Topological centres of weighted convolution algebras

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Abstract

Let $G$ be a non-compact locally compact group with a continuous submultiplicative weight function $\omega$ such that $\omega(e) = 1$ and $\omega$ is diagonally bounded with bound $K \geq 1$. When $G$ is $\sigma$-compact, we show that $\lceil K \rceil + 1$ many points in the spectrum of $LUC(\omega^{-1})$ are enough to determine the topological centre of $LUC(\omega^{-1})^*$ and that $\lceil K \rceil + 2$ many points in the spectrum of $L^\infty(\omega^{-1})$ are enough to determine the topological centre of $L^1(\omega)^{**}$ when $G$ is in addition a SIN-group. We deduce that the topological centre of $LUC(\omega^{-1})^*$ is the weighted measure algebra $M(\omega)$ and that of $C_0(\omega^{-1})^\perp$ is trivial for any locally compact group. The topological centre of $L^1(\omega)^{**}$ is $L^1(\omega)$ and that of of $L^\infty_0(\omega)^\perp$ is trivial for any non-compact locally compact SIN-group. The same techniques apply and lead to similar results when $G$ is a weakly cancellative right cancellative discrete semigroup.

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1. Introduction

Investigations related to topological centres go back all the way to Richard Arens [1, 2], who defined two products on the second dual of a Banach algebra (in fact, even in a more general set up). He proved that the two products coincide for the ring $C(X)$ of continuous functions on a compact Hausdorff space $X$ as well as for $\ell^1$ with the pointwise multiplication, but the two products do not coincide for $A = \ell^1$ with convolution as product. Note that the two products coincide on the second dual of a commutative Banach algebra if and only if the second dual itself is commutative under either product. The work of Arens was followed by the seminal articles by Day [8] in 1957 and by Civin and Yood [4] in 1961, where the second duals of group algebras of infinite locally compact

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abelian groups and of some amenable discrete infinite semigroups were shown to be non-commutative. For a more detailed account, see the survey [12].

Grosser and Losert [14] proved that the centre of $UC(G)^*$ is the measure algebra $M(G)$ when $G$ is abelian. A natural extension of this result is the theorem of Lau [16] stating that for any locally compact group $G$, the topological centre of $LUC(G)^*$ is $M(G)$. Then Lau and Losert [17] proved that the topological centre of $L^1(G)^{**}$ is the group algebra $L^1(G)$. These results were followed by numerous papers studying the topological centres of other algebras arising in abstract harmonic analysis such as the second dual of the Fourier algebra $A(G)$ or algebras associated to discrete semigroups.

One of the points which has attracted attention more recently is the number of points necessary to determine the topological centres. This was initiated in [7], where Dales, Lau and Strauss proved that two points are sufficient to determine the topological centre of $LUC(G)^*$. In [10], we also proved that two points are enough to determine the topological centre of $LUC(G)^*$ when $G$ is $\sigma$-compact.

Dales and Dedania showed that there is a set $V$ consisting of $\lfloor K \rfloor + 1$ points such that $V$ is determining for the topological centre of $LUC(G)^*$, where $\lfloor K \rfloor$ denotes the integer part of $K$. More recently, Budak, Işık and Pym proved in their exquisite paper [3] that continuity at one point is enough to determine the topological centres of $LUC(G)^*$ and $L^1(G)^{**}$.

In this paper, we carry on our investigation on topological centres of convolution algebras started in [10]. The method proposed in [10] combined with elements from [7], [5] and [3] leads also to the topological centres in the weighted cases, assuming the weight is diagonally bounded. Moreover, in the $\sigma$-compact case, we find finite sets determining the topological centres (for the case of the second dual $L^1(\omega)^{**}$ of the weighted group algebra $L^1(\omega)$ we need that $G$ is SIN).

Throughout the paper the weight $\omega$ will be diagonally bounded on $E \subseteq G$ with bound $K$, where $E$ has the same compact covering number as $G$ (precise definitions are given in Section 2). We shall consider the C*-algebras $LUC(\omega^{-1})$, $C_0(\omega^{-1})$, $L^\infty(\omega^{-1})$ and $L^\infty_0(\omega^{-1})$, defined as weighted function algebras. We prove in Section 3 that if $G$ is $\sigma$-compact, then $\lfloor K \rfloor + 1$ many points in the spectrum $\Delta$ of $LUC(\omega^{-1})$ are enough to determine the topological centre of $LUC(\omega^{-1})^*$. In general, we show that the topological centre of $LUC(\omega^{-1})^*$ is the weighted measure algebra $M(\omega)$. Similarly, the topological centre of $C_0(\omega^{-1})^*$ is trivial for any locally compact group, and in the case of $\sigma$-compact groups, it is enough to check continuity at $\lfloor K \rfloor + 1$ many points.

The main technique behind the proofs is to use points $x$ in the spectrum $\Delta$ of $LUC(\omega^{-1})$ that satisfy the bi-Lipschitz property

$$\frac{\|\mu\|}{K} \leq \|\mu \square x\| \leq \|\mu\| \quad (1)$$

and that can be separated by slowly oscillating functions. Right isometries and
slowly oscillating functions were the main tools in [10]. The use of points with the bi-Lipschitz property replaces the use of right isometries in the unweighted case. The first inequality in (1) is special for these points: it is due to two crucial facts, first, $x$ is taken from the closure of a specially constructed set $T$ giving $x$ a factorisation property, and second, the weight $\omega$ is diagonally bounded on $T$. These techniques are then combined with methods from [7] and [5]. We should note that right cancellation, factorisation and right isometries are the three basic properties used by different authors to find the topological centres. In [11], they are shown to be basically the same, under certain assumptions.

We continue in Section 4 with $L^1(\omega)^{**}$. With the help of an argument based on [3], we prove that $\lfloor K \rfloor + 2$ many points are enough to determine the topological centres of $L^1(\omega)^{**}$ and $L^\infty_0(\omega-1)\perp$ when $G$ is a $\sigma$-compact SIN-group. Our method shows also that the topological centre of $L^1(\omega)^{**}$ is $L^1(\omega)$ and the topological centre of $L^\infty_0(\omega)^{\perp}$ is trivial for any non-compact locally compact SIN-group.

Our method applies also to weakly cancellative, right cancellative discrete semigroups, as presented in Section 6. We prove that $\lfloor K \rfloor + 1$ is again the number of points enough to determine the centres of $\ell^1(\omega)^{**}$ and $c_0(\omega-1)$ when $S$ is countable; the result for $\ell^1(\omega)^{**}$ is thus similar to the result of Dales and Dedania [5, Theorem 5.6] but our assumptions on the semigroup and the weight are slightly different. Moreover, we also deduce that the topological centres of $\ell^1(\omega)^{**}$ and $c_0(\omega-1)^{\perp}$ are respectively $\ell^1(\omega)$ and $\{0\}$, without assuming that $S$ is countable.

2. Preliminaries

2.1. Function algebras

In this paper, a weight on a locally compact group $G$ means a continuous function $\omega: G \to (0, \infty)$ such that the value of $\omega$ at the identity $e \in G$ is 1 and $\omega$ is submultiplicative, that is,

$$\omega(st) \leq \omega(s)\omega(t) \quad \text{for every } s, t \in G.$$  

For any function space $F(G)$, we let $F(\omega^{-1})$ denote the weighted analogue defined by

$$F(\omega^{-1}) = \{ f : G \to \mathbb{C} ; \omega^{-1} f \in F(G) \}$$

with the norm that makes the map

$$f \mapsto \omega^{-1} f : F(\omega^{-1}) \to F(G)$$

an isometry. We shall apply this construction to the the C*-algebra $C_0(G)$ of all continuous functions vanishing at infinity, to the C*-algebra $LUC(G)$ of the bounded left uniformly continuous functions on $G$, to the C*-algebra $L^\infty(G)$ of essentially bounded locally measurable functions on $G$ and to the C*-algebra $L^\infty_0(G)$. For $f \in L^\infty(G)$, we put $\|f\|_K = \text{esssup}\{ |f(x)| ; x \in K \}$, and define

$$L^\infty_0(G) = \{ f \in L^\infty(G) ; \text{for } K \text{ compact } \|f\|_{G\setminus K} \to 0 \text{ as } K \to G \}.$$
The resulting weighted function spaces are C*-algebras under the weighted pointwise product
\[ f \cdot \omega g(s) = \frac{f(s)g(s)}{\omega(s)}, \]
and the map \( f \mapsto \omega^{-1}f : F(\omega^{-1}) \to F(G) \) is a *-isomorphism.

2.2. Convolution algebras

The space \( M(\omega) \) consists of all Radon measures \( \mu \) on \( G \) such that the weighted measure \( \omega \mu \) is bounded. This space is normed such that the map \( \mu \mapsto \omega \mu \) : \( M(\omega) \to M(G) \) is an isometry, where \( M(G) \) is the usual measure algebra consisting of bounded Radon measures on \( G \). The measure algebra \( M(G) \) is the dual space of \( C_0(G) \), and similarly \( M(\omega) \) is the dual space of \( C_0(\omega^{-1}) \).

