Recovery of singularities from a backscattering Born approximation for a biharmonic operator in 3D

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Abstract

We consider a backscattering Born approximation for a perturbed biharmonic operator in three space dimensions. Previous results on this approach for biharmonic operator use the fact that the coefficients are real-valued to obtain reconstruction of singularities in the coefficients. In this text we drop the assumption about real-valued coefficients and establish the recovery of singularities also for complex coefficients. The proof uses mapping properties of the Radon transform.

1 Introduction

This work is continuation to our study of the backscattering problem for the perturbed biharmonic operator $H_4 u := \Delta^2 u + \vec{q} \cdot \nabla u + V u$ in three dimensions, considered in [15]. Here $\vec{q}$ is a vector-valued function and V is a scalar-valued function from suitable function spaces. We are interested in the scattering problem for H_4 given by the equations

$$\begin{cases} H_4 u = k^4 u, \quad u = u_0 + u_{\mathrm{sc}}, \quad u_0(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(\theta,x)}, \\ \frac{\partial f}{\partial |x|} - \mathrm{i}kf = o\left(|x|^{-\frac{n-1}{2}}\right), \quad |x| \to \infty, \text{ for both } f = u_{\mathrm{sc}} \text{ and } f = \Delta u_{\mathrm{sc}}, \end{cases} \tag{1}$$

where $(\cdot,\cdot)$ denotes the usual inner product in $\mathbb{R}^3$, $\theta \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 \mid |x| = 1\}$ and the parameter $k > 0$ is usually called the wavenumber. The second line of (1) is an analogue of Sommerfeld's radiation condition at infinity for this biharmonic operator [16].

The solution to the above scattering problem fulfils the Lippmann-Schwinger integral equation

$$u(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(x,\theta)} - \int_{\mathbb{R}^3} G_k^+(|x-y|)\,(\vec{q}(y)\cdot\nabla u(y,k,\theta) + V(y)u(y,k,\theta))\,\mathrm{d}y, \tag{2}$$

under some integrability conditions for the coefficients [16]. Here

$$G_k^+(|x|) := \frac{\mathrm{e}^{\mathrm{i}k|x|} - \mathrm{e}^{-k|x|}}{8\pi k^2|x|}$$

is an outgoing fundamental solution to $H_0 := \Delta^2 - k^4$ in three dimensions.

For the author a motivating starting point for inverse scattering problems for biharmonic operators was articles by K. Iwasaki [7]. In these texts the inverse problem is formulated as a Riemann-Hilbert boundary value problem. From suitable smoothness assumptions for the coefficients Iwasaki then proved that given suitable scattering data (the reflection and connection coefficients) it is possible to uniquely recover the coefficients $\vec{q}$ and V. We approach the inverse problem similarly as [11, 12, 13, 14] (among many others) do for the the Schrödinger operator and the magnetic Schrödinger operator. The aforementioned texts expand the solution u to the scattering problem into several terms and then study the smoothness of certain inverse Born approximation term-by-term. In particular, we use the backscattering Born approximation where the data is simply obtained by taking the measurement in the opposing angle of the incident wave.

As possible applications to biharmonic problems we mention the theory of vibrations of beams and the study of elasticity. For example the time-dependent beam equation

$$\partial_t^2 U + \Delta^2 U + mU = 0$$

with the time-harmonic ansatz $U(x,t) = u(x)\mathrm{e}^{-\mathrm{i}\omega t}$ yields the equation

$$\Delta^2 u + mu = \omega^2 u.$$

More concretely, one can model hinged plate configurations by the equations

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

In this context the quantities u, ∇u and Δu are known as the displacement, the slope and the bending moment of the beam. In the operator H_4 the functions $\vec{q}$ and V can be considered as perturbations of the slope and displacement. For more theory, see for example [3]. A different kind of example is given in [9], where the wave scattering by grating stacks is considered.

