ON THE PARABOLIC HARNACK INEQUALITY FOR NON-LOCAL DIFFUSION EQUATIONS

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ABSTRACT. We settle the open question concerning the Harnack inequality for globally positive solutions to non-local in time diffusion equations by constructing a counter-example for dimensions $d \geq \beta$, where $\beta \in (0, 2]$ is the order of the equation with respect to the spatial variable. The equation can be non-local both in time and in space but for the counter-example it is important that the equation has a fractional time derivative. In this case, the fundamental solution is singular at the origin for all times $t > 0$ in dimensions $d \geq \beta$. This underlines the markedly different behavior of time-fractional diffusion compared to the purely space-fractional case, where a local Harnack inequality is known.

The key observation is that the memory strongly affects the estimates. In particular, if the initial data $u_0 \in L^q_{loc}$ for q larger than the critical value $\frac{d}{\beta}$ of the elliptic operator $(-\Delta)^{\beta/2}$, a non-local version of the Harnack inequality is still valid as we show.

We also observe the *critical dimension phenomenon* already known from other contexts: the diffusion behavior is substantially different in higher dimensions than $d = 1$ provided $\beta > 1$, since we prove that the local Harnack inequality holds if $d < \beta$.

1. INTRODUCTION

We construct a counter-example for the local Harnack inequality for globally positive solutions to the time-fractional heat equation

$$\partial_t^\alpha (u - u_0) + (-\Delta)^{\beta/2} u = 0, \quad 0 < \alpha < 1, \quad 0 < \beta \leq 2, \tag{1.1}$$

in $(0, T) \times \mathbb{R}^d$ for $d \geq \beta$. We also show that the result is still valid in one dimension provided $\beta > 1$, at least for global solutions. Instead of the local formulation, a non-local version of the Harnack inequality is true also in higher dimensions. Indeed, the initial data plays a role, as the result remains true if the superharmonic function induced by the nonnegative initial data satisfies the Harnack principle. Thus, the memory has a substantial effect on the properties of the solutions, a phenomenon not seen in the case of the space-fractional diffusion

equation

$$\partial_t u + (-\Delta)^{\beta/2} u = 0. \tag{1.2}$$

In particular, if $u_0 \in L^q_{loc}$ for $q > \frac{d}{\beta}$, we prove a version of the classical local Harnack principle with the constant depending on u_0. We also observe the *critical dimension phenomenon*, which is not seen in the case of equation (1.2), but is known for time-fractional diffusion from [18, 19]. These results underline the fact that equation (1.1) has a substantially different behavior from that of (1.2). The phenomena we see, as well as the methods used, are often completely different from the ones for equation (1.2).

In the purely space-fractional case the Harnack inequality is known and was first proven by heat kernel estimates by Bass and Levin in [2] as well as Chen and Kumagai in [10]. Very recently a new proof for viscosity solutions was given by Chang-Lara and Dávila in [9]. In the variational formulation the question has been studied for kernels vanishing outside the local diagonal in [1] and for viscosity solutions when the Harnack constant is allowed to depend on the non-local tail in [22]. For elliptic theory we refer to the work of Caffarelli and Silvestre in [5, 6] and Kassmann in [17] and the references given therein. See also [4, 7].

The study of equation (1.1) is firmly rooted in applications. The time-fractional diffusion equation is closely related to a so-called continuous time random walk (CTRW) model for particle diffusion and it has become one of the standard physics approaches to model anomalous diffusion processes [14, 23, 25, 24]. For a detailed derivation of this equation from physics principles and for further applications of such models we refer to the expository review article of Metzler and Klafter in [25]. For the connection of (1.1) to the related stochastistic process we refer to monograph [23] and the extensive bibliography therein. Moreover, equations of the form (1.1) and nonlinear variants of them appear also in modelling of dynamic processes in materials with *memory*. Examples are given by the theory of heat conduction with memory, see [26] and the references therein, and the diffusion of fluids in porous media with memory [8, 15].

We point out that the non-local in time term in (1.1), with ∂_t^α being the Riemann-Liouville fractional derivation operator, coincides (for sufficiently smooth u) with the Caputo fractional derivative of u, see (2.1) below. The formulation with the Riemann-Liouville fractional derivative has the advantage that a priori less regularity is required on u to define the nonlocal operator. In particular, our formulation is exactly the one which naturally arises from physics applications, see for instance [25, equation (40)].

2. Preliminaries and main results

2.1. Notations and definitions. Let us first fix some notations. The Riemann-Liouville fractional integral of order $\alpha \geq 0$ is defined for $\alpha = 0$ as $J^0 := I$, where

I denotes the identity operator, and for $\alpha > 0$ as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau = (g_\alpha * f)(t),$$

where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

is the Riemann-Liouville kernel and $*$ denotes the convolution in time on the positive half-line. We denote the convolution in space by $\star$.

The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined by

$$\partial_t^\alpha f(t) = \frac{d}{dt} J^{1-\alpha} f(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau.$$

Observe that for sufficiently smooth f and $\alpha \in (0,1)$

$$\partial_t^\alpha (f - f(0))(t) = (J^{1-\alpha} f')(t) =: {}^c D_t^\alpha f(t), \tag{2.1}$$

the so-called Caputo fractional derivative of f. In case $\alpha = 1$, we have the standard time derivative.

Let

$$\widehat{u}(\xi) = \mathcal{F}(u)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx$$

and

$$\mathcal{F}^{-1}(u)(\xi) := \mathcal{F}(u)(-\xi)$$

denote the Fourier and inverse Fourier transform of u, respectively.

2.2. The fundamental solution. We recall that under suitable conditions on the initial data the classical solution of the Cauchy problem

$$\begin{cases} \partial_t^\alpha (u - u_0) + (-\Delta)^{\beta/2} u = 0 \\ \qquad\qquad\qquad u(0,x) = u_0(x) \end{cases} \tag{2.2}$$

is given by the representation formula

$$u(t,x) = \int_{\mathbb{R}^d} Z(t, x-y) u_0(y) \, dy \tag{2.3}$$

where Z is the fundamental solution of the problem. We call the function u defined by (2.3) the *mild solution* of the Cauchy problem (2.2) whenever the integral in (2.3) is well defined. We remark that under appropriate regularity conditions on the data, existence and uniqueness of strong L^p-solutions of the Cauchy problem (2.2) follows from the results in [27], which are formulated in the framework of abstract parabolic Volterra equations, see also the monograph [26].

The fundamental solution Z can be expressed in terms of Fox H-functions which are quite a general class of special functions. Without going to excessive details concerning the theory of Fox H-functions, for which we refer the reader to [20], we just give a very brief review, see also [19].

