1-|\varphi(w)|^2} \ge v(w)|u'(w)|\cdot\|\delta_{\varphi(w)}\| - CN_2.$$

Therefore $N_1 \lesssim \|uC_\varphi\|_{\mathcal{X}\to\mathcal{B}_v}$.

Also $|u(0)|\cdot||\delta_{\varphi(0)}|| = \|(uC_\varphi)^*(\delta_0)\| \le \|(uC_\varphi)\|_{\mathcal{X}\to\mathcal{B}_v}$, because $\|\delta_0\| \le 1$. □

4. Essential Norm Estimates

We will use the following result. It is an extension of Theorem 3 in [3], whose proof follows immediately by inspection of the original one.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach spaces such that $\mathcal{A}^*$ has the λ-metric approximation property. Then*

$$\|T\|_{e,\mathcal{A}\to\mathcal{B}} \ge \|T^*\|_{e,\mathcal{B}^*\to\mathcal{A}^*} \ge \frac{1}{1+\lambda}\|T\|_{e,\mathcal{A}\to\mathcal{B}}.$$

We next give a new formula to estimate the essential norm of uC_φ when acting between analytic function spaces that makes it easier to find test functions for estimating the lower bound of $\|uC_\varphi\|_e$ for several concrete spaces.

Theorem 4.2. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces of analytic functions on the open unit disc and such that they both satisfy condition (I). Moreover, suppose that $\mathcal{Y}$ has the λ-metric approximation property. If the weighted composition operator $uC_\varphi : \mathcal{X} \to \mathcal{Y}$ is bounded, then the essential norm of uC_φ satisfies*

$$\|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{Y}} \asymp \inf\{\|uC_\varphi - K^*\| : K : {}^*\mathcal{Y} \to {}^*\mathcal{X} \text{ compact}\}. \tag{4.1}$$

Proof. By Lemma 2.1 there exists an operator ${}^*(uC_\varphi) : {}^*\mathcal{Y} \to {}^*\mathcal{X}$ such that $({}^*(uC_\varphi))^* = uC_\varphi$. Take $\mathcal{A}$ as the predual ${}^*\mathcal{Y}$, $\mathcal{B}$ as the predual ${}^*\mathcal{X}$ and $T = {}^*(uC_\varphi)$ in Theorem 4.1. Then $\|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{Y}} \asymp \|{}^*(uC_\varphi)\|_{e,{}^*\mathcal{Y}\to{}^*\mathcal{X}}$. Since

$$\begin{aligned} &\inf\{\|uC_\varphi - K^*\| : K : {}^*\mathcal{Y} \to {}^*\mathcal{X} \text{ compact}\} \\ = \; &\inf\{\|{}^*(uC_\varphi) - K\| : K : {}^*\mathcal{Y} \to {}^*\mathcal{X} \text{ compact}\} = \|{}^*(uC_\varphi)\|_{e,{}^*\mathcal{Y}\to{}^*\mathcal{X}}, \end{aligned}$$

we are done. □

The next result should be compared with Theorem 3.4 in [7]. Notice that it holds without assuming the reflexivity of $\mathcal{X}$.

Theorem 4.3. *Assume that $\mathcal{X}$ is a Banach space of analytic functions on the open unit disc satisfying conditions (I), (II), (III), (V) and (VI). Let $uC_\varphi \colon \mathcal{X} \to H_v^\infty$ be bounded. Then*

$$\|uC_\varphi\|_{e,\mathcal{X}\to H_v^\infty} \asymp \lim_{s\to 1} \sup_{|\varphi(z)|>s} v(z)|u(z)| \cdot \|\delta_{\varphi(z)}\|.$$

Proof. Following the first part of the proof of Theorem 3.4 in [7] we get that

$$\|uC_\varphi\|_{e,\mathcal{X}\to H_v^\infty} \lesssim A(u,\varphi),$$

where

$$A(u,\varphi) := \lim_{s\to 1} \sup_{|\varphi(z)|>s} v(z)|u(z)| \cdot \|\delta_{\varphi(z)}\|.$$