2 Preliminaries

Before going into the main text we recall our notation. The Fourier transform of a function f is defined by

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}(x,\xi)} f(x)\mathrm{d}x.$$

The weighted Lebesgue spaces (see eg. [1], [6]) $L^p_\delta(\mathbb{R}^3)$ are defined by the norm

$$\|f\|_{L^p_\delta(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3} (1+|x|)^{\delta p} |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}$$

and the weighted Sobolev spaces $W^k_{p,\delta}(\mathbb{R}^3)$ are the spaces of those functions whose weak derivatives up to order $k \geq 0$ belong to $L^p_\delta(\mathbb{R}^3)$. The symbol $H^t(\mathbb{R}^3)$ is used to denote the L^2-based Sobolev space of order $t \in \mathbb{R}$ defined by the norm

$$\|f\|_{H^t(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3} (1+|x|^2)^t |\widehat{f}(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Finally, let

$$\chi_c(x) := \begin{cases} 1, & \text{if } |x| \geq 2, \\ 0, & \text{if } |x| < 2. \end{cases}$$

(There is a misprint in [15] and the function χ should be the characteristic function of $\mathbb{R} \setminus [-k_0, k_0]$.) As usual, the symbol C denotes a positive constant whose value can change from line to line.

To continue with the inverse problem we recall the relevant definitions and results from [15, 16]. If the coefficients satisfy $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^3)$ and $V \in L^p_{2\delta}(\mathbb{R}^3)$, with $3 < p \leq \infty$ and $2\delta > 3 - \frac{3}{p}$ then for fixed and large enough $k > 0$ the equation (2) has a unique solution with $u_{\mathrm{sc}} \in H^1_{-\delta}(\mathbb{R}^3)$. This solution has the following asymptotic representation

$$u(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(\theta,x)} - \frac{\mathrm{e}^{\mathrm{i}k|x|}}{8\pi k^2 |x|} A(k,\theta,\theta') + o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

where

$$A(k,\theta,\theta') = \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta',y)} \left[\vec{q} \cdot \nabla u + V u\right] \mathrm{d}y$$

is called the scattering amplitude and $\theta' = x/|x|$ is the direction of observation. Our data, the backscattering amplitude, is obtained by taking the measurement in the opposing direction of the incident wave ($\theta' = -\theta$) and is given by

$$A_{\mathrm{b}}(k,\theta) := A(k,\theta,-\theta) = \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}k(\theta,y)} [\vec{q}(y) \cdot \nabla u(y,k,\theta) + V(y) u(y,k,\theta)] \mathrm{d}y$$

for $k \geq k_0 > 0$ and is defined $A_\mathrm{b}(k,\theta) = 0$ otherwise. By using the first order Born approximation $u(x,k,\theta) \approx u_0(x,k,\theta)$ and the divergence theorem we obtain the approximation

$$\begin{aligned} A_\mathrm{b}(k,\theta) \approx A_0(k,\theta) &:= \int_{\mathbb{R}^3} \mathrm{e}^{2\mathrm{i}k(\theta,y)} \left[\mathrm{i}k\theta \cdot \vec{q}(y) + V(y)\right] \mathrm{d}y \\ &= \int_{\mathbb{R}^3} \mathrm{e}^{2\mathrm{i}k(\theta,y)} \left[-\frac{1}{2}\nabla \cdot \vec{q}(y) + V(y)\right] \mathrm{d}y \\ &= (2\pi)^{\frac{3}{2}} F^{-1}(\beta)(2k\theta), \end{aligned}$$

where we denote $\beta := -\frac{1}{2}\nabla \cdot \vec{q} + V$. This motivates the definition of the inverse backscattering Born approximation q_B of β by Fourier inversion as the integral

$$q_\mathrm{B}(x) := \frac{1}{(2\pi)^3} \int_0^\infty k^2 \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)} A_\mathrm{b}\left(\frac{k}{2},\theta\right) \mathrm{d}\theta \mathrm{d}k.$$

This function β is the quantity related to the coefficients $\vec{q}$ and V which we hope to recover.

