

Fixed energy problem for nonlinear Schrödinger operator

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Abstract. This work studies the inverse fixed energy scattering problem for the generalised nonlinear Schrödinger operators. We prove that in a three-dimensional case the unknown compactly supported generalised nonlinear potential (with some restriction for this potential) from L^2 space can be uniquely determined by the scattering data with fixed positive energy (meaning that we have the knowledge of the scattering amplitude with fixed non-zero spectral parameter). The results are based on the new estimates for the Faddeev's Green function in L^∞ . These results may have applications in nonlinear optics for the saturation model. In particular, the constant coefficients of this model can be uniquely reconstructed by the scattering data with fixed energy.

1. Introduction

We investigate the generalized nonlinear Schrödinger equation

$$i \frac{\partial}{\partial t} E(x, t) = -\Delta E(x, t) + h(x, |E|)E(x, t),$$

where E denotes the electromagnetic field, Δ is an n -dimensional Laplacian and h describes in a general form the nonlinear contribution to the index of refraction. Considering harmonic time-dependence $E(x, t) = e^{-i\omega t}u(x)$ with frequency $\omega > 0$ we obtain the steady-state nonlinear equation with fixed energy

$$-\Delta u(x) + h(x, |u|)u(x) = k^2 u(x), \quad (1)$$

where $k^2 = \omega$ and fixed, and u here denotes the complex amplitude of the field. Concerning the nonlinearity $h(x, s)$ we assume that its support in x lies in bounded domain $\Omega \subset \mathbb{R}^n$ and

- 1) there is a function $\alpha(x) \in L^2(\Omega)$ such that

$$|h(x, s)| \leq \alpha(x), \quad s \in \mathbb{R}_+,$$

- 2) there is a function $\beta(x) \in L^2(\Omega)$ such that the inequality

$$|h(x, |e^{i(x,z)}u_1|) - h(x, |e^{i(x,z)}u_2|)| \leq \beta(x)|u_1 - u_2|$$

holds for any complex vector $z \in C^n$ and for any $u_1, u_2 \in L^\infty(\mathbb{R}^n)$ with L^∞ norms which semi-bounded from below by some positive constants,

3) Function $h(x, s)$ is continuous in $s \in R_+$ a.e. for x in Ω .

The main practical example (it can be considered as the motivation of this research) is an equation of the form

$$-\Delta u(x) + q(x)u(x) + \chi(x) \frac{|u(x)|^2}{1 + r|u(x)|^2} u(x) = k^2 u(x), \quad x \in R^n \quad (2)$$

with real number $k \neq 0$, complex-valued function $q(x) \in L^2(\Omega)$ and characteristic function χ of some subdomain of Ω , and parameter $r \geq 0$. A particular nonlinearity in (2) of cubic type ($r = 0$) can be met in the context of a Kerr-like nonlinear dielectric film, while the case when $r > 0$ corresponds to the saturation model (see [1], [2], [4], [10]). We consider inverse fixed energy ($k^2 > 0$ and fixed) scattering problem for this equation (1).

In scattering theory one considers solutions to (1) of the form

$$u(x, k, \theta) = u_0(x, k, \theta) + u_{sc}(x, k, \theta),$$

where $\theta \in S^{n-1}$ - unit sphere in R^n , $u_0(x, k, \theta) = \exp(ik(x, \theta))$ is the incident wave and $u_{sc}(x, k, \theta)$ is the scattered wave. The scattered wave must satisfy the Sommerfeld radiation condition at the infinity. These solutions are the unique solutions of the Lippmann-Schwinger equation

$$u(x) = u_0(x) - \int_{R^n} G_k^+(|x - y|) h(y, |u|) u(y) dy, \quad (3)$$

where G_k^+ is the outgoing fundamental solution of the operator $(-\Delta - k^2)$ in R^n .

It can be proved that if $k > 0$ is large enough for $n = 2, 3$ (see [19], [8]) then there is a unique solution of nonlinear equation (3) such that

$$\|u_{sc}\|_{L^\infty(R^n)} \rightarrow 0, \quad k \rightarrow +\infty. \quad (4)$$

uniformly in $\theta \in S^{n-1}$.