In order to prove that

$$\|uC_\varphi\|_{e,\mathcal{X}\to H_v^\infty} \gtrsim A(u,\varphi), \tag{4.2}$$

we may assume that $\|\varphi\|_\infty = 1$, since otherwise both sides in (4.2) are zero. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ and

$$A(u,\varphi) = \lim_{n\to\infty} v(z_n)|u(z_n)| \cdot \|\delta_{\varphi(z_n)}\|.$$

By going to a subsequence if necessary, we can assume that the sequence $\{\varphi(z_n)\}$ is interpolating for H^∞, so there are functions $\{g_n\} \subset H^\infty$ and a constant $C > 0$ such that

$$\sum_{n=1}^\infty |g_n(z)| \le C$$

for every $z \in \mathbb{D}$ and

$$g_n(\varphi(z_k)) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n. \end{cases} \tag{4.3}$$

See [14] and [25].

By condition (I), for each $n \in \mathbb{N}$ we can also find a function $f_n \in \mathcal{X}$ such that $\|f_n\|_{\mathcal{X}} \leq 1$ and that

$$\|\delta_{\varphi(z_n)}\| = |f_n(\varphi(z_n))|. \tag{4.4}$$

Appealing to Theorem 4.2 we may only consider compact operators $T\colon \mathcal{X} \to H_v^\infty$ that are dual operators, that is $T = K^*$. Now consider $F_n := f_n g_n$. According to condition (VI), the sequence $\{F_n\} \subset \mathcal{X}$, and

$$\|F_n\|_{\mathcal{X}} = \|M_{g_n} f_n\|_{\mathcal{X}} \leq \|g_n\|_\infty \|f_n\|_{\mathcal{X}} \leq C.$$

Hence $\{F_n\} \subset \mathcal{X} = ({}^*\mathcal{X})^*$, is a weak*-relatively compact set. Moreover $\{F_n\}$ is pointwise null, thus it has to be weak* null. Consequently, $\big\{K^*(F_n)\big\}$ is weak* null as well; since K^* is a compact operator, we can assure that $\big\{\|K^*(F_n)\|\big\}$ converges to 0. Thus we get

$$\begin{aligned}\|uC_\varphi - K^*\|_{\mathcal{X}\to H_v^\infty} &\gtrsim \|(uC_\varphi - K^*)F_n\|_{H_v^\infty}\\ &\geq \|uC_\varphi(F_n)\|_{H_v^\infty} - \|K^*(F_n)\|_{H_v^\infty}\\ &\geq v(z_n)|u(z_n)| \cdot |F_n(\varphi(z_n))| - \|K^*(F_n)\|_{H_v^\infty}\\ &= v(z_n)|u(z_n)| \cdot \|\delta_{\varphi(z_n)}\| - \|K^*(F_n)\|_{H_v^\infty}.\end{aligned}$$

Letting n tend to infinity and taking the infimum over all compact operators $K\colon {}^*H_v^\infty \to {}^*\mathcal{X}$ gives (4.2). □

Remark. The above proof can also be ended without using Theorem 4.2. In fact, since the sequence $\{F_n\}$ is bounded in $\mathcal{X}$, we can find a subsequence $\{F_{n_k}\}$ such that $T(F_{n_k})$ converges in H_v^∞. We may assume that the whole sequence $\{T(F_n)\}$ converges to some function G in H_v^∞. Define $H_n := F_{2n} - F_{2n-1}$ and note that $\|H_n\|_{\mathcal{X}} \leq 2C$ and $\lim_{n\to\infty} \|T(H_n)\|_{H_v^\infty} = 0$. By (4.4) and the interpolation property (4.3) we have that $F_{2n}(\varphi(z_{2n})) = \|\delta_{\varphi(z_{2n})}\|$ and $F_{2n-1}(\varphi(z_{2n})) = 0$ for every $n \in \mathbb{N}$, and thus as above

