Scattering problems for perturbations of the multidimensional biharmonic operator

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Abstract

Some scattering problems for the multidimensional biharmonic operator are studied. The operator is perturbed by first and zero order perturbations, which maybe complex-valued and singular. We show that the solutions to direct scattering problem satisfy a Lippmann-Schwinger equation, and that this integral equation has a unique solution in the weighted Sobolev space $H^2_{-\delta}$. The main result of this paper is the proof of Saito's formula, which can be used to prove a uniqueness theorem for the inverse scattering problem. The proof of Saito's formula is based on norm estimates for the resolvent of the direct operator in $H^1_{-\delta}$.

1 Introduction

We consider the following n-dimensional ($n \geq 2$) biharmonic operator

$$H_4 u = \Delta^2 u + \vec{q} \cdot \nabla u + Vu, \tag{1}$$

where Δ is the Laplacian and $\cdot$ denotes the dot-product $x \cdot y = \sum_{j=1}^{n} x_j y_j$ for $x, y \in \mathbb{C}^n$. The bi-Laplacian is perturbed by first and zero order perturbations, vector-valued function $\vec{q}$ and a scalar function V, that may be complex-valued.

The motivation to study operators of order 4 appears for example in the study of elasticity and the theory of vibrations of beams. As a concrete example, the (linear) beam equation [9]

$$\partial_t^2 U + \Delta^2 U + mU = 0$$

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under time-harmonic assumption $U(x,t) = u(x)\mathrm{e}^{-\mathrm{i}\omega t}$ results in the equation

$$\Delta^2 u + mu = \omega^2 u.$$

For scattering in the nonlinear case, see e.g. [20] and the references therein. Other examples of biharmonic problems include hinged plate configurations, described by equations of form

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where u satisfies Navier boundary conditions [9] and scattering by grating stacks, described by the equation $\Delta^2 u - \beta^4 u = 0$ [17].

In terms of inverse problems for bi- and poly-harmonic operators we mention some solutions to inverse boundary value problems, see e.g., [14] and also [3, 4]. In these papers the aim is to recover the coefficients $\vec{q}$ and V in operator $\Delta^m + \vec{q} \cdot \nabla + V$, $m \geq 2$ in some bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. The main idea is to define the so-called Dirichlet-to-Neumann map and then to recover the coefficients $\vec{q}$ and V from it. In [22] it was proved that the Dirichlet-to-Neumann map uniquely corresponds to $\vec{q}$ and V and also to a second order perturbation F. The motivation for the approach in the above texts is based on the fundamental papers [24, 25].

The present work is concerned with the following scattering problem for H_4 given by

$$\begin{cases} H_4 u = k^4 u, \quad k > 0, \quad u = u_0 + u_{\mathrm{sc}}, \quad u_0(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(x,\theta)}, \\ \frac{\partial}{\partial n} f - \mathrm{i}kf = o\left(|x|^{-\frac{n-1}{2}}\right), \quad |x| \to \infty, \text{ for both } f = u_{\mathrm{sc}} \text{ and } f = \Delta u_{\mathrm{sc}}, \end{cases} \tag{2}$$

where the scattered wave u_{sc} should be from some suitable function space. Here $\theta \in S^{n-1}$ is the angle of incident wave and $(\cdot, \cdot)$ denotes the usual real inner product. The second line in (2) is interpreted as an analogue of Sommerfeld's radiation condition at infinity for this biharmonic operator. To the best of our knowledge this radiation condition has not appeared in the literature before. We use it to reduce (2) to a related integral equation.

The authors were originally motivated to start studying scattering for fourth order operators by the article of Aktosun and Papanicolaou [2], where the time-evolution of several scattering coefficients for the 1D biharmonic operator was studied. In terms of inverse scattering problems for fourth order operator we mention K. Iwasaki's results [12, 13]. Iwasaki studied the scattering problem in one-dimension and considered the inverse problem as a Riemann-Hilbert boundary value problem with respect to the wavenumber k in the complex cone $\arg([0, \pi/4]) \setminus \{0\}$. Given certain reflection and connection coefficients he then showed that it is possible to uniquely recover the potentials $\vec{q}$ and V of the scattering operator. In abstract setting, Kuroda [15] has studied scattering for zero-order perturbations for self-adjoint operators.

Let us now turn to (2) and define $H_0 := \Delta^2 - k^4$. Due to the equality $H_0 = (-\Delta - k^2)(-\Delta + k^2)$ the operator H_4 inherits some Schrödinger-like properties which allows us to conclude that a fundamental solution to H_0 is given by

$$G_k^+(|x|) = \frac{\mathrm{i}}{8k^2}\left(\frac{|k|}{2\pi|x|}\right)^{\frac{n-2}{2}}\left(H^{(1)}_{\frac{n-2}{2}}(|k||x|) + \frac{2\mathrm{i}}{\pi}K_{\frac{n-2}{2}}(|k||x|)\right).$$

Here $H^{(1)}_{\frac{n-2}{2}}$ and $K_{\frac{n-2}{2}}$ are the Hankel function of the first kind and Macdonald's function of orders $\frac{n-2}{2}$. The function G_k^+ is also the kernel of the integral operator $(\Delta^2 - k^4 - \mathrm{i}0)^{-1}$. By applying the fundamental solution to (1) we obtain a Lippmann–Schwinger integral equation

$$u(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(x,\theta)} - \int_{\mathbb{R}^n} G_k^+(|x-y|)\left[\vec{q}(y)\cdot\nabla u(y,k,\theta) + V(y)u(y,k,\theta)\right]\mathrm{d}y. \quad (3)$$

Since $u_0 = \mathrm{e}^{\mathrm{i}k(\theta,x)}$ is just a bounded function it is more convenient to study the equivalent integral equation for the scattered wave, namely

$$\begin{aligned} u_{\mathrm{sc}} &= -\int_{\mathbb{R}^n} G_k^+(|x-y|)\left[\vec{q}(y)\cdot\nabla(u_0 + u_{\mathrm{sc}}) + V(y)(u_0 + u_{\mathrm{sc}})\right]\mathrm{d}y \\ &= \widetilde{u_0} - \int_{\mathbb{R}^n} G_k^+(|x-y|)\left[\vec{q}(y)\cdot\nabla u_{\mathrm{sc}} + V(y)u_{\mathrm{sc}}\right]\mathrm{d}y =: \widetilde{u_0} + L_k(u_{\mathrm{sc}}), \end{aligned} \quad (4)$$

where $\widetilde{u_0} = L_k u_0$.

The aim of the present text is to study the classical scattering theory for (1) and in particular to prove an analogue of so-called Saito's formula. The approach is similar to scattering theory of Schrödinger operators see e.g., books by Cakoni and Colton [6] and Eskin [7] and to scattering theory of operators with constant coefficients, see the classic book of Hörmander [11]. We start by showing that a solution to the scattering problem (2) indeed also satisfies equation (3). This translates the study of the scattering problem to the somewhat simpler study of an integral equation for which we can then show the unique solvability by extending some well-known estimates for the Schrödinger operator to biharmonic case. This solution admits the asymptotic representation

$$u(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(x,\theta)} - C_n\frac{k^{\frac{n-7}{2}}\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}}A(k,\theta,\theta') + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right), \quad |x| \to \infty,$$

where $\theta' \in S^{n-1}$ is the angle of measurement and the function

$$A(k,\theta,\theta') = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta',y)}\left[\vec{q}\cdot\nabla u + Vu\right]\mathrm{d}y,$$

is called the scattering amplitude. From the point of view of inverse problems we regard this scattering amplitude as the relevant scattering data. For our purposes we require the scattering amplitude to be known for all possible incident and measurement angles and all arbitrarily high frequencies ($k > 0$ large). Then Saito's formula is given by the following

Theorem 1.1 (Saito's formula). *Assume that $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, where $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. Then the limit*

$$\lim_{k\to\infty} k^{n-1} \int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} A(k,\theta,\theta')\mathrm{d}\theta\mathrm{d}\theta' = 2^n\pi^{n-1} \int_{\mathbb{R}^n} \frac{\beta(y)}{|x-y|^{n-1}}\mathrm{d}y$$

holds uniformly in x. Here $\beta := -\frac{1}{2}\nabla\cdot\vec{q} + V$.

To the best of our knowledge a formula of this type first appeared in [21] for the Schrödinger operator with real short-range potential $|V(x)| \leq C(1+|x|)^{-\mu}$, $\mu > 1$ in $\mathbb{R}^3$. This formula has since then been generalized to Schrödinger operators in any dimensions with more general singular potentials and in some cases to nonlinear coefficients, see e.g., [18, 10, 23]. The significance of Saito's formula for inverse problems is apparent from its corollaries.

Corollary 1.2 (Uniqueness). *Let $\vec{q_1}, V_1$ and $\vec{q_2}, V_2$ be as in Theorem 1.1. If the corresponding scattering amplitudes for these coefficients coincide for some sequence $k_j \to \infty$ then the corresponding coefficients β_1 and β_2 are equal in the sense of distributions.*

Corollary 1.3 (Representation formula). *Under the same assumptions as in Theorem 1.1 we have*

$$\beta(x) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2^{n+1}\pi^{\frac{3n-1}{2}}} \lim_{k\to\infty} k^{n-1} \int_{S^{n-1}\times S^{n-1}} A(k,\theta,\theta')|\theta-\theta'|\mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}\mathrm{d}\theta\mathrm{d}\theta'$$

in the sense of tempered distributions.

This paper is organized as follows. In Section 2 we fix some notations and recall asymptotic formulas for Hankel and Macdonald functions. Then in Section 3 we show that a solution to (2) satisfies (3). In Section 4 we prove the existence and the uniqueness of the solution to (3). Several estimates for the solution and its related operators are given. Section 5 concerns the asymptotic behaviour of the solution u. In particular, we define the scattering amplitude which is regarded as the scattering data for the inverse problem. Finally, in Section 6 we give a proof for Saito's formula.

2 Preliminaries

We use the following notations throughout the text. The letter C will be used to denote a generic constant $C > 0$ which may have different values from step to step. The symbol p' denotes the Hölder conjugate of p, i.e. $1/p' + 1/p = 1$. The weighted Lebesgue spaces L^p_δ are defined by the norm

$$\|f\|_{L^p_\delta(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} (1 + |x|)^{\delta p} |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

The Sobolev spaces are defined by $W^1_{p,\delta}(\mathbb{R}^n) := \{f \in L^p_\delta(\mathbb{R}^n) \mid \nabla f \in L^p_\delta(\mathbb{R}^n)\}$. We say that $\vec{q} \in W^1_{p,\delta}(\mathbb{R}^n)$, if each scalar-valued component of $\vec{q}$ belongs to $W^1_{p,\delta}(\mathbb{R}^n)$ in the above sense. As usual, in the case of L^2_δ-based Sobolev space, we write $W^s_{2,\delta}(\mathbb{R}^n) =: H^s_\delta(\mathbb{R}^n)$.