These properties (4) allow us to conclude that the solutions $u(x, k, \theta)$ for fixed $k > 0$ admit asymptotically as $|x| \rightarrow \infty$ uniformly with respect to θ a representation

$$u = e^{ik(x, \theta)} + C_n(k) \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} A(k, \theta', \theta) + O\left(\frac{1}{|x|^{\frac{n+1}{2}}}\right),$$

where $\theta, \theta' = \frac{x}{|x|} \in S^{n-1}$, $C_n(k)$ is known constant and the function $A(k, \theta', \theta)$ is called a scattering amplitude and defined by

$$A(k, \theta', \theta) := \int_{R^n} e^{-ik(\theta', y)} h(y, |u|) u(y) dy. \quad (5)$$

The following main result holds. Assume that function $\chi(x)$ is given and the domain Ω is given also. Let for $q_1(x), q_2(x) \in L^2(\Omega)$ and for some sequence $r_j \rightarrow +\infty$ the corresponding scattering amplitudes satisfy the equality

$$A_{q_1, \chi, r_j}(k_0, \theta', \theta) = A_{q_2, \chi, r_j}(k_0, \theta', \theta)$$

for fixed $k_0 > 0$ and for all $\theta', \theta \in S^n$. Then

$$q_1(x) = q_2(x)$$

a.e. in Ω .

Fixed energy problem for linear Schrödinger equation is well investigated. In dimensions higher than 2 it is well known that the scattering amplitude at fixed positive energy uniquely determines a compactly supported potential (bounded or singular) [13], [15], [16], [5] in the linear Schrödinger operator. There is one special result of the uniqueness for nonlinear Schrödinger equation (see [9]). The two-dimensional problem even in linear case is much more difficult (see, for example, [14], [17]) and was not so investigated until recently. But first Bukhgeim [3], and then Lakshtanov and Vainberg [11] proved the uniqueness result for a bounded potential from W_p^1 and for a singular potential from L^p , respectively. It should be noted that the reconstruction of singularities in two-dimensional case for linear and nonlinear Schrödinger operator (using the Born approximation) is known much earlier (see [22], [23], [24], [18] and [19]). Note that Grinevich and Novikov [7] showed that in two dimensions there are nonzero real potentials (at least two) from the Schwartz class without compact support with zero amplitude at fixed energy. Thus, the compactness of the supports of potentials is very natural condition in the present consideration.

2. Faddeev's Green function

We concentrate our attention to three-dimensional nonlinear Schrödinger equation of the form mentioned above. In order to prove main result we need to investigate the Faddeev's Green function

$$g_z(x) := \frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{i(x,\xi)}}{\xi^2 + 2(z,\xi)} d\xi,$$

where $z \in C^3$ is a 3-dimensional complex vector with $(z, z) = 0$. Here and later on the symbol (\cdot, \cdot) denotes the inner product in R^3 . It can be mentioned that $g_z(x)$ is the fundamental solution of the following operator with constant coefficients

$$(-\Delta - 2i(z, \nabla))g_z(x) = \delta(x).$$

We assume as before that Ω is bounded domain in R^n . We extend f by zero outside of Ω . Then the following estimate holds

There exists constant $c > 0$ depending on γ such that for any $f \in L^2(\Omega)$ and for $|z| > 1$

$$\|g_z \star f\|_{L^\infty(R^3)} \leq \frac{c}{|z|^\gamma} \|f\|_{L^2(\Omega)},$$

where symbol $g_z \star f$ denotes the convolution of g_z and f , $\gamma < \frac{1}{2}$.