Since \( \omega \) is submultiplicative, \( M(\omega) \) is a Banach algebra under the convolution product
\[
\langle \mu * \nu, f \rangle = \int \int f(st) \, d\mu(s) \, d\nu(t), \quad (\mu, \nu \in M(\omega), f \in C_0(\omega^{-1})).
\]

Note that although \( M(\omega) \) is isometric to \( M(G) \) their Banach algebra structures may be very different, depending on \( \omega \).

Let \( L^1(G) \) be the group algebra of \( G \), consisting of those measures in \( M(G) \) that are absolutely continuous with respect to the left Haar measure. The weighted group algebra \( L^1(\omega) \subseteq M(\omega) \) is defined via the isometry \( \mu \mapsto \omega \mu \).

Then \( L^1(\omega) \) is a closed two-sided ideal in \( M(\omega) \), similarly as \( L^1(G) \) in \( M(G) \).

When \( \omega \geq 1 \), we have \( L^1(\omega) \subseteq L^1(G) \) (with different norms); in this well-studied case \( L^1(\omega) \) is called a Beurling algebra.

Let \( \pi : LUC(G)^* \to LUC(\omega^{-1})^* \) be the adjoint of the isometry given in (2) for \( F(G) = LUC(G) \); that is,
\[
\langle \pi(\mu), f \rangle = \langle \mu, \omega^{-1}f \rangle, \quad (\mu \in LUC(G)^*, f \in LUC(\omega^{-1})).
\]

Since \( f \mapsto \omega^{-1}f \) is a *-isomorphism, \( \pi \) maps the spectrum \( G^{LUC} \) of \( LUC(G) \) onto the spectrum \( \Delta \) of \( LUC(\omega^{-1}) \). Define \( \epsilon : G \to \Delta \) by
\[
\langle \epsilon(s), f \rangle = \frac{f(s)}{\omega(s)} \quad (f \in LUC(\omega^{-1})).
\]

Then \( \epsilon \) is a homeomorphism, and in fact
\[
\epsilon(s) = \pi(s),
\]
where \( s \) in the right-hand side is considered as an element of \( G^{LUC} \) (the point evaluation at \( s \)). We shall always use the identification \( G \subseteq G^{LUC} \). We may also identify \( G \) with its image in \( \Delta \) (topologically), but keep writing \( \epsilon(s) \) in the weighted case to deter confusion.

Let \( A \) be a Banach algebra; in our interests \( A \) is either \( L^1(\omega) \) or \( \ell^1(\omega) \), the latter being the weighted semigroup algebra of a discrete semigroup \( S \) (see
Section 6 for more details). The first Arens product on the second dual of $A$ is defined by the following formulas:

\[
\langle fa, b \rangle = \langle f, ab \rangle, \quad a, b \in A, f \in A^*; \\
\langle νf, a \rangle = \langle ν, fa \rangle, \quad ν \in A^{**}; \\
\langle μ □ ν, f \rangle = \langle μ, νf \rangle, \quad μ \in A^{**}.
\]

This makes $A^{**}$ a Banach algebra.

There is also the second Arens product on the second dual of any Banach algebra $A$; this is defined by the following formulas:

\[
\langle af, b \rangle = \langle f, ba \rangle, \quad a, b \in A, f \in A^*; \\
\langle fμ, a \rangle = \langle μ, af \rangle, \quad μ \in A^{**}; \\
\langle μ □ ν, f \rangle = \langle ν, fμ \rangle, \quad ν \in A^{**}.
\]

The product on $LUC(ω^{-1})^*$ is defined by

\[
L_s f(t) = f(st), \quad s, t \in G, f \in LUC(ω^{-1}), \\
νf(s) = \langle ν, L_s f \rangle, \quad ν \in LUC(ω^{-1})^*, \\
\langle μ □ ν, f \rangle = \langle ν, fμ \rangle, \quad μ \in LUC(ω^{-1})^*.
\]

Let $C_0(ω^{-1})^\perp$ denote the functionals in $LUC(ω^{-1})^*$ that annihilate $C_0(ω^{-1}) \subseteq LUC(ω^{-1})$. It is easily seen that $C_0(ω^{-1})^\perp$ is a weak*-closed ideal in $LUC(ω^{-1})^*$. Moreover,

\[
LUC(ω^{-1})^* \cong M(ω) \oplus C_0(ω^{-1})^\perp
\]

as an $ℓ^1$-direct sum.

Let $Φ: L^1(ω)^{**} \to LUC(ω^{-1})^*$ denote the natural quotient map, i.e., the adjoint of the inclusion $LUC(ω^{-1}) \hookrightarrow L^∞(ω^{-1})$. Then $Φ$ is a weak*-continuous homomorphism with respect to the first Arens product $□$ on $L^1(ω)^{**}$ and the product $□$ on $LUC(ω^{-1})^*$. Moreover, $Φ$ maps $L^∞_0(ω^{-1})^*$ onto $M(ω)$, $L^∞_0(ω^{-1})^\perp$ onto $C_0(ω^{-1})^\perp$ and the spectrum $Ω$ of $L^∞(ω^{-1})$ onto $Δ$ (the last statement follows from [21, Lemma 4.1.7] for example).

It is a consequence of Cohen’s factorisation theorem that $LUC(ω^{-1}) = L^∞(ω^{-1})L^1(ω)$, where the action of $L^1(ω)$ on $L^∞(ω^{-1})$ defined by the first formula of (3) (see for example [13], there is a small gap in the proof of [13, Proposition 1.3], which is mended in [6, Proposition 7.15]). We therefore see that the Banach algebra $LUC(ω^{-1})^*$ acts on $L^∞(ω^{-1})$ and on $L^1(ω)^{**}$ by

\[
\langle x \bullet f, φ \rangle = \langle x, fφ \rangle \quad \text{and} \quad \langle μ \bullet x, f \rangle = \langle μ, x \bullet f \rangle,
\]

where $μ \in L^∞(ω^{-1})^*$, $x \in LUC(ω^{-1})^*$, $φ \in L^1(ω)$ and $f \in L^∞(ω^{-1})$. This leads immediately to the identity

\[
μ □ ν = μ \bullet Φ(ν) \quad (μ, ν \in L^1(ω)^{**}),
\]

and the spectrum $Ω$ of $L^∞(ω^{-1})$ onto $Δ$ (the last statement follows from [21, Lemma 4.1.7] for example).

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\]

where $μ \in L^∞(ω^{-1})^*$, $x \in LUC(ω^{-1})^*$, $φ \in L^1(ω)$ and $f \in L^∞(ω^{-1})$. This leads immediately to the identity

\[
μ □ ν = μ \bullet Φ(ν) \quad (μ, ν \in L^1(ω)^{**}),
\]
which implies that the right shift by \( \nu \) in \( L^1(\omega)^{**} \) depends only on the restriction of \( \nu \) to \( \text{LUC}(\omega^{-1}) \). Moreover, note that \( x \bullet f = xf \) if \( f \in \text{LUC}(\omega^{-1}) \) and \( x \in \text{LUC}(\omega^{-1})^* \).

The right translations on \( (L^1(\omega)^{**}, \Box) \) and on \( \text{LUC}(\omega^{-1})^* \) are weak*-continuous, but left translations need not be. On these algebras, the topological centre is the collection of all \( \mu \) such that the left translation by \( \mu \), i.e. \( \nu \mapsto \mu \Box \nu \), is weak*-continuous.

It is not difficult to show that \( M(\omega), L^1(\omega) \) and \( \ell^1(\omega) \) are contained in the topological centres of \( \text{LUC}(\omega^{-1})^*, L^1(\omega)^{**} \) and \( \ell^1(\omega)^{**} \), respectively.

2.3. More definitions

For \( E \subseteq G \), let \( \kappa(E) \) denote the compact covering number of \( E \), i.e., the minimal cardinality of a covering of \( E \) by compact subsets of \( G \). Then the height of \( x \in \Delta \) is defined to be

\[
\rho(x) = \min \{ \kappa(E); E \subseteq G, x \in \overline{e(E)} \},
\]

where \( \overline{e(E)} \) denotes the closure in \( \Delta \). Finally, let \( \mathcal{U}_\Delta \) denote the points in \( \Delta \) with height \( \kappa(G) \).

The weight \( \omega \) is diagonally bounded on \( E \subseteq G \) with bound \( K > 0 \) if

\[
\sup_{s \in E} \omega(s) \omega(s^{-1}) \leq K.
\]

Necessarily, \( K \geq 1 \) (assuming \( E \neq \emptyset \)).