$$\begin{aligned}\|uC_\varphi - T\|_{\mathcal{X}\to H_v^\infty} &\gtrsim \|(uC_\varphi - T)H_n\|_{H_v^\infty}\\ &\geq v(z_{2n})|u(z_{2n})| \cdot |H_n(\varphi(z_{2n}))| - \|T(H_n)\|_{H_v^\infty}\\ &= v(z_{2n})|u(z_{2n})| \cdot \|\delta_{\varphi(z_{2n})}\| - \|T(H_n)\|_{H_v^\infty},\end{aligned}$$

and we are done.

The following result shows that Theorem 5.6 in [7] holds as well without the reflexivity of $\mathcal{X}$.

Theorem 4.4. *Assume that $\mathcal{X}$ is a Banach space of analytic functions on the open unit disc satisfying conditions (I), (II), (III), (IV), (V) and (VI). Let $uC_\varphi\colon \mathcal{X} \to \mathcal{B}_v$ be bounded. Then*

$$\|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{B}_v} \asymp D(u,\varphi) + E(u,\varphi), \tag{4.5}$$

where

$$D(u,\varphi) = \lim_{s\to 1} \sup_{|\varphi(z)|>s} v(z)|u'(z)| \cdot \|\delta_{\varphi(z)}\|,$$

$$E(u,\varphi) = \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{v(z)|u(z)\varphi'(z)| \cdot \|\delta_{\varphi(z)}\|}{1 - |\varphi(z)|^2}.$$

Proof. Following the first part of the proof of Theorem 5.6 in [7] and taking v instead of the standard weight v_α, we get that

$$\|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{B}_v} \lesssim D(u,\varphi) + E(u,\varphi) < \infty.$$

For the lower estimate we first prove that

$$\|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{B}_v} \gtrsim E(u,\varphi). \tag{4.6}$$

We may assume that $\|\varphi\|_\infty = 1$, since otherwise both sides in (4.5) are zero. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ and

$$E(u,\varphi) = \lim_{n\to\infty} \frac{v(z_n)|u(z_n)\varphi'(z_n)| \cdot \|\delta_{\varphi(z_n)}\|}{1-|\varphi(z_n)|^2}.$$

Now let $\{F_n\}$ be as in the proof of Theorem 4.3, and define $Q_n(z) := \sigma_{\varphi(z_n)}(z)F_n(z) \in \mathcal{X}$. Then $\{Q_n\}$ is bounded in $\mathcal{X}$ and $\{Q_n\}$ is weak* null. Moreover,

$$Q_n'(\varphi(z_n)) = \sigma_{\varphi(z_n)}'(\varphi(z_n))F_n(\varphi(z_n)) = \frac{\|\delta_{\varphi(z_n)}\|}{1-|\varphi(z_n)|^2}.$$

Again, by Theorem 4.2, we only need to consider compact dual operators $L^*: \mathcal{X} \to \mathcal{B}_v$, and get as in the proof of Theorem 4.3 that

$$\begin{aligned}\|uC_\varphi - L^*\|_{\mathcal{X}\to\mathcal{B}_v} &\gtrsim \|(uC_\varphi - L^*)Q_n\|_{\mathcal{B}_v}\\ &\geq \|uC_\varphi(Q_n)\|_{\mathcal{B}_v} - \|L^*(Q_n)\|_{\mathcal{B}_v}\\ &\geq \frac{v(z_n)|u(z_n)\varphi'(z_n)| \cdot \|\delta_{\varphi(z_n)}\|}{1-|\varphi(z_n)|^2} - \|L^*(Q_n)\|_{\mathcal{B}_v}.\end{aligned}$$

Hence (4.6) holds.