Let us now elaborate on the Born inversion scheme. The solution of the Lippmann-Schwinger integral equation (2) can be expressed as the series

$$u(x,k,\theta) = \sum_{j=0}^{\infty} u_j(x,k,\theta)$$

(see [16]). Here the iterations u_j are defined by

$$u_j(x,k,\theta) := -\int_{\mathbb{R}^n} G_k^+(|x-y|)[\vec{q}(y) \cdot \nabla u_{j-1}(y,k,\theta) + V(y) u_{j-1}(y,k,\theta)] \mathrm{d}y,$$

for $j = 1, 2, \ldots$ and u_0 as before. Then the backscattering Born series has the representation

$$q_\mathrm{B} = q_0 + q_1 + q_\mathrm{rest},$$

where $q_\mathrm{rest} := \sum_{j=2}^{\infty} q_j$ and

$$\begin{aligned} q_j(x) &:= \frac{1}{(2\pi)^3} \int_0^\infty k^2 \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)} A_j\left(\frac{k}{2},\theta\right) \mathrm{d}\theta \mathrm{d}k, \quad j = 0, 1, \ldots, \\ A_j(k,\theta) &:= \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}k(\theta,y)} [\vec{q} \cdot \nabla u_j + V u_j] \mathrm{d}y, \quad j = 0, 1, \ldots. \end{aligned}$$

It was shown in [15] that $q_0 = \beta + \widetilde{q}$, where $\widetilde{q} \in C^\infty(\mathbb{R}^3)$ and also that $q_\mathrm{rest} \in H^t(\mathbb{R}^3)$ for all $t < \frac{3}{2}$.

The main difficulty in the inverse Born approximation is to obtain good estimates for the first nonlinear term q_1. The approach in [15] was to split $q_1 = q_{1,\mathrm{M}} + q_{1,\mathrm{E}}$, where the term $q_{1,\mathrm{E}}$ corresponds to the exponentially decaying part of the fundamental solution G_k^+ and satisfies $q_{1,\mathrm{E}} \in H^t(\mathbb{R}^3)$ for all $t < \frac{5}{2}$. The term $q_{1,\mathrm{M}}$ corresponds to the oscillating main part of G_k^+ and is more difficult to estimate. The following lemma was proved in [15].

Lemma 2.1. *Let $n \geq 2$. If $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$ with $n < p \leq \infty$ and $2\delta > n - \frac{n}{p}$, then*

$$\begin{aligned} q_{1,\mathrm{M}}(x) = &\frac{(2\pi)^{\frac{n}{2}}}{2}\mathcal{F}^{-1}\left(\frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2}\frac{\widehat{\nabla\cdot\vec{q}}(\eta)\widehat{\nabla\cdot\vec{q}}(\mu)}{(\mu,\eta)+\mathrm{i}0}\right)(x,x) \\ &-\frac{(2\pi)^{\frac{n}{2}}}{2}\mathcal{F}^{-1}\left(\frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2}\frac{\sum_{j,k=1}^n\widehat{\partial_j q_k}(\eta)\widehat{\partial_k q_j}(\mu)}{(\mu,\eta)+\mathrm{i}0}\right)(x,x) \\ &-2(2\pi)^{\frac{n}{2}}\mathcal{F}^{-1}\left(\frac{\chi_c(|\mu+\eta|/2)}{|\mu+\eta|^2}\frac{\widehat{\widetilde{V}}(\eta)\widehat{V}(\mu)}{(\mu,\eta)+\mathrm{i}0}\right)(x,x) \end{aligned}$$

in the sense of distributions. Here $\mathcal{F}^{-1}$ is the $2n$-dimensional inverse Fourier transform and $\widetilde{V} := \nabla\cdot\vec{q} - V$.