A function \( f: G \to \mathbb{C} \) is said to be slowly oscillating if for every \( \epsilon > 0 \) and \( s \in G \) there is \( A \subseteq G \) with \( \kappa(A) < \kappa(G) \) such that

\[
|f(st) - f(t)| < \epsilon \quad \text{and} \quad |f(ts) - f(t)| < \epsilon \quad \text{for every} \quad t \in G \setminus A.
\]

Slowly oscillating functions were constructed in [10, Lemma 1]. If \( f \in \text{LUC}(G) \) is slowly oscillating, then

\[
\langle y \Box s, f \rangle = \langle y, f \rangle = \langle x \Box y, f \rangle
\]

whenever \( s \in G, x \in G^{\text{LUC}} \) and the height of \( y \in G^{\text{LUC}} \) is \( \kappa(G) \).

3. Sets determining topological centres

The notion of sets determining for the (left) topological centre (dtc) of \( A^{**} \) was introduced by Dales, Lau, and Strauss [7]. A set \( V \subset A^{**} \) is a dtc set if

\[
\mu \Box y = \mu \circ y \quad \text{for every} \quad y \in V \quad \text{implies that} \quad \mu \in A.
\]

Note that since \( \mu \Box a = \mu \circ a \) for every \( a \in A \), we have \( \mu \Box y = \mu \circ y \) if and only if \( \mu \circ y = \lim_\alpha \mu \Box y_\alpha \) whenever \( (y_\alpha) \) is a net in \( A \) such that \( y_\alpha \to y \) in \( A^{**} \).

Recall that \( G \) is an SIN group if it has a neighbourhood base at the identity consisting of symmetric, invariant neighbourhoods. When \( G \) is an SIN group,
the left and right uniformities coincide and \( LUC(G) \) is the C*-algebra of all bounded uniformly continuous functions. So in this case, a second product may be defined on \( LUC(\omega^{-1})^* \) as follows:

\[
R_s f(t) = f(ts), \quad f\nu(s) = \langle \nu, R_s f \rangle, \quad (\mu \Diamond \nu, f) = \langle \mu, f\nu \rangle,
\]

where \( s, t \in G, f \in LUC(\omega^{-1}) \), and \( \mu, \nu \in LUC(\omega^{-1})^* \). In this case, we say that a subset \( V \) of \( \Delta \) is a dtc set of \( LUC(\omega^{-1})^* \) if \( \mu \in M(\omega) \) whenever

\[
\mu \Diamond x = \mu \circ x \quad \text{for every} \quad x \in V.
\]

Equivalently, \( \mu = 0 \) is the only element in \( C_0(\omega^{-1})^\perp \) with property (7). We remark at this point that when \( G \) is a SIN group, the map \( \Phi: L^1(\omega)^{**} \to LUC(\omega^{-1})^* \) is a homomorphism also with respect to the second products.

**Lemma 1.** Suppose that \( G \) is a locally compact SIN group. Let \( \mu \in LUC(\omega^{-1})^* \) and \( x \in \Delta \). Then \( \mu \Diamond x = \mu \circ x \) if and only if \( \mu \Diamond x = \lim_\alpha \mu \Diamond \epsilon(s_\alpha) \) whenever \( (s_\alpha) \) is a net in \( G \) such that \( \epsilon(s_\alpha) \to x \) in \( \Delta \).

**Proof.** Let \( (s_\alpha) \) be a net in \( G \) such that \( \epsilon(s_\alpha) \to x \) in \( \Delta \). As \( \mu \Diamond \epsilon(s) = \mu \circ \epsilon(s) \) for every \( s \in G \), we have

\[
\lim_\alpha \mu \Diamond \epsilon(s_\alpha) = \lim_\alpha \mu \circ \epsilon(s_\alpha) = \mu \circ x.
\]

The preceding lemma justifies the following definition which applies to all locally compact groups. A subset \( V \) of \( \Delta \) is **determining for the topological centre of** \( LUC(\omega^{-1})^* \) **(a dtc set)** if \( \mu \in M(\omega) \) whenever

\[
\mu \Diamond x = \lim_\alpha \mu \Diamond \epsilon(s_\alpha)
\]

for every \( x \in V \) and for every net \( (s_\alpha) \) in \( G \) such that \( \epsilon(s_\alpha) \to x \).

The approach in this paper shall lead to a subset \( Y \) of \( \Delta \) of cardinality \( 2^\kappa \) such that any \( \{K\} + 1 \) points from \( Y \) form a dtc set for the algebra \( LUC(\omega^{-1})^* \) when \( G \) is \( \sigma \)-compact. Moreover, it will then be enough to check the continuity of the left translations at \( \{K\} + 1 \) points to deduce that the topological centre of \( C_0(\omega^{-1})^\perp \) is \( \{0\} \).

When \( G \) is in addition an SIN group, we find a subset \( Y \) in the spectrum \( \Omega \) of cardinality \( 2^\kappa \) such that any \( \{K\} + 1 \) points from \( Y \) together with any right identity in \( \Omega \) form a dtc set for the algebra \( L^1(\omega)^{**} \). It will also be enough then to check the continuity of the left translations at \( \{K\} + 2 \) points to deduce that the topological centre of \( L_0^\infty(\omega^{-1})^\perp \) is \( \{0\} \).

In the general case when \( G \) is not necessarily \( \sigma \)-compact, the methods still lead to the topological centres of any of these algebras.

The dtc sets considered in [3] (in the unweighted case) are different than the dtc sets defined above. Budak, Işık and Pym find one point \( x \) together with a net \( (x_\alpha) \) both in the remainder \( G(LUC \setminus G) \) such that \( x_\alpha \to x \), and
\( \mu \sqcap x_\alpha \to \mu \sqcap x \) happens only for \( \mu = 0 \) in \( C_0(\omega^{-1})^* \). So their dtc set is a singleton in \( G^{LUC} \setminus G \) but the whole net of points in \( G^{LUC} \setminus G \) is also needed in the process, while our dtc set consists simply of two points in \( G^{LUC} \setminus G \). In fact, with our definition, 2 is the smallest cardinality dtc sets may have in general: take \( G \) abelian, \( \omega = 1, x \in G^{\text{LUC}} \setminus G \) and \( (s_\alpha) \) any net in \( G \) converging to \( x \). Then \( x \sqcap x = \lim_\alpha s_\alpha \sqcap x = \lim_\alpha x \sqcap s_\alpha \), and so a dtc set (even for \( G^{\text{LUC}} \)) must have more than one point.

4. Dtc sets for \( LUC(\omega^{-1})^* \)

This section forms the core of the paper. Throughout this section we assume that \( \omega \) is a weight on a non-compact locally compact group \( G \), which is diagonally bounded, with bound \( K \geq 1 \), on a subset \( E \subseteq G \) with \( \kappa(E) = \kappa(G) \).

The proof of the following lemma builds upon elements from [10] and [5, Theorem 5.6] (the latter of which is adapted from [7]).

Recall that \( \mathcal{U}_\Delta \) denotes the set of uniform points in \( \Delta \), i.e., points with height \( \kappa(G) \).

**Lemma 2.** There exist \( n = |K| + 1 \) points \( x_1, x_2, \ldots, x_n \) in \( \Delta \) such that \( \mu = 0 \) is the only element in \( LUC(\omega^{-1})^* \) with \( \text{supp} \mu \subseteq \mathcal{U}_\Delta \) having the property that \( \lim_\alpha \mu \sqcap \epsilon(s_\alpha) = \mu \sqcap x_k \) whenever \( k = 1, \ldots, n \) and \( (s_\alpha) \) is a net in \( G \) with \( \lim_\alpha \epsilon(s_\alpha) = x_k \) in \( \Delta \).

**Proof.** We start by constructing points \( x_1, x_2, \ldots, x_n \) in \( \mathcal{U}_\Delta \) that can be separated by slowly oscillating functions and that have the following factorisation property (obtained with the help of [20]): for any \( f \in LUC(\omega^{-1}) \) there is \( g \in LUC(\omega^{-1}) \) such that \( f = x_k g \).

Let \( T \subseteq E \) be a set as constructed in Lemma 1 of [10]. Let \( f \in LUC(\omega^{-1}) \) and \( \epsilon > 0 \) be arbitrary. Since \( L^1(\omega) \) has a bounded approximate identity with bound 1 (as \( \omega(\epsilon) = 1 \)), it follows from the Cohen factorisation theorem [15, Theorem 32.22] that \( f = h \varphi \) where \( h \in L^\infty(\omega^{-1}) \) and \( \varphi \in L^1(\omega) \) are such that \( \|h - f\|_\omega < \epsilon \) and \( \|\varphi\|_\omega \leq 1 \). Comparing the construction of our set \( T \) in Lemma 1 of [10] to the construction leading to Lemma 8 of [9], we see that the set \( T \) satisfies the properties needed for the latter result. Therefore, any \( x \) in the closure of \( \epsilon(T) \) with height \( \kappa(G) \) has the factorisation property that any \( h \in L^\infty(\omega^{-1}) \) can be written as \( h = x \cdot g' \) for some \( g' \in L^\infty(\omega^{-1}) \). (This factorisation result goes back to Neufang [20].) Put \( g = g' \varphi \in LUC(\omega^{-1}) \) so that

\[
\begin{align*}
    xg = x(g' \varphi) = (x \cdot g') \varphi = h \varphi = f,
\end{align*}
\]

as required. Since \( \omega \) is diagonally bounded on \( E \) with bound \( K \), an inspection of the argument in [9] shows that \( \|g'\|_\omega \leq K\|h\|_\omega \) and so

\[
\|g\|_\omega \leq \|g'\|_\omega \|\varphi\|_\omega \leq K\|h\|_\omega \leq K(\|f\|_\omega + \epsilon).
\]

Therefore, for every \( \mu \in LUC(\omega^{-1})^*, f \in LUC(\omega^{-1}) \) and \( \epsilon > 0 \), writing \( f = xg \) as above, we have

\[
|\langle \mu, f \rangle| = |\langle \mu \sqcap x, g \rangle| \leq K\|\mu \sqcap x\|\|f\|_\omega + \epsilon).
\]
It follows that
\[
\frac{\|\mu\|}{K} \leq \|\mu \square x\| \tag{8}
\]
for every \(\mu\) in \(LUC(\omega^{-1})^*\).