Finally, we prove that

$$\|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{B}_v} \gtrsim D(u,\varphi). \tag{4.7}$$

So let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ and

$$D(u,\varphi) = \lim_{n\to\infty} v(z_n)|u'(z_n)| \cdot \|\delta_{\varphi(z_n)}\|.$$

For $L^*: \mathcal{X} \to \mathcal{B}_v$ compact, we get

$$\begin{aligned}\|uC_\varphi - L^*\|_{\mathcal{X}\to\mathcal{B}_v} &\gtrsim \limsup_{n\to\infty} \|(uC_\varphi - L^*)F_n\|_{\mathcal{B}_v}\\ &\geq \limsup_{n\to\infty} \|uC_\varphi(F_n)\|_{\mathcal{B}_v}\\ &\geq \limsup_{n\to\infty} v(z_n)|u'(z_n)| \cdot \|\delta_{\varphi(z_n)}\|\\ &\quad - \liminf_{n\to\infty} v(z_n)|u(z_n)\varphi'(z_n)F_n'(\varphi(z_n))|.\end{aligned}$$

By condition (IV), $|F_n'(\varphi(z_n))| \leq C_1\|F_n\| \cdot \|\delta_{\varphi(z_n)}\|(1-|\varphi(z_n)|^2)^{-1}$, so

$$\begin{aligned}&\limsup_{n\to\infty} v(z_n)|u'(z_n)| \cdot \|\delta_{\varphi(z_n)}\|\\ &\lesssim \|uC_\varphi - L^*\|_{\mathcal{X}\to\mathcal{B}_v} + C_1 \liminf_{n\to\infty} \frac{v(z_n)|u(z_n)\varphi'(z_n)| \cdot \|\delta_{\varphi(z_n)}\|}{1-|\varphi(z_n)|^2}.\end{aligned}$$

Hence, we get by using (4.6) that $\|uC_\varphi - L^*\|_{\mathcal{X}\to\mathcal{B}_v} + \|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{B}_v} \gtrsim D(u,\varphi)$, and the statement (4.7) follows. □

5. Weak compactness

In the next theorem, we obtain a result that turns to be very useful for our purposes.

Theorem 5.1. *Assume that $\mathcal{Y}$ is a Banach space of analytic functions on the open unit disc and that it satisfies condition (I). Suppose that the weight v is normal. If the weighted composition operator $uC_\varphi : \mathcal{B}_v \to \mathcal{Y}$ is weakly compact, then it is also compact. The analogous statement holds also for H_v^∞ instead of $\mathcal{B}_v$.*

Proof. By Lemma 2.1 there exists an operator ${}^*(uC_\varphi) : {}^*\mathcal{Y} \to {}^*\mathcal{B}_v$ such that $({}^*(uC_\varphi))^* = uC_\varphi$. Therefore also ${}^*(uC_\varphi) : {}^*\mathcal{Y} \to {}^*\mathcal{B}_v$ is weakly compact by Gantmacher's theorem. In the preliminaries it was mentioned that ${}^*H_v^\infty \approx l^1$. To show that $l^1 \approx {}^*\mathcal{B}_v$, we first prove that ${}^*H_v^\infty \approx {}^*\tilde{\mathcal{B}}_v$. Let $D : \tilde{\mathcal{B}}_v \to H_v^\infty$ the differentiation operator. Notice that the continuous mapping between compact spaces $D_{|B_{\tilde{\mathcal{B}}_v}} : (B_{\tilde{\mathcal{B}}_v}, co) \to (B_{H_v^\infty}, co)$ is actually a bijection. Hence it is a homeomorphism. Since D is an isomorphism, also $D^* : (H_v^\infty)^* \to \tilde{\mathcal{B}}_v^*$ is an isomorphism. Moreover $D^* : {}^*H_v^\infty \to {}^*\tilde{\mathcal{B}}_v$ is an onto mapping since for $u \in {}^*\tilde{\mathcal{B}}_v$, $u \circ D^{-1}$ is *co*-continuous on $B_{H_v^\infty}$. Thus, ${}^*H_v^\infty \approx {}^*\tilde{\mathcal{B}}_v$. Moreover, it can be seen that the map $l \mapsto (l_{|\tilde{\mathcal{B}}_v}, l_{|\mathbb{C}})$ is an isomorphism between ${}^*\mathcal{B}_v$ and ${}^*\tilde{\mathcal{B}}_v \oplus_1 \mathbb{C}$. Consequently, ${}^*\mathcal{B}_v \approx l^1 \oplus_1 \mathbb{C} \approx l^1$, which implies that ${}^*(uC_\varphi) : {}^*\mathcal{Y} \to {}^*\mathcal{B}_v$ is compact. Hence we conclude that uC_φ is compact. □

