LOCAL HIGHER INTEGRABILITY OF THE GRADIENT OF A QUASIMINIMIZER UNDER GENERALIZED ORLICZ GROWTH CONDITIONS

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ABSTRACT. We study local quasiminimizers of the Dirichlet energy under generalized growth conditions. Special cases include standard, variable exponent and double phase growths. We show that the gradient of a local quasiminimizer has local higher integrability.

1. INTRODUCTION

In this paper we study local quasiminimizers of the minimazation problem

$$\min_{u\in W^{1,1}} \int_\Omega \varphi(x, |\nabla u|)\, dx.$$

Here φ satisfies generalized Orlicz type conditions, see Section 2. Our result covers, for example, $p(\cdot)$-type growth $\varphi(x,t) = t^{p(x)}$ and its perturbations $\varphi(x,t) = t^{p(x)}\log(e+t)$ [6, 7, 18, 27, 30], and double phase growth $\varphi(x,t) = t^p + a(x)t^q$ [1, 3, 8, 10, 11]. Other properties in the general case have been studied e.g. in [14, 21, 25, 28, 29]. More examples can be found from Section 2.

Our main result is the following.

Theorem 1.1 (Local higher integrability of the gradient)**.** *Let $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfy assumptions (A0), (A1), (aInc) and (aDec)$^\infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose $u \in W^{1,\varphi}_{\rm loc}(\Omega)$ is a local quasiminimzer of the φ-energy. Then there exists $\varepsilon > 0$ such that*

$$\varphi(\cdot, |\nabla u|) \in L^{1+\varepsilon}_{\rm loc}(\Omega).$$

The idea of the proof is to combine a Sobolev–Poincaré inequality (Proposition 3.6), a Caccioppoli inequality (Lemma 4.2) and Gehring's lemma [17]. The hardest part is to prove a suitable modular type Sobolev–Poincaré inequality. Here we need to use equivalent more regular weak Φ-functions.

In the variable exponent case, $\varphi(x,t) = t^{p(x)}$, higher integrability was proved by X.-L. Fan and D. Zhao in [16]. They assumed that $1 < \inf p \leqslant \sup p < \infty$ and p is log-Hölder continuous. These are special cases of our assumptions. In the double phase case, $\varphi(x,t) = t^p + a(x)t^q$ where $1 < p < q$, Colombo and Mingione [9] proved the higher integrability of the gradient under assumption $\frac{q}{p} < 1 + \frac{\alpha}{n}$, where α is the Hölder exponent of the function a. Again, this is a special case of our assumption (A1). Our result also contains as special cases the perturbed variable exponent and the degenerate double phase cases (cf. (2.3)) where higher integrability was not previously know, as well as many other cases. Further information of our assumptions and related special cases are collected as a table in the next section.

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2. Properties of generalized Φ-functions

By $\Omega \subset \mathbb{R}^n$ we denote a bounded domain, i.e. an open and connected set. The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leqslant Cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. By c we denote a generic constant whose value may change between appearances. A function f is *almost increasing* if there exists a constant $L \geqslant 1$ such that $f(s) \leqslant Lf(t)$ for all $s \leqslant t$ (more precisely, L-almost increasing). *Almost decreasing* is defined analogously.

Definition 2.1. We say that $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ is a *weak Φ-function*, and write $\varphi \in \Phi_w(\Omega)$, if

- For every $t \in [0,\infty)$ the function $x \mapsto \varphi(x,t)$ is measurable and for every $x \in \Omega$ the function $t \mapsto \varphi(x,t)$ is increasing.
- $\varphi(x,0) = \lim_{t\to 0^+} \varphi(x,t) = 0$ and $\lim_{t\to\infty} \varphi(x,t) = \infty$ for every $x \in \Omega$.
- The function $t \mapsto \frac{\varphi(x,t)}{t}$ is L-almost increasing for $t > 0$ for some $L \geqslant 1$ and every $x \in \Omega$.
- The function $t \mapsto \varphi(x,t)$ is left-continuous for $t > 0$ and $x \in \Omega$.

We denote $\varphi \in \Phi_c(\Omega)$ and say that φ is a convex Φ-function if, additionally, $t \mapsto \varphi(x,t)$ is convex.

Two functions φ and ψ are *equivalent*, $\varphi \simeq \psi$, if there exists $L \geqslant 1$ such that $\psi(x, \frac{t}{L}) \leqslant \varphi(x,t) \leqslant \psi(x, Lt)$ for every $x \in \Omega$ and every $t > 0$. Equivalent Φ-functions give rise to the same space with comparable norms. By φ^{-1} we denote the left-continuous inverse of a weak Φ-function φ,

$$\varphi^{-1}(x,\tau) := \inf\{t \geqslant 0 \,:\, \varphi(x,t) \geqslant \tau\}.$$

We say that φ is *doubling* if there exists a constant $L \geqslant 1$ such that $\varphi(x,2t) \leqslant L\varphi(x,t)$ for every $x \in \Omega$ and every $t \geqslant 0$. If φ is doubling with constant L, then by iteration

$$\varphi(x,t) \leqslant L^2 \Big(\frac{t}{s}\Big)^Q \varphi(x,s) \tag{2.2}$$

for every $x \in \Omega$ and every $0 < s < t$, where $Q = \log_2(L)$. For the proof see for example [5, Lemma 3.3 , p. 66]. If φ is doubling, then (2.2) shows that $\simeq$ implies $\approx$. On the other hand, $\approx$ always implies $\simeq$ since the function $t \mapsto \frac{\varphi(x,t)}{t}$ is almost increasing; hence $\simeq$ and $\approx$ are equivalent in the doubling case. Note that doubling also yields that $\varphi(x,t+s) \leqslant L\varphi(x,t) + L\varphi(x,s)$.