Now let \(x_1, x_2, \ldots, x_n\) be \(n\) distinct points in \(e(T)\) with height \(\kappa(G)\), and pick \(n\) distinct points \(y_1, y_2, \ldots, y_n\) in \(\mathcal{T} \subseteq G^{LUC}\) (necessarily with height \(\kappa(G)\)) such that \(x_k = \pi(y_k)\) for \(1 \leq k \leq n\). The construction in [10, Lemma 1] gives slowly oscillating functions \(f_j \in LUC(G)\), \(1 \leq j \leq n\), such that \(0 \leq f_j \leq 1\), \(\langle y_k, f_j \rangle = \delta_{k,j}\) and the supports of \(f_j\)'s are disjoint (regarded as continuous functions on \(G^{LUC}\)). For every \(1 \leq j \leq n\), let \(g_j = \omega f_j\) and note that \(g_j\) is in the unit ball of \(LUC(\omega^{-1})\) and \(\langle x_k, g_j \rangle = \delta_{k,j}\).

Choose \(\epsilon > 0\) such that \(\frac{1}{K} - \epsilon \geq \frac{1}{n}\). Let \(z \in \Delta\) and let \(h\) be an arbitrary function in the unit ball of \(LUC(\omega^{-1})\). Now for every \(1 \leq k \leq n\) and \(s, t \in G\),

\[
(\epsilon(s) (h \cdot \omega g_k))(t) = \langle \epsilon(s), L_t(h \cdot \omega g_k) \rangle = \frac{(h_t \omega g_k)(ts)}{\omega(s)} = \frac{h(t) g_k(t)}{\omega(s) \omega(ts)}. \tag{9}
\]

Then
\[
|\langle z \square \epsilon(s), h \cdot \omega g_k \rangle| = |\langle z, \epsilon(s)(h \cdot \omega g_k) \rangle| = \lim_{\epsilon(t) \to z} \left| \frac{\langle h(t) g_k(t) \rangle}{\omega(t) \omega(s) \omega(ts)} \right| = \lim_{t \to \pi^{-1}(z)} |f_k(t)|
\]

where the right slow oscillation of \(f_k\) is used at (*). Since the supports of the functions \(g_k\) (regarded as continuous functions on \(\Delta\)) are disjoint, there is at most one \(k\) such that \(\langle z, g_k \rangle\) is non-zero. It follows that if \(\mu\) is a finite sum of the form \(\sum_{i \in I} \alpha_i z_i\) such that \(z_i \in \Delta\) and \(\alpha_i \in \mathbb{R}\) with \(\sum_{i \in I} |\alpha_i| \leq 1\), then for some \(1 \leq k \leq n\)

\[
|\langle \mu \square \epsilon(s), h \cdot \omega g_k \rangle| \leq \frac{1}{n} \leq \frac{1}{K} - \epsilon. \tag{10}
\]

Indeed, if \(I_k = \{i \in I; \langle z_i, g_k \rangle \neq 0\}\), then the sets \(I_k\) are disjoint, and by (9),

\[
|\langle \mu \square \epsilon(s), h \cdot \omega g_k \rangle| \leq \sum_{i \in I} |\alpha_i| |\langle z_i \square \epsilon(s), h \cdot \omega g_k \rangle| \leq \sum_{i \in I} |\alpha_i| |\langle z_i, g_k \rangle| = \sum_{i \in I_k} |\alpha_i| |\langle z_i, g_k \rangle| \leq \sum_{i \in I_k} |\alpha_i| \tag{11}
\]

Since
\[
\sum_{k=1}^{n} \sum_{i \in I_k} |\alpha_i| \leq 1,
\]
inquality (10) is clear.

By weak*-approximation, (10) holds for any \(\mu\) in the unit ball of \(LUC(\omega^{-1})^*\) that is supported by \(\Delta\) (as a measure on \(\Delta\)).
Suppose that $\|\mu\| = 1$. Now we show that for every $1 \leq k \leq n$, we can find $h_k$ in the unit ball of $LUC(\omega^{-1})$ such that
\[
|\langle \mu \boxtimes x_k, h_k \cdot \omega g_k \rangle| > \frac{1}{K} - \epsilon. \tag{12}
\]

Since $\|\mu \boxtimes x_k\| \geq \|\mu\|/K = 1/K$ by (8), we can already pick $h_k$ in the unit ball of $LUC(\omega^{-1})$ such that
\[
|\langle \mu \boxtimes x_k, h_k \rangle| > \frac{1}{K} - \epsilon. \tag{13}
\]

Next, for each $1 \leq k \leq n$, we let
\[
F_k = \{ z \in \Delta; \langle z, g_k \rangle = 1 \}
\]
and show that supp($\mu \boxtimes x_k$) $\subseteq F_k$ (recall that $\Delta$ is usually not closed under the multiplication of $LUC(\omega^{-1})^\ast$). We first claim that for every $z \in \Delta$, the support of $z \boxtimes x_k$ is contained in $F_k$. For every $s, t \in G$,
\[
\epsilon(s) \boxtimes \epsilon(t) = \frac{\omega(st)}{\omega(s)\omega(t)} \epsilon(st).
\]
Noting that $\omega(st)/\omega(s)\omega(t)$ is bounded by 1, and taking first $\epsilon(t) \to x_k$ then $\epsilon(s) \to z$, the preceding identity leads to
\[
z \boxtimes x_k = \lambda u
\]
where $\lambda \in (0, 1]$ and $u \in \Delta$. Since
\[
\langle \lambda u, g_k \rangle = \lim_{\epsilon(s) \to z, \epsilon(t) \to x_k} \frac{\omega(st)}{\omega(s)\omega(t)} g_k(st) = \lim_{s \to \pi^{-1}(z)} \frac{\omega(st)}{\omega(s)\omega(t)} f_k(st) \tag{**} \xi \langle x_k, g_k \rangle = \lambda,
\]
where as in (9), the left slow oscillation of $f_k$ is used at (**). We have therefore $\langle u, g_k \rangle = 1$, and so supp($z \boxtimes x_k$) = \{u\} $\subseteq F_k$. Taking linear combinations of elements of the form $z \boxtimes x_k$, $z \in \Delta$, and then weak*-limits, we see that supp($\mu \boxtimes x_k$) $\subseteq F_k$ (as $F_k \subseteq \Delta$ is closed). Now, since $\tilde{g}_k = 1$ on $F_k$, it is easy to check that $h_k \cdot \omega g_k = h_k$ on $F_k$. Accordingly, inequality (12) follows from inequality (13).

Suppose now that $\mu \in LUC(\omega^{-1})^\ast$ is supported by $\mathcal{U}_\Delta$ and has norm 1. Since both (10) and (12) hold for $\mu$, we see that
\[
\lim_{\alpha} \mu \boxtimes \epsilon(s_\alpha) \neq \mu \boxtimes x_k
\]
even if $(s_\alpha)$ is a net in $G$ such that $\lim_{\alpha} \epsilon(s_\alpha) = x_k$. This proves the claim. \qed
Theorem 3. Suppose that $G$ is $\sigma$-compact. Then $\Delta$ contains a dtc set for $LUC(\omega^{-1})^*$ of cardinality $|K|+1$. In other words, there exist $n = |K|+1$ points $x_1, x_2, \ldots, x_n$ in $\Delta$ such that if $\mu \in LUC(\omega^{-1})^*$ and $\lim_{s_\alpha} \epsilon(s_\alpha) = \mu \sqcap x_k$ whenever $k = 1, \ldots, n$ and $(s_\alpha)$ is a net in $G$ with $\lim_{s_\alpha} \epsilon(s_\alpha) = x_k$ in $\Delta$, then $\mu \in M(\omega)$.

In particular, the topological centre of $LUC(\omega^{-1})^*$ is $M(\omega)$.

