RESEARCH ARTICLE

On the Statistics of the Ratio of Non-Constrained Arbitrary α - μ Random Variables: a General Framework and Applications

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Summary

In this paper, we derive closed-form exact expressions for the main statistics of the ratio of two squared α - μ random variables, which are of interest in many scenarios for future wireless networks where generalized distributions are more suitable to fit with field data. Importantly, different from previous proposals, our expressions are general in the sense that are valid for non-constrained arbitrary values of the parameters of the α - μ distribution. Thus, the probability density function, cumulative distribution function, moment generating function, and higher order moments are given in terms of both (i) the Fox H-function for which we provide a portable and efficient Wolfram Mathematica code and (ii) easily computable series expansions. Our expressions can be used straightforwardly in the performance analysis of a number of wireless communication systems, including either interference-limited scenarios, spectrum sharing, full-duplex or physical-layer security networks, for which we present the application of the proposed framework. Moreover, closed-form expressions for some classical distributions, derived as special cases from the α - μ distribution, are provided as byproducts. The validity of the proposed expressions is confirmed via Monte Carlo simulations.

KEYWORDS:

α - μ distribution, cognitive radio networks, full-duplex relaying networks, generalized fading channels, physical layer security, ratio of random variables

1 | INTRODUCTION

The accurate wireless channel characterization is, currently, of paramount importance for the understanding, evaluation and design of future wireless communication systems, which will operate under stringent requirements of capacity, reliability, latency, and scalability, to enable the new applications for machine-type communications (MTC) and mission-critical communications envisioned for the scenarios of 5G and beyond. Accordingly, wireless propagations models are crucial to compare potential candidate technologies that will be used for the deployment of these networks¹. Particularly, classical small-scale fading models have shown to be limited to adequately fit experimental data from practical scenarios. Then, more general fading models, which better capture the wireless channel statistics, are specially useful for modeling future wireless systems.

Precisely, the α - μ fading distribution, first proposed in², is a more general and flexible model to better fit with field data in cases where other widely-used classical distributions are not accurate. Furthermore, the applicability of this model has been

proven in practical scenarios. For instance, field measurements of diverse propagation environments investigated in^{3,4,5} have shown that the α - μ model suits better to statistical variations of the propagated signal than the commonly used fading models. Moreover, this model presents a simpler mathematical tractability and includes other important distributions as special cases, such as Gamma, Nakagami- m , Exponential, Weibull, one-sided Gaussian, and Rayleigh. These well-known fading distributions have been derived assuming a homogeneous diffuse scattering field, resulting from randomly distributed point scatterers. The following homogeneity assumptions are usually to be valid in these models: 1) large number of partial waves; 2) no correlation between different partial waves; 3) no correlation between phase and amplitude of one partial wave; 4) homogeneous phase distribution on $[0, 2\pi]$. Physically, these assumptions are equivalent to a homogeneous diffuse scattering field. Based on this facts, the central limit theorem (CLT) leads to a complex Gaussian process with Gaussian distributed variables, from which the well-known models such as Rayleigh and rice are derived⁶. Therefore, the α - μ distribution allows to characterize the non-homogeneity of the propagation medium by considering a signal composed of clusters of multipath waves propagating in a non-homogeneous environment. The clusters of multipath waves are assumed to have the scattered waves with identical power. The resulting envelope is obtained as a non-linear function of the modulus of the sum of the multipath components². Thus, this fading model is described by two physical parameters, namely: α which represents the nonlinearity of the propagation environment, and μ which represents the number of multipath clusters.

The knowledge of the statistics of the sum, product, and ratio of fading random variables (RVs) has a pivotal role in the performance analysis and evaluation of many practical wireless applications. In this context, the distribution of both the sum and the product of α - μ RVs has been extensively studied by many research works, among them, the following are notable:^{7,8,9,10} (for the sum), and^{11,12,13,14, 15} (for the product). On the other hand, the statistics of the ratio of two squared α - μ RVs has been little explored in the literature, as will be shown later. It is noteworthy that, the performance analysis of wireless communications systems, specifically for scenarios considering some of the key technologies for future wireless networks, commonly involves the calculation of the ratio of signals powers, such as the signal-to-interference ratio (SIR), for instance. Therefore, the distribution of the ratio of two squared RVs is of particular interest, and it has a pivotal role in the analytical performance evaluation of those scenarios.

Different approaches concerning the statistics of the ratio between RVs with well-known distributions such as Gamma, Exponential, Weibull, Normal and the most recent Fluctuating Two-Ray model, are presented in^{16,17,18,19, 20}, where some application uses are also provided. Moreover, regarding generalized distributions, the statistics of the ratio of independent and arbitrary squared α - μ RVs, via series representation, was proposed in²¹. However, in that work, the convergence of the power series was attained by making an strong assumption, more specifically: the values related to the non-linearity of the environment (i.e., to the α parameter, also referred as shape parameter) of each α - μ RV involved in the quotient must be co-prime integers. Further, under the same consideration, the work in²² provides closed-form expressions for the statistics of the ratio of products of an arbitrary number of independent and non-identically distributed α - μ variates. Thus, an important constraint on the results of²¹ and²² is that the shape parameter (or, equivalently, the α parameter) of the α - μ RVs involved on the ratio cannot take non-constrained arbitrary values. This fact hinders a more comprehensive insight into the performance analysis of different wireless communication systems.

In light of the above considerations, in this paper we derive closed-form expressions for the main statistics of the ratio of independent and non-identically distributed (i.n.i.d.) squared α - μ RVs, for which all the fading parameters of both distributions can be non-constrained arbitrary positive real numbers (thus including the special case of positive integers). This way, our expressions relieve the strong assumption considered in²¹ and²². Also, our results can be employed as a powerful tool for the performance evaluation of different scenarios. The following are our main contributions:

- Novel closed-form expressions for the probability density function (PDF), cumulative distribution function (CDF), moment generating function (MGF), and higher order moments of the ratio of two squared α - μ RVs are derived in terms of the Fox H-function.
- The statistics of the ratio of two squared RVs for some special cases of classical fading distributions are also provided as byproducts.
- Application uses in wireless networks are proposed in the context of Physical Layer Security (PLS), Cognitive Radio (CR), and Full-Duplex (FD) relaying, where the obtained analytical expressions can be used straightforwardly.
- A simple, efficient and useful algorithm for the implementation of the univariate Fox H-function is also provided.

The remainder of this paper is organized as follows. Section II revisits preliminaries on the α - μ distribution. In Section III, the statistics of the ratio of non-constrained two squared α - μ RVs are derived. Section IV presents some application uses of the derived expressions, while Section V shows some illustrative numerical results and draws some discussions. Finally, some concluding remarks are presented in Section VI.

In what follows, we use the following notation: $f_A(\cdot)$ and $F_A(\cdot)$ for the PDF and CDF of a RV A , respectively; $\mathbb{E}[\cdot]$ for expectation; $\mathbb{V}[\cdot]$ for variance; $\Pr\{\cdot\}$ for probability; and $|\cdot|$ for absolute value. In addition, $\Gamma(\cdot)$ is the gamma function^{23, Eq. (6.1.1)}; $P(z, y) = \frac{1}{\Gamma(z)} \int_0^y t^{z-1} \exp(-t) dt$ is the regularized lower incomplete gamma function^{23, Eq. (6.5.1)}; $H_{p,q}^{m,n}[\cdot]$ is the Fox H-function^{24, Eq. (1.1)}; and $G_{p,q}^{m,n}[\cdot]$ is the Meijer G-function^{25, Eq. (7.82)}. We also use $i = \sqrt{-1}$ for the imaginary unit; $\mathbb{N}^0$ for natural numbers including zero; $\mathbb{C}$ for complex numbers; $\mathbb{R}$ for real numbers; $\mathbb{R}^+$ for positive real numbers; $\approx$ to denote “approximately equal to”; and $\propto$ to denote “proportionally to”.

2 | PRELIMINARIES

The PDF of the envelope R of a signal propagating on a fading channel with distribution α - μ is given by²

$$f_R(r) = \frac{\alpha \mu^\mu r^{\alpha\mu-1}}{\hat{r}^{\mu\alpha} \Gamma(\mu)} \exp\left(-\frac{\mu r^\alpha}{\hat{r}^\alpha}\right), \quad (1)$$

where α denotes the non-linearity of the environment, $\hat{r} = \sqrt[\alpha]{\mathbb{E}[R^\alpha]}$ is the α -root mean value of the channel envelope, and $\mu = \hat{r}^{2\alpha} \mathbb{V}^{-1}[R^\alpha]$ is related to the number of multipath clusters. Therefore, some special cases for the parameters α and μ , such that the α - μ distribution reduces to well-known distributions commonly used in wireless application scenarios, are specified in Table 1².