In order to prove it let us denote by $\delta = |z|$, and $T_\delta f(x) = f(\delta x)$. Following Chanillo (see [5]) we can get by a change of variables

$$g_z \star f(x) = \delta^{-2} T_\delta g_{\frac{z}{\delta}} \star \left(T_{\frac{1}{\delta}} f\right)(x). \quad (6)$$

Moreover, we have

$$\|T_\delta f\|_{L^p(R^3)} = \delta^{-\frac{3}{p}} \|f\|_{L^p(R^3)}. \quad (7)$$

Thus,

$$\|g_z \star f\|_\infty = \delta^{-2} \|g_{\frac{z}{\delta}} \star \left(T_{\frac{1}{\delta}} f\right)\|_\infty \quad (8)$$

We consider g_z with $|z| = 1$. Let $z = a + ib$, where $a, b \in R^3$. Since $(z, z) = 0$ then $(a, b) = 0$ and $a^2 - b^2 = 0$. We set $a = te_2$ and $b = te_1$, where e_1, e_2, e_3 is the orthonormal basis in R^3 , and $t = \frac{1}{\sqrt{2}}$. It is easy to see that for any $\xi \in R^3$

$$\xi^2 + 2(z, \xi) = \xi_1^2 + (\xi_2 + t)^2 + \xi_3^2 + 2i\xi_1 t - t^2.$$

Denoting by \hat{f} the Fourier transform of f we can rewrite $g_z \star f(x)$ as the following integral:

$$\frac{1}{(2\pi)^3} \int_{R^3} \hat{f}(\xi) \frac{e^{i(x,\xi)}}{\xi^2 + 2(z,\xi)} d\xi. \quad (9)$$

Thus, our task is to estimate this integral.

By the symbol B we denote the circle of radius $\frac{1}{\sqrt{2}}$ with center $(0, -\frac{1}{\sqrt{2}}, 0)$ on the plane $(0, \xi_2, \xi_3)$. And by the symbol U_σ we denote the beam which is obtained by the movement of the square with side 2σ and with center on B , and perpendicular to the plane $(0, \xi_2, \xi_3)$, over B . If we choose now σ small enough (for example $\sigma < \frac{1}{2}$) then we can get again that for any $\xi \notin U_\sigma$ and $|z| = 1$

$$|\xi^2 + 2(z,\xi)| > \sigma.$$

Using this fact we can obtain easily for any $p \leq 2$ the inequality

$$\begin{aligned} |g_z \star f(x)| &\leq \left| \int_{R^3 \setminus U_\sigma} \hat{f}(\xi) \frac{e^{i(x,\xi)}}{\xi^2 + 2(z,\xi)} d\xi \right| + \left| \int_{U_\sigma} \hat{f}(\xi) \frac{e^{i(x,\xi)}}{\xi^2 + 2(z,\xi)} d\xi \right| \leq \\ &\leq c \left(\|f\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)} \left(\int_{U_\sigma} \frac{1}{|\xi^2 + 2(z,\xi)|^p} d\xi \right)^{\frac{1}{p}} \right). \end{aligned} \quad (10)$$

Using the definition of U_σ and the polar coordinates we can estimate the integral from (10) as follows (taking $\sigma < \frac{1}{4}$)

$$\begin{aligned} \int_{U_\sigma} \frac{1}{|\xi^2 + 2(\zeta,\xi)|^p} d\xi &\leq c \int_0^\sigma d\xi_1 \int_{-\sigma + \frac{1}{\sqrt{2}}}^{\sigma + \frac{1}{\sqrt{2}}} \frac{r dr}{(2\xi_1^2 + (\xi_1^2 + r^2 - \frac{1}{2})^2)^{\frac{p}{2}}} \leq \\ &\leq c \int_0^\sigma d\xi_1 \int_{(-\sigma + \frac{1}{\sqrt{2}})^2 + \xi_1^2 - \frac{1}{2}}^{(\sigma + \frac{1}{\sqrt{2}})^2 + \xi_1^2 - \frac{1}{2}} \frac{d\mu}{(2\xi_1^2 + \mu^2)^{\frac{p}{2}}} \leq c \int_0^\sigma d\xi_1 \int_0^1 \frac{d\mu}{(2\xi_1^2 + \mu^2)^{\frac{p}{2}}}. \end{aligned}$$

Considering now $1 < p < 2$ and using the latter estimate, estimate (10) and the equalities (8) and (9) we obtain

$$\|g_z \star f\|_\infty \leq c_p |z|^{-2} \left(|z|^{\frac{3}{2}} \|f\|_{L^2(\Omega)} + |z|^{\frac{3}{p}} \|f\|_{L^p(\Omega)} \right) \leq c_p |z|^{-2 + \frac{3}{p}} \|f\|_{L^2(\Omega)}. \quad (11)$$

The inequality (11) means that the needed result is proved.