Proof. Let $x_1, x_2, \ldots, x_n$ be the points in $\overline{\epsilon(T)}$ as given by Lemma 2. Let $\mu \in LUC(\omega^{-1})^*$ and write it as $\mu = \mu_0 + \mu_1$ where $\mu_0 \in M(\omega)$ and $\mu_1 \in C_0(\omega^{-1})^\perp$. Suppose that $\mu$ has the property that $\lim_{s_\alpha} \mu \sqcap \epsilon(s_\alpha) = \mu \sqcap x_k$ whenever $k = 1, \ldots, n$ and $(s_\alpha)$ is a net in $G$ with $\lim_{s_\alpha} \epsilon(s_\alpha) = x_k$ in $\Delta$. Since $M(\omega)$ is contained in the topological centre, it is clear that $\mu_0$ satisfies also the property, and so does $\mu_1$. But as $G$ is $\sigma$-compact, $\mu_1$ is supported by $\Delta \setminus \epsilon(G) = U_\Delta$. It then follows from Lemma 2 that $\mu_1 = 0$. \hfill $\square$

Next we shall show that the topological centre of $LUC(\omega^{-1})^*$ is $M(\omega)$ for any locally compact group $G$. Let $H$ be an open subgroup of $G$. Restricting $\omega$ to $H$ gives a weight on $H$ and we denote the associated weighted $LUC$-space by $LUC(H, \omega^{-1})$. For every $f$ in $LUC(H, \omega^{-1})$, let $\hat{f}$ denote extension of $f$ to $G$ by 0 and note that $\hat{f} \in LUC(\omega^{-1})$. Then if $\mu \in LUC(\omega^{-1})^*$, let $\hat{\mu}$ denote the functional in $LUC(H, \omega^{-1})^*$ defined by

$$\langle \hat{\mu}, f \rangle = \langle \mu, \hat{f} \rangle \quad (f \in LUC(H, \omega^{-1})).$$

Note that the spectrum $\Delta(H)$ of $LUC(H, \omega^{-1})$ may be identified with the closure of $\epsilon(H)$ in $\Delta$. Moreover, denote the set of points in $\overline{\epsilon(H)} \subseteq \Delta$ with height $\kappa(H)$ by $U_\Delta(H)$.

Lemma 4. Let $H$ be an open subgroup of $G$. Let $y \in \overline{\epsilon(H)} \cong \Delta(H)$ where the closure is taken in $\Delta$. If the map

$$L_\mu: x \mapsto \mu \sqcap x: \Delta \rightarrow LUC(\omega^{-1})^*$$

is continuous at $y$, then also the map

$$L_{\hat{\mu}}: x \mapsto \hat{\mu} \sqcap x: \Delta(H) \rightarrow LUC(H, \omega^{-1})^*$$

is continuous at $y$.

Proof. We begin by checking that $(xf)^\sim = x\hat{f}$ whenever $x \in \overline{\epsilon(H)}$ and $f \in LUC(H, \omega^{-1})$. For every $s \in G$,

$$x\hat{f}(s) = \langle x, L_s\hat{f} \rangle = \lim_{\epsilon(h) \rightarrow x} \hat{f}(sh) = \begin{cases} xf(s) & \text{if } s \in H \\ 0 & \text{if } s \notin H, \end{cases}$$

and so $(xf)^\sim = x\hat{f}$. Now if $(y_\alpha)$ is a net in $\overline{\epsilon(H)}$ converging to $y$, then

$$\langle L_{\hat{\mu}}(y_\alpha), f \rangle = \langle \hat{\mu}, y_\alpha f \rangle = \langle \mu, (y_\alpha f)^\sim \rangle = \langle \mu, y_\alpha \hat{f} \rangle \rightarrow \langle \mu \sqcap y, \hat{f} \rangle = \langle L_\mu(y), f \rangle$$

correspondence because $L_\mu$ is continuous at $y$. Hence $L_{\hat{\mu}}$ is continuous at $y$. \hfill $\square$
The following result is known from [20] and [6], but our methods are very different.

**Theorem 5.** The topological centre of \( LUC(\omega^{-1})^* \) is \( M(\omega) \).

**Proof.** Suppose that \( \mu \) is in the topological centre of \( LUC(\omega^{-1})^* \). Since \( M(\omega) \) is contained in the topological centre, we may assume without loss of generality that \( \mu \in C_0(\omega^{-1})^\perp \). Suppose that \( \mu \neq 0 \). Pick \( \xi \) from \( \text{supp} \mu \subseteq \Delta \) with minimal height and construct an open subgroup \( H \) of \( G \) such that \( \xi \in U_\Delta(H) \) and \( \kappa(E \cap H) = \kappa(H) \) (this is obtained by taking first a subset \( A \) of \( G \) such that \( \xi \in \overline{\epsilon(A)} \) and \( \kappa(A) \) equals the height of \( \xi \), then any subset \( B \) of \( E \) with \( \kappa(B) = \kappa(A) \) and a compact neighbourhood \( V \) of the identity, and finally defining \( H \) as the subgroup generated by \( A \cup B \cup V \)). Then \( \hat{\mu} \neq 0 \) and \( \text{supp} \hat{\mu} \subseteq U_\Delta(H) \) due to the minimality of the height of \( \xi \). Since \( \mu \) is in the topological centre of \( LUC(\omega^{-1})^* \), the map

\[
L_{\hat{\mu}}: x \mapsto \hat{\mu} \cdot x: \Delta(H) \to LUC(H, \omega^{-1})^*
\]

is continuous by Lemma 4. This contradicts Lemma 2 when applied to \( H \). \( \square \)

**Theorem 6.** The topological centre of \( C_0(\omega^{-1})^\perp \) is \( \{0\} \). If \( G \) is \( \sigma \)-compact, then it is enough to check the continuity of left translations at \( [K] + 1 \) points.

**Proof.** The argument is similar to the proof of [10, Theorem 17]. For the first statement, it is enough to show that any element \( \mu \) in the topological centre of \( C_0(\omega^{-1})^\perp \) is in the topological centre of \( LUC(\omega^{-1})^* \). To this end, fix a right cancellable point \( x \) in \( C_0(\omega^{-1})^\perp \). For the definition of right cancellable, see section 6. Any point in \( \overline{T(\mu)} \), where \( T \) is the set used throughout the paper starting from Lemma 2, is right cancellable, see [9, Theorem 10]. If \( (\nu_\alpha) \) is a bounded net in \( LUC(\omega^{-1})^* \) converging to \( \nu \) in the weak*-topology of \( LUC(\omega^{-1})^* \), then \( \nu_\alpha \cdot x \rightarrow \nu \cdot x \) in \( C_0(\omega^{-1})^\perp \) with respect to the relative weak*-topology. Hence \( \mu \cdot \nu_\alpha \cdot x \rightarrow \mu \cdot \nu \cdot x \). On the other hand, the net \((\mu \cdot \nu_\alpha)\) clusters at some \( \eta \in LUC(\omega^{-1})^* \) due to boundedness. But then \( \eta \cdot x = \mu \cdot \nu \cdot x \) and since \( x \) is right cancellable, we have \( \eta = \mu \cdot \nu \). Therefore \( \mu \cdot \nu \) is the unique cluster point of \((\mu \cdot \nu_\alpha)\), and so \( \mu \cdot \nu_\alpha \rightarrow \mu \cdot \nu \). This shows that the left translation by \( \mu \) is weak*-continuous on bounded sets of \( LUC(\omega^{-1})^* \). Since bounded nets (in fact just nets from the group) were enough to deduce the topological centre in the argument of Theorem 5, the left translation by \( \mu \) is weak*-continuous on all of \( LUC(\omega^{-1})^* \). (It should be mentioned that this passage from bounded nets to general nets was wrongly argued in the proof of [10, Theorem 17].)

For the second statement, note that we only need to check the continuity of the left translation by \( \mu \) at the points \( x_1 \sqcup x, \ldots, x_n \sqcup x \), where \( x_1, \ldots, x_n \) are as in Theorem 3. \( \square \)

5. Dtc sets for \( L^1(\omega)^{**} \)

Again we assume that \( \omega \) is a weight on a non-compact locally compact group \( G \) and that \( \omega \) is diagonally bounded, with bound \( K \geq 1 \), on a subset \( E \subseteq G \) with \( \kappa(E) = \kappa(G) \).
Recall that $L_0^\infty(\omega^{-1})$ is the closure of the compactly supported functions in $L^\infty(\omega^{-1})$, and that $L_1^1(\omega)^\ast$ has an $\ell^1$-direct sum decomposition

$$L_1^1(\omega)^\ast = L_0^\infty(\omega^{-1})^* \oplus L_0^\infty(\omega^{-1})^\perp$$

(see [9, 18]). Note that $L_0^\infty(\omega^{-1})^\perp$ is a weak*-closed ideal in $L_1^1(\omega)^\ast$ consisting of the functionals annihilating $L_0^\infty(\omega^{-1})$.