TABLE 1 Particular cases of the α - μ distribution

Distribution	α - μ fading values
Nakagami- m	$\alpha = 2, \mu = m$
Weibull	$\alpha = \alpha, \mu = 1$
Rayleigh	$\alpha = 2, \mu = 1$

From (1), the n -th moment $\mathbb{E}[R^n]$ can be obtained as

$$\mathbb{E}[R^n] = \hat{r}^n \frac{\Gamma(\mu + n/\alpha)}{\mu^{n/\alpha} \Gamma(\mu)}. \quad (2)$$

Let $\Upsilon \triangleq \gamma_t R^2$ be the instantaneous received signal-to-noise ratio (SNR) through an α - μ fading channel, where $\gamma_t \triangleq P_T/N_0$ is defined as the transmit SNR, with P_T being the transmit power and N_0 being the mean power of the additive white Gaussian noise (AWGN)^{7,26}. Hence, the corresponding PDF and CDF can be obtained from (1) by performing a transformation of variables as in^{7, Eqs. (8) and (10)}

$$f_Y(\gamma) = \frac{\alpha \gamma^{(\alpha\mu/2)-1}}{2\beta^{\alpha\mu/2} \Gamma(\mu)} \exp\left[-\left(\frac{\gamma}{\beta}\right)^{\alpha/2}\right], \quad (3)$$

$$F_Y(\gamma) = P\left(\mu, \left(\frac{\gamma}{\beta}\right)^{\alpha/2}\right), \quad (4)$$

where $\beta = \bar{\Upsilon} \Gamma(\mu) / \Gamma(\mu + 2/\alpha)$, with $\bar{\Upsilon}$ being the average received SNR, so that

$$\begin{aligned} \bar{\Upsilon} &= \mathbb{E}[\Upsilon] d^{-\eta} \\ &= \hat{r}^2 \frac{\Gamma(\mu + 2/\alpha)}{\mu^{2/\alpha} \Gamma(\mu)} \gamma_t d^{-\eta}, \end{aligned} \quad (5)$$

in which d is the propagation distance, and η is path-loss exponent. Now, by using^{27, Eq. (01.03.26.0004.01)}, we can express the exponential function in (3) in terms of the Meijer G-function, so that the PDF of Y can be rewritten as

$$f_Y(\gamma) = \frac{\alpha \gamma^{(\alpha\mu/2)-1}}{2\beta^{\alpha\mu/2}\Gamma(\mu)} G_{0,1}^{1,0} \left[\left(\frac{\gamma}{\beta} \right)^{\alpha/2} \middle| 0 \right]. \quad (6)$$

Likewise, using^{27, Eq. (06.09.26.0006.01)}, the regularized lower incomplete gamma function in (4) can be expressed in terms of the Meijer G-function. Thus, the CDF of Y can be rewritten as

$$F_Y(\gamma) = \frac{1}{\Gamma(\mu)} \left(\frac{\gamma}{\beta} \right)^{\frac{\mu\alpha}{2}} G_{1,2}^{1,1} \left[\left(\frac{\gamma}{\beta} \right)^{\frac{\alpha}{2}} \middle| 1-\mu, -\mu \right]. \quad (7)$$

3 | STATISTICS OF THE RATIO OF INDEPENDENT AND ARBITRARY SQUARED α - μ RVs

In this section, we derive closed-form expressions for the PDF, CDF, MGF and higher order moments of the ratio $X=Y_1/Y_2$, where Y_1 and Y_2 are i.n.i.d. RVs following an α - μ distribution. Moreover, hereafter we assume that $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $k = \frac{\alpha_1}{\alpha_2}$, $\mu_1, \mu_2 \in \mathbb{R}^+$, and $x \in \mathbb{R}^+$.

3.1 | PDF, CDF and MGF of X

Herein, the PDF and CDF of the ratio of two independent squared α - μ RVs are given in the following proposition. Besides, as one of the most important characterizations of a RV, the corresponding MGF is also provided.

Proposition 1. Let Y_1 and Y_2 be i.n.i.d. squared α - μ distributed RVs with probability functions given as in (6) and (7). The PDF, CDF, and MGF of the ratio $X=Y_1/Y_2$ are respectively given by

$$f_X(x) = \frac{\alpha_1 x^{\frac{\alpha_1\mu_1}{2}-1} \beta_2^{\frac{\alpha_1\mu_1}{2}}}{2\beta_1^{\frac{\alpha_1\mu_1}{2}} \Gamma(\mu_2)\Gamma(\mu_1)} \underbrace{H_{1,1}^{1,1} \left[\left(\frac{x\beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle| \begin{matrix} (1-\mu_2-k\mu_1, k) \\ (0, 1) \end{matrix} \right]}_{H_1}, \quad (8)$$

$$F_X(x) = \frac{1}{\Gamma(\mu_2)\Gamma(\mu_1)} \left(\frac{x\beta_2}{\beta_1} \right)^{\frac{\alpha_1\mu_1}{2}} \underbrace{H_{2,2}^{1,2} \left[\left(\frac{x\beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle| \begin{matrix} (1-\mu_1, 1), (1-\mu_1 k-\mu_2, k) \\ (0, 1), (-\mu_1, 1) \end{matrix} \right]}_{H_2}, \quad (9)$$

$$\mathcal{M}_X(s) = \frac{\alpha_1}{2\Gamma(\mu_2)\Gamma(\mu_1)} \left(\frac{\beta_2}{s\beta_1} \right)^{\frac{\alpha_1\mu_1}{2}} \underbrace{H_{2,1}^{1,2} \left[\left(\frac{\beta_2}{s\beta_1} \right)^{\frac{\alpha_1}{2}} \middle| \begin{matrix} (1-\mu_2-k\mu_1, k), (1-\frac{\mu_1\alpha_1}{2}, \frac{\alpha_1}{2}) \\ (0, 1) \end{matrix} \right]}_{H_3}. \quad (10)$$

Proof. See Appendix A. □

Remark 1. Notice that contrary to previous works^{21,22}, the results of Proposition 1 are general, since no constraints are imposed on the parameters of Y_1 and Y_2 .

Remark 2. It is worth mentioning that currently the Fox H-function is not implemented in mathematical software packages such as Wolfram Mathematica. However, the Fox H-function can be evaluated using either numerical evaluations in the form of a Mellin–Barnes integral²⁴ or by applying calculus of residues. In the former, we provide a portable implementation of the Fox H-function in MATHEMATICA®Wolfram. The code is simple, efficient, highly accurate and its convergence is always attained. Here, it is worth mentioning that the overall convergence speed of the proposed code is achieved faster for small values of the argument, namely “z”, and influenced by the fading parameters of the distribution in matter. Moreover, tests were carried out for different fading conditions, from which the average elapsed time to obtain a desired accuracy (e.g., 10^{-8} in our tests) was ~ 4.5 s. Unlike other codes available in the literature for the implementation of the Fox H-function, our algorithm is valid for the evaluation of any univariate Fox H-function without any mathematical constraint. In the latter, an alternative method to compute the results presented here is given by the series representation for the Fox H-functions H_1 , H_2 , and H_3 as in (11), (12)

and (13), respectively, shown at the bottom of this page. It is important to emphasize that (11), (12) and (13) converge adequately and correctly as long as the convergence condition with regard to parameter k is fulfilled. Moreover, these series yield accurate results and the evaluation time is negligible. For instance, exhaustive tests have shown that the number of terms to arrive at the desired accuracy (e.g., 10^{-9}) were around 40 and the time taken varied approximately from 0.8 s to 6.5 s depending on the choice of the fading parameters. The mathematical derivation of the referred expressions is provided in Appendix C.

3.2 | Higher Order Moments

The n th order moment for a RV X is defined as $\mathbb{E}[X^n] \triangleq \int_0^\infty x^n f_X(x) dx$. Then, to calculate the n th moment of the ratio of squared α - μ distributed RVs, $X = Y_1/Y_2$, we resort to the identity for the product of two statistically independent RVs, i.e., $\mathbb{E}[(Y_1 Y_2)^n] = \mathbb{E}[Y_1^n] \mathbb{E}[Y_2^n]$ ²⁸. However, for the case of the ratio of two squared RVs, we are interested in solving $\mathbb{E}[(Y_1/Y_2)^n]$, thus being necessary to determine the n th moment of the inverse of a RV. To this end, let us define $Z = 1/Y_2$, such that $\mathbb{E}[(Y_1 Z)^n] = \mathbb{E}[Y_1^n] \mathbb{E}[Z^n]$. Thus, by determining the moments of $\mathbb{E}[Y_1^n]$ and $\mathbb{E}[Z^n]$, then the higher order moments of $\mathbb{E}[X^n]$ can be found. The moments of Z are determined from the distribution of the inverse of R_2 by considering $\mathbb{E}[(Y_1 R_2^2)^n] = \mathbb{E}[Y_1^n]$ ⁷. From this consideration, the higher order moments $\mathbb{E}[X^n]$ can be obtained as in the following proposition.