The estimate (11) allows us to prove the existence of complex geometric optics (CGO) solutions for nonlinear equation

$$-\Delta u + q(x)u + \chi(x) \frac{|u|^2}{1 + r|u|^2} u = 0 \quad (12)$$

with $r > 0$. By CGO solutions we mean the solutions of this equation in the form

$$u(x, z) = e^{i(x,z)}(1 + R(x, z)),$$

where $z \in C^3$ with $(z, z) = 0$.

For parameter $r > 0$ and for $q, \in L^2(\Omega)$, and $|z|$ large enough there exists unique CGO solution of the nonlinear Schrödinger equation (12) such that

$$\|R\|_{L^\infty(R^n)} \leq \frac{c}{|z|^\gamma},$$

with some constant $c > 0$ and γ as before.

We observe that this function $R(x, z)$ is the solution of the following integral equation

$$R(x, z) = R_0 - \int_{\Omega} g_z(x-y) \left(q(y)R + \chi(y)(1+R) \frac{|1+R|^2}{e_0(y) + r|1+R|^2} \right) dy, \quad (13)$$

where $e_0(y) = e^{2(y, Imz)} > 0$, R_0 is given by

$$R_0(x, z) = - \int_{\Omega} g_z(x-y)q(y) dy$$

and $g_z(x)$ is the Faddeev's Green function. In the Banach space $L^\infty(R^3)$ we rewrite equation (13) as

$$R = F(R), \quad (14)$$

where nonlinear operator F defines in the right hand-side of (13), and apply Banach fixed point theorem (see, for example, [25]). In order to prove the existence of solutions for equation (14) it suffices to check first that whether this operator F maps the ball $B_\rho(0) = \{u : \|u\|_{L^\infty} < \rho\} \subset L^\infty(R^3)$ to itself. Indeed, if $R \in B_\rho(0)$ then using Lemma 1 we can easily obtain that

$$\|R\|_{L^\infty(R^3)} \leq \frac{c_\gamma}{|z|^\gamma} (1 + \rho) \left(\|q\|_{L^2(\Omega)} + \frac{\|\chi\|_{L^2(\Omega)}}{r} \right).$$

Thus, the following inequality must be valid

$$\frac{c_\gamma}{|z|^\gamma} (1 + \rho) \left(\|q\|_{L^2(\Omega)} + \frac{\|\chi\|_{L^2(\Omega)}}{r} \right) < \rho.$$

But the latter inequality holds if we choose $|z|$ so large that

$$|z|^\gamma > c_\gamma (1 + \rho^{-1}) \left(\|q\|_{L^2(\Omega)} + \frac{\|\chi\|_{L^2(\Omega)}}{r} \right). \quad (15)$$

It means that for the values of $|z|$ and ρ (15) continuous map F transfers ball $B_\rho(0)$ in itself. Hence, equation (14) has at least one solution inside $B_\rho(0)$. For uniqueness of this solution it remains to prove that F is contractive (see [25]). To prove the contraction of F we consider the difference $F(R_1) - F(R_2)$ and obtain

$$\begin{aligned} \|F(R_1) - F(R_2)\|_{L^\infty(R^3)} &\leq \frac{c_\gamma}{|z|^\gamma} (\|q\|_{L^2(\Omega)} \|R_1 - R_2\|_{L^\infty(R^3)} + \\ &+ \|\chi\|_{L^2(\Omega)} (\|1 + R_1\|_{L^\infty(R^3)} \left\| \frac{|1 + R_1|^2}{e_0 + r|1 + R_1|^2} - \frac{|1 + R_2|^2}{e_0 + r|1 + R_2|^2} \right\|_{L^\infty(R^3)} + \end{aligned}$$

$$+\|R_1 - R_2\|_{L^\infty(\mathbb{R}^3)} \left\| \frac{|1 + R_2|^2}{e_0 + r|1 + R_2|^2} \right\|_{L^\infty(\mathbb{R}^3)}).$$