To simplify the notation we introduce

$$(a_i, \alpha_i)_{1,p} := ((a_1, \alpha_1), (a_2, \alpha_2), \dots , (a_p, \alpha_p))$$

for the set of parameters appearing in the definition of Fox H-functions, which are defined via Mellin-Barnes type integrals as

$$H_{pq}^{mn}(z) := H_{pq}^{mn}\left[z \Big|\, {}^{(a_i,\alpha_i)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}_{pq}^{mn}(s) z^{-s} ds,$$

where

$$\mathcal{H}_{pq}^{mn}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + \beta_j s)\prod_{i=1}^{n}\Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i + \alpha_i s)\prod_{j=m+1}^{q}\Gamma(1 - b_j - \beta_j s)}$$

is the Mellin transform of H_{pq}^{mn} and $\mathcal{L}$ is an infinite contour in the complex plane which separates the poles

$$b_{jl} = \frac{-b_j - l}{\beta_j} \quad (j = 1, \dots , m;\ l = 0, 1, 2, \dots)$$

of the Gamma function $\Gamma(b_j + \beta_j s)$ to the left of $\mathcal{L}$ and the poles

$$a_{ik} = \frac{1 - a_i + k}{\alpha_i} \quad (i = 1, \dots , n;\ k = 0, 1, 2, \dots)$$

of $\Gamma(1 - a_i - \alpha_i s)$ to the right of $\mathcal{L}$.

The fundamental solution satisfying the Cauchy problem (2.2) with the initial datum u_0 given by the Dirac delta distribution, $u_0(x) = \delta(x)$, can be represented in terms of a Fox H-function as

$$Z(t, x) = \pi^{-d/2}|x|^{-d} H_{23}^{21}\left[2^{-\beta} t^{-\alpha}|x|^{\beta} \Big|\, {}^{(1,1),\ (1,\alpha)}_{(\frac{d}{2},\frac{\beta}{2}),\ (1,1),\ (1,\frac{\beta}{2})}\right]. \tag{2.4}$$

Our analysis requires the sharp asymptotics of the fundamental solution, which is collected into the following lemma. For the proof we refer to [19, Lemma 3.3]. See also [11, equation (3.7)] and [20, Theorem 1.10].

Lemma 2.1. *Let $\alpha \in (0, 1]$, $\beta \in (0, 2]$ and $d \in \mathbb{Z}_+$. Denote $R := 2^{-\beta}|x|^{\beta} t^{-\alpha}$. The function Z has the following asymptotic behavior:*

(i) *If $R \leq 1$, then*

$$Z(t, x) \sim \begin{cases} t^{-\alpha d/\beta}, & \text{if } \alpha = 1, \text{ or } \beta > d \text{ and } 0 < \alpha < 1, \\ t^{-\alpha}(|\log(|x|^{\beta} t^{-\alpha})| + 1), & \text{if } \beta = d \text{ and } 0 < \alpha < 1, \\ t^{-\alpha}|x|^{-d+\beta} & \text{if } 0 < \beta < d \text{ and } 0 < \alpha < 1. \end{cases}$$

(ii) *If $R \geq 1$, then*

$$Z(t, x) \sim \begin{cases} t^{\alpha}|x|^{-d-\beta}, & \text{if } \beta < 2, \\ t^{-\frac{\alpha d}{2}} R^{\frac{d(\alpha-1)}{2(2-\alpha)}} e^{-\sigma R^{\frac{1}{2-\alpha}}}, & \text{if } \beta = 2, \end{cases}$$

where $\sigma = (2 - \alpha)\left(\alpha^{\alpha}/4\right)^{\frac{1}{2-\alpha}}$.

In Lemma 2.1 and in the sequel we use the notation

$$a \sim b \quad \text{if there exists a constant } c > 0 \text{ such that} \quad \frac{1}{c}a \leq b \leq ca.$$

Lemma 2.1 guarantees that we have not only upper bounds but also lower bounds for the fundamental solution Z. The upper and the lower bounds are the same up to some multiplicative constants. In our analysis we are not interested in the exact constants. Therefore we give our estimates in a form $a \leq Cb$, or even more briefly in a form $a \lesssim b$.

Although the Fox H-functions are rather complicated objects, the Fourier transform with respect to the spatial variable of the fundamental solution $Z(t, \cdot)$ takes a much simpler form. In fact,

$$\widehat{Z}(t, \xi) = (2\pi)^{-d/2} E_\alpha(-|\xi|^\beta t^\alpha), \tag{2.5}$$

where E_α is the Mittag-Leffler function, which is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}, \quad z \in \mathbb{C}.$$

It is known that for $0 < \alpha < 1$ the Mittag-Leffler function behaves as

$$E_\alpha(-x) \sim \frac{1}{1 + x}, \quad x \in \mathbb{R}_+. \tag{2.6}$$

For details we refer to [12, Chapter 18].

2.3. Main results. The question about a full, local Harnack inequality

$$\sup_{Q^-} u \leq C \inf_{Q^+} u, \tag{2.7}$$

for nonnegative solutions of equations of type (1.1) and for some time-lagged properly scaled cylinders Q^- and Q^+ has been a somewhat longstanding open question. The constant $C > 0$ in the classical Harnack estimate (2.7) should neither depend on the initial datum u_0 and the solution u nor on the scaling. It turns out that for dimensions $d \geq \beta$ this is not the case for the solution of (1.1). Our first result is the construction of a counter-example for such a Harnack principle. We have the following theorem.

Theorem 2.2. *Let* $\alpha \in (0, 1)$, $\beta \in (0, 2]$ *and* $d \geq \beta$. *There exists a sequence* $(u_n)_{n=1}^{\infty}$ *of smooth, positive solutions to equation* (1.1) *in* $(0, \infty) \times \mathbb{R}^d$ *(with* $u_n|_{t=0}$ *instead of* u_0*) such that the ratio*

$$\frac{u_n(t_1, 0)}{u_n(t_2, x_0)} \to \infty \quad \text{as } n \to \infty,$$

for each pair of time levels $t_1, t_2 > 0$ *and for each* $x_0 \neq 0$. *This contradicts the local Harnack inequality* (2.7).

The proof of Theorem 2.2 is based on the detailed asymptotics of the fundamental solution Z. We also provide another counter-example, which uses the Fourier transform.

Theorem 2.3. *Let $\alpha \in (0,1)$ and $\beta \in (0,2]$. There exists a sequence $(u_n)_{n=1}^\infty$ of smooth, positive solutions to equation (1.1) in $(0,\infty) \times \mathbb{R}^d$ (with $u_n|_{t=0}$ instead of u_0) such that for all $t_1, t_2 > 0$ there holds*

$$\frac{u_n(t_1, 0)}{\|u_n(t_2, \cdot)\|_{L^p(\mathbb{R}^d)}} \to \infty \quad as\ n \to \infty,$$

if

    (i) $d > \beta$ and $1 \le p < \frac{d}{\beta}$, or

    (ii) $d = \beta$ and $p = 1$.

In the classical local case the proof of the Harnack inequality consists of the following steps:

    (a) boundedness of subsolutions ('$L^\infty - L^p-$ estimate'),

    (b) the weak Harnack inequality for supersolutions ('$L^p - L^{-\infty}-$ estimate').