Proposition 2. The n th order moment for the ratio of squared α - μ distributed RVs, $X = Y_1/Y_2$, is given by

$$\mathbb{E}[X^n] = \frac{(\hat{r}_1 \hat{r}_2)^{2n} \Gamma\left(\mu_1 + \frac{2n}{\alpha_1}\right) \Gamma\left(\mu_2 - \frac{2n}{\alpha_2}\right)}{\mu_1^{2n/\alpha_1} \mu_2^{2n/\alpha_2} \Gamma(\mu_1) \Gamma(\mu_2)}, \quad \text{for } n > \mu_i \alpha_i, \quad i \in \{1, 2\}. \quad (14)$$

Proof. See Appendix D. □

Remark 3. An equivalent expression for the n th order moment in (14) can be obtained by applying the Mellin transform²⁹, Eq. (6.3.3.c) to the PDF in (8).

The formulations derived in (8) to (14) are general results that can be reduced to other distributions for different channel models, such as Rayleigh, Nakagami- m , and Weibull, by considering the corresponding parameters as in Table 1. Therefore, the PDF, CDF, and MGF for the distribution of the ratio of two squared of the aforementioned distributions are given in Table 2, 3 and 4, respectively.

$$H_1 = \begin{cases} \sum_{h=0}^{\infty} \frac{z^h \Gamma(k(h+\mu_1)+\mu_2)}{(-1)^h h!}, & k \leq 1, \text{ if } k = 1 \rightarrow |z| < 1. \\ \sum_{h=0}^{\infty} \frac{z^{-\frac{h+k\mu_1+\mu_2}{k}} \Gamma\left(\frac{h+k\mu_1+\mu_2}{k}\right)}{(-1)^h k h!}, & k \geq 1, \text{ if } k = 1 \rightarrow |z| > 1. \end{cases} \quad (11)$$

$$H_2 = \begin{cases} \sum_{h=0}^{\infty} \frac{z^h \Gamma(k(h+\mu_1)+\mu_2)}{(-1)^h (h+\mu_1) \Gamma(1+h)}, & k \leq 1, \text{ if } k = 1 \rightarrow |z| < 1. \\ \sum_{h=0}^{\infty} \frac{z^{-h-\mu_1} \Gamma(h+\mu_1) \Gamma(-hk+\mu_2)}{(-1)^{h-2} \Gamma(1-h) h!} + \sum_{h=0}^{\infty} \frac{z^{-\frac{h}{k}-\mu_1-\frac{\mu_2}{k}} \Gamma\left(\frac{-h-\mu_2}{k}\right) \Gamma\left(\frac{h+k\mu_1+\mu_2}{k}\right)}{(-1)^{h-2} \Gamma\left(\frac{-h+k-\mu_2}{k}\right) k h!}, & k \geq 1, \text{ if } k = 1 \rightarrow |z| > 1. \end{cases} \quad (12)$$

$$H_3 = \begin{cases} \sum_{h=0}^{\infty} \frac{\Gamma(hk+k\mu_X+\mu_Y) z^h}{(-1)^h h!}, & k \leq 1, \text{ if } k = 1 \rightarrow |z| < 1. \\ \sum_{h=0}^{\infty} \frac{\Gamma\left(\frac{h+k\mu_X+\mu_Y}{k}\right) z^{-\frac{h+k\mu_X+\mu_Y}{k}}}{(-1)^h k h!}, & k \geq 1, \text{ if } k = 1 \rightarrow |z| > 1. \end{cases} \quad (13)$$

TABLE 2 PDF of the Ratio for Different Distributions as Special Cases

Ratio	PDF
Nakagami- m /Nakagami- m	$f_X(x) = \frac{x^{\mu_1-1} \beta_2^{\mu_1}}{\beta_1^{\mu_1} \Gamma(\mu_2) \Gamma(\mu_1)} G_{1,1}^{1,1} \left[\frac{x\beta_2}{\beta_1} \middle 1 - \mu_2 - \mu_1 \right]$
Nakagami- m /Weibull	$f_X(x) = \frac{x^{\mu_1-1} \beta_2^{\mu_1}}{\beta_1^{\mu_1} \Gamma(\mu_2)} H_{1,1}^{1,1} \left[\frac{x\beta_2}{\beta_1} \middle \left(-\frac{2\mu_1}{\alpha_2}, \frac{2}{\alpha_2} \right) \right]$
Nakagami- m /Rayleigh	$f_X(x) = \frac{x^{\mu_1-1} \beta_2^{\mu_1}}{\beta_1^{\mu_1} \Gamma(\mu_1)} G_{1,1}^{1,1} \left[\frac{x\beta_2}{\beta_1} \middle -\mu_1 \right]$
Weibull/Weibull	$f_X(x) = \frac{a_1 x^{\frac{\alpha_1}{2}-1} \beta_2^{\frac{\alpha_1}{2}}}{2\beta_1^{\frac{\alpha_1}{2}}} H_{1,1}^{1,1} \left[\left(\frac{x\beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle \left(-k, k \right) \right]$
Weibull/Nakagami- m	$f_X(x) = \frac{a_1 x^{\frac{\alpha_1}{2}-1} \beta_2^{\frac{\alpha_1}{2}}}{2\beta_1^{\frac{\alpha_1}{2}} \Gamma(\mu_2)} H_{1,1}^{1,1} \left[\left(\frac{x\beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle \left(1 - \mu_2 - \frac{\alpha_1}{2}, \frac{\alpha_1}{2} \right) \right]$
Weibull/Rayleigh	$f_X(x) = \frac{a_1 x^{\frac{\alpha_1}{2}-1} \beta_2^{\frac{\alpha_1}{2}}}{2\beta_1^{\frac{\alpha_1}{2}}} H_{1,1}^{1,1} \left[\left(\frac{x\beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle \left(-\frac{\alpha_1}{2}, \frac{\alpha_1}{2} \right) \right]$
Rayleigh/Rayleigh	$f_X(x) = \frac{\beta_2}{\beta_1} G_{1,1}^{1,1} \left[\frac{x\beta_2}{\beta_1} \middle -1 \right]$
Rayleigh/Nakagami- m	$f_X(x) = \frac{\beta_2}{\beta_1 \Gamma(\mu_2)} G_{1,1}^{1,1} \left[\frac{x\beta_2}{\beta_1} \middle -\mu_2 \right]$
Rayleigh/Weibull	$f_X(x) = \frac{\beta_2}{\beta_1} H_{1,1}^{1,1} \left[\frac{x\beta_2}{\beta_1} \middle \left(-\frac{2}{\alpha_2}, \frac{2}{\alpha_2} \right) \right]$

4 | APPLICATIONS

In this section, we present some illustrative application uses of our analytical expressions in the context of key enabling technologies for 5G and beyond networks, including PLS, CR, and FD relaying. Henceforth, we assume that all wireless links undergo flat fading where channel coefficients remain constants during a block period.

4.1 | Physical Layer Security and Secrecy Outage Probability

In the context of physical layer security, a widely used metric to evaluate the secrecy performance of wireless networks is the secrecy outage probability. Thus, let us consider the Wyner's wiretap channel as depicted in Fig. 1, where a legitimate transmitter (Alice) sends confidential messages to the legitimate receiver (Bob) through the main channel, while the eavesdropper (Eve) tries to intercept these messages from its received signal over the eavesdropper channel. Furthermore, assume that both the main and eavesdropper channels experience independent α - μ distributed fading.

FIGURE 1 The system model of a wiretap channel consisting of two legitimate correspondents and one eavesdropper.