Let us write $\varphi_B^+(t) := \sup_{x\in B} \varphi(x,t)$ and $\varphi_B^-(t) := \inf_{x\in B} \varphi(x,t)$; and abbreviate $\varphi^\pm := \varphi_\Omega^\pm$. Assume that there exists a constant $\sigma > 0$ such that the following two conditions hold.

(A0) There exists $\beta \in (0,1)$ such that $\varphi^+(\beta\sigma) \leqslant 1 \leqslant \varphi^-(\sigma)$.

(A1) There exists $\beta \in (0,1)$ such that

$$\varphi_B^+(\beta t) \leqslant \varphi_B^-(t)$$

for every $t \in \big[\sigma, (\varphi_B^-)^{-1}(\frac{1}{|B|})\big]$ and every ball B with $\big(\varphi_B^-\big)^{-1}\big(\frac{1}{|B|}\big) \geqslant \sigma$.

We also introduce the following assumptions, which are of different nature. They are related to the Δ_2 and ∇_2 conditions from Orlicz space theory.

$(\mathrm{aInc})_p$ There exists $L \geqslant 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^p}$ is L-almost increasing in $(0,\infty)$.

$(\mathrm{aDec})_q^\infty$ There exists $L \geqslant 1$ such that $t \mapsto \frac{\varphi(x,t)+1}{t^q}$ is L-almost decreasing in $(0,\infty)$.

We write (aInc) if there exists $p > 1$ such that $(\mathrm{aInc})_p$ holds, similarly for $(\mathrm{aDec})^\infty$. Further, we write $(\mathrm{aDec})_q$ if $(\mathrm{aDec})_q^\infty$ has no "+1" term. This is equivalent to doubling [25, Lemma 2.6]. These conditions are invariant under equivalence of Φ-functions.

The condition (aDec)$_q^\infty$ is new to this paper, but it corresponds to doubling at infinity, $\Delta_2(\infty)$, which is the base form of doubling in Rao–Ren [31]. (What is here called doubling, they call globally doubling.) It can equivalently be written as the requirement that $t \mapsto \frac{\varphi(x,t)}{t^q}$ is L-almost decreasing for values $t > 0$ such that $\varphi(x,t) \geqslant 1$. The constant 1 has been chosen for convenience and the choice does not play a significant role.

The reason for considering doubling at infinity rather than the full range is that some researchers [12, 13] have recently considered the following variant of the double phase functional,

$$F(x,t) \approx (t-1)_+^p + a(x)(t-1)_+^q, \tag{2.3}$$

with $(s)_+ := \max\{s, 0\}$, which is singular even for positive values of the gradient. Clearly, this is not doubling, but it does satisfy the condition (aDec)$_q^\infty$.

In results to come, positive constants, such as c, might depend on the weak Φ-function φ. This means that they may depend on all or some of the following parameters: β, σ, p and q. In such cases we denote for example $c = c(n, \varphi)$ if c depends on the dimension n and some of the parameters of φ.

The next table interprets the assumptions in the context of variable exponent and double phase growth.

$\varphi(x,t)$	(A0)	(A1)	(aInc)	(aDec)	(aDec)$^\infty$
$t^{p(x)}a(x)$	$a \approx 1$	$p \in C^{\log}$	$p^- > 1$	$p^+ < \infty$	$p^+ < \infty$
$t^p + a(x)t^q$	$a \in L^\infty$	$a \in C^{\frac{n}{p}(q-p)}$	$p > 1$	$q < \infty$	$q < \infty$
$(t-1)_+^p + a(x)(t-1)_+^q$	$a \in L^\infty$	$a \in C^{\frac{n}{p}(q-p)}$	$p > 1$	false	$q < \infty$

Remark 2.4. In the double phase case, the assumption $a \in C^{\frac{n}{p}(q-p)}$ or (A1) is related to the boundedness of the maximal operator and several other properties that can be obtained through it. These kind of properties are also used in this paper. However, Baroni–Colombo–Mingione [2, 3, 4] have shown that a weaker assumption suffices if one has additional information about the minimizer u. Namely, if u is locally bounded then $a \in C^{q-p}$ suffices for higher integrability, whereas if $u \in C^\gamma$, then $a \in C^{(1-\gamma)(q-p)}$ suffices. With the method of this paper, it is not possible to use possible additional information about u to recover these results, so this remains for future research.

Generalized Orlicz and Orlicz–Sobolev spaces have been studied with our assumptions in [22, 23, 24, 25]. We recall some definitions. We denote by $L^0(\Omega)$ the set of measurable functions in Ω and the integral average of a function f over a set A is denoted by $\fint_A f(x)\,dx =: f_A$. Also, if B is a ball with radius r, then tB is a concentric ball with radius tr.