The main part of the section is inspired by the work of Budak, İsk and Pym [3]. We shall first show that if $\mu$ is in the topological centre of $L_1^1(\omega)^\ast$ and $\mu = \mu_0 + \mu_1$ is the decomposition of $\mu$ according to (14), then $\mu_0$ is in $L_1^1(\omega)$. To this end, we say that $\mu \in L_1^1(\omega)^\ast$ is singular if for every $f \in L_1^1(\omega)$ we have $\mu \perp f$ as measures on $\Omega$. Since $L_1^1(G)$ is a band in $L_1^1(G)^\ast$ (by Lemma 3.5 of [3]) and the isometry $L_1^1(G)^\ast \to L_1^1(\omega)^\ast$ is a lattice isomorphism, also $L_1^1(\omega)$ is a band in $L_1^1(\omega)^\ast$. Hence $L_0^\infty(\omega^{-1})^\ast$ has an orthogonal decomposition $L_0^\infty(\omega^{-1})^\ast = L_1^1(\omega) \oplus L_0^\infty(\omega^{-1})^\perp$ where $L_0^\infty(\omega^{-1})^\perp$ denotes the singular elements in $L_0^\infty(\omega^{-1})^\ast$ (as argued in [3]; for more details on Banach lattices, see [19], in particular Theorem 1.2.9).

The following lemma is a weighted version of [3, Lemma 5.3]. The result can be deduced from [3], because the proof there relies only on the lattice structure of $L_1^1(G)^\ast$, which is the same as that of $L_1^1(\omega)^\ast$. For completeness, here is another proof directly for $L_1^1(\omega)^\ast$. For compact $F \subseteq G$, define

$$\omega_F^{-1}(s) = \begin{cases} \omega(s)^{-1} & \text{if } s \in F \\ 0 & \text{otherwise,} \end{cases}$$

and note that $\omega_F^{-1} \in L_1^1(\omega)$.

For $\mu \in L_1^1(\omega^{-1})^\ast$ and compact set $K \subseteq G$, let $\mu|_K$ denote the functional defined by $\langle \mu|_K, f \rangle = \langle \mu, 1_K f \rangle$ for $f \in L_1^1(\omega^{-1})$ (note that $1_K f$ denotes the pointwise product).

**Lemma 7.** Let $\mu \in L_0^\infty(\omega^{-1})^\ast$. Suppose that $\{K_n\}$ is a sequence of increasing compact sets such that $\|\mu|_{K_n}\| \to \|\mu\|$ as $n \to \infty$ (such a sequence always exists). Then there is a sequence of functions $(f_n) \subseteq L_1^1(\omega^{-1})$ such that

1. $\|f_n\|_\omega \leq 1$,
2. $f_n = 0$ off $K_n$,
3. $\lim_{n \to \infty} \langle \phi, f_n \rangle = 0$ whenever $\phi \in L_1^1(\omega)$,
4. $\lim_{n \to \infty} \langle \mu, f_n \rangle = \|\mu_s\|$.

**Proof.** Write $\mu = \mu_{ab} + \mu_s$ where $\mu_{ab} \in L_1^1(\omega)$ and $\mu_s$ is singular. Fix a natural number $n$. Identify elements of $L_1^1(\omega)^\ast$ with bounded Radon measures on the spectrum $\Omega$ of $L_1^1(\omega^{-1})$; note that this identification preserves the lattice structure, and so $\mu_s \perp \omega_{K_n^{-1}}$ as measures on $\Omega$. To simplify notation, write $\nu_n = \mu_s|_{K_n}$. Since $\nu_n \perp \omega_{K_n^{-1}}$, there exists a Borel measurable set $A \subseteq \Omega$ such that $\|\nu_n\| = \|\nu_n\|$ and $\omega_{K_n^{-1}}(A) = 0$. By regularity, there is $g \in L_1^1(\omega^{-1})$ with $0 \leq g \leq \omega$ such that $\langle |\nu_n|, g \rangle \geq \|\nu_n\| - 1/n$ and $\langle \omega_{K_n^{-1}}, g \rangle \leq 1/n$. Choose $h \in
$L^\infty(\omega^{-1})$ such that $|h| \leq g$ and $\langle \nu_n, h \rangle \geq |\nu_n|, g - 1/n$. Define $f_n \in L^\infty(\omega^{-1})$ by putting $f_n = h$ on $K_n$ and $f_n = 0$ off $K_n$. The first two statements are then immediate.

To see that the third statement holds, note that for $\psi \in C_c(G)$ (the compactly supported continuous functions on $G$)

\[
\left| \int_G \psi(s) f_n(s) \, ds \right| = \left| \int_{K_n} \psi(s) f_n(s) \, ds \right|
\leq \sup_{s \in G} |\psi(s)\omega(s)| \int_{K_n} \omega^{-1}(s)|f_n(s)| \, ds
\leq \sup_{s \in G} |\psi(s)\omega(s)| \langle \omega^{-1} K_n, g \rangle \leq \frac{\sup_{s \in G} |\psi(s)\omega(s)|}{n}.
\]

So $\int_G \psi(s) f_n(s) \, ds \to 0$ for every $\psi \in C_c(G)$ and it follows that $\langle \phi, f_n \rangle \to 0$ whenever $\phi \in L^1(\omega)$.

As for the fourth statement, note that

\[
|\langle \mu, f_n \rangle| \geq |\langle \nu_n, f_n \rangle| - |\langle \mu_{ab}, f_n \rangle| \geq \|\nu_n\| - 2/n - |\langle \mu_{ab}, f_n \rangle| \to \|\mu\|
\]

as $n \to \infty$. \hfill \Box

**Lemma 8.** If $\mu \in L_0^\infty(\omega^{-1})^*$ is in the topological centre of $L^1(\omega)^{**}$, then $\mu$ is in $L^1(\omega)$.

**Proof.** Let $\{K_n\}$ be a sequence of increasing compact sets such that $\mu|_{K_n} \to \mu$ in norm, and let by Lemma 7, $(f_n) \subseteq L^\infty(\omega^{-1})$ be the sequence of functions obtained for $\mu$ and $\{K_n\}$. Then pick a sequence $(y_n) \subseteq E$ such that

\[
K_n y_n \cap K_m y_m = \emptyset
\]

whenever $n \neq m$. Then the function

\[
h = \sum_{n=1}^\infty \omega(y_n) R_{y_n^{-1}} f_n
\]

is in $L^\infty(\omega^{-1})$. For every $\phi \in L^1(\omega)$ supported by $K_m$, we have

\[
\langle \phi, \epsilon(y_m) \cdot h \rangle = \frac{h\phi(y_m)}{\omega(y_m)} = \sum_{n=1}^\infty \frac{\omega(y_n)}{\omega(y_m)} \int f_n(sy_n y_n^{-1})\phi(s) \, ds
\]

\[
= \langle \phi, f_m \rangle,
\]

as $\phi$ is supported by $K_m$ and $K_n y_n \cap K_m y_m = \emptyset$ for $n \neq m$. It then follows that

\[
\langle \mu|_{K_n}, \epsilon(y_m) \cdot h \rangle = \langle \mu|_{K_n}, f_m \rangle
\]

for $m \geq n$. Now

\[
|\langle \mu, \epsilon(y_m) \cdot h \rangle - \langle \mu, f_m \rangle| \leq 2\|\mu - \mu|_{K_n}\| + |\langle \mu|_{K_n}, \epsilon(y_m) \cdot h - f_m \rangle|
\]

\[
= 2\|\mu - \mu|_{K_n}\|
\]

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for $m \geq n$, and so
\begin{equation}
\lim_{m \to \infty} \langle \mu, \epsilon(y_m) \cdot h \rangle = \lim_{m \to \infty} \langle \mu, f_m \rangle = \|\mu_s\|
\end{equation}
(16)
by Lemma 7.

Let $y$ be a cluster point of the sequence $(\epsilon(y_n))_{n=1}^{\infty}$ in $\Delta$. Since $\mu$ is in the
topological centre of $L^1(\omega)^{**}$, it follows from (16) that
\begin{equation}
\langle \mu \cdot y, h \rangle = \|\mu_s\|.
\end{equation}
(17)

On the other hand, it follows from (15) and Lemma 7 that
\begin{equation}
\langle \phi \cdot y, h \rangle = \lim_{m \to \infty} \langle \phi, f_m \rangle = 0
\end{equation}
for every $\phi \in L^1(\omega)$ supported by any $K_n$. As $\mu|_{K_n} \to \mu$, we may take $\phi$ to $\mu$
in the weak*-topology, and so
\begin{equation}
\langle \mu \cdot y, h \rangle = 0.
\end{equation}
Combining this with (17) we see that $\mu_s = 0$, and so $\mu \in L^1(\omega)$.

Recall that $\Phi: L^1(\omega)^{**} \to LUC(\omega^{-1})^*$ is the natural quotient map, which
maps $L_0^\infty(\omega^{-1})^*$ onto $M(\omega)$ and $L_0^\infty(\omega^{-1})^\perp$ onto $C_0(\omega^{-1})^\perp$.