TABLE 3 CDF of the Ratio for Different Distributions as Special Cases

Ratio	CDF
Nakagami- m /Nakagami- m	$F_X(x) = \frac{1}{\Gamma(\mu_2)\Gamma(\mu_1)} \left(\frac{x\beta_2}{\beta_1}\right)^{\mu_1} G_{2,2}^{1,2} \left[\frac{x\beta_2}{\beta_1} \middle \begin{matrix} 1-\mu_1, 1-\mu_1-\mu_2 \\ 0, -\mu_1 \end{matrix} \right]$
Nakagami- m /Weibull	$F_X(x) = \frac{1}{\Gamma(\mu_1)} \left(\frac{x\beta_2}{\beta_1}\right)^{\mu_1} H_{2,2}^{1,2} \left[\frac{x\beta_2}{\beta_1} \middle \begin{matrix} (1-\mu_1, 1), (-\frac{2\mu_1}{\alpha_2}, \frac{2}{\alpha_2}) \\ (0, 1), (-\mu_1, 1) \end{matrix} \right]$
Nakagami- m /Rayleigh	$F_X(x) = \frac{1}{\Gamma(\mu_1)} \left(\frac{x\beta_2}{\beta_1}\right)^{\mu_1} G_{2,2}^{1,2} \left[\frac{x\beta_2}{\beta_1} \middle \begin{matrix} 1-\mu_1, -\mu_1 \\ 0, -\mu_1 \end{matrix} \right]$
Weibull/Weibull	$F_X(x) = \left(\frac{x\beta_2}{\beta_1}\right)^{\frac{\alpha_1}{2}} H_{2,2}^{1,2} \left[\left(\frac{x\beta_2}{\beta_1}\right)^{\frac{\alpha_1}{2}} \middle \begin{matrix} (0, 1), (-k, k) \\ (0, 1), (-1, 1) \end{matrix} \right]$
Weibull/Nakagami- m	$F_X(x) = \frac{1}{\Gamma(\mu_2)} \left(\frac{x\beta_2}{\beta_1}\right)^{\frac{\alpha_1}{2}} H_{2,2}^{1,2} \left[\left(\frac{x\beta_2}{\beta_1}\right)^{\frac{\alpha_1}{2}} \middle \begin{matrix} (0, 1), (1-k-\mu_2, k) \\ (0, 1), (-1, 1) \end{matrix} \right]$
Weibull/Rayleigh	$F_X(x) = \left(\frac{x\beta_2}{\beta_1}\right)^{\frac{\alpha_1}{2}} H_{2,2}^{1,2} \left[\left(\frac{x\beta_2}{\beta_1}\right)^{\frac{\alpha_1}{2}} \middle \begin{matrix} (0, 1), (-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}) \\ (0, 1), (-1, 1) \end{matrix} \right]$
Rayleigh/Rayleigh	$F_X(x) = \frac{x\beta_2}{\beta_1} G_{2,2}^{1,2} \left[\frac{x\beta_2}{\beta_1} \middle \begin{matrix} 0, -1 \\ 0, -1 \end{matrix} \right]$
Rayleigh/Nakagami- m	$F_X(x) = \frac{x\beta_2}{\beta_1 \Gamma(\mu_2)} G_{2,2}^{1,2} \left[\frac{x\beta_2}{\beta_1} \middle \begin{matrix} 0, -\mu_2 \\ 0, -1 \end{matrix} \right]$
Rayleigh/Weibull	$F_X(x) = \frac{x\beta_2}{\beta_1} H_{2,2}^{1,2} \left[\frac{x\beta_2}{\beta_1} \middle \begin{matrix} (0, 1), (-\frac{2}{\alpha_2}, \frac{2}{\alpha_2}) \\ (0, 1), (-1, 1) \end{matrix} \right]$

According to³⁰, the secrecy capacity is obtained as

$$\begin{aligned}
C_s &= \max \{C_B - C_E, 0\} \\
&= \max \left\{ \log_2 \left(1 + \frac{|h_{AB}|^2 P_A}{N_0} \right) - \log_2 \left(1 + \frac{|h_{AE}|^2 P_A}{N_0} \right), 0 \right\} \\
&= \max \{ \log_2(1 + \gamma_B) - \log_2(1 + \gamma_E), 0 \} \\
&= \begin{cases} \log_2 \left(\frac{1+\gamma_B}{1+\gamma_E} \right), & \text{if } \gamma_B > \gamma_E \\ 0, & \text{if } \gamma_B \leq \gamma_E, \end{cases} \quad (15)
\end{aligned}$$

where P_A is the transmit power at Alice, and C_k and h_{Ak} , with $k \in \{B, E\}$, are the capacity and the channel coefficient of the main channel and the wiretap channel, respectively. Hence, the secrecy outage probability (SOP) is defined as the probability that the instantaneous secrecy capacity falls below a target secrecy rate threshold R_{th} ³⁰, thus being given by

$$\begin{aligned}
\text{SOP} &= \Pr \{C_s(\gamma_B, \gamma_E) < R_{th}\} = \Pr \left\{ \left(\frac{1+\gamma_B}{1+\gamma_E} \right) < 2^{R_{th}} \right\} \\
&\stackrel{(a)}{\geq} \Pr \left\{ \frac{\gamma_B}{\gamma_E} < 2^{R_{th}} \triangleq \tau_1 \right\} \\
&= F_{X_1}(\tau_1) \quad (16)
\end{aligned}$$

where $X_1 = \gamma_B/\gamma_E$ and $F_{X_1}(\cdot)$ is a CDF given as in (9). In step (a), we have considered a lower bound of the SOP, which results very tight at high SNR, as shall be shown in Section 5. It is noteworthy that, our formulation for the lower bound of the SOP is valid for non-constrained arbitrary values of the fading parameters corresponding to the main channel and eavesdropper channel (i.e., α_i and μ_i , for $i \in \{B, E\}$). This is in contrast to previous works^{31,32} related to the performance analysis of physical layer security over single-input single-output (SISO) α - μ fading channels, where constraints on the fading parameter values were considered (more specifically, $\alpha_B = \alpha_E$ in³², and α_B, α_E must be co-prime integers in³¹). Therefore, our expressions are a generalization of the aforementioned approaches. It is worth mentioning that the secrecy performance analysis over multiple-input multiple-output (MIMO) α - μ fading channels by considering transmit antenna selection was derived in terms of the Fox H-function in³³.

TABLE 4 MGF of the Ratio for Different Distributions as Special Cases

Ratio	MGF
Nakagami- m /Nakagami- m	$\mathcal{M}_X(s) = \frac{1}{2\Gamma(\mu_2)\Gamma(\mu_1)} \left(\frac{\beta_2}{s\beta_1}\right)^{\mu_1} G_{2,1}^{1,2} \left[\frac{\beta_2}{s\beta_1} \middle \begin{matrix} 1-\mu_2-\mu_1, 1-\mu_1 \\ 0 \end{matrix} \right]$
Nakagami- m /Weibull	$\mathcal{M}_X(s) = \frac{1}{\Gamma(\mu_1)} \left(\frac{\beta_2}{s\beta_1}\right)^{\mu_1} H_{2,1}^{1,2} \left[\frac{\beta_2}{s\beta_1} \middle \begin{matrix} (-\frac{2\mu_1}{\alpha_2}, \frac{2}{\alpha_2}), (1-\mu_1, 1) \\ (0, 1) \end{matrix} \right]$
Nakagami- m /Rayleigh	$\mathcal{M}_X(s) = \frac{1}{\Gamma(\mu_1)} \left(\frac{\beta_2}{s\beta_1}\right)^{\mu_1} G_{2,1}^{1,2} \left[\frac{\beta_2}{s\beta_1} \middle \begin{matrix} -\mu_1, 1-\mu_1 \\ 0 \end{matrix} \right]$
Weibull/Weibull	$\mathcal{M}_X(s) = \frac{\alpha_1}{2} \left(\frac{\beta_2}{s\beta_1}\right)^{\frac{\alpha_1}{2}} H_{2,1}^{1,2} \left[\left(\frac{\beta_2}{s\beta_1}\right)^{\frac{\alpha_1}{2}} \middle \begin{matrix} (-k, k), (1-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}) \\ (0, 1) \end{matrix} \right]$
Weibull/Nakagami- m	$\mathcal{M}_X(s) = \frac{\alpha_1}{2\Gamma(\mu_2)} \left(\frac{\beta_2}{s\beta_1}\right)^{\frac{\alpha_1}{2}} H_{2,1}^{1,2} \left[\left(\frac{\beta_2}{s\beta_1}\right)^{\frac{\alpha_1}{2}} \middle \begin{matrix} (1-\mu_2-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}), (1-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}) \\ (0, 1) \end{matrix} \right]$
Weibull/Rayleigh	$\mathcal{M}_X(s) = \frac{\alpha_1}{2} \left(\frac{\beta_2}{s\beta_1}\right)^{\frac{\alpha_1}{2}} H_{2,1}^{1,2} \left[\left(\frac{\beta_2}{s\beta_1}\right)^{\frac{\alpha_1}{2}} \middle \begin{matrix} (-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}), (1-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}) \\ (0, 1) \end{matrix} \right]$
Rayleigh/Rayleigh	$\mathcal{M}_X(s) = \frac{\beta_2}{s\beta_1} G_{2,1}^{1,2} \left[\frac{\beta_2}{s\beta_1} \middle \begin{matrix} -1, 0 \\ 0 \end{matrix} \right]$
Rayleigh/Nakagami- m	$\mathcal{M}_X(s) = \frac{\beta_2}{s\beta_1\Gamma(\mu_2)} G_{2,1}^{1,2} \left[\frac{\beta_2}{s\beta_1} \middle \begin{matrix} -\mu_2, 0 \\ 0 \end{matrix} \right]$
Rayleigh/Weibull	$\mathcal{M}_X(s) = \frac{\beta_2}{s\beta_1} H_{2,1}^{1,2} \left[\frac{\beta_2}{s\beta_1} \middle \begin{matrix} (-\frac{2}{\alpha_2}, \frac{2}{\alpha_2}), (0, 1) \\ (0, 1) \end{matrix} \right]$