Later we will need some knowledge about the behaviour of the function G^+_k. We start by recalling that the asymptotic behaviour of Hankel functions is

$$H^{(1)}_\nu(x) = \begin{cases} O(|x|^{-\nu}), & \nu > 0, \\ O(\log(2x)), & \nu = 0 \end{cases}$$

as $x \to 0+$ (see e.g., [26, 16]). The Macdonald function K_ν has the same behaviour at $x \to 0+$. Moreover, for all $\nu \geq 0$

$$H^{(1)}_\nu(x) = \sqrt{\frac{2}{\pi x}} \mathrm{e}^{\mathrm{i}\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)} + O(x^{-\frac{3}{2}}), \quad x \to +\infty.$$

Correspondingly, Macdonald's function has the asymptotic representation

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} \mathrm{e}^{-x} + O\left(\frac{\mathrm{e}^{-x}}{x^{\frac{3}{2}}}\right), \quad x \to +\infty. \tag{5}$$

The explicit form of the fundamental solution G^+_k allows us to conclude that for $k > 0$ the function G^+_k behaves like

$$G^+_k(|x|) = \begin{cases} O\left(\frac{1}{k^2}(1 + |\log(k|x|)|)\right), & n = 2, \\ O\left(\frac{1}{k^2|x|^{n-2}}\right), & n \geq 3 \end{cases}$$

as $k|x| \to 0$ and

$$G^+_k(|x|) = O\left(\frac{k^{\frac{n-7}{2}}}{|x|^{\frac{n-1}{2}}}\right), \quad k|x| \to +\infty.$$

On the other hand, the relations

$$\frac{\mathrm{d}}{\mathrm{d}z} z^{-\nu} H_\nu^{(1)}(z) = -z^{-\nu} H_{\nu+1}^{(1)}(z), \quad \frac{\mathrm{d}}{\mathrm{d}z} z^{-\nu} K_\nu(z) = -z^{-\nu} K_{\nu+1}(z)$$

and chain rule imply that

$$\nabla G_k^+(|x|) = -\frac{\mathrm{i}x}{8k^2(2\pi)^{\frac{n-2}{2}}} \left(\frac{k}{|x|}\right)^{\frac{n}{2}} \left(H_{\frac{n}{2}}^{(1)}(k|x|) + \frac{2\mathrm{i}}{\pi} K_{\frac{n}{2}}(k|x|)\right)$$

for all $n \geq 2$. This leads to the asymptotic behaviour

$$\nabla G_k^+(|x|) = \begin{cases} O\left(\frac{1}{k^2|x|^{n-1}}\right), & k|x| \to +0, \\ O\left(\frac{k^{\frac{n-5}{2}}}{|x|^{\frac{n-1}{2}}}\right), & k|x| \to +\infty. \end{cases}$$

Remark 2.1. In the three dimensional case the fundamental solution G_k^+ ($k > 0$) has a compact explicit representation

$$G_k^+(|x|) = \frac{\mathrm{e}^{\mathrm{i}k|x|} - \mathrm{e}^{-k|x|}}{8\pi k^2|x|},$$

see e.g., [16].

3 From scattering problem to Lippmann-Schwinger equation

This section is devoted to showing that a solution to the scattering problem (2) must satisfy the Lippmann-Schwinger equation. The proof is classical and uses similar techniques as Cakoni and Colton [6] and Eskin [7].

Lemma 3.1. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, where $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. If $f \in H^1_{-\delta}(\mathbb{R}^n)$ then $\vec{q} \cdot \nabla f + Vf \in L^2_\delta(\mathbb{R}^n)$.*

Proof. By Hölder's inequality we obtain

$$\begin{aligned} \|Vf\|^2_{L^2_\delta(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|x|)^{2\delta} |V(x)|^2 |f(x)|^2 \mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^n} (1+|x|)^{4\delta r} |V(x)|^{2r} \mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} (1+|x|)^{-2\delta r'} |f(x)|^{2r'} \mathrm{d}x\right)^{\frac{1}{r'}}. \end{aligned}$$

We choose $2r = p$ which implies $2r' = \frac{2p}{p-2}$. By the Sobolev embedding theorem we have the continuous embedding

$$f \in H^1_{-\delta}(\mathbb{R}^n) \subset L^{\frac{2p}{p-2}}_{-\delta}(\mathbb{R}^n),$$

when $p > n$, with the norm estimate

$$\|f\|_{L^{\frac{2p}{p-2}}_{-\delta}(\mathbb{R}^n)} \leq c_0 \|f\|_{H^1_{-\delta}(\mathbb{R}^n)}$$

for some constant $c_0 > 0$ depending only on p and n. Since $W^1_{p,2\delta}(\mathbb{R}^n) \subset L^\infty_{2\delta}(\mathbb{R}^n)$ we conclude that

$$\|\vec{q} \cdot \nabla f + V f\|_{L^2_\delta(\mathbb{R}^n)} \leq \|\vec{q}\|_{L^\infty_{2\delta}(\mathbb{R}^n)} \|\nabla f\|_{L^2_{-\delta}(\mathbb{R}^n)} + c_0 \|V\|_{L^p_{2\delta}(\mathbb{R}^n)} \|f\|_{H^1_{-\delta}(\mathbb{R}^n)} < \infty.$$

□

Lemma 3.2. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, where $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. If $u = u_0 + u_{\mathrm{sc}}$, $u_{\mathrm{sc}} \in H^4_{\mathrm{loc}}(\mathbb{R}^n) \cap H^2_{-\delta}(\mathbb{R}^n)$, solves (2) then*

$$\lim_{R \to \infty} \int_{|y|=R} (|\Delta u_{\mathrm{sc}}|^2 + |u_{\mathrm{sc}}|^2) \mathrm{d}\sigma(y) \leq C$$

for some constant $C > 0$ and for fixed $k > 0$.

Proof. By using the radiation condition for u_{sc} and Δu_{sc} and then applying Green's second identity we calculate (with slight abuse of o-notation)

$$\begin{aligned}
&2\mathrm{i}k \int_{|y|=R} (|\Delta u_{\mathrm{sc}}|^2 + k^4 |u_{\mathrm{sc}}|^2) \mathrm{d}\sigma(y) \\
&= \int_{|y|=R} (\overline{\Delta u_{\mathrm{sc}}}(\mathrm{i}k \Delta u_{\mathrm{sc}}) - \Delta u_{\mathrm{sc}}(\overline{\mathrm{i}k \Delta u_{\mathrm{sc}}}) + k^4 \overline{u_{\mathrm{sc}}}(\mathrm{i}k u_{\mathrm{sc}}) - k^4 u_{\mathrm{sc}}(\overline{\mathrm{i}k u_{\mathrm{sc}}})) \mathrm{d}\sigma(y) \\
&= \int_{|y|=R} \left(\overline{\Delta u_{\mathrm{sc}}} \frac{\partial}{\partial n} \Delta u_{\mathrm{sc}} - \Delta u_{\mathrm{sc}} \frac{\partial}{\partial n} \overline{\Delta u_{\mathrm{sc}}} + k^4 \overline{u_{\mathrm{sc}}} \frac{\partial}{\partial n} u_{\mathrm{sc}} - k^4 u_{\mathrm{sc}} \frac{\partial}{\partial n} \overline{u_{\mathrm{sc}}} \right) \mathrm{d}\sigma(y) \\
&\quad + o(R^{-\frac{n-1}{2}}) \int_{|y|=R} (\Delta u_{\mathrm{sc}} + \Delta \overline{u_{\mathrm{sc}}} + k^4 u_{\mathrm{sc}} + k^4 \overline{u_{\mathrm{sc}}}) \mathrm{d}\sigma(y) \\
&= \int_{|y| \leq R} (\overline{\Delta u_{\mathrm{sc}}} \Delta^2 u_{\mathrm{sc}} - \Delta u_{\mathrm{sc}} \overline{\Delta^2 u_{\mathrm{sc}}} + k^4 \overline{u_{\mathrm{sc}}} \Delta u_{\mathrm{sc}} - k^4 u_{\mathrm{sc}} \overline{\Delta u_{\mathrm{sc}}}) \mathrm{d}y \\
&\quad + o(R^{-\frac{n-1}{2}}) \int_{|y|=R} (\Delta u_{\mathrm{sc}} + \Delta \overline{u_{\mathrm{sc}}} + k^4 u_{\mathrm{sc}} + k^4 \overline{u_{\mathrm{sc}}}) \mathrm{d}\sigma(y).
\end{aligned}$$

Note that $\Delta^2 u_{\rm sc} - k^4 u_{\rm sc} = -\vec{q}\cdot\nabla u - Vu$. Therefore

$$
\begin{aligned}
&2\mathrm{i}k \int_{|y|=R} (|\Delta u_{\rm sc}|^2 + k^4 |u_{\rm sc}|^2)\mathrm{d}\sigma(y) \\
&\qquad = \int_{|y|\le R} (\overline{\Delta u_{\rm sc}}(-\vec{q}\cdot\nabla u - Vu) - \Delta u_{\rm sc}(\overline{-\vec{q}\cdot\nabla u - Vu}))\mathrm{d}y \\
&\qquad\qquad + o(R^{-\frac{n-1}{2}}) \int_{|y|=R} (\Delta u_{\rm sc} + \Delta\overline{u_{\rm sc}} + k^4 u_{\rm sc} + k^4 \overline{u_{\rm sc}})\mathrm{d}\sigma(y).
\end{aligned}
$$

Using the Cauchy-Schwartz inequality in both terms gives

$$
\begin{aligned}
\int_{|y|=R} (|\Delta u_{\rm sc}|^2 + |u_{\rm sc}|^2)\mathrm{d}\sigma(y) &\le C\|u_{\rm sc}\|_{H^2_{-\delta}(\mathbb{R}^n)} \|\vec{q}\cdot\nabla u + Vu\|_{L^2_\delta(\mathbb{R}^n)} \\
&\quad + o(1)\left(\int_{|y|=R} (|\Delta u_{\rm sc}|^2 + |u_{\rm sc}|^2)\mathrm{d}\sigma(y)\right)^{\frac{1}{2}},
\end{aligned}
$$

where the first term on the right-hand side is finite according to Lemma 3.1. This inequality then implies the claim. $\square$

Theorem 3.3. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$ where $2\delta > n - \frac{n}{p}$ and $n < p \le \infty$. If $u = u_0 + u_{\rm sc}$, $u_{\rm sc} \in H^4_{\rm loc}(\mathbb{R}^n) \cap H^2_{-\delta}(\mathbb{R}^n)$, solves (2) then it also solves (3).*

Proof. Fix a point $x \in \mathbb{R}^n$ and choose $R > 0$ so that $x \in B_R(0)$, where $B_R(0)$ denotes the open ball of radius R centered at origin. Let $\varepsilon > 0$ be such that $B_\varepsilon(x) \subset B_R(0)$ and denote $\Omega_{R,\varepsilon} := B_R(0)\backslash B_\varepsilon(x)$. Then by Lemma A.1 we calculate

$$
\begin{aligned}
&\int_{\Omega_{R,\varepsilon}} (u_{\rm sc}(y)(\Delta_y^2 - k^4)G_k^+(|x-y|) - G_k^+(|x-y|)(\Delta^2 - k^4)u_{\rm sc}(y))\mathrm{d}y \\
&= \int_{\Omega_{R,\varepsilon}} (u_{\rm sc}(y)\Delta_y^2 G_k^+(|x-y|) - G_k^+(|x-y|)\Delta^2 u_{\rm sc}(y))\mathrm{d}y \\
&= \int_{\partial\Omega_{R,\varepsilon}} \Big[u_{\rm sc}\frac{\partial}{\partial n}\Delta_y G_k^+(|x-y|) + \Delta u_{\rm sc}\frac{\partial}{\partial n}G_k^+(|x-y|) \\
&\quad - G_k^+(|x-y|\frac{\partial}{\partial n}\Delta u_{\rm sc} - \Delta G_k^+(|x-y|)\frac{\partial}{\partial n}u_{\rm sc}\Big]\mathrm{d}\sigma(y) \\
&= \int_{\partial\Omega_{R,\varepsilon}} \Big[u_{\rm sc}\left(\frac{\partial}{\partial n} - \mathrm{i}k\right)\Delta_y G_k^+(|x-y|) + \Delta u_{\rm sc}\left(\frac{\partial}{\partial n} - \mathrm{i}k\right)G_k^+(|x-y|) \\
&\quad - G_k^+(|x-y|)\left(\frac{\partial}{\partial n} - \mathrm{i}k\right)\Delta u_{\rm sc} - \Delta G_k^+(|x-y|)\left(\frac{\partial}{\partial n} - \mathrm{i}k\right)u_{\rm sc}\Big]\mathrm{d}\sigma(y).
\end{aligned}
$$