In [29] the weak Harnack inequality is proved in the case $\beta = 2$ for all spatial dimensions d and all positive $p < \frac{2+d\alpha}{2+d\alpha-2\alpha}$, with the upper bound for p being optimal. In [16] the weak Harnack inequality is proved in the case $\frac{d}{2} < \beta < 2$ for the spatial dimensions $d = 2, 3$. Theorem 2.3 reveals that the boundedness estimate not only of subsolutions but also of solutions fails in the cases

    (1) $1 < \beta \le 2$ and $d \ge 2$,

    (2) $0 < \beta \le 1$ and $d \ge 1$.

The boundedness of solutions is an open problem in the case $1 < \beta \le 2$ and $d = 1$. Hence for subdiffusion there are cases in which the weak Harnack inequality is true but the boundedness of solutions fails. Apart from the cases $\beta = 2$ and $d \ge 1$ or $\frac{d}{2} < \beta < 2$ and $d = 2, 3$ even the weak Harnack inequality is open.

Concerning the Harnack inequality for non-local diffusion equations, there are some positive results. In [9] the Harnack inequality is proved for a generalization of (1.2) in the case $1 \le \beta < 2$. For the non-local in time equations, the only positive result up to now is the Harnack inequality for the Riemann-Liouville fractional derivation operator proved in [28], which can be regarded as the case '$d = 0$'. The case $0 < \alpha < 1$ in dimensions $d \ge 1$ has been an open problem until now. We show in Theorem 2.7 that the Harnack inequality remains to hold for (1.1) for $0 < \alpha < 1$ in the one-dimensional case provided $\beta > 1$. Having positive results, one could think that the Harnack inequality could hold also for the whole range of parameters α and β, but this is not the case as Theorem 2.2 shows in dimensions $d \ge \beta$.

The local Harnack inequality being settled for dimensions $d \ge \beta$, it is a natural question, what is a proper replacement of this result. Unlike for most non-local equations, the non-local memory term in the time-fractional diffusion equation

has a substantial effect on the behavior of solutions. Indeed, large enough historical events affect the solution indefinitely for all positive times. From the construction of Theorem 2.2 it is evident that if the initial data is pathological enough, then the solution will never satisfy a local Harnack inequality. If the initial data, however, is good enough, then a Harnack principle still holds.

Moreover, the theory is also very different depending on the dimension. In one dimension, we are still able to prove the classical Harnack inequality for global solutions, but in higher dimensions the counter-example prevails. This corresponds to the *critical dimension phenomenon* observed in the case of the decay estimates for the equation (1.1) in [18]. See also [19].

For dimensions $d \geq \beta$ the equation (1.1) preserves the properties of the initial data in the sense that if the superharmonic function induced by the nonnegative initial data satisfies a Harnack principle, also the solution of the Cauchy problem will have a similar behavior. For making this rigorous, we consider the potential

$$u \mapsto G \star u$$

with the kernel

$$G(t, x) = \frac{1}{c(d, \beta)} \begin{cases} |x|^{\beta - d}, & \text{for } d > \beta \\ \log |x|, & \text{for } d = \beta, \end{cases} \tag{2.8}$$

where $c(d, \beta)$ is a normalization constant. Note that for $\beta = 2$ the function G is nothing but the Newtonian kernel and the corresponding potential is the Newtonian potential. Now $G \star u_0$ induces a superharmonic function in $\mathbb{R}^d$, that is, setting $v = G \star u_0$ for a nonnegative and sufficiently smooth initial data u_0 we obtain that $(-\Delta)^{\beta/2} v \geq 0$ in $\mathbb{R}^d$.

We have the following theorem. Here by $\chi_A(x)$ we denote the characteristic function of the set $A \subset \mathbb{R}^d$.

Theorem 2.4. *Let $\alpha \in (0, 1)$, $\beta \in (0, 2]$ and $r > 0$. Suppose u is a mild solution of the Cauchy problem (2.2) in $[0, T] \times \mathbb{R}^d$ with the nonnegative and sufficiently smooth initial data $u_0 \not\equiv 0$. Suppose further that t_1 and t_2 satisfy condition*

$$(2r)^{\beta/\alpha} \leq t_1 < t_2 \leq t_1 + (2r)^{\beta/\alpha} \leq T. \tag{2.9}$$

Then there exists a constant $C = C(d, \alpha, \beta, t_2/t_1) > 0$ such that for all $x_1, x_2 \in B_r(0)$ there holds

$$u(t_1, x_1) \leq C \left[1 + \frac{[G \star (u_0 \chi_{B_{r_1}(x_1)})](x_1)}{[G \star (u_0 \chi_{B_{r_2}(x_2)})](x_2)} \right] u(t_2, x_2), \quad \text{if } d > \beta,$$

and, if $d = \beta$,

$$u(t_1, x_1) \leq C \left[1 + \frac{[(c + \alpha \log t_1 + \beta c(d, \beta)G) \star (\chi_{B_{r_1}(x_1)} u_0)](x_1)}{[(c + \alpha \log t_2 + \beta c(d, \beta)G) \star (\chi_{B_{r_2}(x_2)} u_0)](x_2)} \right] u(t_2, x_2),$$

where $c = 1 + \beta \log 2$, $c(d, \beta)$ is the constant in (2.8) *and $r_i = 2t_i^{\alpha/\beta}, i = 1, 2$.*
Moreover, the constant C blows up, as $t_2 \to t_1$, when $\beta = 2$.

Remark 2.5. Observe that by elliptic regularity theory, if $u_0 \in L_{loc}^q(\mathbb{R}^d)$ with
$q > \frac{d}{\beta}$, then $G \star (u_0 \chi_{B_{r_1}(x_1)})$ is locally bounded and continuous. In this case the
convolution factor above is always bounded as long as $B_{r_1}(x_1) \subset B_{r_2}(x_2)$ which
is true for all $(t_1, x_1), (t_2, x_2)$ satisfying

$$t_2^{\alpha/\beta} \geq t_1^{\alpha/\beta} + \frac{|x_1 - x_2|}{2}$$

and (2.9). Therefore, the Harnack inequality is always satisfied for such an initial
data after a proper time-lag corresponding to the scaling of the equation.

We would also like to point out that the proof of the above Theorem is rather
sharp consisting almost exclusively of identities rather than estimates. Indeed,
the crucial estimates rely on the asymptotic behavior of the fundamental solution
and these estimates provide sharp behavior not only as upper bounds, but also
as lower bounds.

In the above formulation the effect of the initial data is readily observable.
The superharmonic function induced by the initial data fully determines the
Harnack constant. Moreover, it is easy to see that if this superharmonic function
satisfies a proper Harnack inequality, then this is true also for the solution of the
evolution equation. In particular, if one is willing to make some more additional
assumptions on the initial data, we are able to obtain a more classical formulation
of the Harnack inequality in the form of the following corollary.