If we choose $\rho = \frac{1}{2}$ in (23) then for this values of $|z|$ we obtain from the latter inequality

$$\|F(R_1) - F(R_2)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{c_\gamma}{|z|^\gamma} \left(\|q\|_{L^2(\Omega)} + \frac{7}{r} \|\chi\|_{L^2(\Omega)} \right) \|R_1 - R_2\|_{L^\infty(\mathbb{R}^3)}.$$

So that F is contractive if

$$\frac{c_\gamma}{|z|^\gamma} \left(\|q\|_{L^2(\Omega)} + \frac{7}{r} \|\chi\|_{L^2(\Omega)} \right) < 1. \quad (16)$$

Combining (15) and (16) we obtain the needed result.

3. The uniqueness result

The proof consists of two steps, following classical lines. The first step is to prove that the equality of the Dirichlet-to-Neumann maps

$$\Lambda_{q_1, \chi, r_j} = \Lambda_{q_2, \chi, r_j}$$

for boundary data on $\partial\Omega$ and a sequence of r_j with $r_j \rightarrow +\infty$ implies the equality of potentials $q_1 = q_2$. The second step is to show that the equality of the scattering amplitudes

$$A_{q_1, \chi}(k_0, \theta', \theta) = A_{q_2, \chi}(k_0, \theta', \theta)$$

for fixed $k_0 > 0$ implies the equality of the Dirichlet-to-Neumann maps

$$\Lambda_{q_1 - k_0^2, \chi} = \Lambda_{q_2 - k_0^2, \chi},$$

where $r > 0$ is fixed.

For the first step we consider the Dirichlet boundary value problem for homogeneous nonlinear Schrödinger equation

$$-\Delta u(x) + q(x)u(x) + \chi(x) \frac{|u|^2}{1 + r|u|^2} u(x) = 0, \quad x \in \Omega,$$

$$u(x) = f(x), \quad x \in \partial\Omega, \quad (17)$$

with function f from Sobolev space $W_2^t(\partial\Omega)$ for some $t > 1$. Following Gilbarg and Trudinger (see Theorems 11.4 and 11.8, pp. 281-288, [6]) we can get that there is a unique solution u of this nonlinear boundary value problem which belongs to Sobolev space $W_2^s(\Omega)$ for some $\frac{3}{2} < s \leq 2$. Thus, we may define the Dirichlet-to-Neumann map $\Lambda_{q,r}$ as follows:

$$\Lambda_{q,r} f(x) := \frac{\partial u}{\partial \nu}(x), \quad x \in \partial\Omega,$$

where ν is outward normal vector at the boundary $\partial\Omega$. This map acts here as

$$\Lambda_{q,r} : W_2^t(\partial\Omega) \rightarrow W_2^{t-1}(\partial\Omega), \quad t > 1.$$

Denoting

$$h(x, |u|) := q(x) + \chi(x) \frac{|u|^2}{1 + r|u|^2}$$

we can easily get the following fact:

If u_1 and u_2 are the solutions of nonlinear Dirichlet boundary value problem (17) with (q_1, q_2) and (f_1, f_2) , respectively, then

$$\int_{\Omega} (h_1(x, |u_1|) - h_2(x, |u_2|)) u_1 u_2 dx = \int_{\partial\Omega} \left(f_1 \frac{\partial u_2}{\partial \nu} - f_2 \frac{\partial u_1}{\partial \nu} \right) d\sigma(x).$$

In particular, if $f_1 = f_2$ and $\Lambda_{q_1, r} = \Lambda_{q_2, r}$ then

$$\int_{\Omega} (h_1(x, |u_1|) - h_2(x, |u_2|)) u_1 u_2 dx = 0. \quad (18)$$

This orthogonality condition (18) holds for any two solutions of the Dirichlet boundary value problem (17). But we need to use it also for CGO solutions. We proceed as follows.