Letting $\varepsilon \to 0+$ we obtain the equality

$$\begin{aligned} u_{\mathrm{sc}}(x) = &- \int_{\Omega_R} G_k^+(|x-y|)(\vec{q}\cdot\nabla u + Vu)\mathrm{d}y \\ &+ \int_{\partial\Omega_R} \left[u_{\mathrm{sc}} \left(\frac{\partial}{\partial n} - \mathrm{i}k\right) \Delta_y G_k^+(|x-y|) + \Delta u_{\mathrm{sc}} \left(\frac{\partial}{\partial n} - \mathrm{i}k\right) G_k^+(|x-y|) \right. \\ &\left. - G_k^+(|x-y|)\left(\frac{\partial}{\partial n} - \mathrm{i}k\right)\Delta u_{\mathrm{sc}} - \Delta G_k^+(|x-y|)\left(\frac{\partial}{\partial n} - \mathrm{i}k\right) u_{\mathrm{sc}} \right] \mathrm{d}\sigma(y) \end{aligned}$$

Note that for fixed $x \in \mathbb{R}^n$ and large $R > 0$ we have $G_k^+(|x-y|) = O\left(R^{-\frac{n-1}{2}}\right)$ (same for ΔG_k^+). Therefore, letting $R \to +\infty$ we see that the latter integral tends to zero. This follows from Lemma 3.2 and the fact that all $u_{\mathrm{sc}}, \Delta u_{\mathrm{sc}}, G_k^+$ and ΔG_k^+ satisfy the radiation condition. □

Corollary 3.4. *If $v \in H^4_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies the radiation condition at infinity and fulfils the homogeneous equation $\Delta^2 v - k^4 v = 0$, then $v = 0$.*

4 Solution to Lippmann-Schwinger equation

Next we show that when $k > 0$ the integral equation (4) can be solved by iterations defined by $u_{\mathrm{sc}}^{(j)} = L_k^j \widetilde{u_0}$ for $j = 0, 1, \ldots$ We denote by $\widehat{G_k^+}$ the convolution operator with kernel G_k^+. Our first lemma gives the required mapping properties for this integral operator. Similar estimates for Schrödinger operator are familiar in the literature and can be met for example in [7] and [19]. The first lemma is analogous to the limiting absorption principle.

Lemma 4.1. *The operator $\widehat{G_k^+}$ maps from $L^2_\delta(\mathbb{R}^n)$ to $H^2_{-\delta}(\mathbb{R}^n)$ with the estimates*

$$\|\widehat{G_k^+} f\|_{H^j_{-\delta}(\mathbb{R}^n)} \le \frac{C_0}{k^{3-j}} \|f\|_{L^2_\delta(\mathbb{R}^n)}, \quad j = 0, 1, 2$$

when $k > 1$ and $\delta > \frac{1}{2}$ for $n \ge 2$. Here the constant C_0 only depends on n and δ.

Proof. The claim follows from Agmon's estimate [1, Appendix A, Remark 2] for $p = 2$, $\delta > \frac{1}{2}$ and $k > 1$ given by

$$\sum_{|\alpha|\le 4} k^{3-|\alpha|} \|D^\alpha f\|_{L^2_{-\delta}(\mathbb{R}^n)} \le C_0 \|(\Delta^2 - k^4) f\|_{L^2_\delta(\mathbb{R}^n)}$$

for any $f \in H^4(\mathbb{R}^n)$, where the constant $C_0 > 0$ only depends on n and δ. We use this result to the opposite direction and take only the terms where $|\alpha|$ equals 0,1 or 2 to get the claim. □

We now proceed to prove norm estimates for the operator L_k.

Theorem 4.2. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$ where $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. Then the following properties are satisfied*

1. *The function $\widetilde{u_0}$ belongs to $H^2_{-\delta}(\mathbb{R}^n)$ with the estimates*
$$\|\widetilde{u_0}\|_{H^j_{-\delta}(\mathbb{R}^n)} \leq \frac{C_1}{k^{2-j}}, \quad j = 0, 1, 2$$
2. *The operator $L_k : H^1_{-\delta}(\mathbb{R}^n) \to H^2_{-\delta}(\mathbb{R}^n)$ is bounded and satisfies the norm estimates*
$$\|L_k f\|_{H^j_{-\delta}(\mathbb{R}^n)} \leq \frac{C_2}{k^{3-j}} \|f\|_{H^1_{-\delta}(\mathbb{R}^n)}, \quad j = 0, 1, 2$$

for $k > 1$ and for some $C_1, C_2 > 0$ independent of k.

Proof. In view of Lemma 4.1 we consider only the case $j = 0$ and the estimates for derivatives follow. We apply Lemma 4.1 to estimate

$$\begin{aligned}
\|\widetilde{u_0}\|_{L^2_{-\delta}(\mathbb{R}^n)} &= \|\widehat{G^+_k}(\vec{q} \cdot \nabla u_0 + V u_0)\|_{L^2_{-\delta}(\mathbb{R}^n)} \leq \frac{C_0}{k^3} \|\vec{q} \cdot \nabla u_0 + V u_0\|_{L^2_{\delta}(\mathbb{R}^n)} \\
&\leq \frac{C_0}{k^3} \left(k \|\vec{q}\|_{L^2_\delta(\mathbb{R}^n)} + \|V\|_{L^2_\delta(\mathbb{R}^n)} \right) \leq \frac{C_1}{k^2},
\end{aligned}$$

where the constant $C_0 > 0$ comes from Lemma 4.1 and $C_1 := C_0(\|\vec{q}\|_{L^2_\delta(\mathbb{R}^n)} + \|V\|_{L^2_\delta(\mathbb{R}^n)})$. Next, suppose that $f \in H^1_{-\delta}(\mathbb{R}^n)$. By Lemmata 3.1 and 4.1 we see that

$$\begin{aligned}
\|L_k f\|_{L^2_{-\delta}(\mathbb{R}^n)} &\leq \frac{C_0}{k^3} \left(\|\vec{q}\|_{L^\infty_{2\delta}(\mathbb{R}^n)} \|\nabla f\|_{L^2_{-\delta}(\mathbb{R}^n)} + c_0 \|V\|_{L^p_{2\delta}(\mathbb{R}^n)} \|f\|_{H^1_{-\delta}(\mathbb{R}^n)} \right) \\
&\leq \frac{C_2}{k^3} \|f\|_{H^1_{-\delta}(\mathbb{R}^n)},
\end{aligned}$$

where $C_2 := C_0(\|\vec{q}\|_{L^\infty_{2\delta}(\mathbb{R}^n)} + c_0 \|V\|_{L^p_{2\delta}(\mathbb{R}^n)})$. The proof is then finished. $\square$

Corollary 4.3. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$ where $n < p \leq \infty$ and $2\delta > n - \frac{n}{p}$. Then there exists a constant $k_0 > 1$ such that the function $u_{\mathrm{sc}}(x, k, \theta)$ defined by the series*

$$u_{\mathrm{sc}}(x, k, \theta) = \sum_{j=0}^{\infty} L^j_k \widetilde{u_0}(x, k, \theta) \tag{6}$$

solves integral equation (4) uniquely in $H^2_{-\delta}(\mathbb{R}^n)$, when $k \geq k_0$. Moreover,

$$\|u_{\mathrm{sc}}\|_{H^1_{-\delta}(\mathbb{R}^n)} \leq \frac{C}{k},$$

when $k \geq k_0$ is large enough.

Proof. We show that

$$\|L_k^j \widetilde{u_0}\|_{H^1_{-\delta}(\mathbb{R}^n)} \leq \frac{C_1 (C_2)^j}{k^{2j+1}}, \quad j = 0, 1, \ldots$$

By Theorem 4.2 the claim holds for $j = 0$. Suppose then that the claim is proved for some $j > 0$. By induction hypothesis

$$\|L_k^{j+1} \widetilde{u_0}\|_{H^1_{-\delta}(\mathbb{R}^n)} \leq \frac{C_2}{k^2} \|L_k^j \widetilde{u_0}\|_{H^1_{-\delta}(\mathbb{R}^n)} \leq \frac{C_2}{k^2} \cdot \frac{C_1 (C_2)^j}{k^{2j+1}} = \frac{C_1 (C_2)^{j+1}}{k^{2(j+1)+1}},$$

so the claim holds for all $j = 0, 1, \ldots$ We may now choose any $k_0 > \max\{1, \sqrt{C_2}\}$ to conclude that the series (6) converges in $H^1_{-\delta}(\mathbb{R}^n)$. Because the operator L_k is linear and maps continuously in $H^1_{-\delta}(\mathbb{R}^n)$ the series solves (4). Similar calculation shows that $u_{\mathrm{sc}} \in H^2_{-\delta}(\mathbb{R}^n)$.

To verify the uniqueness of the solution, suppose that $\widetilde{u}$ and $\widetilde{v}$ are solutions to (4). Then by Theorem 4.2 we have

$$\|\widetilde{u} - \widetilde{v}\|_{H^1_{-\delta}(\mathbb{R}^n)} \leq \frac{C_2}{k^2} \|\widetilde{u} - \widetilde{v}\|_{H^1_{-\delta}(\mathbb{R}^n)}.$$

Our assumption $k \geq k_0 > \sqrt{C_2}$ means that $\frac{C_2}{k^2} \leq \frac{C_2}{k_0^2} < 1$ and therefore $\widetilde{u} = \widetilde{v}$ in $H^2_{-\delta}(\mathbb{R}^n) \subset H^1_{-\delta}(\mathbb{R}^n)$. $\square$

With these results the limiting absorption principle can be extented for the full operator H_4.

Corollary 4.4. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$ with $n < p \leq \infty$ and $2\delta > n - \frac{n}{p}$. Then the operator*

$$\widehat{G_{\mathrm{p}}} := \lim_{\varepsilon \to 0^+} (H_4 - k^4 - \mathrm{i}\varepsilon)^{-1}$$

exists in the uniform operator topology from $L^2_\delta(\mathbb{R}^n)$ to $H^1_{-\delta}(\mathbb{R}^n)$ with the norm estimates

$$\|\widehat{G_{\mathrm{p}}} f\|_{H^j_{-\delta}(\mathbb{R}^n)} \leq \frac{C}{k^{3-j}} \|f\|_{L^2_\delta(\mathbb{R}^n)}, \quad j = 0, 1,$$

for $k > 0$ large enough.