The notation $\frac{1}{x+\mathrm{i}0}$ is to be understood in the sense of tempered distributions as the limit

$$\frac{1}{x+\mathrm{i}0} := \lim_{\varepsilon\to 0^+}\frac{1}{x+\mathrm{i}\varepsilon}.$$

By Lemma 2.1 we see that it suffices to study the behaviour of the bilinear form

$$I(f,g) = (2\pi)^n\mathcal{F}^{-1}\left(\frac{\chi_c(|\mu+\eta|)}{|\mu+\eta|^2}\frac{\widehat{f}(\eta)\widehat{g}(\mu)}{(\mu,\eta)+\mathrm{i}0}\right)(x,x)$$

for functions $f, g \in L^p_{2\delta}(\mathbb{R}^3)$ with $n < p \leq \infty$ and $2\delta > n - \frac{n}{p}$. The above formula can be further expanded by the Sokhotski-Plemelj formula (cf. [8])

$$\frac{1}{x\pm\mathrm{i}0} = \mathrm{p.v.}\frac{1}{x} \mp \mathrm{i}\pi\delta_0,$$

which in distributional form reads

$$\left\langle\frac{1}{x+\mathrm{i}0},\varphi\right\rangle = \lim_{\rho\to 0^+}\int_{|x|>\rho}\frac{\varphi(x)}{x}\mathrm{d}x \mp \mathrm{i}\pi\varphi(0), \quad \varphi\in C_0^\infty(\mathbb{R}^n),$$

by pulling back with the quadratic form (η,μ). This way we obtain

$$I(f,g) := \text{p.v.}\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x,\eta+\mu)}\frac{\chi_c(|\eta+\mu|)}{|\eta+\mu|^2}\frac{\widehat{f}(\eta)\widehat{g}(\mu)}{(\eta,\mu)}\mathrm{d}\eta\mathrm{d}\mu$$
$$-\,\mathrm{i}\pi\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x,\eta+\mu)}\frac{\chi_c(|\eta+\mu|)}{|\eta+\mu|^2}\widehat{f}(\eta)\widehat{g}(\mu)\delta_0((\eta,\mu)=0)\mathrm{d}\eta\mathrm{d}\mu =: I' + I''. \quad (3)$$

The notation $\delta_0(H(x)=0)$ denotes the pullback of δ_0-distribution by H, i.e.

$$\int_{\mathbb{R}^n} f(x)\delta_0(H(x)=0)\mathrm{d}x = \int_{H(x)=0} f(x)\frac{\mathrm{d}\sigma(x)}{|\nabla H|},$$

where $\mathrm{d}\sigma(x)$ is the surface measure on the surface $\{x \mid H(x)=0\}$ (see e.g. Theorem 6.1.5 of [6]).

At this point in [15] in the 3-dimensional case the assumption that $\vec{q}$ and V are real-valued was used in quite essential way: by extending the solutions to (1) for $k<0$ and real $\vec{q}$ and V via formulae $u(x,k,\theta) := \overline{u(x,-k,\theta)}$ and $\nabla u(x,k,\theta) := \overline{\nabla u(x,-k,\theta)}$ it turns out that the term I'' can be eliminated. This simplifies the calculations considerably, but restricts the recovery of singularities to the form: $\mathrm{Re}(q_\mathrm{B}) - \beta$ (where β is also real-valued) belongs to $H^t(\mathbb{R}^3)$ (mod $C(\mathbb{R}^3)$) for all $t<\frac{3}{2}$.

In this text we drop the assumption about real-valued coefficients and analyze the term I'' more carefully. Along the lines of proofs of [11, Proposition 3.2] or [15, Lemma 3.3] in the 2-dimensional case one could write

$$I'' = \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{e}^{\mathrm{i}(x,\eta+\mu)}\frac{\chi_c(|\mu+\eta|)}{|\mu+\eta|^2}\widehat{f}(\eta)\widehat{g}(\mu)\delta_0((\eta,\mu)=0)\mathrm{d}\eta\mathrm{d}\mu$$
$$= \int_{\mathbb{R}^2}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x,\eta+t\eta^\perp)}\frac{\chi_c(|\eta+t\eta^\perp|)}{|\eta|^2+t^2}\widehat{f}(\eta)\widehat{g}(t\eta^\perp)|\eta|^{-1}\mathrm{d}t\mathrm{d}\eta,$$