Corollary 2.6. *Let the assumptions of Theorem* 2.4 *hold. Suppose also that the
initial data satisfies the Harnack inequality*

$$\sup_{B_{r_1}(x_1)} u_0 \leq H_0 \inf_{B_{r_2}(x_2)} u_0, \tag{2.10}$$

for $x_1, x_2 \in B_r(0)$ and for some constant H_0. Then there exists a constant $C = C(d, \alpha, \beta, H_0, t_2/t_1) > 0$ such that there holds

$$u(t_1, x_1) \leq Cu(t_2, x_2).$$

In the case $\beta > d$ the situation is simpler as the counter-example we build in
Theorem 2.2 is not valid anymore. In this case the Harnack inequality (2.11)
takes the standard form. We have the following theorem. We formulate the
result in a form, which covers also the space-fractional case (1.2), since $\alpha = 1$ is
included in Lemma 2.1.

Theorem 2.7. *Let $\alpha \in (0, 1], \beta \in (0, 2), r > 0$ and $x_1, x_2 \in B_r(0)$. Assume that
either*

   (a) *$\alpha \neq 1$ and $1 = d < \beta$, or*
   (b) *$\alpha = 1$, $\beta \neq 2$ and $d \in \mathbb{Z}_+$.*

Suppose u is a mild solution of the Cauchy problem (2.2) *in* $[0, T] \times \mathbb{R}^d$. *Then for all t_1, t_2 satisfying*

$$(2r)^{\beta/\alpha} \le t_1 < t_2 \le t_1 + (2r)^{\beta/\alpha} \le T.$$

there exists a constant $C = C(t_2/t_1, \alpha, \beta) > 0$ such that

$$u(t_1, x_1) \le Cu(t_2, x_2), \tag{2.11}$$

Moreover, the constant blows up, as $t_2 \to t_1$, when $\beta = 2$.

Remark 2.8. (i) The proofs of Theorems 2.4 and 2.7 reveal that the time-lag is needed only for handling the exponential term, which is missing in the case $\beta \ne 2$. In particular, the time lag is not needed in the case $\alpha = 1$ and $\beta \ne 2$, which was observed also in [3]. Heuristically this means that the diffusion is so fast that a lot of heat is diffused far away from the source at all times $t > 0$. In the probabilistic framework this is a consequence of a fat tail at infinity of the probability distribution $Z(t, \cdot)$.

(ii) In the fully non-local case $\alpha \ne 1$ and $\beta \ne 2$ the diffusion is at the same time slow and fast. It is slow, since the fundamental solution has singularity at $x = 0$ for all times $t > 0$. On the other hand, it is fast since the fundamental solution $Z(t, \cdot)$ has a fat tail at infinity.

3.   THE COUNTER-EXAMPLES

We begin with the proof of Theorem 2.2. The proof relies on the properties of the fundamental solution. Indeed, the fundamental solution is not a smooth function and is merely superparabolic instead of being a proper solution of the problem. This resembles the elliptic Newtonian potential and the case of some nonlinear parabolic equations, such as the porous medium equation (and the corresponding Barenblatt solution), which have been used to model similar phenomena as equation (1.1). Equation (1.1) can be viewed as interpolation between the parabolic and elliptic cases.

We need the sharp behavior of the fundamental solution given in Lemma 2.1. It seems that in the literature only upper bounds (see, e.g. [11, Proposition 1] and [21]) are available although the sharp asymptotics of the Fox H-functions is known. For the convenience of the reader we provide here an easy argument that the fundamental solution is actually strictly positive on the whole space $\mathbb{R}^d$ for all times $t > 0$. It is well-known that the fundamental solution is a probability density and hence nonnegative everywhere for all $t > 0$ but it does not exclude the possibility of the fundamental solution having zeros at some points.

We first prove the following auxiliary result. The proof is based on the Mellin transform of the Fox H-function appearing in the representation of the fundamental solution and the calculus of residues. For details we refer to [19, 20].

Lemma 3.1. *Let $\alpha \in (0, 1]$ and $\beta \in (0, 2]$. The fundamental solution $Z(t, \cdot)$ given by* (2.4) *is radially decreasing.*

Proof. We proceed similarly as in the proof of [19, Lemma 4.7].

The fundamental solution given by (2.4) is clearly radial. With a slight abuse of notation we denote $Z(t, x) = Z(t, r)$, where $r = |x|$. Using the differentiation rule [19, Lemma 2.14] of the Fox H-functions we have

$$\frac{\partial}{\partial r} Z(t,r) = \pi^{-d/2} r^{-d-1} [\beta H_{34}^{22}(2^{-\beta} t^{-\alpha} r^\beta) - d H_{23}^{21}(2^{-\beta} t^{-\alpha} r^\beta)], \qquad (3.1)$$

where we have noted that Z is a radial function and denoted $r = |x|$, and we have omitted the parameters appearing in the H-functions.

We may combine the linear combination

$$\beta H_{34}^{22}(z) - d H_{23}^{21}(z) \qquad (3.2)$$

as follows. Let $\mathcal{H}_{23}^{21}$ and $\mathcal{H}_{34}^{22}$ be the Mellin transforms of the functions H_{23}^{21} and H_{34}^{22} appearing in (3.1). Using the definition of the Fox H-function [19, Formulae (2.10) and (2.11)] we have

$$\beta \mathcal{H}_{34}^{22}(s) - d \mathcal{H}_{23}^{21}(s) = \beta \frac{\Gamma(\frac{d}{2} + \frac{\beta}{2} s)\Gamma(1+s)\Gamma(1-s)}{\Gamma(1+\alpha s)\Gamma(-\frac{\beta}{2} s)} - d \frac{\Gamma(\frac{d}{2} + \frac{\beta}{2} s)\Gamma(1+s)\Gamma(-s)}{\Gamma(1+\alpha s)\Gamma(-\frac{\beta}{2} s)}.$$

Using the property $\Gamma(z+1) = z\Gamma(z)$ of the Gamma function we may proceed as

$$\begin{aligned}
\beta \mathcal{H}_{34}^{22}(s) - d \mathcal{H}_{23}^{21}(s) &= -2\left(\frac{d}{2} + \frac{\beta}{2} s\right) \frac{\Gamma(\frac{d}{2} + \frac{\beta}{2} s)\Gamma(1+s)\Gamma(-s)}{\Gamma(1+\alpha s)\Gamma(-\frac{\beta}{2} s)} \\
&= -2 \frac{\Gamma(\frac{d+2}{2} + \frac{\beta}{2} s)\Gamma(1+s)\Gamma(-s)}{\Gamma(1+\alpha s)\Gamma(-\frac{\beta}{2} s)} \\
&= -2\mathcal{H}_{23}^{21}\left[s \Big|_{(\frac{d+2}{2}, \frac{\beta}{2}),\ (1,1),\ (1,\frac{\beta}{2})}^{(1,1),\quad (1,\alpha)}\right],
\end{aligned}$$

which implies

$$\frac{\partial}{\partial r} Z(t,r) = -2\pi^{-d/2} r^{-d-1} H_{23}^{21}\left[2^{-\beta} t^{-\alpha} r^\beta \Big|_{(\frac{d+2}{2}, \frac{\beta}{2}),\ (1,1),\ (1,\frac{\beta}{2})}^{(1,1),\quad (1,\alpha)}\right].$$

which is nothing but

$$-2\pi r Z_{d+2}(t,r),$$

where the subscript $d+2$ means that Z_{d+2} is the fundamental solution $Z(t, \cdot)$ in dimensions $d + 2 \geq 3$. Since $Z(t, \cdot)$ is a probability density, the result follows. $\square$