Let q_1, q_2 and χ be extended to be zero for $x \in R^3 \setminus \Omega$. Suppose also that $\Lambda_{q_1, r, \chi} = \Lambda_{q_2, r, \chi}$. If now u_1 and u_2 are two CGO solutions then it can be proved (see, for example, [12]) that $u_1 = u_2$ in $R^3 \setminus \Omega$. Thus

$$u_1(x) = u_2(x), \quad x \in \partial\Omega$$

and we can use (18) also for CGO solutions. Denoting now by u_1 and u_2 CGO solutions for (q_1, χ, r) and (q_2, χ, r) respectively, we have

$$u_1(x, z) = e^{i(x, z)} (1 + R_1(x, z)), \quad u_2(x, \tilde{z}) = e^{i(x, \tilde{z})} (1 + R_2(x, \tilde{z}))$$

with

$$iz = l + i(k + m), \quad i\tilde{z} = -l + i(k - m),$$

where $l, k, m \in R^3$ and mutually orthogonal. Fix arbitrary k and choose $l, m \rightarrow \infty$, such that

$$|l|^2 = |k|^2 + |m|^2.$$

We obtain from the orthogonality condition (18)

$$\int_{\Omega} \left(q_1(x) - q_2(x) + \chi(x) \left(\frac{e^{2(x, l)} |1 + R_1|^2}{1 + r e^{2(x, l)} |1 + R_1|^2} - \frac{e^{-2(x, l)} |1 + R_2|^2}{1 + r e^{-2(x, l)} |1 + R_2|^2} \right) \right) \times \\ \times (1 + R_1)(1 + R_2) e^{2i(x, k)} dx = 0.$$

It can be rewritten as

$$\int_{\Omega} (q_1(x) - q_2(x) + M(x, l, m, k)) (1 + R_1)(1 + R_2) e^{2i(x, k)} dx = 0,$$

where M has the form

$$M = \chi(x) \frac{e^{2(x, l)} |1 + R_1|^2 - e^{-2(x, l)} |1 + R_2|^2}{(e^{-2(x, l)} + r |1 + R_1|^2)(e^{2(x, l)} + r |1 + R_2|^2)}.$$

It is very important (and quite remarkable) that this value M satisfies the inequality

$$|M(x, l, m, k)| \leq \frac{|1 + R_1|^2}{e^{-2(x, l)} + r |1 + R_1|^2} + \frac{|1 + R_2|^2}{e^{2(x, l)} + r |1 + R_2|^2} \leq \frac{2}{r}$$

which holds uniformly in x, l, m, k . That's why we have also

$$\overline{\lim}_{l \rightarrow \infty} M \leq \frac{2}{r}, \quad \overline{\lim}_{l \rightarrow \infty} |M| \leq \frac{2}{r}.$$

Since

$$\|R_1\|_{L^\infty(\mathbb{R}^3)} \rightarrow 0, \quad \|R_2\|_{L^\infty(\mathbb{R}^3)} \rightarrow 0, \quad l \rightarrow \infty \quad (m \rightarrow \infty)$$

we have that uniformly in k

$$\lim_{l \rightarrow \infty} \int_{\Omega} (q_1(x) - q_2(x))(R_1 + R_2 + R_1 R_2) e^{2i(x,k)} dx = 0$$

as well as

$$\lim_{l \rightarrow \infty} \int_{\Omega} M(R_1 + R_2 + R_1 R_2) e^{2i(x,k)} dx = 0.$$

Taking into account these three facts and using Lebesgue theorem about dominated convergence we obtain

$$\int_{\Omega} (q_1(x) - q_2(x)) e^{2i(x,k)} dx = - \int_{\Omega} \overline{\lim}_{l \rightarrow \infty} M e^{2i(x,k)} dx.$$

This equality holds for all $k \in \mathbb{R}^3$. Even more is true, $\overline{\lim}_{l \rightarrow \infty} M$ does not depend on k . Hence,

$$q_1(x) - q_2(x) = - \overline{\lim}_{l \rightarrow \infty} M, \quad a.e.$$

Moreover, the estimates for M immediately imply the following inequality

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq \frac{2}{r} (\text{Vol}(\Omega))^{\frac{1}{2}}.$$

Since parameter r can be chosen large enough and since the norms in the latter inequality do not depend on r we may conclude that

$$q_1(x) = q_2(x)$$

a.e. in Ω . So, the first step is done. Namely, we proved that the equality of the Dirichlet-to-Neumann maps imply the equality of the potentials.