Proof. Consider the equation $\widehat{G_{\mathrm{p}}} f = \widehat{G_k^+} f - \widehat{G_k^+}(\vec{q} \cdot \nabla + V)\widehat{G_{\mathrm{p}}} f$ in $H^1_{-\delta}(\mathbb{R}^n)$. By (formally) applying this equation repeatedly we obtain

$$\widehat{G_{\mathrm{p}}} f = \widehat{G_k^+} f + \sum_{j=1}^{\infty} \left(-\widehat{G_k^+}(\vec{q} \cdot \nabla + V) \right)^j \widehat{G_k^+} f = \widehat{G_k^+} f + \sum_{j=1}^{\infty} L_k^j \widehat{G_k^+} f.$$

The convergence of this series in $H^1_{-\delta}$ follows from the estimates of Theorem 4.2 and the norm estimates follow from those of $\widehat{G_k^+}$, so there exists a unique solution. Then it is straight-forward to confirm that $\widehat{G_\mathrm{p}}$ satisfies $(H_4 - k^4)\widehat{G_\mathrm{p}} = I$. □

Remark 4.5. The previous proof yields a useful identity

$$u_{\mathrm{sc}} = -\widehat{G_k^+}(\vec{q}\cdot\nabla u + Vu) = -\widehat{G_\mathrm{p}}(\vec{q}\cdot\nabla u_0 + Vu_0).$$

Proposition 4.6. *Suppose $n = 2$ or 3 and let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, with $2\delta > n - \frac{n}{p}$ and $n < p \leq \infty$. Then the operator L_k maps $W^1_\infty(\mathbb{R}^n)$ to $W^1_\infty(\mathbb{R}^n)$ with the estimates*

$$\|L_k f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{k^2}\|f\|_{W^1_\infty(\mathbb{R}^3)} \text{ and } \|\nabla L_k f\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{k}\|f\|_{W^1_\infty(\mathbb{R}^n)}$$

for $k \geq 1$.

Proof. We start from the case $n = 3$ and choose some $f \in W^1_\infty(\mathbb{R}^3)$. Then a direct estimation by using Remark 2.1 yields

$$\begin{aligned}|L_k f(x)| \leq{}& \frac{1}{4\pi k^2}\int_{\mathbb{R}^3}\frac{|\vec{q}(y)|}{|x-y|}\mathrm{d}y\|\nabla f\|_{L^\infty(\mathbb{R}^3)}\\ &+\frac{1}{4\pi k^2}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\mathrm{d}y\|f\|_{L^\infty(\mathbb{R}^3)}.\end{aligned}$$

For the gradient of $L_k f$ we have

$$\begin{aligned}\frac{\partial}{\partial x_i}L_k f(x) = -\int_{\mathbb{R}^3}&\frac{x_i-y_i}{8\pi k^2|x-y|^3}\big((\mathrm{i}k|x-y|-1)\mathrm{e}^{\mathrm{i}|k||x-y|}+\\ &+(k|x-y|+1)\mathrm{e}^{-|k||x-y|}\big)\left[\vec{q}(y)\cdot\nabla f(y) + V(y)f(y)\right]\mathrm{d}y,\end{aligned}$$

so that

$$\begin{aligned}|\nabla L_k f(x)| \leq{}& \frac{\sqrt{3}}{4\pi k}\int_{\mathbb{R}^3}\frac{|\vec{q}(y)|}{|x-y|}\mathrm{d}y\|\nabla f\|_{L^\infty(\mathbb{R}^3)} + \frac{\sqrt{3}}{4\pi k}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\mathrm{d}y\|f\|_{L^\infty(\mathbb{R}^3)}\\ &+\frac{\sqrt{3}}{4\pi k^2}\int_{\mathbb{R}^3}\frac{|\vec{q}(y)|}{|x-y|^2}\mathrm{d}y\|\nabla f\|_{L^\infty(\mathbb{R}^3)} + \frac{\sqrt{3}}{4\pi k^2}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|^2}\mathrm{d}y\|f\|_{L^\infty(\mathbb{R}^3)}.\end{aligned}$$

These integrals converge uniformly in x when $n < p \leq \infty$, which yields the claim in $\mathbb{R}^3$.

Consider next the case $n = 2$. Suppose that $f \in W^1_\infty(\mathbb{R}^2)$. Then by using the asymptotic representation obtained in Section 2 we get

$$\begin{aligned}|L_k f(x)| \leq{}& \frac{C}{k^2}\int_{k|x-y|<1}(1+|\log(k|x-y|)|)|\vec{q}\cdot\nabla f + Vf|\mathrm{d}y\\ &+\frac{C}{k^2}\int_{k|x-y|>1}\frac{|\vec{q}\cdot\nabla f + Vf|}{\sqrt{k|x-y|}}\mathrm{d}y =: I_1 + I_2.\end{aligned}$$

Since $\vec{q}, V \in L^1(\mathbb{R}^2)$ then I_2 satisfies the claim. In the first integral we may use Hölder's inequality to get

$$\begin{aligned} I_1 &\leq \frac{C}{k^2}\left(\|\vec{q}\|_{L^p(\mathbb{R}^2)}\|\nabla f\|_{L^\infty(\mathbb{R}^2)} + \|V\|_{L^p(\mathbb{R}^2)}\|f\|_{L^\infty(\mathbb{R}^2)}\right) \\ &\quad \times \left(\int_{k|x-y|<1}(1+|\log(k|x-y|)|)^{p'}\mathrm{d}y\right)^{\frac{1}{p'}} \\ &= \frac{C}{k^{2+\frac{2}{p'}}}\|f\|_{W^1_\infty(\mathbb{R}^2)}\left(\int_{|z|<1}(1+|\log(|z|)|)^{p'}\mathrm{d}y\right)^{\frac{1}{p'}}. \end{aligned}$$

The above integral converges when $1 \leq p' < \infty$, that is, $1 < p \leq \infty$, and therefore the claim holds also for I_1. The gradient of $L_k f$ can be estimated straight-forwardly as

$$\begin{aligned} |\nabla L_k f(x)| \leq \frac{C}{k^2}&\int_{k|x-y|<1}\frac{|\vec{q}\cdot\nabla f + Vf|}{|x-y|}\mathrm{d}y \\ &+ \frac{C}{k}\int_{k|x-y|>1}\frac{|\vec{q}\cdot\nabla f + Vf|}{\sqrt{k|x-y|}}\mathrm{d}y =: J_1 + J_2. \end{aligned}$$

Again, J_2 clearly satisfies the claimed estimate. In the term J_1 Hölder's inequality yields

$$\begin{aligned} J_1 &\leq \frac{C}{k^2}\|f\|_{W^1_\infty(\mathbb{R}^2)}\left(\int_{k|x-y|<1}\frac{1}{|x-y|^{p'}}\mathrm{d}y\right)^{\frac{1}{p'}} \\ &= \frac{C}{k^{1+\frac{2}{p'}}}\|f\|_{W^1_\infty(\mathbb{R}^2)}\left(\int_{|z|<1}\frac{1}{|z|^{p'}}\mathrm{d}z\right)^{\frac{1}{p'}}. \end{aligned}$$

This integral converges when $1 \leq p' < 2$, or $2 < p \leq \infty$, concluding the proof. □

Corollary 4.7. *Let $n = 2$ or 3 and $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$, $V \in L^p_{2\delta}(\mathbb{R}^n)$ with $2\delta > n + \frac{n}{p}$ and $n < p \leq \infty$. Then there exists $k_0' > 0$ such that the unique solution u to* (3) *is bounded uniformly in $k \geq k_0'$.*

Proof. The proof is essentially the same as the proof of Corollary 4.3, but uses the estimates of Proposition 4.6. □

5 Asymptotics for Lippmann-Schwinger equation

Let us first prove an elementary fact about an integral that appears in the sequel.

Lemma 5.1. *Let $\psi \in L^1(\mathbb{R}^n)$ and $0 < a < 1$. Then*

$$\mathrm{e}^{-k|x|}\int_{|y|\le|x|^a}\mathrm{e}^{k(\theta',y)}\psi(y)\mathrm{d}y = o(1), \quad |x|\to+\infty$$

for all $\theta' \in S^{n-1}$ and fixed $k > 0$.

Proof. For all $\theta' \in S^{n-1}$ and $y \in \mathbb{R}^n$ it holds that $(\theta', y) \le |(\theta', y)| \le |y|$. Therefore $\mathrm{e}^{k(\theta',y)} \le \mathrm{e}^{k|y|}$. Then we estimate the integral

$$\begin{aligned}\left|\mathrm{e}^{-k|x|}\int_{|y|\le|x|^a}\mathrm{e}^{k(\theta',y)}\psi(y)\mathrm{d}y\right| &\le \mathrm{e}^{-k|x|}\int_{|y|\le|x|^a}\mathrm{e}^{k|y|}|\psi(y)|\mathrm{d}y\\ &\le \mathrm{e}^{-k|x|}\mathrm{e}^{k|x|^a}\int_{|y|\le|x|^a}|\psi(y)|\mathrm{d}y \le \mathrm{e}^{-k|x|}\mathrm{e}^{k|x|^a}\|\psi\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Since $0 < a < 1$ then

$$\mathrm{e}^{-k|x|}\mathrm{e}^{k|x|^a} = \mathrm{e}^{-k(|x|-|x|^a)} = o(1), \quad |x|\to+\infty.$$

This concludes the proof. $\square$

Next we study the asymptotic behaviour of the solution u obtained in the previous section.

Theorem 5.2. *Let $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$ with $2\delta > n - \frac{n}{p}$ and $n < p \le \infty$. Then for fixed $k > 0$ the solution $u(x,k,\theta)$ to (3) admits the representation*

$$u(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(x,\theta)} - C_n\frac{k^{\frac{n-7}{2}}\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}}A(k,\theta,\theta') + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right), \quad |x|\to\infty,$$

if $n = 2$ or 3. The same representation holds for any $n \ge 4$ if we assume in addition that $\vec{q}$ and V are compactly supported. The constant C_n is given by

$$C_n := \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{n-1}{4}\pi}}{4(2\pi)^{\frac{n-1}{2}}}.$$

The function $A(k,\theta',\theta)$ is called the scattering amplitude and is defined by

$$A(k,\theta,\theta') = \int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}k(\theta',y)}\left[\vec{q}\cdot\nabla u + Vu\right]\mathrm{d}y,$$

where $\theta' = \frac{x}{|x|}$ is the direction of observation.