Corollary 3.2. *The fundamental solution $Z(t, r)$ has no positive zeros $r \in \mathbb{R}_+$.*

Proof. Since $r \mapsto Z(t, r)$ is decreasing and nonnegative in $\mathbb{R}_+$, it cannot have zeros on the positive real axis. Otherwise $Z(t, \cdot)$ would have compact support, which would imply that the analytic function $H_{23}^{21}(z)$, $z \neq 0$, would have zeros in a continuum, which is impossible. $\square$

Having Lemma 3.1 and Corollary 3.2 in our hands, we can use the sharp behavior of the fundamental solution given by the asymptotics formulated in terms of the similarity variable

$$R = \frac{|x|^\beta}{2^\beta t^\alpha}.$$

Note that also the classical heat kernel contains a function depending on R with $\alpha = 1$ and $\beta = 2$.

Proof of Theorem 2.2. First of all, we note that by scaling it is enough to consider $x_0 \in \partial B(0,1)$. Let $\psi(x)$ be the standard mollifier

$$\psi(x) := \begin{cases} c_d e^{-\frac{1}{1-|x|^2}}, & \text{for } |x| < 1 \\ 0, & \text{for } |x| \geq 1, \end{cases}$$

where c_d is a dimension-dependent normalization constant to guarantee the integral of ψ to equal one. We choose $u_0^\varepsilon(x) = \varepsilon^{-d}\psi(\frac{x}{\varepsilon})$ as the initial data for our problem. We will find a lower bound for the left hand side of (2.7), which will tend to infinity as $\varepsilon \to 0$, and an upper bound of the right hand side of (2.7), which stays bounded uniformly in ε. We start with the lower bound, where we consider the cases $d > \beta$ and $d = \beta$ separately, since the asymptotics of the fundamental solution is different in these cases.

The lower bound. Case: $d > \beta$. Let ψ and u_0^ϵ be as above and let $t > 0$ be fixed. Then

$$u^\varepsilon(t,0) = \int_{\mathbb{R}^d} Z(t,y)u_0^\varepsilon(y)dy = \int_{\mathbb{R}^d} Z(t,\varepsilon z)\psi(z)dz,$$

where we made the change of variables $y \leftrightarrow \varepsilon z$. Take now

$$R_0 = \min\{t^{\alpha/\beta}, \frac{1}{2}\},$$

whence for $|z| \leq R_0$ there holds

$$R := \frac{|z|^\beta}{2^\beta t^\alpha} \leq \frac{1}{2^\beta} \quad \text{and} \quad \psi(z) \geq e^{-\frac{4}{3}}c_d.$$

Note that $R \leq \frac{1}{2^\beta}$ implies $R_\varepsilon := \frac{|\varepsilon z|^\beta}{2^\beta t^\alpha} \leq \frac{1}{2^\beta}$ for all $\varepsilon \leq 1$. Hence the estimate

$$Z(t,x) \sim t^{-\alpha}|x|^{\beta-d}, \quad R \leq \frac{1}{2^\beta}$$

given by Lemma 2.1 allows us to estimate u^ϵ from below as

$$u^\varepsilon(t,0) \gtrsim \int_{|z| \leq R_0} Z(t,\varepsilon z)dz \gtrsim \varepsilon^{\beta-d}t^{-\alpha}\int_{|z| \leq R_0} |z|^{\beta-d}dz.$$

Since $z \mapsto |z|^{\beta-d}$ is locally integrable,

$$u^\varepsilon(t,0) \gtrsim \varepsilon^{\beta-d} \to \infty, \quad \varepsilon \to 0.$$

The lower bound. Case: $d = \beta$. We proceed as above. Again, Lemma 2.1 gives the estimate

$$Z(t,z) \sim t^{-\alpha}(1 + |\log(R)|) \sim -t^{-\alpha}\log(R), \quad R \leq \frac{1}{2^\beta},$$

which implies

$$u^\varepsilon(t,0) \gtrsim \int_{|z| \leq R_0} Z(t, \varepsilon z)dz \gtrsim t^{-\alpha} \int_{|z| \leq R_0} \left| \log\left(\frac{|\varepsilon z|^\beta}{2^\beta t^\alpha} \right) \right| dz.$$

Substituting $y = \frac{\varepsilon z}{2t^{\alpha/\beta}}$ in the last integral gives the lower bound

$$u^\varepsilon(t,0) \gtrsim \varepsilon^{-d} \int_{|y| \leq \frac{\varepsilon R_0}{2t^{\alpha/\beta}}} |\log(|y|)| dy.$$

But the integral on the right hand side behaves like

$$\varepsilon^d |\log \varepsilon| \quad \text{as } \varepsilon \to 0,$$

whence

$$u^\varepsilon(t,0) \to \infty \quad \text{as } \varepsilon \to 0.$$

The upper bound. We consider a point $x_0 \in \partial B_1(0)$. We have

$$u^\varepsilon(t, x_0) = \int_{\mathbb{R}^d} Z(t, x_0 - y)u_0^\varepsilon(y)\, dy$$

$$= \varepsilon^{-d} \int_{B_\varepsilon(0)} Z(t, x_0 - y)\psi\left(\tfrac{y}{\varepsilon}\right) dy$$

$$= \int_{B_1(0)} Z(t, x_0 - \varepsilon z)\psi(z)\, dz.$$

For $0 < \varepsilon \leq \frac{1}{2}$, we have that $x_0 - \varepsilon z \in B_{1/2}^c(0)$ for $z \in \partial B_1(0)$. Since $Z(t, \cdot)$ is continuous and bounded in $B_{1/2}^c(0)$ uniformly with respect to ε and ψ is integrable, we have by the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon \to 0} u^\varepsilon(t, x_0) = \int_{B_1(0)} \lim_{\varepsilon \to 0} Z(t, x_0 - \varepsilon z)\psi(z)\, dz = Z(t, x_0) < \infty,$$

for all $t > 0$. Since for all $t > 0$ we have that $u^\varepsilon(t,0) \to \infty$ as $\varepsilon \to 0$, the Harnack inequality cannot hold even after an arbitrarily long time lag. This finishes the proof. $\qquad\square$

Proof of Theorem 2.3. Here we do not need the exact form of the fundamental solution in terms of a quite complicated H-functions. Instead, working in the frequency domain we can use the simpler Mittag-Leffler function E_α.