Definition 2.5. Let $\varphi \in \Phi_w(\Omega)$ and define the *modular* ϱ_φ for $f \in L^0(\Omega)$ by

$$\varrho_\varphi(f) := \int_\Omega \varphi(x, |f(x)|)\,dx.$$

The *generalized Orlicz space*, also called Musielak–Orlicz space, is defined as the set

$$L^\varphi(\Omega) := \{f \in L^0(\Omega) : \lim_{\lambda \to 0^+} \varrho_\varphi(\lambda f) = 0\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{L^\varphi(\Omega)} := \inf\left\{\lambda > 0 : \varrho_\varphi\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

If the set is clear from the context we abbreviate $\|f\|_{L^\varphi(\Omega)}$ by $\|f\|_\varphi$.

A function $u \in L^\varphi(\Omega)$ belongs to the *Orlicz–Sobolev space* $W^{1,\varphi}(\Omega)$ if its weak partial derivatives $\partial_1 u, \ldots, \partial_n u$ exist and belong to $L^\varphi(\Omega)$.

3. Auxiliary results

We start with three lemmas regarding Jensen type inequalities for $\Phi_w(\Omega)$-functions. The first one concerns a Φ-prefunction φ, which is a weak Φ-function without the left-continuity. Note also that first we also consider φ independent of x.

Lemma 3.1. *Let φ be a prefunction that satisfies (aInc)$_p$, $p \geqslant 1$. Then there exists $\beta_0 > 0$ such that the following Jensen-type inequality holds for every $f \in L^1_{\rm loc}(\Omega)$ and every ball $B \subset \Omega$:*

$$\varphi\bigg(\beta_0 \fint_B f\,dx\bigg)^{\frac1p} \leqslant \fint_B \varphi(f)^{\frac1p}\,dx.$$

Proof. Since $\varphi^{1/p}$ satisfies (aInc)$_1$, there exists $\psi \in \Phi_c(\Omega)$ such that $\psi \simeq \varphi^{1/p}$, by [24, Lemma 2.2]. By Jensen's inequality for ψ,

$$\varphi\bigg(\tfrac{1}{L^2} \fint_B f\,dx\bigg)^{\frac1p} \leqslant \psi\bigg(\fint_B \tfrac1L f\,dx\bigg) \leqslant \fint_B \psi(\tfrac1L f)\,dx \leqslant \fint_B \varphi(f)^{\frac1p}\,dx. \qquad \square$$

The proof of the next lemma is slightly modified version of Lemma 4.4 of [26]. The original was only stated for convex φ. For completeness, we present a proof of the slight generalization. Since we are interested in bounded domain Ω, the assumption (A2) from the original proof can be omitted.

Lemma 3.2. *Let $\varphi \in \Phi_w(B)$ satisfy assumptions (A0), (A1) and (aInc)$_p$, $p \geqslant 1$. There exists $\beta_1 = \beta_1(\varphi) > 0$ such that*

$$\varphi\bigg(x, \beta_1 \fint_B |f|\,dy\bigg)^{\frac1p} \leqslant \fint_B \varphi(y,f)^{\frac1p}\,dy + 1,$$

for every ball B and $f \in L^\varphi(B)$ with $\varrho_\varphi(f\chi_{\{|f|>\sigma\}}) < 1$, where σ is the constant in (A0) and (A1).

Proof. We may assume without loss of generality that $f \geqslant 0$. Fix a ball B and $x \in B$. Denote $f_1 := f\chi_{\{f>\sigma\}}$, $f_2 := f - f_1$, and $A_i := \fint_B f_i\,dy$. Since $\varphi^{1/p}$ is increasing,

$$\varphi\bigg(x, \beta_1 \fint_B f\,dy\bigg)^{\frac1p} \leqslant \varphi\big(x, 2\beta_1 \max\{A_1, A_2\}\big)^{\frac1p} \leqslant \varphi\big(x, 2\beta_1 A_1\big)^{\frac1p} + \varphi\big(x, 2\beta_1 A_2\big)^{\frac1p}.$$

Since φ satisfies (A0), a short calculation gives that φ_B^- satisfies the conditions of Lemma 3.1 with $p = 1$. Thus Lemma 3.1 and $\varrho_\varphi(f\chi_{\{|f|>\sigma\}}) < 1$ yield that

$$\varphi_B^-(\beta_0 A_1) \leqslant \fint_B \varphi_B^-(f_1)\,dy \leqslant \fint_B \varphi(y, f_1)\,dy < \tfrac{1}{|B|}. \tag{3.3}$$

Suppose first that $\beta_0 A_1 \geqslant \sigma$. This assumption and (3.3) yield that $\beta_0 A_1$ is in the allowed range of (A1). Thus

$$\varphi\big(x, \beta\beta_0 A_1\big)^{\frac1p} \leqslant \varphi_B^-\big(\beta_0 A_1\big)^{\frac1p} \leqslant \fint_B \varphi_B^-(f_1)^{\frac1p}\,dy \leqslant \fint_B \varphi(y, f_1)^{\frac1p}\,dy.$$

Next consider $\beta_0 A_1 \leqslant \sigma$. Using (aInc)$_p$ and (A0), we conclude that

$$\varphi\big(x, \beta^2\beta_0 A_1\big)^{\frac1p} \leqslant \varphi\big(x, \beta\sigma\big)^{\frac1p} \frac{L\beta\beta_0}{\sigma} A_1 \leqslant c \fint_B \frac{f_1}{\sigma}\,dy. \tag{3.4}$$