4.2 | Outage Performance of Cognitive Relaying Networks

Cognitive relaying networks is another application where the statistics of the ratio of two squared RVs appear^{34,35}. In particular, consider the cognitive relaying network depicted in Fig. 2. In this system, a secondary network consisting of one secondary source (S), one secondary half-duplex (HD) decode-and-forward (DF) relay (R), and one secondary destination (D), operate by sharing the spectrum belonging to a primary network. Thus, the secondary transmissions are subject to power constraints inflicted by a primary destination (P) in an underlay spectrum-sharing scenario, so that a predetermined level of interference temperature on the primary receiver is satisfied³⁶. In this system, the direct link is neglected, as it is considered to be extremely attenuated, and all terminals are assumed to be equipped with a single antenna. The channel coefficients of the data links $S \rightarrow R$ and $R \rightarrow D$ are denoted by h_{SR} and h_{RD} , respectively; and the channel coefficients of the interference links $S \rightarrow P$ and $R \rightarrow P$ are denoted by h_{SP} and h_{RP} , respectively. Thus, the corresponding channel power gains $g_{i,j} = |h_{i,j}|^2$, with $i \in \{R, S\}$ and $j \in \{D, P, R\}$, are subject to block α - μ fading. The maximum interference power tolerated at P, coming from the cognitive network, is denoted

FIGURE 2 System model of an underlay cognitive relaying network. The data links are represent by solid lines, while the interference links are represented by dashed lines.

by I . Thus, for the half-duplex relaying mode, in which the relay receives information from the source and forwards it to the destination in orthogonal channels, the transmit powers at the secondary source and relay, P_S and P_R , must satisfy the following constraints, respectively: $g_{SP}P_S \leq I$ and $g_{RP}P_R \leq I$. Herein, we assume that the transmission power at the source and relay is determined only by the interference temperature, denoted as I (we refer the interested reader to^{35,37} for further guidance on the assumptions made here). Hence, the transmit powers at the secondary source and relay are given by $P_S = I/g_{SP}$ and $P_R = I/g_{RP}$, respectively. In addition, $\bar{\gamma}_I \triangleq I/N_0$ is defined as the maximum interference-to-noise ratio tolerated at the primary destination. Then, the instantaneous received signal-to-noise ratio (SNR) at the secondary relay and the secondary destination are given, respectively, by

$$\gamma_{SR} = \frac{g_{SR}P_S}{N_0} = \frac{g_{SR}I}{g_{SP}N_0} = \frac{g_{SR}\bar{\gamma}_I}{g_{SP}}, \quad (17)$$

$$\gamma_{RD} = \frac{g_{RD}P_R}{N_0} = \frac{g_{RD}I}{g_{RP}N_0} = \frac{g_{RD}\bar{\gamma}_I}{g_{RP}}. \quad (18)$$

The outage probability of the secondary network for the DF relaying protocol can be written as³⁵

$$\begin{aligned} P_{\text{out}} &= \Pr \left(\min \left\{ \gamma_{SR}, \gamma_{RD} \right\} < 2^{2R} - 1 \triangleq \tau_2 \right) \\ &= F_{X_2}(\tau_2) + F_{X_3}(\tau_2) - F_{X_2}(\tau_2) F_{X_3}(\tau_2), \end{aligned} \quad (19)$$

where $F_{X_2}(\cdot)$ and $F_{X_3}(\cdot)$ are the CDFs for the RVs $X_2 = \gamma_{SR}$ and $X_3 = \gamma_{RD}$, respectively, which can be evaluated as in (9), R is the target rate and τ_2 is the target SNR threshold.

4.3 | Outage Performance of Full-Duplex Relaying Networks

Another application where the statistics of the ratio of independent squared α - μ random variables are considered is in FD relaying systems^{38,39}. Let us consider the system depicted in Fig. 3, which illustrates a two-hop FD relaying network composed of three nodes: one single-antenna source (S), one single-antenna destination (D), and one DF relay (R) equipped with one transmit antenna and one receive antenna to operate in full-duplex mode. In this system, it is assumed that the direct link is highly attenuated, thus being neglected. Furthermore, for the particular case of FD relaying mode, in which the relay receives information from the source and forwards it to the destination, simultaneously, we consider that, during each channel use, the total transmit power of the system is given by P_p . Thus, as source and relay transmit signals concurrently, P_S and P_R are halved to meet such restriction on the total transmit power of the system, such that $P_S = P_R = P_p/2$. Thus, $\gamma_{SR} = |h_{SR}|^2 \gamma_p/2$ and $\gamma_{RD} = |h_{RD}|^2 \gamma_p/2$ are the instantaneous received SNRs for the first- and second-hop relaying links, respectively, where h_{SR} and h_{RD} are the corresponding channel coefficients, and γ_p is the transmit system SNR. Moreover, due to imperfect stages of interference cancellation at the FD relay, a residual self-interference (RSI) is considered, which can be modeled as a Rayleigh fading loop back channel^{39,38}, with channel coefficient $h_{RR} \sim \mathcal{CN}(0, \sigma^2)$, such that $\gamma_{RR} = |h_{RR}|^2 \gamma_p/2$ is the instantaneous received SNR.

Considering the DF relaying protocol, the outage probability of the system under study can be formulated as³⁹

$$\begin{aligned} P_{\text{out}} &= \Pr \left(\min \left\{ \frac{\gamma_{SR}}{\gamma_{RR} + 1}, \gamma_{RD} \right\} < 2^R - 1 \triangleq \tau_3 \right) \\ &\approx F_{X_4}(\tau_3) + F_{X_5}(\tau_3) - F_{X_4}(\tau_3) F_{X_5}(\tau_3), \end{aligned} \quad (20)$$

FIGURE 3 System model of a three-node FD relaying network (data link: solid line; interference link: dashed line).

where, by considering an interference-limited scenario, such that $\gamma_{\text{SR}}/(\gamma_{\text{RR}} + 1) \approx \gamma_{\text{SR}}/\gamma_{\text{RR}}$, $F_{X_4}(\cdot)$ is the CDF of the RV $X_4 = \gamma_{\text{SR}}/\gamma_{\text{RR}}$ and $F_{X_5}(\cdot)$ is the CDF of the RV γ_{RD} , both of which being straightforwardly evaluated as in (9).

5 | NUMERICAL RESULTS AND DISCUSSIONS

In this section, we validate the accuracy of the proposed expressions for some representative cases via Monte Carlo simulations. Without loss of generality, we normalize the propagation distance d_i , with $i \in \{1, 2\}$, to unity.

Figs. 4 and 5 respectively show the PDF and CDF obtained for the ratio of two squared α - μ RVs, by considering different values of fading parameters. In both figures, the values of the fading parameters are chosen to show the wide range of shapes that the distribution of the ratio can assume. Fig. 4 illustrates the resulting PDF for different values of $\{\mu_1, \mu_2\}$, with $\{\alpha_1, \alpha_2\} = \{1.5, 1.1\}$ and $\bar{\gamma}_1 = \bar{\gamma}_2 = 0$ dB. It can be observed that our expressions perfectly match the Monte Carlo simulations, thus validating our results.

Fig. 5 shows the resulting CDF for distinct values of $\{\alpha_1, \alpha_2\}$, with $\{\mu_1, \mu_2\} = \{3.5, 2.8\}$ and $\bar{\gamma}_1 = \bar{\gamma}_2 = 0$ dB. Once again, it is observed that our expressions perfectly match the Monte Carlo simulations. It can also be noticed from the cases presented in those figures that our expressions allow non-constrained arbitrary values of fading.

FIGURE 4 PDF of the ratio of two squared α - μ RVs for different values of $\{\mu_1, \mu_2\}$, with $\{\alpha_1, \alpha_2\} = \{1.5, 1.1\}$ and $\bar{\gamma}_1 = \bar{\gamma}_2 = 0$ dB.

Fig. 6 shows the SOP versus $\bar{\gamma}_B$ for different combinations of fading parameters, with $\bar{\gamma}_B = 1$ dB and $\tau_1 = 0$ dB. More specifically, we set the fading parameters to the following cases:

- *Case 1 (Nakagami-m)*: $\{\alpha_B, \mu_B\} = \{2, 4.5\}$, $\{\alpha_E, \mu_E\} = \{2, 0.6\}$.
- *Case 2 (Weibull)*: $\{\alpha_B, \mu_B\} = \{3.9, 1\}$, $\{\alpha_E, \mu_E\} = \{1.3, 1\}$.
- *Case 3 (Rayleigh)*: $\{\alpha_B, \mu_B\} = \{2, 1\}$, $\{\alpha_E, \mu_E\} = \{2, 1\}$.
- *Case 4 (Weibull)*: $\{\alpha_B, \mu_B\} = \{1.2, 1\}$, $\{\alpha_E, \mu_E\} = \{4.5, 1\}$.
- *Case 5 (Nakagami-m)*: $\{\alpha_B, \mu_B\} = \{2, 0.5\}$, $\{\alpha_E, \mu_E\} = \{2, 3.1\}$.