From this result we get immediately the following results for weighted composition operators acting from the Bloch type space $\mathcal{B}_v$ into the spaces $\mathcal{D}_\mu^p$, $\mathcal{M}(\mathcal{D}_\mu^p)$ and $\mathcal{M}_0(\mathcal{D}_\mu^p)$ respectively. These results were proved for composition operators from the Bloch space by completely different methods in [10].

Corollary 5.2. *Let v be a normal weight.*

(*a*) *The operator $uC_\varphi : \mathcal{B}_v \to \mathcal{D}_\mu^p$ is bounded if and only if $uC_\varphi : \mathcal{B}_v \to \mathcal{D}_\mu^p$ is compact.*

(*b*) *The operator $uC_\varphi : \mathcal{B}_v \to \mathcal{M}(\mathcal{D}_\mu^p)$ is weakly compact if and only if $uC_\varphi : \mathcal{B}_v \to \mathcal{M}(\mathcal{D}_\mu^p)$ is compact.*

(*c*) *Assume that $\mathcal{M}_0(\mathcal{D}_\mu^p)$ does not contain a copy of l^∞. Then $uC_\varphi : \mathcal{B}_v \to \mathcal{M}_0(\mathcal{D}_\mu^p)$ is bounded if and only if $uC_\varphi : \mathcal{B}_v \to \mathcal{M}_0(\mathcal{D}_\mu^p)$ is compact.*

Proof. Theorem 5.1 yields (*b*) right away. As pointed out in the preliminaries, $\mathcal{D}_\mu^p$ is a weakly sequentially complete Banach space, hence it does not contain copies of c_0, neither of ℓ_∞. Since every bounded operator from $\mathcal{B}_v \approx l^\infty$ into a Banach space not containing a copy of l^∞ is weakly compact (see [8], Theorem 5.9.9), then (*a*) follows from Theorem 5.1 and (*c*) from (*b*). □

Example. For $\varphi(z) = \frac{sz}{1-(1-s)z}$, $z \in \mathbb{D}$, $0 < s < 1$, the composition operator $C_\varphi : \mathcal{B}_{v_\alpha} \to \mathcal{B}_{v_\alpha}$ is bounded for all $\alpha > 0$ and $||C_\varphi||_{e,\mathcal{B}_{v_\alpha}\to\mathcal{B}_{v_\alpha}} = s^{\alpha-1} \neq 0$ (see [2]). Thus $C_\varphi : \mathcal{B}_{v_\alpha} \to \mathcal{B}_{v_\alpha}$ is bounded but not weakly compact by Theorem 5.1.

6. Applications

In this section we obtain formulas of the norm and the essential norm of weighted composition operators acting on Hardy spaces H^p and weighted Bergman spaces A^p_α directly from our previous results. The case $p = 1$ for the essential norm was left open by Colonna and Tjani in [7]. Since these spaces have all properties (I)-(VI), the next results follow immediately.