By (A0), $1 \leqslant \varphi(y,\sigma)$. If $f_1 > \sigma$, it follows from (aInc)$_p$ and (A0) that

$$\frac{f_1}{\sigma} \leqslant L\frac{\varphi(y,f_1)^{\frac{1}{p}}}{\varphi(y,\sigma)^{\frac{1}{p}}} \leqslant L\varphi(y,f_1)^{\frac{1}{p}}.$$

The inequality is trivial when $f_1(x)=0$, and by definition f_1 does not take values in $(0,\sigma)$. Thus the inequality holds in all cases. From (3.4) we then deduce

$$\varphi(x,\beta^2\beta_0 A_1)^{\frac{1}{p}} \leqslant c \fint_B \varphi(y,f_1)^{\frac{1}{p}}\,dy.$$

In view of this and the conclusion of previous paragraph, we find that

$$\varphi\big(x,\tfrac{1}{Lc}\beta^2\beta_0 A_1\big)^{\frac{1}{p}} \leqslant \tfrac{1}{c}\varphi\big(x,\beta^2\beta_0 A_1\big)^{\frac{1}{p}} \leqslant \fint_B \varphi(y,f_1)^{\frac{1}{p}}\,dy \leqslant \fint_B \varphi(y,f)^{\frac{1}{p}}\,dy,$$

where we also used (aInc)$_p$ for the first inequality.

For f_2, we note that $A_2 \leqslant \sigma$, since $f_2 \leqslant \sigma$. Thus it follows from (A0) that

$$\varphi\big(x,\beta A_2\big) \leqslant \varphi(x,\beta\sigma) \leqslant 1.$$

Adding the estimates for f_1 and f_2, we obtain the claim with constant $\beta_1 = \min\big\{\frac{\beta}{2},\frac{\beta^2\beta_0}{2Lc}\big\}$. □

Lemma 3.5. *Let $\varphi \in \Phi_w(B)$ satisfy assumptions (A0), (A1) and (aInc)$_p$, $p \geqslant 1$. Then there exists $\beta_2 = \beta_2(n,\varphi) > 0$ such that*

$$\varphi\left(x,\beta_2\int_B \frac{|f(y)|}{\operatorname{diam} B\,|x-y|^{n-1}}\,dy\right)^{\frac{1}{s}} \leqslant \int_B \frac{\varphi(y,|f(y)|)^{\frac{1}{s}}}{\operatorname{diam} B\,|x-y|^{n-1}}\,dy + 1$$

for almost every x in the ball B, $1 \leqslant s \leqslant p$ and every $f \in L^{\varphi}(B)$ with $\varrho_\varphi(f\chi_{\{|f|>\sigma\}}) < 1$.

Proof. We may assume without loss of generality that $f \geqslant 0$. Fix $r > 0$ and let B be a ball with radius r. Define annuli $A_k := \{y \in B : 2^{-k}r \leqslant |x-y| \leqslant 2^{1-k}r\}$ for $k \geqslant 1$. We split B into annuli A_k and obtain

$$\int_B \frac{f(y)}{2r\,|x-y|^{n-1}}\,dy \leqslant c_1 \sum_{k=1}^{\infty} 2^{-k} \fint_{B(x,2^{1-k}r)} \chi_{A_k} f(y)\,dy.$$

By [26, Lemma 3.2] there exists a Φ-function ψ such that $\varphi \simeq \psi$ and $\psi^{1/p}$ is convex. Fix $s \in [1,p]$. Since $\sum_{k=1}^{\infty} 2^{-k} = 1$, it follows by convexity of $\psi^{1/s}$ that

$$\begin{aligned}
\varphi\left(x,\frac{1}{L^2}\sum_{k=1}^{j} 2^{-k}\fint_{B(x,2^{1-k}r)} \chi_{A_k} f(y)\,dy\right)^{\frac{1}{s}} &\leqslant \psi\left(x,\frac{1}{L}\sum_{k=1}^{j} 2^{-k}\fint_{B(x,2^{1-k}r)} \chi_{A_k} f(y)\,dy\right)^{\frac{1}{s}} \\
&\leqslant \sum_{k=1}^{j} 2^{-k}\psi\left(x,\frac{1}{L}\fint_{B(x,2^{1-k}r)} \chi_{A_k} f(y)\,dy\right)^{\frac{1}{s}} \\
&\leqslant \sum_{k=1}^{\infty} 2^{-k}\varphi\left(x,\fint_{B(x,2^{1-k}r)} \chi_{A_k} f(y)\,dy\right)^{\frac{1}{s}}
\end{aligned}$$

and by left continuity of φ

$$\varphi\left(x,\frac{1}{L^2}\sum_{k=1}^{\infty} 2^{-k}\fint_{B(x,2^{1-k}r)} \chi_{A_k} f(y)\,dy\right)^{\frac{1}{s}} \leqslant \sum_{k=1}^{\infty} 2^{-k}\varphi\left(x,\fint_{B(x,2^{1-k}r)} \chi_{A_k} f(y)\,dy\right)^{\frac{1}{s}}.$$