For all cases, it can be noticed that the proposed lower bound is very tight to the exact SOP obtained by Monte Carlo simulations. Also, it is observed that, in general, the secrecy performance worsens as α_B, μ_B decrease and α_E, μ_E increase (see, e.g.,

FIGURE 5 CDF of the ratio of two squared α - μ RVs for different values of $\{\alpha_1, \alpha_2\}$, with $\{\mu_1, \mu_2\} = \{3.5, 2.8\}$ and $\bar{\gamma}_1 = \bar{\gamma}_2 = 0$ dB.

FIGURE 6 Secrecy outage probability versus $\bar{\gamma}_B$ for different combinations of $\{\alpha_B, \mu_B, \alpha_E, \mu_E\}$, and $\bar{\gamma}_E = 1$ dB and $\tau_1 = 1$.

cases 2, 3, and 4), which are the fading parameters of the main and eavesdropper channels, respectively. In contrast, note that the secrecy performance improves as α_B, μ_B increase and α_E, μ_E decrease (i.e., the eavesdropper channel is in a worse channel condition). Importantly, this fact implies that the fading conditions can be exploited to prevent the information from being overheard by an eavesdropper.

Fig. 7 illustrates the influence of the fading parameters on the outage performance of a cognitive relaying network. This figure shows the outage probability versus the maximum interference power constraint at the primary receiver, $\bar{\gamma}_I$, for $\bar{\gamma}_{SP} = \bar{\gamma}_{SR} = \bar{\gamma}_{RP} = \bar{\gamma}_{RD} = 1$ dB and a target SNR threshold $\tau_2 = 0$ dB. For these scenarios, the fading parameters are set to the next cases:

- Case 6: $\{\alpha_{SR}, \mu_{SR}\} = \{4.2, 1\}$, $\{\alpha_{SP}, \mu_{SP}\} = \{2, 4.1\}$, $\{\alpha_{RD}, \mu_{RD}\} = \{3.9, 1\}$, $\{\alpha_{RP}, \mu_{RP}\} = \{2, 3.8\}$.
- Case 7: $\{\alpha_{SR}, \mu_{SR}\} = \{2, 1\}$, $\{\alpha_{SP}, \mu_{SP}\} = \{2, 1\}$, $\{\alpha_{RD}, \mu_{RD}\} = \{2, 1\}$, $\{\alpha_{RP}, \mu_{RP}\} = \{2, 1\}$.
- Case 8: $\{\alpha_{SR}, \mu_{SR}\} = \{2, 0.6\}$, $\{\alpha_{SP}, \mu_{SP}\} = \{0.8, 1\}$, $\{\alpha_{RD}, \mu_{RD}\} = \{2, 0.9\}$, $\{\alpha_{RP}, \mu_{RP}\} = \{0.7, 1\}$.

FIGURE 7 Outage performance versus interference power constraint $\bar{\gamma}_I$ of a cognitive relaying network, for different values of fading parameters. Notation: Ry $\rightarrow$ Rayleigh, Wb $^+$ $\rightarrow$ severe Weibull, Wb $^-$ $\rightarrow$ weak Weibull, Nak $^+$ $\rightarrow$ severe Nakagami- m , Nak $^-$ $\rightarrow$ weak Nakagami- m .

- Case 9: $\{\alpha_{SR}, \mu_{SR}\} = \{0.6, 1\}$, $\{\alpha_{SP}, \mu_{SP}\} = \{2, 4.2\}$, $\{\alpha_{RD}, \mu_{RD}\} = \{4.1, 1\}$, $\{\alpha_{RP}, \mu_{RP}\} = \{2, 0.8\}$.

It can be observed from all the curves that our analytical expression matches the Monte Carlo simulations. Moreover, note that, as the fading parameters increase, i.e., for better channel conditions (see, e.g., Case 6, and Case 7), the outage performance improves, as expected. In the opposite scenario, i.e., signals with lower values of the fading parameters (see, e.g., Case 8, and Case 9), the outage performance worsens, due to poor channel conditions. In addition, the outage behavior improves as $\bar{\gamma}_I$ increases, as expected.

FIGURE 8 Outage performance versus $\bar{\gamma}_P$ of a FD relaying network, by considering distinct values of average channel power gain at the RSI link, $\bar{\gamma}_{RR}$. Two cases are considered: severe fading (case 10) and weak fading (case 11).

Fig. 8 shows the outage performance of a FD relaying network versus the transmit system SNR for $\bar{Y}_{SR} = \bar{Y}_{RD} = 0$ dB, $\tau_3 = 0$ dB, and different values of average channel power gain at the RSI link, namely, $\bar{Y}_{RR} = -10, -20, -30$ dB. For these scenarios, the values of the fading parameters are set to the next cases:

- *Case 10: Severe fading*
 $\{\alpha_{SR}, \mu_{SR}\} = \{1.8, 0.8\}$, $\{\alpha_{RR}, \mu_{RR}\} = \{2.2, 0.7\}$, $\{\alpha_{RD}, \mu_{RD}\} = \{2.1, 0.6\}$.
- *Case 11: Weak fading*
 $\{\alpha_{SR}, \mu_{SR}\} = \{1.9, 2.3\}$, $\{\alpha_{RR}, \mu_{RR}\} = \{2.1, 2.8\}$, $\{\alpha_{RD}, \mu_{RD}\} = \{2.2, 2.9\}$.

In a similar manner, note that our analytical results are highly accurate with respect to the Monte Carlo simulations, thus confirming the correctness of our derivations. We can also observe that the average channel power gain of the RSI link affects the system performance in a different manner according to the fading parameters of the channel. For instance, when dealing with weak fading (e.g., Case 11), the outage performance shows significant improvements as the level of RSI decreases. On the other hand, for a severe fading case (e.g., Case 10), it is observed that, even though for lower values of the level of RSI, the improvement on the performance is not significant. Also, it is observed a performance floor in the medium-to-high SNR regime. This behavior is caused by the RSI at the FD relay. In this context, it is worth mentioning that self-interference mitigation techniques play a pivotal role in exploiting the potential benefits of FD relaying, mainly at the medium-to-high SNR region. Additionally, it can be noticed that the consideration of a more general distribution in the case of FD, lead us to a more comprehensive analysis of different scenarios according to the severity of fading. Also, interested readers can revise^{40,41} for further guidance about self-interference cancellation on FD relay systems.

6 | CONCLUSIONS

In this paper, novel exact analytical expressions for the PDF, CDF, MGF, and higher order moments of the ratio of two squared α - μ RVs in terms of the Fox H-function were derived. Importantly, this expressions, unlike previous related works, are valid for any values of the fading parameters α and μ . Additionally, a series representation for the formulations are also provided. Based on these results, analytical expressions for the statistics of the ratio of well-known distributions, such as Nakagami- m , Weibull, and Rayleigh, were also provided as byproducts. These novel statistics represent a useful tool to assess the performance of wireless communication schemes considering generalized fading-channel models with applicability in scenarios for next-generation wireless networks. For illustration purposes, we analyze three application uses by analyzing (i) the secrecy outage probability for PLS-based wireless networks, (ii) the outage performance for cognitive relaying networks, and (iii) the outage performance for FD relaying networks. The obtained analytical expressions were validated by Monte Carlo simulations. It is worthwhile to mention that the analytical results presented in this work can be evaluated in a straightforward and efficient manner through mathematical software packages. For this purpose, we have also provided an implementation of the Fox H-function. Finally, it is important to remark that the ratio of correlated squared RVs becomes a particular challenge, since correlated fading has gained great attention because of their usefulness in many practical wireless communications. In this context, the ratio of squared distributions such as κ - μ shadowed, α - η - κ - μ , and Fluctuating Two-Ray can be investigated, since these new generalized fading models, as well as the α - μ model, provide a good fit to real measurements observed in different scenarios for next generation wireless networks. The authors believe that the ratio of the aforementioned RVs require a special treatment, thus being scenarios suggested for future works.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

APPENDIX

A PROOF OF PROPOSITION 1

Assuming that Y_1 and Y_2 are statistically independent, the PDF of X can be obtained as²¹

$$f_X(x) = \int_0^{\infty} y f_{Y_1}(xy) f_{Y_2}(y) dy. \quad (A1)$$

Now, by substituting (6) into (A1), it follows that

$$f_X(x) = \frac{\alpha_1 \alpha_2 x^{\frac{\alpha_1 \mu_1}{2} - 1}}{4 \beta_2^{\frac{\alpha_2 \mu_2}{2}} \beta_1^{\frac{\alpha_1 \mu_1}{2}} \Gamma(\mu_2) \Gamma(\mu_1)} \int_0^{\infty} y^{\frac{\alpha_2 \mu_2}{2} + \frac{\alpha_1 \mu_1}{2} - 1} G_{0,1}^{1,0} \left[\left(\frac{xy}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle| 0 \right] G_{0,1}^{1,0} \left[\left(\frac{y}{\beta_2} \right)^{\frac{\alpha_2}{2}} \middle| 0 \right] dy. \quad (A2)$$