Now we are in the position to make the second step. In order to finish the proof of our main result we assume that the support of q and the support of χ belong to the ball $B_R(0) = \{x \in \mathbb{R}^3 : |x| < R\}$, so the same is true for the support of $h(x, |u|)$. The scattering solutions u_1, u_2 of nonlinear Schrödinger equation (2) satisfy the asymptotic representation ($k_0 > 0$ and fixed) as $|x| \rightarrow \infty$

$$u_j(x, k_0, \theta) = e^{ik_0(x, \theta)} + C_3 \frac{e^{ik_0|x|}}{|x|} A_j(k_0, \theta', \theta) + O\left(\frac{1}{|x|^2}\right), \quad j = 1, 2.$$

Since

$$A_{q_1, \chi}(k_0, \theta', \theta) = A_{q_2, \chi}(k_0, \theta', \theta)$$

then for $|x| \rightarrow \infty$

$$u_1(x, k_0, \theta) - u_2(x, k_0, \theta) = O\left(\frac{1}{|x|^2}\right).$$

At the same time these solution satisfy

$$(\Delta + k_0^2)(u_1 - u_2) = h_1(x, |u_1|u_1 - h_2(x, |u_2|)).$$

Due to the assumptions we have

$$\text{supp}(h_1u_1 - h_2u_2) \subset B_R(0).$$

Applying some modification of Rellich lemma (see, for example, [20], [21]) we obtain

$$\text{supp}(u_1 - u_2) \subset B_R(0)$$

also. This fact implies that

$$u_1(x) = u_2(x), \quad x \in \partial B_R(0)$$

and

$$\frac{\partial u_1}{\partial \nu}(x) = \frac{\partial u_2}{\partial \nu}(x), \quad x \in \partial B_R(0).$$

It remains only to remark that the latter equality is equivalent to the equality

$$\Lambda_{q_1 - k_0^2, \chi} = \Lambda_{q_2 - k_0^2, \chi}$$

for the ball $B_R(0)$. Applying now the result of the first step we finally obtain that the equality

$$A_{q_1, \chi, r_j}(k_0, \theta', \theta) = A_{q_2, \chi, r_j}(k_0, \theta', \theta)$$

implies

$$q_1(x) = q_2(x)$$

a.e. in Ω . This finishes the prove of main result.

Let for $q_1(x), q_2(x) \in L^2(\Omega)$ and for one value $r > 0$ the corresponding scattering amplitudes satisfy the equality

$$A_{q_1, \chi}(k_0, \theta', \theta) = A_{q_2, \chi}(k_0, \theta', \theta)$$

for fixed $k_0 > 0$ and for all $\theta', \theta \in S^2$. Then

$$q_1(x) = q_2(x) \quad \text{mod}(-\overline{\lim}_{l \rightarrow \infty} M)$$

a.e. in Ω . Thus, we have a uniqueness result in fixed energy problem up to known function.

If we assume now that the potential q is given (and fixed), but the coefficient χ (it can be considered as a inclusion) is a characteristic function of some subdomain of Ω then the following "uniqueness" result w.r.t. this inclusion is valid. Namely, let for given $q(x) \in L^2(\Omega)$ and for one value $r > 0$ the corresponding scattering amplitudes satisfy the equality

$$A_{q, \chi_1}(k_0, \theta', \theta) = A_{q, \chi_2}(k_0, \theta', \theta)$$

for fixed $k_0 > 0$ and for all $\theta', \theta \in S^2$. Then

$$\overline{\lim}_{l \rightarrow \infty} \left(\frac{\chi_1(x)}{e^{-2(x,l)} + r} - \frac{\chi_2(x)}{e^{2(x,l)} + r} \right) = 0,$$

a.e. for x in Ω .

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