Let $\beta_1 > 0$ be from Lemma 3.2. We obtain

$$(I) := \varphi\bigg(x, \frac{\beta_1}{L^2 c_1}\int_B \frac{f(y)}{r\,|x-y|^{n-1}}\,dy\bigg)^{\frac{1}{s}} \leqslant \sum_{k=1}^{\infty} 2^{-k}\varphi\bigg(x, \beta_1 \fint_{B(x,2^{1-k}r)} \chi_{A_k} f(y)\,dy\bigg)^{\frac{1}{s}},$$

so Lemma 3.2 yields

$$\begin{aligned}
(I) &\leqslant \sum_{k=1}^{\infty} 2^{-k}\bigg(\fint_{B(x,2^{1-k}r)} \chi_{A_k}\varphi(y, f(y))^{\frac{1}{s}}\,dy + 1\bigg)\\
&\leqslant \frac{1}{2}\sum_{k=1}^{\infty} \frac{2^{1-k}}{|B(x,2^{1-k}r)|}\int_{B(x,2^{1-k}r)} \chi_{A_k}\varphi(x, f(y))^{\frac{1}{s}}\,dy + 1\\
&\leqslant \sum_{k=1}^{\infty} \frac{1}{2r(2^{1-k}r)^{n-1}}\int_{B(x,2^{1-k}r)} \chi_{A_k}\varphi(y, f(y))^{\frac{1}{s}}\,dy + 1 \leqslant \int_B \frac{\varphi(y,f(y))^{\frac{1}{s}}}{2r\,|x-y|^{n-1}}\,dy + 1.
\end{aligned}$$

This is the claim for $\beta_2 = \frac{\beta_1}{L^2 c_1}$. □

The next proposition is a Sobolev–Poincaré inequality for weak Φ-functions and yields an exponent strictly less than 1. This is the main requirement for Gehring's lemma later on. The proof introduces a probability measure that allows Jensen's inequality to be used in the usual setting. This technique was used in [15]. The rest of the proof consists of handling leftover terms and technicalities.

Proposition 3.6. *Let $\varphi \in \Phi_w(B)$ satisfy assumptions (A0), (A1) and (aInc)$_p$, $p \geqslant 1$, and let $s \in [1,p]$ with $s < \frac{n}{n-1}$. Then there exists a constant $\beta_3 = \beta_3(n,s,\varphi)$ such that*

$$\fint_B \varphi\bigg(x, \beta_3\frac{|u-u_B|}{\operatorname{diam} B}\bigg)\,dx \leqslant \bigg(\fint_B \varphi(x, |\nabla u|)^{\frac{1}{s}}\,dx\bigg)^s + 1$$

for every $v \in W^{1,1}(B)$ with $\|\nabla u\|_\varphi < 1$.

Proof. For brevity, we denote $\kappa := \operatorname{diam} B$. Suppose first that $\varphi(x,|\nabla u|) = 0$ for almost every $x \in B$. Then by (A0) we see that $|\nabla u(x)| \leqslant \sigma$ that is $|\nabla u| \in L^\infty(B)$ and thus u is Lipschitz continuous. Hence

$$|u(x) - u_B| = |u(x) - u(y)| \leqslant \sigma\kappa$$

for some $y \in B$ and

$$\fint_B \varphi\left(x, \beta\frac{|u-u_B|}{\kappa}\right)dx \leqslant \fint_B \varphi(x, \beta\sigma)\,dx \leqslant 1.$$

Thus the proposition is true if the integral on the right-hand side is 0.

Assume then that the integral on the right-hand side of inequality in the claim is positive. We have for almost every $x \in B$, by [19, Chapter 7],

$$|u(x) - u_B| \leqslant C_1(n)\int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}dy.$$

The previous inequality and Lemma 3.5 with constant $\beta' = \frac{2\beta_2}{C_1}$ yield

$$\begin{aligned}\varphi\left(x, \beta' \frac{|u(x) - u_B|}{\kappa}\right) &\leqslant \varphi\left(x, \beta_2 \int_B \frac{|\nabla u(y)|}{\kappa|x-y|^{n-1}}\, dy\right) \\ &\leqslant \left(\int_B \frac{\varphi(y, |\nabla u(y)|)^{\frac{1}{s}}}{\kappa|x-y|^{n-1}}\, dy + 1\right)^s \\ &\leqslant 2^{s-1} \left(\int_B \frac{\varphi(y, |\nabla u(y)|)^{\frac{1}{s}}}{\kappa|x-y|^{n-1}}\, dy\right)^s + 2^{s-1}.\end{aligned} \tag{3.7}$$

Set $J := \int_B \varphi(x, |\nabla u|)^{1/s}\, dx > 0$ and define a measure by $d\mu(y) := \frac{1}{J}\varphi(y, |\nabla u|)^{1/s}\, dy$. Then

$$\int_B \frac{\varphi(y, |\nabla u(y)|)^{\frac{1}{s}}}{\kappa|x-y|^{n-1}}\, dy = \int_B \frac{J}{\kappa|x-y|^{n-1}}\, d\mu(y).$$

Since μ is a probability measure, we can use Jensen's inequality for the convex function $t \mapsto t^s$:

$$\left(\int_B \frac{\varphi(y, |\nabla u(y)|)^{\frac{1}{s}}}{\kappa|x-y|^{n-1}}\, dy\right)^s \leqslant \int_B \frac{J^s}{\kappa^s|x-y|^{s(n-1)}}\, d\mu(y) = J^{s-1} \int_B \frac{\varphi(y, |\nabla u|)^{1/s}}{\kappa^s|x-y|^{s(n-1)}}\, dy.$$