After some mathematical manipulations in (A2), we have that

$$f_X(x) = \frac{\alpha_1 x^{\frac{\alpha_1 \mu_1}{2} - 1}}{2 \beta_2^{\frac{\alpha_2 \mu_2}{2}} \beta_1^{\frac{\alpha_1 \mu_1}{2}} \Gamma(\mu_2) \Gamma(\mu_1)} \underbrace{\int_0^{\infty} w^{\mu_2 + k \mu_1 - 1} G_{1,0}^{1,0} \left[\frac{w}{\beta_2^{\frac{\alpha_2}{2}}} \middle| 0 \right] G_{0,1}^{1,0} \left[\frac{w^k}{\left(\frac{x}{\beta_1} \right)^{\frac{\alpha_1}{2}}} \middle| 0 \right] dw}_{I_1}, \quad (A3)$$

where $w = y^{\alpha_2/2}$ and recalling that $k = \frac{\alpha_1}{\alpha_2}$. Then, by using²⁷, Eq. (07.34.21.0012.01), I_1 in (A3) can be solved in a straightforward manner as

$$I_1 = \left(\frac{1}{\beta_2^{\frac{\alpha_2}{2}}} \right)^{-(\mu_2 + k \mu_1)} H_{1,1}^{1,1} \left[\left(\frac{x \beta_2}{\beta_1} \right)^{\frac{\alpha_2}{2}} \middle| \begin{matrix} (1 - \mu_2 - k \mu_1, k) \\ (0, 1) \end{matrix} \right]. \quad (A4)$$

Finally, by replacing I_1 into (A3), we obtain the expression in (8). On the other hand, the CDF of $X = Y_1/Y_2$ can be formulated as

$$\begin{aligned} F_X(x) &= \Pr \{ X \leq x \} \\ &= \Pr \left\{ \frac{Y_1}{Y_2} \leq x \right\} \\ &= \Pr \{ Y_1 \leq x Y_2 \} \\ &= \int_0^{\infty} F_{Y_1}(xy) f_{Y_2}(y) dy. \end{aligned} \quad (A5)$$

Then, by replacing (6) and (7) into (A5), we get

$$F_X(x) = \frac{\alpha_2 x^{\frac{\mu_1 \alpha_1}{2}}}{2 \beta_2^{\frac{\alpha_2 \mu_2}{2}} \beta_1^{\frac{\alpha_1 \mu_1}{2}} \Gamma(\mu_2) \Gamma(\mu_1)} \int_0^{\infty} y^{\frac{\alpha_2 \mu_2}{2} + \frac{\alpha_1 \mu_1}{2} - 1} G_{0,1}^{1,0} \left[\left(\frac{y}{\beta_2} \right)^{\frac{\alpha_2}{2}} \middle| 0 \right] G_{1,2}^{1,1} \left[\left(\frac{xy}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle| \begin{matrix} 1 - \mu_1 \\ 0, -\mu_1 \end{matrix} \right] dy. \quad (A6)$$

Here we proceed by following a similar procedure as in the derivation of the PDF of X . By replacing $w = y^{\alpha_Y/2}$ and $k = \frac{\alpha_1}{\alpha_2}$ into (A6), it follows that

$$F_X(x) = \frac{x^{\frac{\mu_1 \alpha_1}{2}}}{\beta_2^{\frac{\alpha_2 \mu_2}{2}} \beta_1^{\frac{\alpha_1 \mu_1}{2}} \Gamma(\mu_2) \Gamma(\mu_1)} \underbrace{\int_0^\infty w^{\mu_1 k + \mu_2 - 1} G_{I_{20,1}}^{1,0} \left[\frac{w}{\beta_Y^{\frac{\alpha_Y}{2}}} \middle| 0 \right] G_{1,2}^{1,1} \left[\left(\frac{x}{\beta_1} \right)^{\frac{\alpha_1}{2}} w^k \middle| \begin{matrix} 1 - \mu_1 \\ 0, -\mu_X \end{matrix} \right] dw}_{I_2}. \quad (A7)$$

Now, by using²⁷, Eq. (07.34.21.0009.01), I_2 in (A7) can be solved in a straightforward manner as

$$I_2 = \left(\frac{1}{\beta_2^{\frac{\alpha_2}{2}}} \right)^{-(\mu_1 k + \mu_2)} H_{2,2}^{1,2} \left[\left(\frac{x \beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle| \begin{matrix} (1 - \mu_1, 1), (1 - \mu_1 k - \mu_2, k) \\ (0, 1), (-\mu_1, 1) \end{matrix} \right]. \quad (A8)$$

Finally, substituting (A8) into (A7), a closed-form expression for the CDF of $X = Y_1/Y_2$ can be calculated as in (9).

Now, the MGF of $X = Y_1/Y_2$ can be obtained, by definition, as⁴²

$$\mathcal{M}_X(s) \stackrel{\Delta}{=} \mathbb{E} [e^{-sX}] = \int_0^\infty \exp(-sx) f_X(x) dx. \quad (A9)$$

By replacing $f_X(x)$ given as in (8) into (A9), we obtain

$$\mathcal{M}_X(s) = \frac{\alpha_1}{2\Gamma(\mu_2)\Gamma(\mu_1)} \left(\frac{\beta_2}{\beta_1} \right)^{\frac{\alpha_1 \mu_1}{2}} \int_0^\infty x^{\frac{\alpha_1 \mu_1}{2} - 1} e^{-sx} H_{1,1}^{1,1} \left[\left(\frac{x \beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \middle| \begin{matrix} (1 - \mu_2 - k\mu_1, k) \\ (0, 1) \end{matrix} \right] dx. \quad (A10)$$

Substituting the Fox H-function in (A10) by its Mellin-Barnes type contour integral as in²⁴, Eq. (1.2), interchanging the order of integrations, and performing some simplifications, we obtain

$$\begin{aligned} \mathcal{M}_X(s) &= \frac{\alpha_1 \left(\frac{\beta_2}{\beta_1} \right)^{\frac{\alpha_1 \mu_1}{2}}}{2\Gamma(\mu_2)\Gamma(\mu_1)} \int_0^\infty x^{\frac{\alpha_1 \mu_1}{2} - 1} \exp(-sx) \frac{1}{2\pi i} \int_C \Gamma(z) \Gamma(\mu_2 + k\mu_1 - kz) \left[\left(\frac{x \beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \right]^{-z} dz dx \\ &= \frac{\alpha_1 \left(\frac{\beta_2}{\beta_1} \right)^{\frac{\alpha_1 \mu_1}{2}}}{4\Gamma(\mu_2)\Gamma(\mu_1)i} \int_C \Gamma(z) \Gamma(\mu_2 + k\mu_1 - kz) \left[\left(\frac{\beta_2}{\beta_1} \right)^{\frac{\alpha_1}{2}} \right]^{-z} \int_0^\infty x^{\frac{\alpha_1 \mu_1}{2} - \frac{z\alpha_1}{2} - 1} \exp(-sx) dx dz \\ &= \frac{\alpha_1 \left(\frac{\beta_2}{\beta_1} \right)^{\frac{\alpha_1 \mu_1}{2}} s^{-\frac{\alpha_1 \mu_1}{2}}}{2\Gamma(\mu_2)\Gamma(\mu_1)} \underbrace{\frac{1}{2\pi i} \int_C \Gamma(z) \Gamma(\mu_2 + k\mu_1 - kz) \Gamma\left(\frac{\mu_1 \alpha_1}{2} - \frac{\alpha_1 z}{2}\right) \left[\left(\frac{\beta_2}{s\beta_1} \right)^{\frac{\alpha_1}{2}} \right]^{-z} dz}_{I_3}. \end{aligned} \quad (A11)$$

Then, by substituting I_3 in (A11) by its corresponding Fox H-function with the use of²⁴, Eq. (1.1), we obtain the expression in (10), thus accomplishing the proof.