where $\eta^\perp$ is the unit vector perpendicular to η chosen according to any specific orthogonal reference. This choice can be made uniquely for each vector, because each $\eta\in\mathbb{R}^2$ only has two perpendicular unit vectors. The above formula can then be used to obtain the continuity of $q_{1,\mathrm{M}}$ in $x\in\mathbb{R}^2$. However, in three dimensions there is no smooth way to assign for every unit $\eta\in\mathbb{S}^2$ a unique unit $\eta^\perp\in\mathbb{S}^2$ (essentially because of the hairy ball theorem, cf. [5]) so the 2D-approach requires some modifications to work in 3D.

3 Estimates for $I''(f,g)$

Our approach is based on the observation that certain integrals in I'' can be interpreted as Radon transforms of some functions. We can then use the mapping properties of Radon transform as the main tool for smoothness estimates.

The Radon transform of a suitable measurable function f is defined as

$$R(f)(\theta, t) := \int_{(\theta,x)=t} f(x)\mathrm{d}\sigma(x),$$

where $\theta \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$, see e.g., [4]. Here again the measure $\mathrm{d}\sigma(x)$ is the usual Lebesgue surface measure. We use the following theorem, which is proved in [10] by methods of complex interpolation.

Theorem 3.1. *For $n \geq 3$ the inequality*

$$\int_{\mathbb{S}^{n-1}} \sup_{t\in\mathbb{R}} |R(f)(\theta, t)|^\rho \, \mathrm{d}\theta \leq C\|f\|^{\alpha}_{L^a(\mathbb{R}^n)} \|f\|^{1-\alpha}_{L^b(\mathbb{R}^n)}$$

holds with $\rho \leq n$ whenever $1 \leq a < \frac{n}{n-1} < b \leq \infty$ and

$$\frac{\alpha}{a} + \frac{1-\alpha}{b} = \frac{n-1}{n}.$$

We remark that the above inequality does not hold if $n = 2$. For our purposes also Corollary 2 of [10] will be useful.

Corollary 3.2. *If $f \in L^a(\mathbb{R}^n) \cap L^b(\mathbb{R}^n)$ $(n \geq 3)$ with $1 \leq a < \frac{n}{n-1} < b \leq 2$ then for almost all $\theta \in \mathbb{S}^{n-1}$ the Radon transform $R(f)(\theta, t)$ is bounded and continuous as function of $t \in \mathbb{R}$.*

Let us now turn to our more particular case. By Hölder's inequality $L^p_{2\delta}(\mathbb{R}^3) \subset L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ when $3 < p \leq \infty$ and $2\delta > 3 - \frac{3}{p}$ so we may work in the larger space $L^1 \cap L^2$.

Lemma 3.3. *Let $f, g \in L^2(\mathbb{R}^3)$. If f or g is also in $L^1(\mathbb{R}^3)$ then $I'' = I''(f, g)(x)$ defines a bounded and continuous function of $x \in \mathbb{R}^3$.*

Proof. By the symmetry $I''(f, g) = I''(g, f)$ we may assume without loss of generality that $f \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Since $|\nabla_\mu(\eta, \mu)| = |\eta|$ then the application of δ_0-distribution in the μ-variable yields

$$\begin{aligned} |I''| &= \left| \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{e}^{\mathrm{i}(x,\eta+\mu)} \frac{\chi_c(|\eta+\mu|)}{|\eta+\mu|^2} \widehat{f}(\eta)\widehat{g}(\mu)\delta_0((\eta, \mu) = 0)\mathrm{d}\mu\mathrm{d}\eta \right| \\ &\leq \int_{\mathbb{R}^3} \frac{|\widehat{f}(\eta)|}{|\eta|} \int_{(\widehat{\eta},\mu)=0} \frac{\chi_c(|\eta+\mu|)}{|\eta+\mu|^2} |\widehat{g}(\mu)|\mathrm{d}\sigma(\mu)\mathrm{d}\eta, \end{aligned}$$

where $\widehat{\eta} := \frac{\eta}{|\eta|}$. Our plan is to show that the above integral is finite. The claim about continuity then follows from the Lebesgue dominated convergence theorem.