Proof. We begin the proof along the lines of [10, Lemma 3.1]. Let $0 < a < 1$ be a parameter, so that we can decide its value to suit our interests later. Since

$$u_{\mathrm{sc}}(x, k, \theta) = -\int_{\mathbb{R}^n} G_k^+(|x-y|)\,[\vec{q}\cdot\nabla u + Vu]\,\mathrm{d}y$$

we can divide the region of integration in two parts as $|y| \le |x|^a$ and $|y| > |x|^a$. In the former case we apply the Maclaurin series $(1+\omega)^s = 1 + s\omega + O(\omega^2)$ to see that

$$\begin{aligned} |x-y| &= \left[|x|^2 - 2(x,y) + |y|^2\right]^{1/2} \\ &= |x|\left[1 - \frac{(x,y)}{|x|^2} + \frac{|y|^2}{2|x|^2} + O\left(\left\{-\frac{2(x,y)}{|x|^2} + \frac{|y|^2}{|x|^2}\right\}^2\right)\right] \\ &= |x| - (\theta', y) + \frac{|y|^2}{2|x|} + O\left(|x|^{2a-1}\right) \\ &= |x| - (\theta', y) + O\left(|x|^{2a-1}\right). \end{aligned}$$

Similarly

$$\begin{aligned} |x-y|^{-\frac{n-1}{2}} &= \left[|x|^2 - 2(x,y) + |y|^2\right]^{-\frac{n-1}{4}} \\ &= |x|^{-\frac{n-1}{2}}\left[1 + \frac{n-1}{2}\frac{(x,y)}{|x|^2} - \frac{n-1}{4}\frac{|y|^2}{|x|^2} + O\left(\left\{-\frac{2(x,y)}{|x|^2} + \frac{|y|^2}{|x|^2}\right\}^2\right)\right] \\ &= |x|^{-\frac{n-1}{2}}\left[1 + O\left(|x|^{a-1}\right) + O\left(|x|^{2a-2}\right)\right] \\ &= |x|^{-\frac{n-1}{2}}\left[1 + O\left(|x|^{a-1}\right)\right] \end{aligned}$$

and $|x-y|^{-1} = |x|^{-1}[1 + O(|x|^{a-1})]$. Then, for fixed $k > 0$ and $|y| \le |x|^a$ we take $a < \frac{1}{2}$. Because $|y| \le |x|^a \le |x| \to +\infty$ then we may assume that $k|x-y| > 1$. This allows us to use the large argument asymptotic behaviour of Hankel and Macdonald functions and the Maclaurin expansion $\mathrm{e}^{\omega} = 1 + O(\omega)$ to compute

$$\begin{aligned}
G_k^+(|x-y|) &= \frac{\mathrm{i}}{8k^2}\left(\frac{k}{2\pi|x-y|}\right)^{\frac{n-2}{2}}\left(H_{\frac{n-2}{2}}(k|x-y|) + \frac{2\mathrm{i}}{\pi}K_{\frac{n-2}{2}}(k|x-y|)\right) \\
&= \frac{\mathrm{i}}{8k^2}\left(\frac{k}{2\pi|x-y|}\right)^{\frac{n-2}{2}}\left(\sqrt{\frac{2}{\pi k|x-y|}}\mathrm{e}^{\mathrm{i}(k|x-y|-\frac{n-1}{4}\pi)}\right. \\
&\qquad \left. + C\sqrt{\frac{\pi}{2k|x-y|}}\mathrm{e}^{-k|x-y|} + O\left(|x-y|^{-\frac{3}{2}}\right)\right) \\
&= C_n k^{\frac{n-7}{2}}|x|^{-\frac{n-1}{2}}\left[1 + O(|x|^{a-1})\right]\left(\mathrm{e}^{\mathrm{i}k|x|}\mathrm{e}^{-\mathrm{i}k(\theta',y)}\mathrm{e}^{-\mathrm{i}kO(|x|^{2a-1})}\right. \\
&\qquad \left. + C\mathrm{e}^{-k|x|}\mathrm{e}^{k(\theta',y)}\mathrm{e}^{kO(|x|^{2a-1})} + O(|x-y|^{-1})\right) \\
&= C_n\frac{k^{\frac{n-7}{2}}}{|x|^{\frac{n-1}{2}}}\left(\mathrm{e}^{\mathrm{i}k|x|}\mathrm{e}^{-\mathrm{i}k(\theta',y)} + C\mathrm{e}^{-k|x|}\mathrm{e}^{k(\theta',y)}\right) + O(|x|^{2a-1-\frac{n-1}{2}}).
\end{aligned}$$

Let us apply these estimates to u_{sc}. We have

$$\begin{aligned}
u_{\mathrm{sc}}(x,k,\theta) = &-\int_{|y|\le|x|^a} G_k^+(|x-y|)\left[\vec{q}\cdot\nabla u + Vu\right]\mathrm{d}y \\
&- \int_{|y|>|x|^a} G_k^+(|x-y|)\left[\vec{q}\cdot\nabla u + Vu\right]\mathrm{d}y =: I_1 + I_2.
\end{aligned}$$

Now integral I_1 equals

$$\begin{aligned}
I_1 = &-C_n\frac{k^{\frac{n-7}{2}}\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}}\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}k(\theta',y)}\left[\vec{q}\cdot\nabla u + Vu\right]\mathrm{d}y \\
&+ C_n\frac{k^{\frac{n-7}{2}}\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}}\int_{|y|>|x|^a}\mathrm{e}^{-\mathrm{i}k(\theta',y)}\left[\vec{q}\cdot\nabla u + Vu\right]\mathrm{d}y \\
&+ C\frac{k^{\frac{n-7}{2}}\mathrm{e}^{-k|x|}}{|x|^{\frac{n-1}{2}}}\int_{|y|\le|x|^a}\mathrm{e}^{k(\theta',y)}\left[\vec{q}\cdot\nabla u + Vu\right]\mathrm{d}y + O\left(|x|^{2a-1-\frac{n-1}{2}}\right).
\end{aligned}$$

As $|x|\to\infty$ the second integral above is $o\left(|x|^{-\frac{n-1}{2}}\right)$, since $\vec{q}\cdot\nabla u$ and Vu are integrable. On the other hand, by Lemma 5.1 the third term in this expression is also $o\left(|x|^{-\frac{n-1}{2}}\right)$.

Next we consider I_2. We first split the region of integration as

$$
\begin{aligned}
|I_2| &\le \int_{|y|>|x|^a} \left| G_k^+(|x-y|)\,[\vec{q}\cdot\nabla u + Vu]\right| \mathrm{d}y \\
&= \int_{|x|^a<|y|<\frac{|x|}{2}} |G_k^+(|x-y|)||\vec{q}\cdot\nabla u + Vu|\mathrm{d}y \\
&\quad + \int_{|y|\ge\frac{|x|}{2}} |G_k^+(|x-y|)||\vec{q}\cdot\nabla u + Vu|\mathrm{d}y =: J_1 + J_2.
\end{aligned}
$$

In the case J_1 we have $|x-y| \ge |x|-|y| > \frac{|x|}{2}$. Thus as $|x| \to +\infty$ by the asymptotic of G_k^+ we have

$$
J_1 \le \frac{C}{|x|^{\frac{n-1}{2}}} \int_{|x|^a<|y|<\frac{|x|}{2}} |\vec{q}\cdot\nabla u + Vu|\mathrm{d}y = o\left(|x|^{-\frac{n-1}{2}}\right).
$$

Note that so far we have only used the fact that $\vec{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$, $n \ge 2$. If these coefficients are compactly supported, then choosing large enough $|x|$ immediately yields $J_2 = 0$, giving the claimed asymptotic representation. On the other hand, even if the coefficients are not compactly supported, in dimensions $n = 2, 3$ we know that the solution u to (3) is in $W^1_\infty(\mathbb{R}^n)$.

We study first the case $n = 3$. By Remark 2.1 we have for a fixed $k > 0$ that

$$
\begin{aligned}
J_2 &\le C \int_{|y|\ge\frac{|x|}{2}} \frac{|\vec{q}| + |V|}{|x-y|}\mathrm{d}y \\
&\le C \left(\int_{|y|\ge\frac{|x|}{2}} \frac{(1+|y|^{-2\delta p'})}{|x-y|^{p'}}\mathrm{d}y \right)^{1/p'} \left(\|\vec{q}\|_{L^p_{2\delta}(\mathbb{R}^3)} + \|V\|_{L^p_{2\delta}(\mathbb{R}^3)} \right) \\
&\le C \left(\int_{|z|\ge\frac{1}{2}} \frac{(1+|x||z|)^{-2\delta p'}}{|x|^{p'}|(x/|x|)-z|^{p'}} |x|^3 \mathrm{d}z \right)^{1/p'} \\
&\le C|x|^{\frac{3}{p'}-2\delta-1} \left(\int_{|z|\ge\frac{1}{2}} \frac{|z|^{-2\delta p'}}{|(x/|x|)-z|^{p'}}\mathrm{d}y \right)^{1/p'},
\end{aligned}
$$

where we did the change of variables $y = z|x|$. Dividing the region of integration to $2 > |z| > \frac{1}{2}$ and $|z| > 2$ we get

$$
\int_{2>|z|>\frac{1}{2}} \frac{|z|^{-2\delta p'}}{|(x/|x|)-z|^{p'}}\mathrm{d}z \le C \int_{2>|z|>\frac{1}{2}} \frac{1}{|(x/|x|)-z|^{p'}}\mathrm{d}z < \infty,
$$

because $p' < n/2$. In the region $|z| > 2$ we have $|(x/|x|) - z| \ge |z| - 1 > |z|/2$ and therefore

$$
\int_{|z|>2} \frac{|z|^{-2\delta p'}}{|(x/|x|)-z|^{p'}}\mathrm{d}z \le C \int_{|z|>2} \frac{1}{|z|^{2\delta p'+p'}}\mathrm{d}z < \infty.
$$

Recalling that $\frac{3}{p'} - 2\delta < 0$ we conclude that J_2' is $o\left(\frac{1}{|x|}\right)$.

In the case $n = 2$ we use the asymptotic behaviour of G_k^+. We consider the regions $k|x-y| < 1$ and $k|x-y| > 1$ separately, with the respective integrals denoted by J_2' and J_2''. In the first case

$$\begin{aligned}
J_2' &\leq C \int_{\substack{|y| \geq \frac{|x|}{2} \\ k|x-y|<1}} (1 + |\log(k|x-y|)|)\,(|\vec{q}| + |V|)\mathrm{d}y \\
&\leq C \left(\int_{\substack{|y| \geq \frac{|x|}{2} \\ k|x-y|<1}} (1 + |\log(k|x-y|)|)^{p'} (1+|y|)^{-2\delta p'} \mathrm{d}y \right)^{\frac{1}{p'}} \\
&\leq C|x|^{-2\delta} \left(\int_{|z|<1} (1 + |\log(|z|)|)^{p'} \mathrm{d}z \right)^{\frac{1}{p'}} = o(|x|^{-1}).
\end{aligned}$$

In the second case J_2'' we pick some $\sigma > 0$ such that $0 < 2\delta p' - 2 < \sigma < \frac{p'}{2} + 2\delta p' - 2$. Then

$$\begin{aligned}
J_2'' &\leq C \int_{\substack{|y| \geq \frac{|x|}{2} \\ k|x-y|>1}} \frac{|\vec{q}| + |V|}{|x-y|^{\frac{1}{2}}} \mathrm{d}y \\
&\leq C \left(\int_{\substack{|y| \geq \frac{|x|}{2} \\ k|x-y|>1}} \frac{(1+|y|)^{-2\delta p'}}{|x-y|^{\frac{p'}{2}}} \mathrm{d}y \right)^{\frac{1}{p'}} \\
&\leq C|x|^{-\frac{\sigma}{p'}} \left(\int_{\substack{|y| \geq \frac{|x|}{2} \\ k|x-y|>1}} \frac{|y|^{-2\delta p' + \sigma}}{|x-y|^{\frac{p'}{2}}} \mathrm{d}y \right)^{\frac{1}{p'}} .
\end{aligned}$$

By changing the variables $y/|x| = z$ in the above integral we get

$$\left(\int_{\substack{|y| \geq \frac{|x|}{2} \\ k|x-y|>1}} \frac{|y|^{-2\delta p' + \sigma}}{|x-y|^{\frac{p'}{2}}} \mathrm{d}y \right)^{\frac{1}{p'}} = |x|^{-\frac{1}{2} - 2\delta + \frac{\sigma}{p'} + \frac{2}{p'}} \left(\int_{\substack{|z| \geq \frac{1}{2} \\ k|x-|x|z|>1}} \frac{|z|^{-2\delta p' + \sigma}}{|(x/|x|) - z|^{\frac{p'}{2}}} \mathrm{d}z \right)^{\frac{1}{p'}} .$$