We integrate the previous inequality over $x \in B$, and use Fubini's theorem to change the order of integration

$$\fint_B \left(\int_B \frac{\varphi(y, |\nabla u(y)|)^{\frac{1}{s}}}{\kappa|x-y|^{n-1}}\, dy\right)^s dx = J^{s-1} \fint_B \varphi(y, |\nabla u(y)|)^{\frac{1}{s}} \int_B \frac{dx}{\kappa^s|x-y|^{s(n-1)}}\, dy.$$

Finally, we use the assumption $s < \frac{n}{n-1}$ to estimate

$$\int_B \frac{dx}{\kappa^s|x-y|^{s(n-1)}} \leqslant c\kappa^{-s(n-1)+n-s} = c\kappa^{n-sn}$$

for $y \in B$ and conclude, taking into account the definition of J, that

$$\fint_B \left(\int_B \frac{\varphi(y, |\nabla u(y)|)^{\frac{1}{s}}}{\kappa|x-y|^{n-1}}\, dy\right)^s dx \leqslant c \left(\frac{J}{\kappa^n}\right)^s = c \left(\fint_B \varphi(y, |\nabla u(y)|)^{\frac{1}{s}}\, dy\right)^s.$$

Combining this with (3.7), which is integrated over B, we complete the proof as the constant c can be absorbed into β' by (aInc)$_1$. □

4. Higher integrability

Next we turn to properties of the minimizing function u, namely the Caccioppoli inequality for a quasiminimzer.

Definition 4.1. Let $\varphi \in \Phi_w(\Omega)$ and $K \geqslant 1$. A function $u \in W^{1,\varphi}(\Omega)$ is a *local K-quasiminimizer* of the φ-energy in Ω if

$$\int_{\{v \neq 0\}} \varphi(x, |\nabla u|)\, dx \leqslant K \int_{\{v \neq 0\}} \varphi(x, |\nabla (u+v)|)\, dx$$

for all $v \in W^{1,\varphi}(\Omega)$ with $\operatorname{spt} v := \overline{\{v \neq 0\}} \subset \Omega$.

In the next result we need the doubling near infinity.

Lemma 4.2 (Caccioppoli inequality for quasiminimizer). *Let $\varphi \in \Phi_w(\Omega)$ satisfy (aDec)$_q^\infty$, u be a local K-quasiminimizer in Ω and $2B \subset\subset \Omega$. Then we have*

$$\int_B \varphi(x, |\nabla u|)\, dx \lesssim \int_{2B} \varphi\left(x, \frac{|u - u_{2B}|}{\operatorname{diam} B}\right) dx + 1, \tag{4.3}$$

in the ball B, where the implicit constant depends only on n, K and (aDec)$_q^\infty$.

Proof. Denote $\kappa := \operatorname{diam} B$. Let $t, s \in [1, 2]$, $s < t$. Also, let $\eta \in C_0^\infty(tB)$ be such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in sB, and $|\nabla\eta| \leqslant \frac{4}{(t-s)\kappa}$. Denote $w = -\eta(u - u_{2B})$ and $v := u + w$. Since u is a local K-quasiminimizer

$$\int_{tB} \varphi(x, |\nabla u|)\, dx \leqslant K \int_{tB} \varphi(x, |\nabla v|)\, dx.$$

We have

$$|\nabla v| \leqslant (1 - \eta)|\nabla u| + |\nabla\eta||u - u_{2B}|.$$

Denote $a := 2^q L \geqslant 1$. By (aDec)$_q^\infty$ and $|\nabla\eta| \leqslant \frac{4}{(t-s)\kappa}$, we get that

$$\begin{aligned}
\varphi(x, |\nabla v|) &\leqslant \varphi\big(x, 2\max\{(1 - \eta)|\nabla u|, \tfrac{4|u-u_{2B}|}{(t-s)\kappa}\}\big) \\
&\leqslant \big(\varphi(x, 2(1 - \eta)|\nabla u|) + 1\big) + \big(\varphi\big(x, \tfrac{8|u-u_{2B}|}{(t-s)\kappa}\big) + 1\big) \\
&\leqslant a\varphi(x, (1 - \eta)|\nabla u|) + a + a^3\varphi\big(x, \tfrac{|u-u_{2B}|}{(t-s)\kappa}\big) + a^3 \\
&\leqslant a\varphi(x, (1 - \eta)|\nabla u|) + a^3\varphi\big(x, \tfrac{|u-u_{2B}|}{(t-s)\kappa}\big) + c.
\end{aligned}$$

Combining the above inequalities, we find that

$$\int_{tB} \varphi(x, |\nabla u|)\, dx \leqslant aK \int_{tB} \varphi(x, (1 - \eta)|\nabla u|)\, dx + a^3 K \int_{tB} \varphi\big(x, \tfrac{|u-u_{2B}|}{(t-s)\kappa}\big)\, dx + Kc|2B|.$$

By decreasing the set on the left hand side, we obtain

$$\begin{aligned}
&\int_{sB} \varphi(x, |\nabla u|)\, dx \\
&\quad\leqslant aK \int_{tB} \varphi(x, (1 - \eta)|\nabla u|)\, dx + a^3 K \int_{tB} \varphi\big(x, \tfrac{|u-u_{2B}|}{(t-s)\kappa}\big)\, dx + Kc|2B|.
\end{aligned}\tag{4.4}$$