Here one can interpret the innermost integral as a Radon transform and note that $|\eta+\mu|^2 = |\eta|^2 + |\mu|^2$ within the region of integration. Then going over to polar coordinates in η allows us to split

$$
\begin{aligned}
|I''| \leq & \int_0^1 r \int_{\mathbb{S}^2} |\widehat{f}(r\theta)| R\left(\frac{\chi_c(|r\theta+\cdot|)}{r^2+|\cdot|^2}|\widehat{g}|\right)(\theta,0)\mathrm{d}\theta\mathrm{d}r \\
& + \int_1^\infty r \int_{\mathbb{S}^2} |\widehat{f}(r\theta)| R\left(\frac{\chi_c(|r\theta+\cdot|)}{r^2+|\cdot|^2}|\widehat{g}|\right)(\theta,0)\mathrm{d}\theta\mathrm{d}r =: J_1 + J_2.
\end{aligned}
$$

Since the Fourier transform of f is bounded by $\|\widehat{f}\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^1(\mathbb{R}^3)}$ we may estimate J_1 as

$$
J_1 \leq C\|f\|_{L^1(\mathbb{R}^3)} \int_0^1 \int_{\mathbb{S}^2} R\left(\frac{\chi_c(\sqrt{r^2+|\cdot|^2})}{|\cdot|^2}|\widehat{g}|\right)(\theta,0)\mathrm{d}\theta\mathrm{d}r. \tag{4}
$$

The function inside the Radon transform in (4) is integrable since $\chi_c(\sqrt{r^2+|\eta|^2}) = 0$ if $|\eta| < 1$ and by the Cauchy-Schwarz inequality

$$
\int_{\mathbb{R}^3} \frac{\chi_c(\sqrt{r^2+|\mu|^2})}{|\mu|^2}|\widehat{g}(\mu)|\mathrm{d}\mu \leq \|\widehat{g}\|_{L^2(\mathbb{R}^3)} \left(\int_{|\mu|>1} \frac{1}{|\mu|^4}\mathrm{d}\mu\right)^{\frac{1}{2}}
$$

uniformly in $r \in [0,1]$, where in polar coordinates

$$
\int_{|\mu|>1} \frac{1}{|\mu|^4}\mathrm{d}\mu = \int_1^\infty r^2 \int_{\mathbb{S}^2} \frac{1}{r^4}\mathrm{d}\theta\mathrm{d}r = 4\pi.
$$

By assumption $g \in L^2(\mathbb{R}^3)$, so that by Plancherel's theorem $\widehat{g} \in L^2(\mathbb{R}^3)$ and Parseval's equality $\|g\|_{L^2(\mathbb{R}^3)} = \|\widehat{g}\|_{L^2(\mathbb{R}^3)}$ holds (see e.g., [2]). Thus the function inside the Radon transform in (4) also belongs to $L^2(\mathbb{R}^3)$. Then Corollary 3.2 shows that the Radon transform in the first integral is well-defined as a bounded and continuous function of $t \in \mathbb{R}$. By Theorem 3.1 this transform is integrable in $\theta \in \mathbb{S}^2$ and therefore by using Theorem 3.1 on (4) and applying Parseval's equality we have the estimate

$$
\begin{aligned}
J_1 &\leq C\|f\|_{L^1(\mathbb{R}^3)} \int_0^1 \left\|\frac{\chi_c(\sqrt{r^2+|\cdot|^2})}{|\cdot|^2}\widehat{g}\right\|_{L^1(\mathbb{R}^3)}^{\frac{1}{3}} \left\|\frac{\chi_c(\sqrt{r^2+|\cdot|^2}|)}{|\cdot|^2}\widehat{g}\right\|_{L^2(\mathbb{R}^3)}^{\frac{2}{3}} \mathrm{d}r \\
&\leq C\|f\|_{L^1(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)},
\end{aligned}
$$

where in the notation of Theorem 3.1 $a = 1$, $b = 2$, $\rho = 1$ and $\alpha = \frac{1}{3}$.