Let $u_0(x) = e^{-|x|^2}$ and $u_0^n(x) = n^{d/p}u_0(nx)$. Then by [19, Theorem 2.12] the function

$$u_n(t, x) = (Z(t, \cdot) \star u_0^n)(x) = \int_{\mathbb{R}^d} Z(t, x - y)u_0^n(y)\, dy$$

is a classical solution of (1.1). By using the Fourier transform and (2.5) we may represent $u_n(t,0)$ in the form

$$
\begin{aligned}
u_n(t,0) &= (2\pi)^{-d/2}\mathcal{F}^{-1}\left(E_\alpha(-|\xi|^\beta t^\alpha)\widehat{u_0^n}(\xi)\right)(0) \\
&= (2\pi)^{-d}n^{\frac{d}{p}-d}\int_{\mathbb{R}^d}E_\alpha(-|\xi|^\beta t^\alpha)\widehat{u_0}(\xi/n)\,d\xi \\
&= (2\pi)^{-d}n^{\frac{d}{p}}\int_{\mathbb{R}^d}E_\alpha(-|n\xi|^\beta t^\alpha)\widehat{u_0}(\xi)\,d\xi.
\end{aligned}
$$

By (2.6) we have a lower bound

$$
E_\alpha(-\rho) \geq \frac{c_1}{1+\rho}, \quad \rho \geq 0,
$$

for some positive constant c_1. Moreover, since the Fourier transform of the Gaussian is also Gaussian, we have $\widehat{u_0} > 0$ in $B(0,2\delta)$ with $\delta = t^{-\alpha/\beta}$ and fixed $t > 0$. Hence for some positive constant $c = c(t,\alpha,\beta,d)$ there holds

$$
u_n(t,0) \geq cn^{\frac{d}{p}}\int_{B(0,2\delta)}\frac{d\xi}{1+|n\xi|^\beta t^\alpha} \gtrsim n^{\frac{d}{p}-\beta}\int_{\frac{\delta}{n}}^{2\delta}r^{d-1-\beta}dr,
$$

where in the last step we introduced the spherical coordinates. This implies

$$
u_n(t,0) \gtrsim
\begin{cases}
n^{\frac{d}{p}-\beta}, & \text{if } d > \beta, \\
n^{\frac{d}{p}-\beta}\log(n), & \text{if } d = \beta.
\end{cases}
$$

In particular, $u_n(t,0) \to \infty$, if (i) $p < \frac{d}{\beta}$ and if (ii) $p = 1$ and $d = \beta$. On the other hand, Young's inequality for convolutions [13, Theorem 1.2.10] and the fact that $Z(t,\cdot)$ is a probability distribution function for all $t > 0$ (see [19]) imply

$$
\|u_n(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq \|Z(t,\cdot)\|_{L^1(\mathbb{R}^d)}\|u_0^n\|_{L^p(\mathbb{R}^d)} = \|u_0\|_{L^p(\mathbb{R}^d)} < \infty,
$$

which finishes the proof. $\qquad\square$

4. Harnack inequality

Proof of Theorem 2.4. For the Harnack inequality, we need to consider the value of the solution in two separate points, and for this purpose, we denote

$$
R_1 := R_1(y) = \frac{|x_1-y|^\beta}{2^\beta t_1^\alpha} \quad \text{and} \quad R_2 := R_2(y) = \frac{|x_2-y|^\beta}{2^\beta t_2^\alpha}.
$$

We split the integral defining the mild solution into two parts and have

$$u(t_1, x_1) = \int_{\mathbb{R}^d} Z(t_1, x_1 - y)u_0(y)\,dy$$

$$= \int_{\{R_1 > 1\}} Z(t_1, x_1 - y)u_0(y)\,dy + \int_{\{R_1 \leq 1\}} Z(t_1, x_1 - y)u_0(y)\,dy$$

$$=: I_1 + I_2. \tag{4.1}$$

We will use the asymptotics of Z given in Lemma 2.1. We start with the integral I_1 and consider only the case $\beta = 2$, since this case is more difficult. The case $\beta \neq 2$ follows from the argument for $\beta = 2$, since for $\beta \neq 2$ there is no exponential function in the asymptotics for large values of the similarity variable R_1 and we only need the estimate for the ratio

$$\frac{|x_1 - y|}{|x_2 - y|}$$

given below.

By Lemma 2.1 we have

$$Z(t, x) \sim t^{-\frac{\alpha d}{2}} R^{\frac{d(\alpha - 1)}{2(2 - \alpha)}} e^{-\sigma R^{\frac{1}{2 - \alpha}}}, \quad R \geq 1,$$

for a constant $\sigma = \alpha^{\frac{\alpha}{2 - \alpha}}$. Observe that for

$$x_1, x_2 \in B_r(0), \quad R_1 \geq 1 \quad \text{and} \quad t_1 \geq (2r)^{2/\alpha}$$

there holds

$$R_2 = \frac{|x_2 - y|^2}{4t_2^\alpha} \geq \frac{(|x_1 - y| - |x_1 - x_2|)^2}{4t_2^\alpha}$$

$$\geq \frac{(2t_1^{\alpha/2} - 2r)^2}{4t_2^\alpha}$$

$$\geq \frac{1}{4}\left(\frac{t_1}{t_2}\right)^\alpha,$$

which implies

$$\{R_1 \geq 1\} \subset \left\{R_2 \geq \frac{1}{4}\left(\frac{t_1}{t_2}\right)^\alpha\right\}. \tag{4.2}$$

We obtain

$$I_1 = \int_{\{R_1 > 1\}} Z(t_1, x_1 - y)u_0(y)\,dy$$

$$\lesssim t_1^{-\frac{\alpha d}{2}} \int_{\{R_1 > 1\}} R_1^{\frac{d(\alpha - 1)}{2(2 - \alpha)}} e^{-\sigma R_1^{\frac{1}{2 - \alpha}}} u_0(y)\,dy$$

$$= t_1^{-\frac{\alpha d}{2}} \int_{\{R_1 > 1\}} \left(\frac{R_1}{R_2}\right)^{\frac{d(\alpha - 1)}{2(2 - \alpha)}} e^{\sigma\left(R_2^{\frac{1}{2 - \alpha}} - R_1^{\frac{1}{2 - \alpha}}\right)} R_2^{\frac{d(\alpha - 1)}{2(2 - \alpha)}} e^{-\sigma R_2^{\frac{1}{2 - \alpha}}} u_0(y)\,dy.$$