B MATHEMATICA®IMPLEMENTATION OF THE FOX-H FUNCTION

C THE FOX-H FUNCTION AS A SUM OF RESIDUES

Here, for illustration purposes, the Fox H-function in (9) is expressed as a sum of residues¹¹. To this end, we start by defining the Fox H-function as²⁴, Eq. (1.1)

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n} \left[z \middle| \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_C \Theta(s) z^{-s} ds, \quad (C12)$$

TABLE B1 MATHEMATICA®IMPLEMENTATION OF THE FOX-H FUNCTION

```

(*H-Fox Function*)
Clear all;
FOX::InconsistentCoeffs = "Inconsistent coefficients!";
FOX[z_,a_,b_] := Module[{ },
  (*where "z" is the argument of the Fox H Function, "a" and "b" are the
  coefficient sequences of Fox H Function defined as:
  a={{a1,A1},...,{an,An},{an+1,An+1},...,{ap,Ap}}
  b={{b1,B1},...,{bm,Bm},{bm+1,Bm+1},...,{bq,Bq}}*)
  (*Computation of  $\theta(s)$ , See Eq. (32) *)
  Pa=Function[s,Product[Gamma[1-a[[1,n,1]]-s a[[1,n,2]]],{n,1,Length[a[[1]]]}];
  Pb=Function[s,Product[Gamma[b[[1,n,1]]+s b[[1,n,2]]],{n,1,Length[b[[1]]]}];
  Qa=Function[s,Product[Gamma[a[[2,n,1]]+s a[[2,n,2]]],{n,1,Length[a[[2]]]}];
  Qb=Function[s,Product[Gamma[1-b[[2,n,1]]-s b[[2,n,2]]],{n,1,Length[b[[2]]]}];
  Theta=Function[s,Pa[s]Pb[s]/Qa[s]/Qb[s]];
  (*Countour limiters (Depends on numerator of  $\theta(s)$  *)
  var[Gamma[x_]]:=x;
  var[Times[x_,y_]]:={var[Times[x]],var[Times[y]]};
  RPoles=var[Pa[s]];
  LPoles=var[Pb[s]];
  ArrayPa=ConstantArray[0,Length[a[[1]]]];
  ArrayPb=ConstantArray[0,Length[b[[1]]]];
  If[Length[a[[1]]]==1,ArrayPa[[1]]=s/.Solve[RPoles==0,s],For[i=1,i<=Length[a[[1]]],
  i++,ArrayPa[[i]]=s/.Solve[RPoles[[i]]==0,s]];
  If[Length[b[[1]]]==1,ArrayPb[[1]]=s/.Solve[LPoles==0,s],For[j=1,j<=Length[b[[1]]],
  j++,ArrayPb[[j]]=s/.Solve[LPoles[[j]]==0,s]];
  (*Assignments*)
  epsilon=Max[ArrayPb]+0.1;
  R=Min[ArrayPa]-0.1;
  W=80;
  (*Evaluation*)
  Print["Numerical Integration:"];
  value=

$$\frac{1}{2\pi i} \text{NIntegrate}[\text{Theta}[s](z)^{-s},$$

  {s,epsilon-I W ,R-epsilon-I W,R-epsilon+I W ,epsilon+I W},
  MaxRecursion->55];
  (*Returning back the value*)
  Return[value];];

```

where $m, n, p, q \in \mathbb{N}^0$, with $0 \leq n \leq p, 1 \leq m \leq q, z \in \mathbb{C} \setminus \{0\}$. Here

$$\Theta(s) = \frac{\left\{ \prod_{j=1}^m \Gamma(b_j + B_j s) \right\} \left\{ \prod_{j=1}^n \Gamma(1 - a_j - A_j s) \right\}}{\left\{ \prod_{j=m+1}^q \Gamma(1 - b_j - B_j s) \right\} \left\{ \prod_{j=n+1}^p \Gamma(a_j + A_j s) \right\}}. \quad (\text{C13})$$

An empty product is always interpreted as unity, $A_i, B_j \in \mathbb{R}^+$, $a_i, b_j \in \mathbb{C}$, $i = 1, \dots, p$; $j = 1, \dots, q$. In addition, $C = (c - i\infty, c + i\infty)$ is a contour of integration separating the poles of $\Gamma(1 - a_j - A_j s)$, $j = 1, \dots, n$ from those of $\Gamma(b_j + B_j s)$, $j = 1, \dots, m$. On the other hand, the contour integral C in (C12) can be obtained by the sum of residues technique, evaluated at

all poles of $\Theta(s)^{24}$. Hence,

$$\frac{1}{2\pi i} \int_C \Theta(s) z^{-s} ds = \sum_{h=0}^{\infty} \lim_{s \rightarrow \pm \chi(h)} (s \pm \chi(h)) \Theta(s) z^{-s}, \quad (C14)$$

where $\chi(h)$ is a specific pole of $\Theta(s)$. Now, using (C12) and (C13), H_2 in (9) can be rewritten as

$$H_2 = \frac{1}{2\pi i} \int_C \frac{\Gamma(s) \Gamma(\mu_1 - s) \Gamma(k\mu_1 + \mu_2 - ks) z^{-s} ds}{\Gamma(1 + \mu_1 - s)}, \quad (C15)$$

where the suitable contour C separates all the poles of $\Gamma(s)$ to the left from those of $\Gamma(\mu_1 - s)$ and $\Gamma(k\mu_1 + \mu_2 - ks)$ to the right. Then, we can evaluate (C15) as the sum of residues, as follows

$$H_2 = S_1 + S_2, \quad (C16)$$

where we have split the analysis of the Fox H-function given in (C15) into two sums of residues¹, according to the following ranges of values of k : (i) $\chi(h) = -h$, for $k \leq 1$; (ii) $\chi(h) = \mu_1 + h$ and $\chi(h) = \frac{k\mu_1 + \mu_2 + h}{k}$, for $k \geq 1$. Now, by using (C14) and the condition for $k \leq 1$ into (C15), the term S_1 can be formulated as

$$\begin{aligned} S_1 &= \sum_{h=0}^{\infty} \lim_{s \rightarrow -h} \frac{\Gamma(s) \Gamma(\mu_1 - s) \Gamma(k\mu_1 + \mu_2 - ks)}{(s + h)^{-1} \Gamma(1 + \mu_1 - s) z^s} \\ &= \sum_{h=0}^{\infty} \frac{z^h \Gamma(k(h + \mu_1) + \mu_2)}{(-1)^h (h + \mu_1) \Gamma(1 + h)}. \end{aligned} \quad (C17)$$

Likewise, by using (C14) and the condition for $k \geq 1$ into (C15), the term S_2 can be expressed as

$$\begin{aligned} S_2 &= - \sum_{h=0}^{\infty} \lim_{s \rightarrow h + \mu_1} \frac{\Gamma(s) \Gamma(\mu_1 - s) \Gamma(k\mu_1 + \mu_2 - ks)}{(s - h - \mu_1)^{-1} \Gamma(1 + \mu_1 - s) z^s} - \sum_{h=0}^{\infty} \lim_{s \rightarrow \frac{k\mu_1 + \mu_2 + h}{k}} \frac{\Gamma(s) \Gamma(\mu_1 - s) \Gamma(k\mu_1 + \mu_2 - ks)}{\left(s - \frac{k\mu_1 + \mu_2 + h}{k}\right)^{-1} \Gamma(1 + \mu_1 - s) z^s} \\ &= \sum_{h=0}^{\infty} \frac{z^{-h - \mu_1} \Gamma(h + \mu_1) \Gamma(-hk + \mu_2)}{(-1)^{h-2} \Gamma(1 - h) h!} + \sum_{h=0}^{\infty} \frac{z^{-\frac{h}{k} - \mu_1 - \frac{\mu_2}{k}} \Gamma\left(\frac{-h - \mu_2}{k}\right) \Gamma\left(\frac{h + k\mu_1 + \mu_2}{k}\right)}{(-1)^{h-2} \Gamma\left(\frac{-h + k - \mu_2}{k}\right) k h!}. \end{aligned} \quad (C18)$$

By following a similar procedure as in the solution for H_2 , the series representation for H_1 and H_3 can be obtained as in (11) and (13), respectively.

D PROOF OF PROPOSITION 2

Let Y_i be the inverse of R_i , with PDF given by⁴³, Eq. (4)

$$f_{Y_i}(y) = \frac{\alpha_i \hat{r}_i^{\mu_i \alpha_i} y^{-1 - \alpha_i \mu_i}}{\mu_i^{\mu_i} \Gamma(\mu_i)} \exp\left(-\frac{\hat{r}_i^{\alpha_i}}{\mu_i y^{\alpha_i}}\right). \quad (D19)$$

From (D19), the n th moment $\mathbb{E}[Y_i^n]$ can be expressed as

$$\mathbb{E}[Y_i^n] = \hat{r}_i^n \frac{\Gamma(\mu_i - n/\alpha_i)}{\mu_i^{n/\alpha_i} \Gamma(\mu_i)}, \quad n > \mu_i \alpha_i. \quad (D20)$$

Next, substituting $\mathbb{E}[Y_1^n] = \mathbb{E}[R_1^{2n}]$ by (2), and $\mathbb{E}[Z^n] = \mathbb{E}[Y_2^{2n}]$ by (D20) into $\mathbb{E}[X^n] = \mathbb{E}[Y_1^n] \mathbb{E}[Z^n]$, we obtain (14), thus completing the proof.

¹It is worth mentioning that S_1 corresponds to the sum of residues with respect to the pole of $\Gamma(s)$. On the other hand, S_2 corresponds to the sum of residues regarding the poles of $\Gamma(\mu_1 - s)$ and $\Gamma(k\mu_1 + \mu_2 - ks)$.

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