If the above integral converges, we have that $J_2'' = o\left(|x|^{-\frac{1}{2}}\right)$, since $-2\delta + \frac{2}{p'} < 0$. And indeed, the integral converges because

$$\begin{aligned}
\int_{\substack{\frac{1}{2} \leq |z| \\ k|x-|x|z|>1}} \frac{|z|^{-2\delta p' + \sigma}}{|(x/|x|) - z|^{\frac{p'}{2}}} \mathrm{d}z &\leq \int_{\substack{\frac{1}{2} \leq |z| \\ |(x/|x|)-z|>|z|}} \frac{1}{|z|^{2\delta p' - \sigma + \frac{p'}{2}}} \mathrm{d}z \\
&\quad + \int_{\substack{\frac{1}{2} \leq |z| \\ |(x/|x|)-z|<|z|}} \frac{|z|^{-2\delta p' + \sigma}}{|(x/|x|) - z|^{\frac{p'}{2}}} \mathrm{d}z,
\end{aligned}$$

where the first integral is finite due to the choice of σ. The second integral also converges, as can be seen by splitting

$$\int_{\substack{\frac{1}{2}\le|z| \\ |(x/|x|)-z|<|z|}} \frac{|z|^{-2\delta p'+\sigma}}{|(x/|x|)-z|^{\frac{p'}{2}}}\mathrm{d}z \le \int_{\frac{1}{2}\le|z|<2} \frac{C}{|(x/|x|)-z|^{\frac{p'}{2}}}\mathrm{d}z + \int_{2\le|z|} \frac{C}{|(x/|x|)-z|^{\frac{p'}{2}+2\delta p'-\sigma}}\mathrm{d}z < \infty.$$

This concludes the proof. $\square$

Remark 5.3. In similar spirit as in the proof of Theorem 5.2 one can confirm that u_{sc} and Δu_{sc} satisfy the radiation condition at infinity at least if $\vec{q}$ and V are smooth and compactly supported. For such coefficients, in light of Section 3 we can then conclude that the scattering problem (2) and Lippmann-Schwinger equation (3) are equivalent.

6 Proof of Theorem 1.1

Note that (unlike the proof for the asymptotic formula of u) this theorem does not require us to assume that $\vec{q}$ and V have compact supports.

Proof. Let us denote

$$I := k^{n-1}\int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}A(k,\theta,\theta')\mathrm{d}\theta\mathrm{d}\theta'.$$

Then, because $u = u_0 + u_{\mathrm{sc}}$, we can write

$$\begin{aligned} I &= k^{n-1}\int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-ik(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}\left[ik\theta\cdot\vec{q}+V\right]\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta' \\ &\quad + k^{n-1}\int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-ik(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta',y)}\left[\vec{q}\cdot\nabla u_{\mathrm{sc}}+Vu_{\mathrm{sc}}\right]\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta' \\ &=: I_1 + I_2. \end{aligned}$$

Consider I_1 first. We can further split I_1 in two terms as

$$\begin{aligned} I_1 &= k^{n-1}\int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-ik(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}\mathrm{i}k\theta\cdot\vec{q}(y)\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta' \\ &\quad + k^{n-1}\int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-ik(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}V(y)\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta' =: I_1^{(1)} + I_1^{(2)}. \end{aligned}$$

In $I_1^{(1)}$ we can first split the integral into two parts and then integrate the first one by parts and do a change of variables $\theta = -\gamma'$ and $\theta' = -\gamma$ in the second term as follows

$$\begin{aligned}
I_1^{(1)} &= k^{n-1} \int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)} \mathrm{i}k(\theta-\theta')\cdot \vec{q}(y)\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta' \\
&\quad + k^{n-1} \int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)} \mathrm{i}k\theta'\cdot \vec{q}(y)\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta' \\
&= -k^{n-1} \int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)} \nabla\cdot \vec{q}(y)\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta' \\
&\quad - k^{n-1} \int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(-\gamma'+\gamma,x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(-\gamma+\gamma',y)} ik\gamma\cdot \vec{q}(y)\mathrm{d}y\mathrm{d}\gamma'\mathrm{d}\gamma.
\end{aligned}$$

Here $\nabla\cdot\vec{q}$ denotes the divergence of $\vec{q}$. The integration by parts is valid, since $\vec{q}\in W^1_{p,2\delta}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)\cap W^1_p(\mathbb{R}^n) \subset \dot{C}(\mathbb{R}^n)$ when $n < p \le \infty$ (see Theorem A.2). But now the last integral is equal to $-I_1^{(1)}$, so that

$$I_1^{(1)} = -k^{n-1} \int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)} \frac{1}{2}\nabla\cdot \vec{q}(y)\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta'$$

and

$$I_1 = k^{n-1} \int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)} \left(-\frac{1}{2}\nabla\cdot\vec{q}(y) + V(y)\right)\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta'.$$

An integral of this form was studied in [10] and we carry out the analogous analysis here. It is known that

$$\int_{S^{n-1}} \mathrm{e}^{\mathrm{i}k(\theta,x-y)}\mathrm{d}\theta = (2\pi)^{\frac{n}{2}} \frac{J_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{\frac{n-2}{2}}},$$

and one proof for this fact can be found for example from [10, Lemma 3.6]. Here $J_\nu(x)$ is the Bessel function of first kind and of order ν. From [16] we find that

$$J_\nu(x) = \begin{cases} O(x^\nu), & x\to 0, \\ \sqrt{\frac{2}{\pi x}}\cos(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + O(x^{-\frac{3}{2}}), & x\to\infty. \end{cases}$$

We split the region of integration as

$$\begin{aligned}
I_1 &= (2\pi)^n k^{n-1} \int_{k|x-y|<1} \frac{J^2_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{n-2}}\beta(y)\mathrm{d}y \\
&\quad + (2\pi)^n k^{n-1} \int_{k|x-y|>1} \frac{J^2_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{n-2}}\beta(y)\mathrm{d}y =: I_1' + I_1''.
\end{aligned}$$

The term $I_1' \to 0$ as $k \to +\infty$. Indeed,

$$
\begin{aligned}
|I_1'| &\le Ck^{n-1} \int_{k|x-y|<1} \frac{(k|x-y|)^{n-2}}{(k|x-y|)^{n-2}} |\beta(y)| \mathrm{d}y \\
&\le k^{n-1-\frac{n}{p'}} \|\beta\|_{L^p(\mathbb{R}^n)} \left(\int_{|z|<1} \mathrm{d}y \right)^{\frac{1}{p'}},
\end{aligned}
$$

where $n-1-\frac{n}{p'} = \frac{n}{p}-1 < 0$ when $p > n$. The term I_1'' converges to the claimed limit of this theorem as $k \to +\infty$. To see this we utilize the large argument behaviour of Bessel function to get

$$
\begin{aligned}
I_1'' &= (2\pi)^n k \int_{k|x-y|>1} \frac{\beta(y)}{|x-y|^{n-2}} \left(\sqrt{\frac{2}{\pi k|x-y|}} \cos\left(k|x-y| - \frac{n-1}{4}\pi \right) \right. \\
&\qquad \left. + O\left(\frac{1}{(k|x-y|)^{\frac{3}{2}}} \right) \right)^2 \mathrm{d}y \\
&= 2^n \pi^{n-1} \int_{k|x-y|>1} \frac{\beta(y)}{|x-y|^{n-2}} 2\cos^2\left(k|x-y| - \frac{n-1}{4}\pi \right) \mathrm{d}y \\
&\qquad + \frac{1}{k} \int_{k|x-y|>1} \frac{\beta(y)}{|x-y|^{n}} O(1) \mathrm{d}y \\
&= 2^n \pi^{n-1} \int_{k|x-y|>1} \frac{\beta(y)}{|x-y|^{n-1}} \mathrm{d}y \\
&\qquad + C \int_{k|x-y|>1} \frac{\beta(y)}{|x-y|^{n-1}} \cos\left(2k|x-y| - \frac{n-1}{2}\pi \right) \mathrm{d}y \\
&\qquad + \frac{1}{k} \int_{k|x-y|>1} \frac{\beta(y)}{|x-y|^{n}} O(1) \mathrm{d}y.
\end{aligned}
$$

Here the first integral gives the claimed limit as $k \to +\infty$, since

$$
\int_{\mathbb{R}^n} \frac{|\beta(y)|}{|x-y|^{n-1}} \mathrm{d}y < \infty
$$

uniformly in x when $p > n$. The second integral above gives

$$
\begin{aligned}
&\int_{k|x-y|>1} \frac{\beta(y)}{|x-y|^{n-1}} \cos\left(2k|x-y| - \frac{n-1}{2}\pi \right) \mathrm{d}y \\
&\qquad = \int_{\mathbb{R}^n} \frac{\beta(y)}{|x-y|^{n-1}} \cos\left(2k|x-y| - \frac{n-1}{2}\pi \right) \mathrm{d}y \\
&\qquad - \int_{|x-y|<\frac{1}{k}} \frac{\beta(y)}{|x-y|^{n-1}} \cos\left(2k|x-y| - \frac{n-1}{2}\pi \right) \mathrm{d}y =: I_1^* + I_1^{**}.
\end{aligned}
$$

The latter integral $I_1^{**} \to 0$ as $k \to +\infty$, while the former integral requires little more effort. We do the change of variables $y = x - z$ and then swap to polar coordinates to obtain

$$I_1^* = \int_0^\infty \int_{S^{n-1}} \beta(x - r\gamma) \cos(2kr - \frac{n-1}{2}\pi) \mathrm{d}\gamma \mathrm{d}r.$$

Define

$$f_x(r) := \int_{S^{n-1}} \beta(x - r\gamma) \mathrm{d}\gamma.$$

Now it suffices to conclude that $f_x \in L^1(0, \infty)$ uniformly in x and then apply the Riemann-Lebesgue lemma to conclude that $I_1^* \to 0$ as $k \to +\infty$. We calculate the L^1-norm of f_x to see that

$$\begin{aligned}\int_0^\infty |f_x(r)| \mathrm{d}r &= \int_0^\infty \int_{S^{n-1}} |\beta(x - r\gamma)| \mathrm{d}\gamma \mathrm{d}r \\ &= \int_{\mathbb{R}^n} \frac{|\beta(y)|}{|x - y|^{n-1}} \mathrm{d}y < \infty\end{aligned}$$

uniformly in x. To finish the consideration of I_1'' we only need to show that

$$\frac{1}{k} \int_{k|x-y|>1} \frac{\beta(y)}{|x - y|^n} \mathrm{d}y \to 0$$

as $k \to +\infty$. Let $\varepsilon > 0$ be a parameter whose value will be chosen later. Estimate

$$\begin{aligned}\frac{1}{k} \int_{k|x-y|>1} \frac{|\beta(y)|}{|x - y|^n} \mathrm{d}y &\leq k^{\varepsilon - 1} \int_{k|x-y|>1} \frac{|\beta(y)|}{|x - y|^{n-\varepsilon}} \mathrm{d}y \\ &\leq k^{\varepsilon - 1} \left(\int_{|z|>\frac{1}{k}} \frac{1}{|z|^{(n-\varepsilon)p'}} \mathrm{d}z \right)^{\frac{1}{p'}} \|\beta\|_{L^p(\mathbb{R}^n)} \\ &= Ck^{\varepsilon - 1} \left(\int_{\frac{1}{k}}^\infty r^{-(n-\varepsilon)p' + n - 1} \mathrm{d}r \right)^{\frac{1}{p'}} \\ &= Ck^{\varepsilon - 1} \left(-r^{-(n-\varepsilon)p' + n} \Big|_{r=\frac{1}{k}}^\infty \right)^{\frac{1}{p'}} = Ck^{\frac{n}{p} - 1},\end{aligned}$$

where we required $\varepsilon < \frac{n}{p}$. This estimate shows that all of the terms of I_1'' tend to 0 as $k \to \infty$.