**Lemma 9.** Suppose that $G$ is a locally compact SIN group and that $V$ is a dtc
set for $LUC(\omega^{-1})^*$. For every $y \in V$, let $\tilde{y} \in L^1(\omega)^{**}$ such that $\Phi(\tilde{y}) = y$. If
$\mu \in L^1(\omega)^{**}$ and
\begin{equation}
\mu \Box \tilde{y} = \mu \circ \tilde{y}
\end{equation}
for every $y \in V$, then $\Phi(\mu) \in M(\omega)$.

**Proof.** Since $G$ is SIN, there are two well-defined products $\Box$ and $\circ$ on $LUC(\omega^{-1})^*$.
Suppose that
\begin{equation}
\mu \Box \tilde{y} = \mu \circ \tilde{y}
\end{equation}
for every $y \in V$. Since $\Phi$ is a homomorphism with respect to the first products
$\Box$ as well as the second products $\circ$, we have that
\begin{equation}
\Phi(\mu) \Box y = \Phi(\mu) \circ y
\end{equation}
for every $y \in V$. Since $V$ is a dtc set for $LUC(\omega^{-1})^*$, we have that $\Phi(\mu) \in M(\omega)$
(via Lemma 1).

The first part of the following result is known to be true for any locally
compact group; see [20] and [6].

**Theorem 10.** Suppose that $G$ is a locally compact SIN group. The topological
centre of $L^1(\omega)^{**}$ is $L^1(\omega)$. When $G$ is also $\sigma$-compact, there exists a dtc set of
$n = |K| + 2$ points in $\Omega$. 15
Proof. Let \( \mu \) be in the topological centre of \( L^1(\omega)^{**} \), and decompose \( \mu \) as in (14) with \( \mu_0 \) as the local component and \( \mu_1 \) as the component at infinity. It follows from Lemma 9 that \( \Phi(\mu) \in M(\omega) \). Since \( \Phi(\mu_0) \in M(\omega) \), we have \( \Phi(\mu_1) \in M(\omega) \). But since \( \mu_1 \in L_0^\infty(\omega^{-1})^\perp \), we must have \( \Phi(\mu_1) = 0 \). This means that \( \mu_1 = 0 \) on \( LUC(\omega^{-1}) \). Since \( G \) is SIN, there is a central bounded approximate identity \((e_\alpha)\) in \( L^1(\omega) \), with the supports contained in some common compact set. Note that each \( e_\alpha \) is also in the algebraic centre of \( L^1(\omega)^{**} \). In particular, for every \( \alpha \) and for every \( f \in L^\infty(\omega^{-1}) \),

\[
\langle \mu_1 \Box e_\alpha, f \rangle = \langle e_\alpha \Box \mu_1, f \rangle = \langle e_\alpha, \mu_1 f \rangle = \langle \mu_1, fe_\alpha \rangle = 0.
\]

So, if \( \nu \) is a weak* cluster point of \((e_\alpha)\), then \( \nu \) is a right identity in \( L^1(\omega)^{**} \) and

\[
\mu = \mu_0 + \mu_1 = (\mu_0 + \mu_1) \Box \nu = \operatorname{w*-lim}_\alpha (\mu_0 + \mu_1) \Box e_\alpha
\]

\[
= \operatorname{w*-lim}_\alpha (\mu_0 \Box e_\alpha + \mu_1 \Box e_\alpha) = \operatorname{w*-lim}_\alpha \mu_0 \Box e_\alpha = \operatorname{w*-lim}_\alpha \mu_0 \Box \mu_0
\]

Now \( \mu_0 \) is the norm limit of a compactly supported functionals \( \mu_{0,n} \) in \( L_0^\infty(\omega^{-1})^* \), \( n = 1, 2, \ldots \). For every \( n \), there is a compact set \( K_n \) such that the support of \( e_\alpha \Box \mu_{0,n} \) is contained in \( K_n \) for every \( \alpha \). Hence \( \nu \Box \mu_{0,n} = \operatorname{w*-lim}_\alpha e_\alpha \Box \mu_{0,n} \) is also supported by \( K_n \). Therefore \( \mu_0 \) as the norm limit of the sequence \( \nu \Box \mu_{0,n} \) is in \( L_0^\infty(\omega^{-1})^* \), and so \( \mu = \mu_0 \). Consequently, \( \mu_0 \) is in the topological centre of \( L^1(\omega)^{**} \), and by Lemma 8, \( \mu = \mu_0 \in L^1(\omega) \), as required.

Now suppose that \( G \) is also \( \sigma \)-compact. Let \( x_1, x_2, \ldots, x_{n-1} \) be the points in \( \Delta \) given by Theorem 3. For every \( k = 1, 2, \ldots, n-1 \), pick \( y_k \in \Omega \) such that \( \Phi(y_k) = x_k \) (that this is possible, recall that \( \Phi \) maps \( \Omega \) onto \( \Delta \) by [21, Lemma 4.1.7]). If

\[
\mu \Box y_k = \mu \circ y_k \quad \text{for every} \quad k = 1, 2, \ldots, n-1,
\]

then by Lemma 9 \( \Phi(\mu) \in M(\omega) \) and hence \( \Phi(\mu_1) = 0 \). To deduce that \( \mu_1 = 0 \), we need as above one right identity in \( \Omega \), taking the number of necessary points to \([K]+2\). Note that any element in \( \Phi^{-1}(\delta_c) \Box \delta_c \) is a right identity, and \( \Phi^{-1}(\delta_c) \Box \Omega \) is non-empty as \( \delta_c \in \Delta \). Finally, to deduce that \( \mu_0 \in L^1(\omega) \), we need to apply Lemma 8. This will not add to the number of necessary points as the element \( y \) used in Lemma 8 can be one of the \( x_k \)'s (so effectively \( y_k \)). \( \square \)

The same argument, which gives Theorem 6 (using [9, Theorem 10]), proves the following.

**Theorem 11.** Suppose that \( G \) is SIN. The topological centre of \( L_0^\infty(\omega^{-1})^\perp \) is \( \{0\} \). If \( G \) is also \( \sigma \)-compact, it is enough to check the continuity at \([K]+2\) points.
6. Dtc sets for $\ell^1(\omega)^{**}$

Let $S$ be a discrete semigroup and consider the weighted semigroup algebra $\ell^1(\omega)$ where $\omega : S \to (0, \infty)$ is a submultiplicative weight function. We want to show that the topological centre of $\ell^1(\omega)^{**}$ is $\ell^1(\omega)$ under some conditions on $S$ and $\omega$. This case is very similar to the case of $LUC(\omega^{-1})^*$ considered in section 4.

The spaces $\ell^1(\omega)$ and $\ell^\infty(\omega^{-1})$ are defined via isometries

$$f \mapsto \omega f : \ell^1(\omega) \to \ell^1(S)$$

and

$$f \mapsto \omega^{-1} f : \ell^\infty(\omega^{-1}) \to \ell^\infty(S).$$

Then $\ell^1(\omega)$ is a Banach algebra with respect to the convolution product and $\ell^\infty(\omega^{-1})$ is a C*-algebra with respect to the weighted pointwise product. We let $\pi : \ell^1(S)^{**} \to \ell^1(\omega)^{**}$ be the adjoint of the $*$-isomorphism $f \mapsto \omega f$. Similarly to the previous case, we denote the spectrum of $\ell^\infty(\omega^{-1})$ by $\Delta$, and let $\epsilon : S \to \Delta$ be the map

$$\langle \epsilon(s), f \rangle = \frac{f(s)}{\omega(s)} \quad (f \in \ell^\infty(\omega^{-1})).$$

Note that the spectrum of $\ell^\infty(S)$ is the Stone–Čech compactification $\beta S$ of $S$, and so $(\epsilon, \Delta)$ is a realisation of the Stone–Čech compactification of $S$. We define the height of points in $\Delta$ similarly as before, and denote by $U_\Delta$ the set of points with the maximal height $|S|$. We define also slowly oscillating functions as in the group case.

We say that $\omega$ is diagonally bounded on $E \subseteq S$ with bound $K > 0$ if

$$\omega(s) \omega(t) \leq K \omega(st) \quad \text{for every } s \in S \text{ and } t \in E.$$  

(In the case when $S$ is a group this is equivalent to the previous definition.)

We say that an element $s$ in a semigroup $S$ is right cancellable if $t_1 s = t_2 s$ implies $t_1 = t_2$ whenever $t_1, t_2 \in S$; left cancellable elements are defined analogously. A semigroup $S$ is right cancellative if every element in $S$ is right cancellable, and $S$ is weakly left cancellative if for every fixed $s, u \in S$, the equation $st = u$ has finitely many solutions $t \in S$. A weakly cancellative semigroup is both weakly left and weakly right cancellative.

Throughout this last section, $S$ is an infinite discrete, right cancellative, weakly cancellative semigroup and $\omega$ is a weight on $S$ that is diagonally bounded, with bound $K$, on $E \subseteq S$ with $|E| = |S|$.