Next we estimate the exponential factor from above. If $R_2 \leq R_1$, there is nothing to prove, and for $R_2 \geq R_1$ we obtain for $\alpha \in (0,1]$ that

$$R_2^{\frac{1}{2-\alpha}} - R_1^{\frac{1}{2-\alpha}} \leq (R_2 - R_1)^{\frac{1}{2-\alpha}}$$

$$= \left(\frac{|x_2 - y|^2}{4t_2^\alpha} - \frac{|x_1 - y|^2}{4t_1^\alpha} \right)^{\frac{1}{2-\alpha}}$$

$$= \left(\frac{t_1^\alpha |x_2 - y|^2 - t_2^\alpha |x_1 - y|^2}{4t_2^\alpha t_1^\alpha} \right)^{\frac{1}{2-\alpha}}$$

$$= \left(\frac{(t_1^\alpha - t_2^\alpha)|y|^2 - 2y \cdot (t_1^\alpha x_2 - t_2^\alpha x_1) + t_1^\alpha |x_2|^2 - t_2^\alpha |x_1|^2}{4t_2^\alpha t_1^\alpha} \right)^{\frac{1}{2-\alpha}}$$

Since $(2r)^{2/\alpha} \leq t_1 < t_2 \leq t_1 + (2r)^{2/\alpha} \leq 2t_1$ and $x_1, x_2 \in B_r(0)$, we can estimate

$$\frac{t_1^\alpha |x_2|^2 - t_2^2 |x_1|^2}{4t_2^\alpha t_1^\alpha} \leq \frac{r^2}{4t_2^\alpha} \leq \frac{1}{16}.$$

It remains to estimate the sum of the first two terms in the brackets. We estimate the numerator of the second term as

$$|y \cdot (t_1^\alpha x_2 - t_2^\alpha x_1)| \leq |y|(1 + 2^\alpha) r t_1^\alpha \leq 3|y| r t_1^\alpha.$$

Denote $r = |y|$ and study the parabola

$$r \mapsto -ar^2 + br$$

with

$$a = \frac{t_2^\alpha - t_1^\alpha}{4t_1^\alpha t_2^\alpha} \quad \text{and} \quad b = \frac{6r}{4t_2^\alpha}.$$

Since the maximum is attained at $r = \frac{b}{2a}$, the upper bound for the sum of the first two terms is

$$\frac{b^2}{4a} = \frac{36 t_1^\alpha r^2}{16 t_2^\alpha (t_2^\alpha - t_1^\alpha)} \leq \frac{3r^2}{t_2^\alpha - t_1^\alpha} \leq \frac{3t_1^\alpha}{4(t_2^\alpha - t_1^\alpha)} = \left(\left(\frac{t_2}{t_1} \right)^\alpha - 1 \right)^{-1}. \qquad (4.3)$$

Next we estimate the power term. It is enough to estimate the ratio

$$\frac{R_1}{R_2} = \left(\frac{t_2}{t_1} \right)^\alpha \frac{|x_1 - y|^2}{|x_2 - y|^2}.$$

Since the exponent is negative, we need a lower bound. In the set $\{R_1 \geq 1\}$ there holds

$$|x_2 - y| \leq |x_1 - x_2| + |x_1 - y| \leq 2r + |x_1 - y| \leq t_1^{\alpha/2} + |x_1 - y| \leq \frac{3}{2}|x_1 - y|, \quad (4.4)$$

which gives the desired lower bound.

Altogether we obtain

$$I_1 \le C t_2^{-\frac{\alpha d}{2}} \int_{\{R_2 \ge \frac{1}{16}\}} R_2^{\frac{d(\alpha-1)}{2(2-\alpha)}} e^{-\sigma R_2^{\frac{1}{2-\alpha}}} u_0(y)\, dy$$

$$\le C \int_{\{R_2 \ge \frac{1}{16}\}} Z(t_2, x_2 - y) u_0(y)\, dy, \tag{4.5}$$

where we used the inclusion (4.2) and estimated

$$\left(\frac{t_1}{t_2}\right)^\alpha \ge \frac{t_1}{t_2} \ge \frac{1}{2}.$$

The proof shows that C depends only on d, α and t_2/t_1. Moreover, we see from (4.3) that the constant C given by the proof blows up as $t_2 \to t_1$.

For I_2 we divide the treatment into two separate cases depending on the dimension.

Case: $d = \beta$.

In this case we have from Lemma 2.1

$$Z(t_1, x_1 - y) \sim t_1^{-\alpha}\left(1 - \log\left(\frac{|x_1 - y|^\beta}{2^\beta t_1^\alpha}\right)\right), \quad R_1 \le 1$$

Noting that $c(d, \beta) = -(d\pi)^{d/2}$ in (2.8) we can estimate

$$I_2 = \int_{\{R_1 \le 1\}} Z(t_1, x_1 - y) u_0(y)\, dy$$

$$\le C t_1^{-\alpha} \int_{\{R_1 \le 1\}} \left[1 - \log\left(\frac{|x_1 - y|^\beta}{2^\beta t_1^\alpha}\right)\right] u_0(y)\, dy$$

$$= C t_1^{-\alpha} \int_{\mathbb{R}^d} [1 + \beta \log 2 + \alpha \log t_1 - \beta \log|x_1 - y|]\, \chi_{B_{r_1}(x_1)} u_0(y)\, dy$$

$$= C t_1^{-\alpha}[(1 + \beta \log 2 + \alpha \log t_1 + \beta c(d, \beta) G) \star (\chi_{B_{r_1}(x_1)} u_0)](x_1)$$

$$\le C \frac{[(c + \alpha \log t_1 + \beta c(d, \beta) G) \star (\chi_{B_{r_1}(x_1)} u_0)](x_1)}{[(c + \alpha \log t_2 + \beta c(d, \beta) G) \star (\chi_{B_{r_2}(x_2)} u_0)](x_2)} \int_{\{R_2 \le 1\}} Z(t_2, x_2 - y) u_0(y)\, dy.$$
$$\tag{4.6}$$

In the last step we controlled the time factor $\left(\frac{t_2}{t_1}\right)^\alpha$ similarly as before and denoted $c = 1 + \beta \log 2$. Combining (4.1), (4.5) and (4.6) gives the claim.

Case: $d > \beta$.

Now we obtain

$$I_2 = \int_{\{R_1 \leq 1\}} Z(t_1, x_1 - y) u_0(y)\, dy$$

$$\leq C t_1^{-\alpha} \int_{\{R_1 \leq 1\}} |x_1 - y|^{\beta - d} u_0(y)\, dy$$

$$= C \left(\frac{t_2}{t_1}\right)^\alpha \frac{\int_{\{R_1 \leq 1\}} |x_1 - y|^{\beta - d} u_0(y)\, dy}{\int_{\{R_2 \leq 1\}} |x_2 - y|^{\beta - d} u_0(y)\, dy} \int_{\{R_2 \leq 1\}} Z(t_2, x_2 - y) u_0(y)\, dy,$$

where

$$t_i^{-\alpha} \int_{\{R_i \leq 1\}} |x_i - y|^{\beta - d} u_0(y)\, dy = \int_{\mathbb{R}^d} |x_i - y|^{\beta - d} \chi_{\{B_{r_i}(x_i)\}} u_0(y)\, dy$$

$$= c(d, \beta) G * (u_0 \chi_{\{B_{r_i}(x_i)\}})$$

for $r_i = 2 t_i^{\alpha/\beta}, i = 1, 2$. We obtain

$$I_2 \leq C \left[\frac{[G \star (u_0 \chi_{B_{r_1}(x_1)})](x_1)}{[G \star (u_0 \chi_{B_{r_2}(x_2)})](x_2)}\right] \int_{\{R_2 \leq 1\}} Z(t_2, x_2 - y) u_0(y)\, dy.$$

Combining the estimates for I_1 and I_2 we have obtained

$$u(t_1, x_1) \leq C \left[1 + \frac{[G \star (u_0 \chi_{B_{r_1}(x_1)})](x_1)}{[G \star (u_0 \chi_{B_{r_2}(x_2)})](x_2)}\right] u(t_2, x_2), \quad d > \beta,$$

for a constant C depending on d, α, β and t_2/t_1. $\square$

The Corollary 2.6 is now an easy consequence of the above proof.