Next we confirm that the term $I_2 \to 0$ as $k \to \infty$. To achieve this we calculate the integrals with respect to θ and θ' as in [18] and use the mapping properties of

$\widehat{G_\mathrm{p}}$. First we separate the integral into two parts with respect to $\vec{q}$ and V as

$$I_2 = k^{n-1}\int_{S^{n-1}\times S^{n-1}} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}k(\theta',y)}(\vec{q}\cdot\nabla u_{\mathrm{sc}}+Vu_{\mathrm{sc}})\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta' =: I_2^{(1)}+I_2^{(2)}.$$

Recall from Remark 4.5 that $u_{\mathrm{sc}} = -\widehat{G_\mathrm{p}}(\vec{q}\cdot\nabla u_0 + Vu_0)$. This allows us to write

$$\begin{aligned}
I_2^{(2)} &= -k^{n-1}\int_{S^{n-1}\times S^{n-1}}\mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)}\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}k(\theta',y)}V(y)\\
&\quad\times\widehat{G_\mathrm{p}}(\mathrm{i}k\theta\cdot\vec{q}\mathrm{e}^{\mathrm{i}k(\theta,\cdot)}+V\mathrm{e}^{\mathrm{i}k(\theta,\cdot)})(y)\mathrm{d}y\mathrm{d}\theta\mathrm{d}\theta'\\
&= -k^{n-1}\int_{\mathbb{R}^n}\frac{J_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{\frac{n-2}{2}}}V(y)\widehat{G_\mathrm{p}}\left(\int_{S^{n-1}}\mathrm{i}k\theta\cdot\vec{q}\mathrm{e}^{-\mathrm{i}k(\theta,x-z)}\mathrm{d}\theta\right)\mathrm{d}y\\
&\quad- k^{n-1}\int_{\mathbb{R}^n}\frac{J_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|^{\frac{n-2}{2}})}V(y)\widehat{G_\mathrm{p}}\left(\int_{S^{n-1}}V\mathrm{e}^{-\mathrm{i}k(\theta,x-z)}\mathrm{d}\theta\right)\mathrm{d}y =: I_2^{(2)*}+I_2^{(2)**}.
\end{aligned}$$

From [26] we find that $\frac{\mathrm{d}}{\mathrm{d}x}(x^{-\nu}J_\nu(x)) = -x^{-\nu}J_{\nu+1}(x)$ and therefore in $I_2^{(2)*}$ we have

$$\begin{aligned}
\int_{S^{n-1}}\vec{q}(z)\cdot\mathrm{i}k\theta\mathrm{e}^{-\mathrm{i}k(x-z,\theta)}\mathrm{d}\theta &= \vec{q}(z)\cdot\nabla_z\int_{S^{n-1}}\mathrm{e}^{-\mathrm{i}k(x-z,\theta)}\mathrm{d}\theta\\
&= -k(2\pi)^{\frac{n}{2}}\vec{q}(z)\cdot\frac{x-z}{|x-z|}\frac{J_{\frac{n}{2}}(k|x-z|)}{(k|x-z|)^{\frac{n-2}{2}}}.
\end{aligned}\tag{7}$$

Next, we may write $\vec{q} = (\vec{q}/|\vec{q}|)|\vec{q}|^{\frac{1}{2}}|\vec{q}|^{\frac{1}{2}} =: q_{\frac{1}{2}}|\vec{q}|^{\frac{1}{2}}$. Now we define operator $\widehat{K_1}$ as the integral operator with the kernel $|V(y)|^{\frac{1}{2}}\widehat{G_\mathrm{p}}(y-\cdot)q_{\frac{1}{2}}(\cdot)$. Actually, $\widehat{K_1}$: $L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ uniformly in x with the same norm estimates as $\widehat{G_\mathrm{p}}$. Indeed, let $\vec{f}\in L^2(\mathbb{R}^n)$. Then, since $\vec{q}\in W^1_{p,2\delta}(\mathbb{R}^n)$, we have that

$$\|q_{\frac{1}{2}}\cdot\vec{f}\|_{L^2_\delta(\mathbb{R}^n)} \le \|\vec{q}\|^{\frac{1}{2}}_{L^\infty_{2\delta}(\mathbb{R}^n)}\|\vec{f}\|_{L^2(\mathbb{R}^n)}.$$

Due to the Sobolev embedding theorem we have the continuous embedding

$$H^1_{-\delta}(\mathbb{R}^n)\subset L^{\frac{2p}{p-1}}_{-\delta}(\mathbb{R}^n).$$

Thus, taking a function $g\in H^1_{-\delta}(\mathbb{R}^n)$ and applying Hölder's inequality gives

$$\||V|^{\frac{1}{2}}g\|_{L^2(\mathbb{R}^n)} \le \|V\|^{\frac{1}{2}}_{L^p_{2\delta}(\mathbb{R}^n)}\|g\|_{L^{\frac{2p}{p-1}}_{-\delta}(\mathbb{R}^n)} \le C\|V\|^{\frac{1}{2}}_{L^p_{2\delta}(\mathbb{R}^n)}\|g\|_{H^1_{-\delta}(\mathbb{R}^n)}.$$

These estimates and the mapping property of $\widehat{G_{\rm p}}$ finally yield

$$\begin{aligned}\||V|^{\frac12}\widehat{G_{\rm p}}(q_{\frac12}f)\|_{L^2(\mathbb{R}^n)} &\le \|V\|^{\frac12}_{L^p_{2\delta}(\mathbb{R}^n)}\|\widehat{G_{\rm p}}(q_{\frac12}f)\|_{H^1_{-\delta}(\mathbb{R}^n)}\\ &\le \frac{C}{k^2}\|V\|^{\frac12}_{L^p_{2\delta}(\mathbb{R}^n)}\|q_{\frac12}f\|_{L^2_\delta(\mathbb{R}^n)}\\ &\le \frac{C}{k^2}\|V\|^{\frac12}_{L^p_{2\delta}(\mathbb{R}^n)}\|\vec{q}\|^{\frac12}_{L^\infty_{2\delta}(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Next we show that the function from (7) satisfies

$$\left\||\vec{q}|^{\frac12}\frac{J_{\frac n2}(k|x-\cdot|)}{(k|x-\cdot|)^{\frac{n-2}{2}}}\right\|_{L^2(\mathbb{R}^n)} \le Ck^{\frac{1-n}{2}}.$$

This is a straight-forward calculation by using the asymptotic behaviour of the Bessel function. We estimate

$$\begin{aligned}\int_{\mathbb{R}^n}|\vec{q}(y)|\frac{J^2_{\frac n2}(k|x-y|)}{|x-y|^{n-2}}\mathrm{d}y &\le C\int_{k|x-y|<1}|\vec{q}(y)|\frac{(k|x-y|)^n}{(k|x-y|)^{n-2}}\mathrm{d}y\\ &\quad + C\int_{k|x-y|>1}\frac{|\vec{q}(y)|}{(k|x-y|)^{n-1}}\mathrm{d}y\\ &\le \frac{C}{k^{n-1}}\int_{\mathbb{R}^n}\frac{|\vec{q}(y)|}{|x-y|^{n-1}}\mathrm{d}y \le Ck^{1-n}.\end{aligned}$$

In the same manner

$$\left\||V|^{\frac12}\frac{J_{\frac{n-2}{2}}(k|x-\cdot|)}{(k|x-\cdot|)^{\frac{n-2}{2}}}\right\|_{L^2(\mathbb{R}^n)} \le Ck^{\frac{1-n}{2}}.$$

To finish the consideration of the term $I_2^{(2)*}$ we integrate with respect to θ' to obtain

$$\begin{aligned}J_1' &\le k^{n-1}\int_{\mathbb{R}^n}|V(y)|^{\frac12}\frac{|J_{\frac n2}(k|x-y|)|}{(k|x-y|)^{\frac{n-2}{2}}}\left|\widehat{K_1}\left(k|\vec{q}|^{\frac12}\frac{J_{\frac n2}(k|y-\cdot|)}{(k|y-\cdot|)^{\frac{n-2}{2}}}\right)\right|\mathrm{d}y\\ &\le Ck^{n+\frac{1-n}{2}-2}\left(\int_{\mathbb{R}^n}|V(y)|\frac{J^2_{\frac n2}(k|x-y|)}{(k|x-y|)^{n-2}}\mathrm{d}y\right)^{\frac12} \le Ck^{n+\frac{1-n}{2}-2}k^{\frac{1-n}{2}} = Ck^{-1}.\end{aligned}$$

This means that the term $I_2^{(2)*}\to 0$ as $k\to\infty$.

Next we consider the integral $I_2^{(2)**}$. The method is essentially the same as in case of $I_2^{(2)*}$. By computing the integral with respect to θ we get

$$I_2^{(2)**} = -k^{n-1}\int_{\mathbb{R}^n}\frac{J_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{\frac{n-2}{2}}}V(y)\widehat{G_{\rm p}}\left(V\frac{J_{\frac{n-2}{2}}(k|x-\cdot|)}{(k|x-\cdot|)^{\frac{n-2}{2}}}\right)\mathrm{d}y.$$

Here it appears to be simpler to conclude directly that

$$\left\| V \frac{J_{\frac{n-2}{2}}(k|x-\cdot|)}{(k|x-\cdot|)^{\frac{n-2}{2}}} \right\|_{L^2_\delta(\mathbb{R}^n)} \leq Ck^{\frac{2-n}{2}}$$

and to not factorize the integrals into new operators. We estimate the square of the above norm by

$$\begin{aligned}\int_{\mathbb{R}^n} (1+|z|)^{2\delta}|V(z)|^2 \frac{J^2_{\frac{n-2}{2}}(k|x-z|)}{(k|x-z|)^{n-2}}\mathrm{d}z &\leq C\int_{k|x-z|<1}(1+|z|)^{2\delta}|V(z)|^2\mathrm{d}z \\ &\quad + C\int_{k|x-z|>1}(1+|z|)^{2\delta}\frac{|V(z)|^2}{(k|x-z|)^{n-1}}\mathrm{d}z \\ &\leq Ck^{2-n}\int_{\mathbb{R}^n}(1+|z|)^{2\delta}\frac{|V(z)|^2}{|x-z|^{n-2}}\mathrm{d}z.\end{aligned}$$

Dividing the region of integration to $|x-z|<1$ and $|x-z|>1$ we see that

$$\begin{aligned}\int_{\mathbb{R}^n}(1+|z|)^{2\delta}\frac{|V(z)|^2}{|x-z|^{n-2}}\mathrm{d}z &\leq \int_{|x-z|<1}(1+|z|)^{2\delta}\frac{|V(z)|^2}{|x-z|^{n-2}}\mathrm{d}z \\ &\quad + \int_{|x-z|>1}(1+|z|)^{2\delta}|V(z)|^2\mathrm{d}z \\ &\leq \left(\int_{|x-z|<1}\frac{1}{|x-z|^{\frac{(n-2)p}{p-2}}}\mathrm{d}z\right)^{\frac{p-2}{p}} \|V\|_{L^p_\delta(\mathbb{R}^n)} + \|V\|_{L^2_\delta(\mathbb{R}^n)}.\end{aligned}$$

The above integral converges uniformly in x, when $n < p \leq \infty$. Thus, since $\widehat{G_\mathrm{p}} : L^2_\delta(\mathbb{R}^n) \to H^1_{-\delta}(\mathbb{R}^n)$, we have

$$\begin{aligned}|I_2^{(2)**}| &\leq Ck^{n-1}\left(\int_{\mathbb{R}^n}|V(y)|\frac{J^2_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{n-2}}\mathrm{d}y\right)^{\frac{1}{2}} \left\| |V|^{\frac{1}{2}}\widehat{G_\mathrm{p}}\left(V\frac{J_{\frac{n-2}{2}}(k|x-\cdot|)}{(k|x-\cdot|)^{\frac{n-2}{2}}}\right)\right\|_{L^2(\mathbb{R}^n)} \\ &\leq Ck^{n-1+\frac{1-n}{2}+\frac{2-n}{2}-2} \leq Ck^{-\frac{3}{2}} \to 0\end{aligned}$$

as $k \to +\infty$.