On the right-hand side, we have $\varphi(x, (1 - \eta)|\nabla u|) = \varphi(x, 0) = 0$ in sB, and so

$$\int_{tB} \varphi(x, (1 - \eta)|\nabla u|)\, dx = \int_{tB\setminus sB} \varphi(x, (1 - \eta)|\nabla u|)\, dx \leqslant \int_{tB\setminus sB} \varphi(x, |\nabla u|)\, dx.$$

Now we can use the hole-filling trick by adding $aK \int_{sB} \varphi(x, |\nabla u|)\, dx$ to both sides of (4.4), ending with $aK + 1$ of the integral on the left-hand side, and aK on the right. After we divide with $aK + 1$, we have

$$\int_{sB} \varphi(x, |\nabla u|)\, dx \leqslant \frac{aK}{aK + 1} \int_{tB} \varphi(x, |\nabla u|)\, dx + \frac{a^3 K}{aK + 1} \int_{2B} \varphi(x, \tfrac{|u-u_{2B}|}{(t-s)\kappa})\, dx + C.$$

Since $1 \leqslant \frac{aK}{aK+1} + \frac{a^3K}{aK+1}$, this implies that

$$\begin{aligned}
\int_{sB} \varphi(x, |\nabla u|) + 1\, dx &\leqslant \frac{aK}{aK + 1} \int_{tB} \varphi(x, |\nabla u|) + 1\, dx \\
&\quad+ \frac{a^3 K}{aK + 1} \int_{2B} \varphi(x, \tfrac{|u-u_{2B}|}{(t-s)\kappa}) + 1\, dx + C.
\end{aligned}$$

Now, as the function $(\varphi(x,t)+1)$ is doubling for all $t>0$, we can use a variant of the standard iteration lemma, Lemma 4.2 of [25], and get

$$\int_B \varphi(x,|\nabla u|)+1\,dx \lesssim \int_{2B} \varphi\left(x, \frac{|u-u_{2B}|}{\kappa}\right)+1\,dx + C.$$

The result follows after we subtract $|B|$ from both sides. $\square$

Lemma 4.5 (Gehring's lemma, [20])**.** *Let $f \in L^1(B_R)$ be non-negative. Assume that $g \in L^q(2B_R)$ for some $q>1$ and that there exists $s \in (0,1)$ such that*

$$\fint_B f\,dx \lesssim \left(\fint_{2B} f^s\,dx\right)^{\frac{1}{s}} + \fint_{2B} g\,dx$$

for every ball $B \subset\subset B_R$. Then there exists $t>1$ such that

$$\left(\fint_{B_R} f^t\,dx\right)^{\frac{1}{t}} \lesssim \fint_{2B_R} f\,dx + \fint_{2B_R} g^t\,dx.$$

Now we are ready to prove our main result, which follows from the last three results.

Proof of Theorem 1.1. Let $p>1$ be such that φ satisfies (aInc)$_p$. Fix $s \in (1,p]$ with $s < \frac{n}{n-1}$. Choose a ball B_R such that $2B_R \subset\subset \Omega$ and $\|\nabla u\|_{L^\varphi(2B)} < 1$. Now the Caccioppoli inequality (Lemma 4.2) yields for a local quasiminimizer u and $B \subset\subset B_R$ that

$$\fint_B \varphi(x,|\nabla u|)\,dx \lesssim \fint_{2B} \varphi\left(x, \frac{|u-u_{2B}|}{\operatorname{diam} B}\right)dx + C.$$

First adding 1 to the right-hand side, then using (aDec)$_q^\infty$ and finally the Sobolev–Poincaré inequality (Proposition 3.6) we get that

$$\fint_B \varphi(x,|\nabla u|)\,dx \lesssim \fint_{2B} \varphi\left(x, \beta_3\frac{|u-u_{2B}|}{\operatorname{diam} B}\right)+1\,dx + C \lesssim \left(\fint_{2B} \varphi(x,|\nabla u|)^{\frac{1}{s}}\,dx\right)^s + C.$$

With $g = C\chi_\Omega$, Gehring's lemma (Lemma 4.5) yields

$$\left(\fint_{B_R} \varphi(x,|\nabla u|)^t\,dx\right)^{\frac{1}{t}} \lesssim \fint_{2B_R} \varphi(x,|\nabla u|)\,dx + \fint_{2B_R} C^t\,dx < \infty.$$

Writing $\epsilon = t-1$, we see that $\varphi(\cdot,|\nabla u|)$ has higher integrability in the ball B.

Cover $\Omega' \subset\subset \Omega$ with balls B_i that satisfy the assumptions of the first part of the proof. Because Ω' is compact, we can choose a finite subcover $\{B_i\}_{i=1}^N$. Now

$$\left(\fint_{\Omega'} \varphi(x,|\nabla u|)^{1+\varepsilon}\,dx\right)^{\frac{1}{1+\varepsilon}} \lesssim \sum_{i=1}^N \fint_{2B_i} \varphi(x,|\nabla u|)\,dx + \fint_{2B_i} C^r\,dx < \infty$$

and therefore $\varphi(\cdot,|\nabla u|) \in L^{1+\varepsilon}_{\mathrm{loc}}(\Omega)$. $\square$

References

[1] M. Avci and A. Pankov: Multivalued elliptic operators with nonstandard growth, Adv. Nonlinear Anal., to appear. DOI: 10.1515/anona-2016-0043

[2] P. Baroni, M. Colombo and G. Mingione: Harnack inequalities for double phase functionals, Nonlinear Anal. 121 (2015), 206–222.