Next we turn to integral J_2. Choose first some $\frac{6}{5} < a < \frac{3}{2}$ to play the same role as it does in Theorem 3.1. Next, the χ_c-term can be estimated as

$$
\frac{1}{r^2+|\eta|^2} \leq \frac{1}{r(1+|\eta|^2)^{\frac{1}{2}}},
$$

when $r > 1$. Then using Cauchy-Schwarz inequality in the θ-integral yields

$$\begin{aligned}
J_2 &\le \int_1^\infty r \int_{\mathbb{S}^2} |\widehat{f}(r\theta)| \frac{1}{r} R\left(\frac{|\widehat{g}|}{(1+|\cdot|^2)^{\frac{1}{2}}}\right)(\theta,0)\mathrm{d}\theta\mathrm{d}r \\
&\le \int_1^\infty \left(\int_{\mathbb{S}^2} |\widehat{f}(r\theta)|^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^2}\left[R\left(\frac{|\widehat{g}|}{(1+|\cdot|^2)^{\frac{1}{2}}}\right)(\theta,0)\right]^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \mathrm{d}r. \qquad (5)
\end{aligned}$$

By the Riemann-Lebesgue lemma the function $\widehat{f}$ is continuous and therefore the θ-integral over $|\widehat{f}|^2$ is well-defined. Let us show that the Radon transform in (5) is also well-defined. By Hölder's inequality

$$\int_{\mathbb{R}^3} \frac{|\widehat{g}(\mu)|^a}{(1+|\mu|^2)^{\frac{a}{2}}}\mathrm{d}\mu \le \left(\int_{\mathbb{R}^3} \frac{1}{(1+|\mu|^2)^{\frac{at}{2}}}\mathrm{d}\mu\right)^{\frac{1}{t}} \left(\int_{\mathbb{R}^3} |\widehat{g}(\mu)|^{at'}\mathrm{d}\mu\right)^{\frac{1}{t'}}, \qquad (6)$$

where $\frac{1}{t} + \frac{1}{t'} = 1$. The latter integral of (6) converges by assumption if we choose $t' := \frac{2}{a}$. This choice gives $\frac{5}{2} < t < 4$ which further means that $at > 3$ whence it follows that the first integral of (6) also converges. Clearly the function

$$\frac{|\widehat{g}(\mu)|}{(1+|\mu|^2)^{\frac{1}{2}}} \in L^a(\mathbb{R}^3) \qquad (7)$$

is also in $L^2(\mathbb{R}^3)$, because $\widehat{g} \in L^2(\mathbb{R}^3)$ by the previous arguments and the rest is bounded. By Corollary 3.2 the Radon transform in (5) is well-defined. We may now use Theorem 3.1 with $b = 2$ and $\rho = 2$ to the function (7) to see that the θ-integral of its Radon transform in (5) is bounded. By (6) we obtain the estimate

$$\begin{aligned}
J_2 &\le C \left\|\frac{\widehat{g}}{(1+|\cdot|^2)^{\frac{1}{2}}}\right\|_{L^a(\mathbb{R}^3)}^{\alpha} \left\|\frac{\widehat{g}}{(1+|\cdot|^2)^{\frac{1}{2}}}\right\|_{L^2(\mathbb{R}^3)}^{1-\alpha} \int_1^\infty \left(\int_{\mathbb{S}^2} |\widehat{f}(r\theta)|^2\mathrm{d}\theta\right)^{\frac{1}{2}} \mathrm{d}r \\
&\le C\|g\|_{L^2(\mathbb{R}^3)} \int_1^\infty \left(\int_{\mathbb{S}^2} |\widehat{f}(r\theta)|^2\mathrm{d}\theta\right)^{\frac{1}{2}} \mathrm{d}r
\end{aligned}$$