The term $I_2^{(1)}$ can be considered analogously to $I_2^{(2)}$. Compared to the term $I_2^{(2)}$, where we utilized the fact that $\widehat{G_\mathrm{p}}$ maps L^2_δ to $H^1_{-\delta}$, in term $I_1^{(1)}$ we can use the previous calculations and the fact that $\nabla\widehat{G_\mathrm{p}}$ maps L^2_δ to $L^2_{-\delta}$ with the same estimates as earlier. Now for $g \in L^2_{-\delta}(\mathbb{R}^n)$ we have that

$$\| |\vec{q}|^{\frac{1}{2}} g\|_{L^2(\mathbb{R}^n)} \leq \|\vec{q}\|^{\frac{1}{2}}_{L^\infty_{2\delta}(\mathbb{R}^n)}\|g\|_{L^2_{-\delta}(\mathbb{R}^n)}.$$

Replacing the first $|V|^{\frac{1}{2}}$ by $|\vec{q}|^{\frac{1}{2}}$ in the previous calculations and using the above norm estimate we may conclude that $I_2^{(1)}$ satisfies the same estimate in k as $I_2^{(2)}$. Therefore, $I_2 \to 0$ as $k \to +\infty$ and the theorem is proved. $\square$

The proofs of Corollaries 1.2 and 1.3 are well-known in literature and can be found for example in [18] or [10] (case $n = 3$).

Conclusion

A direct scattering problem for a first-order perturbation of the biharmonic operator was studied. It was shown that a solution to scattering problem with certain radiation conditions satisfies the Lippmann-Schwinger equation. This integral equation has a unique solution in the weighted Sobolev space $H^1_{-\delta}$ in any dimension and in particular this solution is bounded if the dimension is 2 or 3. The proofs of these results are based on a Green-type formula for bi-Laplacian and Agmon's estimate [1]. The asymptotic behaviour of the solution for fixed $k > 0$ as $|x| \to +\infty$ was studied and a formula for the scattering amplitude was obtained.

The main result of this paper, Saito's formula, was proved under quite general assumptions on the coefficients. More precisely, the formula for scattering amplitude was proved under the assumption that the coefficients of the direct operator are compactly supported, but this assumption was not necessary for the proof of Saito's formula. The proof itself was based on explicit calculations starting from the formula for scattering amplitude and utilizing the properties of certain resolvent operator related to the direct operator. Some consequences of Saito's formula were discussed. Namely, the scattering amplitude uniquely determines a combination of the coefficients for the direct problem and in turn gives a uniqueness result for the inverse problem. A representation formula for this combination of the coefficients was given.

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A Appendix

Lemma A.1. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial\Omega$. If $f, g \in H^4(\Omega)$ then the following equality holds*

$$\int_\Omega (f\Delta^2 g - g\Delta^2 f)\mathrm{d}x = \int_{\partial\Omega} \left(f\frac{\partial}{\partial n}\Delta g + \Delta f \frac{\partial}{\partial n} g - g\frac{\partial}{\partial n}\Delta f - \Delta g \frac{\partial}{\partial n} f \right) \mathrm{d}\sigma(x).$$

Proof. Let f and g be C^4-functions on Ω and define a C^1-vector field $\vec{F}$ by the formula

$$\vec{F} = \Delta f \nabla g + f\nabla(\Delta g) - \Delta g \nabla f - g\nabla(\Delta f).$$

Then the divergence theorem (see e.g., [8]) gives the claimed formula. To extend the result for functions from $H^4(\Omega)$ we note that left-hand side defines a bounded bilinear form. Similarly, for smooth boundary $\partial\Omega$ the trace operator is bounded from $H^4(U)$ to $H^3(\partial U)$ so the right-hand side is also bounded. Since C^4-functions are dense in H^4 the continuity and a density argument show that the claim holds also for functions from $H^4(\Omega)$. □

Theorem A.2. *If $n < p < \infty$, then $W^1_p(\mathbb{R}^n) \subset \dot{C}(\mathbb{R}^n)$.*

Proof. We start from the known [5] embedding $W^1_p(\mathbb{R}^n) \subset C^{1-\frac{n}{p}}(\mathbb{R}^n)$ for $n < p < \infty$, where $C^\alpha(\mathbb{R}^n)$ is the Hölder space for $0 < \alpha < 1$. It is also known that functions in the space $W^1_p(\mathbb{R}^n)$ are bounded when $p > n$. It remains to verify that the functions in $W^1_p(\mathbb{R}^n)$ vanish at infinity.

Let $f \in W^1_p(\mathbb{R}^n)$. If $a \neq b$ then by the mean value theorem we have the equality

$$|a^p - b^p| = p|\xi|^{p-1}|a-b|$$

for some $\xi \in]a, b[$. Thus for the function $|f|$ we find that

$$||f(x)|^p - |f(y)|^p| \le p\|f\|^{p-1}_{L^\infty(\mathbb{R}^n)}||f(x)| - |f(y)|| \le C|x-y|^{1-\frac{n}{p}} \quad \text{a.e.},$$

and therefore $|f|^p$ can be chosen to be uniformly (even $(1-n/p)$-Hölder) continuous.

Next, assume in contrary that $|f|^p$ does not vanish at infinity. It means that there exists a constant $c > 0$ and points $x_j \in \mathbb{R}^n$ so that $|f(x_j)|^p \ge c$. We may pick a sequence of these points with the property that for each $m \in \mathbb{N}$ we have $|x_m| \ge m$. Since $|f|^p$ is uniformly continuous, there exists $\delta > 0$ (that only depends on c) such that $|f(x)|^p \ge \frac{c}{2}$ when $|x_m - x| < \delta$. This is a contradiction since $|f|^p$ is integrable, but $|x_m| \to \infty$ and

$$\int_{|x-x_m|<\delta} |f(x)|^p \mathrm{d}x \ge \frac{c|B_\delta(x_m)|}{2},$$

where $|B_\delta(x_m)|$ denotes the volume of a ball of radius δ centered at x_m.

Finally, if f did not vanish at infinity then we would find constant $\widetilde{c} > 0$ and points $\widetilde{x}_m \in \mathbb{R}^n$ so that $|f(\widetilde{x}_m)| \geq \widetilde{c}$. But this means that $|f(\widetilde{x}_m)|^p \geq \widetilde{c}^p$, which is a contradiction, as we just saw. $\square$

Remark A.3. In dimension $n = 1$ it is possible to prove straight-forwardly that $W_1^1(\mathbb{R}) \subset \dot{C}(\mathbb{R})$. Moreover, Theorem A.2 does not hold for $p = \infty$, since for example all constant functions belong to $W_\infty^1(\mathbb{R}^n)$, but the only constant function vanishing at infinity is the constant $C \equiv 0$.

References

[1] S. Agmon. Spectral properties of Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa*, 2:151–218, 1975.

[2] T. Aktosun and V. G. Papanicolaou. Time evolution of the scattering data for a fourth-order linear differential operator. *Inverse Problems*, 24(5):055013, 2008.

[3] Y. Assylbekov. Inverse problems for the perturbed polyharmonic operator with coefficients in Sobolev spaces with non-positive order. *Inverse Problems*, 32(105009), 2016.

[4] Y. Assylbekov. Corrigendum: Inverse problems for the perturbed polyharmonic operator with coefficients in Sobolev spaces with non-positive order. *Inverse Problems*, 33(9):099501, 2017.

[5] J. Bergh and J. Löfström. *Interpolation spaces: An introduction.* Springer-Verlag, New York, 1976.

[6] F. Cakoni and D. Colton. *A qualitative approach to inverse scattering theory.* Springer, New York, 2014.

[7] G. Eskin. *Lectures on linear partial differential equations.* American Mathematical Society, 2011.

[8] L. Evans. *Partial differential equations.* American Mathematical Society, 2010.

[9] F. Gazzola, H.-C. Grunau, and G. Sweers. *Polyharmonic boundary value problems.* Springer-Verlag Berlin Heidelberg, 2010.

[10] M. Harju. *On the direct and inverse scattering problems for a nonlinear three-dimensional Schrödinger equation.* PhD thesis, University of Oulu, 2010.

[11] L. Hörmander. *The analysis of linear partial differential operators: Differential operators with constant coefficients.* Springer-Verlag Berlin Heidelberg, 2005.

[12] K. Iwasaki. Scattering theory for 4th order differential operators: I. *Japan. J. Math.*, 14:1–57, 1988.

[13] K. Iwasaki. Scattering theory for 4th order differential operators: II. *Japan. J. Math.*, 14:59–96, 1988.

[14] K. Krupchyk, M. Lassas, and G. Uhlmann. Inverse boundary value problems for the perturbed polyharmonic operator. *Trans.Amer.Math.Sco.*, 366:95–112, 2014.

[15] S.T. Kuroda. Finite-dimensional perturbation and a representation of scattering operator. *Pacific J. Math.*, 13(4):95–112, 1963.

[16] N.N. Lebedev. *Special functions and their applications.* Dover Publications, Inc., New York, 1972.

[17] N.V. Movchan, R.C. McPhedran, A.B. Movchan, and C.G. Poulton. Wave scattering by platonic grating stacks. *Proc. R. Soc. A*, 465:3383–3400, 2009.

[18] L. Päivärinta and V. Serov. Recovery of singularities of a multidimensional scattering potential. *SIAM J. Math. Anal.*, 29:697–711, 1998.

[19] L. Päivärinta and V. Serov. New mapping properties for the resolvent of the Laplacian and recovery of singularities of a multi-dimensional scattering potential. *Inverse Problems*, 17:1321–1326, 2001.

[20] B. Pausader. Scattering for the defocusing beam equation in low dimensions. *Indiana Univ. Math. J.*, 59:791–822, 2010.

[21] Y. Saito. Some properties of the scattering amplitude and the inverse scattering problem. *Osaka J. Math.*, 19:527–547, 1982.

[22] V. Serov. Borg-Levinson theorem for perturbations of the bi-harmonic operator. *Inverse Problems*, 32(4):045002, 2016.

[23] V. Serov and M. Harju. A uniqueness theorem and reconstruction of singularities for a two-dimensional nonlinear Schrödinger equation. *Nonlinearity*, 21:1323–1337, 2008.

[24] Z. Sun. An inverse boundary value problem for Schrödinger operators with vector potentials. *Trans. Amer. Math. Soc*, 338:953–969, 1993.

[25] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math.*, 125:153–169, 1987.

[26] G. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 2nd edition, 1966.