Proof of Corollary 2.6. Since $r_2 \geq r_1$, by using the assumption (2.10) we obtain

$$G \star (u_0 \chi_{B_{r_2}(x_2)})(x_2) = \int_{\mathbb{R}^d} G(y) u_0(x_2 - y) \chi_{B_{r_2}(x_2)}(x_2 - y)\, dy$$

$$= \int_{B_{r_2}(0)} G(y) u_0(x_2 - y)\, dy$$

$$\geq \inf_{B_{r_2}(x_2)} u_0 \int_{B_{r_2}(0)} G(y)\, dy$$

$$\geq \frac{1}{H_0} \sup_{B_{r_1}(x_1)} u_0 \int_{B_{r_1}(0)} G(y)\, dy$$

$$\geq \frac{1}{H_0} \int_{B_{r_1}(0)} G(y) u_0(x_1 - y)\, dy$$

$$= \frac{1}{H_0} \int_{\mathbb{R}^d} G(y) (u_0 \chi_{B_{r_1}(x_1)})(x_1 - y)\, dy$$

$$= G \star (u_0 \chi_{B_{r_1}(x_1)})(x_1).$$

Now the result follows from Theorem 2.4.

$\square$

Finally, we turn into studying the one-dimensional case.

Proof of Theorem 2.7. By the definition of the mild solution we have

$$u(t,x) = \int_{\mathbb{R}^d} Z(t, x-y)u_0(y)\,dy$$

for nonnegative u_0.

We evaluate u at the point (t_1, x_1) and divide the integration into two parts as before depending on whether $R_1 = \frac{|x_1 - y|^\beta}{2^\beta t_1^\alpha} \leq 1$ or $R_1 \geq 1$. The argument in the case $R_1 \leq 1$ is the same for all β, but in the case $R_1 \geq 1$ the argument is different depending on whether $\beta = 2$ or not. We give here the proof in the case $\beta \neq 2$. The case $\beta = 2$ can be handled similarly as in the proof of Theorem 2.4.

By Lemma 2.1

$$Z(t_1, x_1 - y) \sim t_1^{-\frac{\alpha d}{\beta}}, \quad R_1 \leq 1,$$

and

$$Z(t_1, x_1 - y) \sim t_1^\alpha |x_1 - y|^{-d-\beta}, \quad R_1 \geq 1.$$

Write

$$\begin{aligned}
u(t_1, x_1) &= \int_{\mathbb{R}^d} Z(t_1, x_1 - y)u_0(y)\,dy \\
&= \int_{\{R_1 > 1\}} Z(t_1, x_1 - y)u_0(y)\,dy + \int_{\{R_1 \leq 1\}} Z(t_1, x_1 - y)u_0(y)\,dy \\
&=: I_1 + I_2
\end{aligned} \tag{4.7}$$

as in the proof of Theorem 2.4.

The estimate for I_1. The argument is essentially the same as in the proof of Theorem 2.4 for the power term. We will repeat the argument here with more details. Recall

$$(2r)^{\alpha/\beta} \leq t_1 < t_2 \leq t_1 + (2r)^{\beta/\alpha},$$

which implies

$$t_2 \leq t_1 + (2r)^{\beta/\alpha} \leq 2t_1.$$

Then

$$R_2 = \frac{|x_2 - y|^\beta}{2^\beta t_2^\alpha} \geq \frac{(|x_1 - y| - |x_1 - x_2|)^\beta}{2^\beta t_2^\alpha} \geq \frac{(2t_1^{\alpha/\beta} - 2r)^\beta}{2^\beta t_2^\alpha} \geq \left(\frac{t_1}{t_2}\right)^\alpha \geq \frac{1}{2},$$

so

$$\{R_1 \geq 1\} \subset \{R_2 \geq \tfrac{1}{2}\}.$$

Hence

$$
\begin{aligned}
I_1 &\lesssim t_1^\alpha \int_{\{R_1 \geq 1\}} |x_1 - y|^{-d-\beta} u_0(y) dy \\
&\lesssim \left(\frac{t_1}{t_2}\right)^\alpha \int_{\{R_2 \geq \frac{1}{2}\}} \left(\frac{|x_2 - y|}{|x_1 - y|}\right)^{d+\beta} t_2^\alpha |x_2 - y|^{-d-\beta} u_0(y) dy.
\end{aligned}
$$

Since the asymptotics of Z is sharp, it remains to have a uniform bound for the ratio

$$
\frac{|x_2 - y|}{|x_1 - y|}.
$$

But noting that $t_1^{\alpha/\beta} \leq \frac{|x_1 - y|}{2}$ in the set $\{R_1 \geq 1\}$, we may estimate similarly as in (4.4) to get

$$
|x_2 - y| \leq \frac{3}{2}|x_1 - y|,
$$

which gives the desired upper bound

$$
I_1 \lesssim \int_{\{R_2 \geq \frac{1}{2}\}} Z(t_2, x_2 - y) u_0(y) dy \leq u(t_2, x_2). \tag{4.8}
$$

The estimate for I_2. For $R_1 \leq 1$ we have

$$
|x_2 - y| \leq |x_1 - y| + |x_1 - x_2| \leq 2t_2^{\alpha/\beta} + 2r \leq 3t_2^{\alpha/\beta} \Rightarrow R_2 \leq \left(\frac{3}{2}\right)^\beta
$$

and we may estimate

$$
\begin{aligned}
I_2 &= \int_{\{R_1 \leq 1\}} Z(t_1, x_1 - y) u_0(y) dy \\
&\lesssim t_1^{-\frac{\alpha d}{\beta}} \int_{\{R_1 \leq 1\}} u_0(y) dy \\
&\lesssim \left(\frac{t_2}{t_1}\right)^{\frac{\alpha d}{\beta}} \int_{\{R_2 \leq (\frac{3}{2})^\beta\}} Z(t_2, x_2 - y) u_0(y) dy \\
&\lesssim u(t_2, x_2).
\end{aligned} \tag{4.9}
$$

Combining (4.7), (4.8) and (4.9) completes the proof. $\qquad\square$

ACKNOWLEDGEMENTS

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