[3] P. Baroni, M. Colombo and G. Mingione: Non-autonomous functionals, borderline cases and related function classes, St Petersburg Math. J. 27 (2016), 347–379.

[4] P. Baroni, M. Colombo and G. Mingione: Regularity for general functionals with double phase, Preprint (2017).

[5] A. Björn and J. Björn: *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, 17, European Mathematical Society (EMS), Zürich, 2011.

[6] S.-S. Byun, M. Lee and J. Ok: $W^{2,p(\cdot)}$-regularity for elliptic equations in nondivergence form with BMO coefficients, Math. Ann. 363 (2015), no. 3-4, 1023–1052.

[7] S.-S. Byun and Y. Youn: Optimal gradient estimatesvia Riesz potentials for $p(\cdot)$-Laplacian type equations, Quart. J. Math., to appear.

[8] F. Colasuonno and M. Squassina: Eigenvalues for double phase variational integrals, Ann. Mat. Pura Appl. (4) 195 (2016), no. 6, 1917–1959.

[9] M. Colombo and G. Mingione: Regularity for double phase variational problems, Arch. Ration. Mech. Anal. 215 (2015), no. 2, 443–496.

[10] M. Colombo and G. Mingione: Bounded minimisers of double phase variational integrals, Arch. Ration. Mech. Anal. 218 (2015), no. 1, 219–273.

[11] M. Colombo and G. Mingione: Calderón–Zygmund estimates and non-uniformly elliptic operators, J. Funct. Anal. 270 (2016), 1416–1478.

[12] A. Clop, R. Giova, F. Hatami and A. Passarelli di Napoli: Regularity theory and congested traffic dynamics, presented at International Conference on Elliptic and Parabolic Problems Gaeta, May 22–26, 2017.

[13] G. Cupini, F. Giannetti, R. Giova and A. Passarelli di Napoli: Higher integrability and differentiability results for vectorial minimizer, presented at International Conference on Elliptic and Parabolic Problems Gaeta, May 22–26, 2017.

[14] L. Diening: Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces, Bull. Sci. Math. 129 (2005), no. 8, 657–700.

[15] L. Diening and S. Schwarzacher: Global gradient estimates for the $p(\cdot)$-Laplacian, Nonlinear Anal. 106 (2014), 70–85.

[16] X.–L. Fan and D. Zhao: The quasi-minimizer of integral functionals with $m(x)$ growth conditions, J. Math. Anal. Appl. 263 (2001), 424–446.

[17] F. Gehring: The L^p integrability of the partial derivatives of a quasiconformal mapping, Acta Mathematica 130 (1973), 265–277

[18] F. Giannetti and A. Passarelli di Napoli: Regularity results for a new class of functionals with non-standard growth conditions, J. Differential Equations 254 (2013) 1280–1305.

[19] D. Gilbarg and N. S. Trudinger: *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977.

[20] E. Giusti: *Direct Methods in the Calculus of Variations*, World Scientific, Singapore, 2003.

[21] P. Gwiazda, P. Wittbold, A. Wróblewska and A. Zimmermann: Renormalized solutions to nonlinear parabolic problems in generalized Musielak-Orlicz spaces, Nonlinear Anal. 129 (2015), 1–36.

[22] P. Harjulehto and P. Hästö: Boundary regularity under generalized growth conditions, Preprint (2016).

[23] P. Harjulehto and P. Hästö: Riesz potential in generalized Orlicz Spaces, Forum Math. 29 (2017), no. 1, 229–244.

[24] P. Harjulehto, P. Hästö and R. Klén: Generalized Orlicz spaces and related PDE, Nonlinear Anal. 143 (2016), 155–173.

[25] P. Harjulehto, P. Hästö and O. Toivanen: Hölder regularity of quasiminimizers under generalized growth conditions, Calc. Var. Partial Differential Equations 56 (2017), no. 2, article:22.

[26] P. Hästö: The maximal operator on generalized Orlicz spaces, J. Funct. Anal 269 (2015), no. 12, 4038–4048; J. Funct. Anal. 271 (2016), no. 1, 240–243.

[27] V. Latvala and O. Toivanen: Weak Harnack estimates for quasiminimizers with non-standard growth and general structure, Potential Anal. 47 (2017), no. 1, 21–36

[28] F.-Y. Maeda, Y. Mizuta, T. Ohno and T. Shimomura: Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces, Bull. Sci. Math. 137 (2013), 76–96.

[29] T. Ohno and T. Shimomura: Musielak-Orlicz-Sobolev spaces with zero boundary values on metric measure spaces, Czechoslovak Math. J. 66(141) (2016), no. 2, 371–394.

[30] J. Ok: Gradient estimates for elliptic equations with $L^{p(\cdot)} \log L$ growth, Calc. Var. Partial Differential Equations 55 (2016), no. 2, 1–30.

[31] M. Rao and Z. Ren: *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 146, Marcel Dekker, Inc., New York, 1991

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