**Lemma 12.** There is a subset $T \subseteq E$, with $|T| = |E|$, such that

1. the points in $\overline{T} \subseteq \beta S$ can be separated by slowly oscillating functions;
2. for every $x \in \epsilon(T) \cap U_\Delta$ and $\nu \in \ell^1(\omega)^{**}$

$$\frac{\|\nu\|}{K} \leq \|\nu \square x\|.$$
Proof. Let \( \{ S_\alpha \}_{\alpha < |S|} \) be an increasing cover of \( S \) as constructed in [10, Lemma 7]; in particular, \(|S_\alpha|\) is finite when \(|\alpha| \) is finite and \(|S_\alpha| = |\alpha| \) otherwise. We may assume without loss of generality that \( S \) has an identity element \( e \) and \( e \in S_0 \). By transfinite induction, there is a subset \( T = \{ t_\alpha \}_{\alpha < |S|} \) of \( E \) such that \( S_\alpha t_\alpha S_\alpha \cap S_\beta t_\beta S_\beta = \emptyset \) when \( \alpha \neq \beta \) (note that weak cancellation is needed at this point). Then the points in the closure of \( T \) in \( \beta S \) can be separated by slowly oscillating functions, as constructed in [10, Lemma 7], so the first statement holds.

To prove the second statement, it suffices in fact that the set \( T \) satisfies that \( S_\alpha t_\alpha \cap S_\beta t_\beta = \emptyset \) whenever \( \alpha \neq \beta \). We show that each function \( f \in \ell^\infty(\omega^{-1}) \) factorises as \( f = xg \), where \( x \in \overline{\ell(T)} \cap \mathcal{U}_\Delta \) and \( g \in \ell^\infty(\omega^{-1}) \). To see this, consider for a given \( f \in \ell^\infty(\omega^{-1}) \), the function

\[
g(s) = \sum_\alpha \omega(t_\alpha) 1_{S_\alpha t_\alpha}(s) f_\alpha(s) \quad (s \in S)
\]

where \( 1_{S_\alpha t_\alpha} \) is the characteristic function of \( S_\alpha t_\alpha \) and

\[
f_\alpha(s) = \begin{cases} f(u) & \text{if } s = ut_\alpha, \\ 0 & \text{otherwise.} \end{cases}
\]

Note that \( f_\alpha \) is well defined since \( S \) is right cancellative and so is \( g \) since \( \{ S_\alpha t_\alpha \} \) is a disjoint family. Since \( \omega \) is diagonally bounded on \( T \) with bound \( K \), we have \( \omega(t_\alpha) \omega(ut_\alpha)^{-1} \leq K \omega(u)^{-1} \) and so

\[
|g(s)\omega(s)^{-1}| \leq K \sum_\alpha 1_{S_\alpha t_\alpha}(s) \|f\|_\omega \leq K \|f\|_\omega \tag{18}
\]

for every \( s \in S \).

Let now \( x \) be any point in \( \overline{\ell(T)} \cap \mathcal{U}_\Delta \) and let \( \epsilon(t_\alpha, x) \to x \). Let \( s \in S \) and pick \( \beta \) such that \( s \in S_\beta \). We may suppose that \( \alpha_\gamma \geq \beta \) for every \( \gamma \). Then

\[
xg(s) = (x, L_\gamma g) = \lim_\gamma \omega(t_\alpha)^{-1} g(st_\alpha)
\]

\[
= \lim_\gamma \sum_\alpha \omega(t_\alpha)^{-1} \omega(t_\alpha) 1_{S_\alpha t_\alpha}(st_\alpha) f_\alpha(st_\alpha).
\]

Note that \( 1_{S_\alpha t_\alpha}(st_\alpha) = 0 \) if \( \alpha \neq \alpha_\gamma \) and that \( f_\alpha(st_\alpha) = f(s) \) if \( \alpha = \alpha_\gamma \). It follows that \( xg(s) = f(s) \), and so we have our wanted factorisation.

If now \( \mu \in \ell^\infty(\omega^{-1})^* \) is non-zero and \( \epsilon > 0 \) is given, pick \( f \) from the unit ball of \( \ell^\infty(\omega^{-1}) \) such that \( |\langle \mu, f \rangle| > \|\mu\| - \epsilon \). If \( g \) is as above, then

\[
|\langle \mu \square x, g \rangle| = |\langle \mu, f \rangle| \geq \|\mu\| - \epsilon.
\]

As \( \|g\|_\omega \leq K \) by (18), we have

\[
\|\mu \square x\| \geq \frac{\|\mu\|}{K}.
\]

\( \square \)
After choosing $T$ as in the preceding lemma, repeating the proof of Lemma 2 gives the following result.

**Lemma 13.** There exist $n = |K| + 1$ points $x_1, x_2, \ldots, x_n$ in $\Delta$ such that $\mu = 0$ is the only element in $\ell^1(\omega)^{**}$ with $\text{supp} \mu \subseteq U_\Delta$ having the property that $\lim_{\alpha} \mu \square e(s_\alpha) = \mu \square x_k$ whenever $k = 1, \ldots, n$ and $(s_\alpha)$ is a net in $S$ with $\lim_{\alpha} e(s_\alpha) = x_k$ in $\Delta$.

The next result follows immediately from the preceding lemma. Note that Dales and Dedania [5, Theorem 5.6] proved a similar result under the assumptions that $S$ is both left and right cancellative, countable semigroup and that the weight $\omega$ is weakly diagonally bounded on an infinite $E \subseteq S$.

**Theorem 14.** Suppose in addition that $S$ is countable. Then $\Delta$ contains a dtc set for $\ell^1(\omega)^{**}$ of cardinality $|K| + 1$. In other words, there exist $n = |K| + 1$ points $x_1, x_2, \ldots, x_n$ in $\Delta$ such that if $\mu \in \ell^1(\omega)^{**}$ and $\lim_{\alpha} \mu \square e(s_\alpha) = \mu \square x_k$ whenever $k = 1, \ldots, n$ and $(s_\alpha)$ is a net in $S$ with $\lim_{\alpha} e(s_\alpha) = x_k$ in $\Delta$, then $\mu \in \ell^1(\omega)$.

**Theorem 15.** The topological centre of $\ell^1(\omega)^{**}$ is $\ell^1(\omega)$.

**Proof.** Let $\mu$ be in the topological centre of $\ell^1(\omega)^{**}$. Suppose that $\mu \in c_0(\omega^{-1})^\perp$ so that it is enough to show that $\mu = 0$. Assume towards contradiction that $\mu \neq 0$ and pick an element $\xi$ from the support of $\mu$ in $\Delta$ such that the height of $\xi$ is minimal (but note that the height of $\xi$ is infinite). There is a subtlety in the construction of a subsemigroup $S_0$ such that an analogue of Lemma 4 holds for $S_0$. Let $A \subseteq S$ such that $|A| = |A \cap E|$ is equal to the height of $\xi$ and $\xi$ is in the closure of $e(A)$. Put $A_0 = A$ and define inductively $A_{n+1} = A_n \cup A_n^2 \cup A_n A_n^{-1}$ (where $A_n A_n^{-1}$ denotes those $t \in S$ such that $ts \in A_n$ for some $s \in A_n$). Since $S$ is right cancellative, $|A_n| = |A|$ for every $n$ and so $S_0 := \bigcup_{n=0}^{\infty} A_n$ is a subsemigroup of $S$ with $|S_0|$ equal to the height of $\xi$. Moreover, Lemma 4 applies when $G$ and $H$ are replaced by $S$ and $S_0$, respectively (that $xf = (xf)^\sim$ requires that $S_0 S_0^{-1} \subseteq S_0$, which is guaranteed by the construction of $S_0$). Identifying the closure of $e(S_0)$ in $\Delta$ with the spectrum of $\ell^\infty(S_0, \omega^{-1})$, we have $\xi \in U_\Delta(S_0)$ and $\omega$ is diagonally bounded on the set $E_0 := E \cap S_0$ of cardinality $|S_0|$. Therefore we may apply Lemma 13 to see that the restriction of $\mu$ to $e(S_0)$ is 0. This contradicts the fact that $\xi \subseteq \text{supp} \mu$. $\Box$

The following result is proved similarly as Theorem 6.

**Theorem 16.** The topological centre of $c_0(\omega^{-1})^\perp$ is trivial. If $S$ is in addition countable, then it is enough to check the continuity at $|K| + 1$ points.

**Remark 17.** In Theorems 3 and 14, the dtc sets are picked from $\overline{e(T)} \cap U_\Delta$. Since $T$ is right uniformly discrete and countably infinite, this set has the same cardinality as the set of points in $\overline{T \setminus T}$ (the closure in $G^{LUC}$) and $\overline{T}$ may be identified with the Stone–Čech compactification $\beta T$ of $T$. Thus $\overline{e(T)} \cap U_\Delta$ has cardinality $2^c$. Any $|K| + 1$ points from this set form a dtc set in the above-mentioned results. In Theorem 10, we further need one right identity.
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References