Corollary 6.1. *Let $1 \le p < \infty$, $\alpha > -1$ and $\mathcal{X}$ be any of the spaces H^p with $s = \frac{1}{p}$ or A^p_α with $s = \frac{\alpha+2}{p}$. If $uC_\varphi : \mathcal{X} \to H^\infty_v$ is bounded, then*

$$\|uC_\varphi\|_{\mathcal{X}\to H^\infty_v} \asymp \sup_{z\in\mathbb{D}} \frac{v(z)|u(z)|}{(1-|\varphi(z)|^2)^s},$$

$$\|uC_\varphi\|_{e,\mathcal{X}\to H^\infty_v} \asymp \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{v(z)|u(z)|}{(1-|\varphi(z)|^2)^s}.$$

Corollary 6.2. *Let $1 \le p < \infty$, $\alpha > -1$ and $\mathcal{X}$ be any of the spaces H^p with $s = \frac{1}{p}$ or A^p_α with $s = \frac{\alpha+2}{p}$. If $uC_\varphi \colon \mathcal{X} \to \mathcal{B}_v$ is bounded, then*

$$\|uC_\varphi\|_{\mathcal{X}\to B_v} \asymp \sup_{z\in\mathbb{D}} \frac{v(z)|u'(z)|}{(1-|\varphi(z)|^2)^s} + \sup_{z\in\mathbb{D}} \frac{v(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+s}},$$

$$\|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{B}_v} \asymp \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{v(z)|u'(z)|}{(1-|\varphi(z)|^2)^s} + \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{v(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+s}}.$$

Finally, we will give estimates of the essential norms of $uC_\varphi \colon \mathcal{X} \to H^\infty_v, \mathcal{B}_v$, respectively, with $\mathcal{X}$ as above, in terms of u, u', φ' and the n^{th} power φ^n of φ.

For weights v and w, it is known (see [20], p. 875, [9] and [15], Theorem 2.4) that the essential norm of $uC_\varphi : H^\infty_w \to H^\infty_v$ is given by

$$\|uC_\varphi\|_{e,H^\infty_w\to H^\infty_v} = \lim_{t\to 1} \sup_{|\varphi(z)|>t} \frac{|u(z)|v(z)}{\tilde{w}(\varphi(z))} = \limsup_{n\to\infty} \frac{\|u\varphi^n\|_{H^\infty_v}}{\|z^n\|_{H^\infty_w}}.$$

Since $\|z^n\|_{H^\infty_{v_s}} = \left(\frac{2s}{n+2s}\right)^s \left(\frac{n}{n+2s}\right)^{\frac{n}{2}}$, $n \in \mathbb{N}$, ([26]), $\lim_{n\to\infty}(n+1)^s\|z^n\|_{H^\infty_{v_s}} = \left(\frac{2s}{e}\right)^s$. This means that for $\mathcal{X}$ any of the spaces H^p with $s = \frac{1}{p}$ or A^p_α with $s = \frac{\alpha+2}{p}$ one has

$$\|uC_\varphi\|_{e,\mathcal{X}\to H^\infty_v} \asymp \|uC_\varphi\|_{e,H^\infty_{v_s}\to H^\infty_v} = \left(\frac{e}{2s}\right)^s \limsup_{n\to\infty} \, (n+1)^s \|u\varphi^n\|_{H^\infty_v}, \tag{6.1}$$

and

$$\|uC_\varphi\|_{e,\mathcal{X}\to\mathcal{B}_v} \asymp \|u'C_\varphi\|_{e,H^\infty_{v_s}\to H^\infty_v} + \|u\varphi'C_\varphi\|_{e,H^\infty_{v_{s+1}}\to H^\infty_v} = \tag{6.2}$$

$$\left(\frac{e}{2s}\right)^s \limsup_{n\to\infty} \, (n+1)^s\|u'\varphi^n\|_{H^\infty_v} + \left(\frac{e}{2(s+1)}\right)^{s+1} \limsup_{n\to\infty} \, (n+1)^{s+1}\|u\varphi'\varphi^n\|_{H^\infty_v}.$$

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