where $\alpha = \frac{a}{3(2-a)}$. Further, by Cauchy-Schwarz inequality and Parseval's equality

$$\begin{aligned}
\int_1^\infty \left(\int_{\mathbb{S}^2} |\widehat{f}(r\theta)|^2\mathrm{d}\theta\right)^{\frac{1}{2}} \mathrm{d}r &\le \left(\int_1^\infty \frac{1}{r^2}\mathrm{d}r\right)^{\frac{1}{2}} \left(\int_1^\infty r^2 \int_{\mathbb{S}^2} |\widehat{f}(r\theta)|^2 \mathrm{d}\theta\mathrm{d}r\right)^{\frac{1}{2}} \\
&\le \|f\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Therefore

$$J_2 \le C\|g\|_{L^2(\mathbb{R}^3)}\|f\|_{L^2(\mathbb{R}^3)}.$$

Combining the estimates for J_1 and J_2 yields

$$|I''(f,g)(x)| \le J_1 + J_2 \le C\|g\|_{L^2(\mathbb{R}^3)} \left(\|f\|_{L^1(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}\right)$$

uniformly in $x \in \mathbb{R}^3$ and this estimate concludes the proof. □

4 Recovery of singularities

To conclude the recovery of singularities of H_4 from the backscattering data for complex-valued coefficients we collect the results in the following

Theorem 4.1 (Recovery of singularities)**.** *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^3)$ and $V \in L^p_{2\delta}(\mathbb{R}^3)$ with $3 < p \le \infty$ and $2\delta > 3 - \frac{3}{p}$. Then the difference $q_{\mathrm{B}} - \beta \in H^t(\mathbb{R}^3) \pmod{C(\mathbb{R}^3)}$ for all $t < \frac{3}{2}$.*

Proof. From the discussion in Section 1 we know that in the expansion

$$q_{\mathrm{B}} = \beta + q_{1,\mathrm{M}} + q_{1,\mathrm{E}} + q_{\mathrm{rest}} + \widetilde{q},$$

the terms $q_{1,\mathrm{E}}, q_{\mathrm{rest}} \in H^t(\mathbb{R}^3)$, $t < \frac{3}{2}$ and $\widetilde{q} \in C^\infty(\mathbb{R}^3)$. The term $q_{1,\mathrm{M}}$ is the main culprit of problems, but due the conditions on the coefficients $\vec{q}$ and V it suffices to use Lemma 2.1 and then properties of $I = I' + I''$ in (3). Now [15, Lemma 3.7] shows that I' belongs to $H^t(\mathbb{R}^3)$. Then the use of Lemma 3.3 to I'' concludes the proof. □

If $\vec{q}$ and V are as before then $\beta \in L^p_{2\delta}(\mathbb{R}^3)$. Theorem 4.1 allows us to conclude that the difference $q_{\mathrm{B}} - \beta$ is smoother than β itself in the sense that the Sobolev space $H^t(\mathbb{R}^3)$ embeds into $L^q(\mathbb{R}^3)$ for any given $2 \le q < \infty$, by choosing $t < \frac{3}{2}$ suitably. This means that if β contains any (local) infinite singularities in the sense that $\beta \notin L^p_{\mathrm{loc}}(\mathbb{R}^3)$ (for some p) then q_{B} has precisely those same singularities.

Corollary 4.2. *Under the same assumptions as in Theorem 4.1 the infinite singularities of $\beta = -\frac{1}{2}\nabla \cdot \vec{q} + V$ over boundaries of smooth domains in three dimensions are uniquely determined by the backscattering data A_{b} and can be recovered from q_{B}.*

To apply these results in practise one needs to know only the backscattering amplitude $A_{\mathrm{b}}(k,\theta)$ for all angles $\theta \in \mathbb{S}^2$ and all arbitrarily high frequencies $k > 0$. A numerical scheme for an approach is treated in [15] in the 2D case.

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