SOLUTION TO PROBLEM 11959

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Set $\sigma_0 = 1$ and $a_{i0} = t_i$, so that the i:th factor in the product is $\sum_{j=0}^{n} \sigma_i \sigma_j a_{ij}$. Then the product expands to

$$\sum_{\varphi} \prod_{i=1}^{n} \sigma_i \sigma_{\varphi(i)} a_{i\varphi(i)},$$

where the sum is over all functions $\varphi\colon \{1, \ldots, n\} \to \{0, \ldots, n\}$, choosing one term for each i. Using this and interchanging the sums,

$$\sum_{\sigma} \prod_{i=1}^{n} \sigma_i \left(t_i + \sum_{j=1}^{n} \sigma_j a_{ij} \right) = \sum_{\varphi} \sum_{\sigma} \prod_{i=1}^{n} \sigma_i \sigma_{\varphi(i)} a_{i\varphi(i)},$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ runs through $\{\pm 1\}^n$. If there is some $k \in \{1, \ldots, n\} \setminus \{\varphi(1), \ldots, \varphi(n)\}$, then changing the sign σ_k changes the sign of exactly one factor in the last product, and hence of the product. Thus for such φ, the sums over $\{\sigma;\, \sigma_k = -1\}$ and $\{\sigma;\, \sigma_k = 1\}$ cancel. If on the other hand φ is a permutation of $\{1, \ldots, n\}$ then each σ_j appears exactly twice in the product for $j = 1, \ldots, n$. Therefore,

$$\begin{aligned} \sum_{\varphi} \sum_{\sigma} \prod_{i=1}^{n} \sigma_i \sigma_{\varphi(i)} a_{i\varphi(i)} &= \sum_{\pi} \sum_{\sigma} \prod_{i=1}^{n} \sigma_i \sigma_{\pi(i)} a_{i\pi(i)} \\ &= 2^n \sum_{\pi} \prod_{i=1}^{n} a_{i\pi(i)} = 2^n \operatorname{perm}(A), \end{aligned}$$

where π runs through the permutations of $\{1, \ldots, n\}$, and the definition of the permanent is used in the last step.

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