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Topological centres of weighted convolution algebras

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Abstract

Let G be a non-compact locally compact group with a continuous submultiplicative weight function ω such that $\omega(e) = 1$ and ω is diagonally bounded with bound $K \geq 1$. When G is σ-compact, we show that $\lfloor K \rfloor + 1$ many points in the spectrum of $LUC(\omega^{-1})$ are enough to determine the topological centre of $LUC(\omega^{-1})^*$ and that $\lfloor K \rfloor + 2$ many points in the spectrum of $L^\infty(\omega^{-1})$ are enough to determine the topological centre of $L^1(\omega)^{**}$ when G is in addition a SIN-group. We deduce that the topological centre of $LUC(\omega^{-1})^*$ is the weighted measure algebra $M(\omega)$ and that of $C_0(\omega^{-1})^\perp$ is trivial for any locally compact group. The topological centre of $L^1(\omega)^{**}$ is $L^1(\omega)$ and that of of $L_0^\infty(\omega)^\perp$ is trivial for any non-compact locally compact SIN-group. The same techniques apply and lead to similar results when G is a weakly cancellative right cancellative discrete semigroup.

Keywords: Arens product, weighted algebra, topological centre, dtc set
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1. Introduction

Investigations related to topological centres go back all the way to Richard Arens [1, 2], who defined two products on the second dual of a Banach algebra (in fact, even in a more general set up). He proved that the two products coincide for the ring $C(X)$ of continuous functions on a compact Hausdorff space X as well as for ℓ^1 with the pointwise multiplication, but the two products do not coincide for $A = \ell^1$ with convolution as product. Note that the two products coincide on the second dual of a *commutative* Banach algebra if and only if the second dual itself is commutative under either product. The work of Arens was followed by the seminal articles by Day [8] in 1957 and by Civin and Yood [4] in 1961, where the second duals of group algebras of infinite locally compact

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abelian groups and of some amenable discrete infinite semigroups were shown to be non-commutative. For a more detailed account, see the survey [12].

Grosser and Losert [14] proved that the centre of $UC(G)^*$ is the measure algebra $M(G)$ when G is abelian. A natural extension of this result is the theorem of Lau [16] stating that for any locally compact group G, the topological centre of $LUC(G)^*$ is $M(G)$. Then Lau and Losert [17] proved that the topological centre of $L^1(G)^{**}$ is the group algebra $L^1(G)$. These results were followed by numerous papers studying the topological centres of other algebras arising in abstract harmonic analysis such as the second dual of the Fourier algebra $A(G)$ or algebras associated to discrete semigroups.

One of the points which has attracted attention more recently is the number of points necessary to determine the topological centres. This was initiated in [7], where Dales, Lau and Strauss proved that two points are sufficient to determine the topological centre of $LUC(G)^*$. In [10], we also proved that two points are enough to determine the topological centre of $LUC(G)^*$ when G is σ-compact. In [5], Dales and Dedania considered the weighted semigroup algebra $\ell^1(\omega)$ of a discrete, countably infinite, cancellative semigroup S with a weight ω on S that is weakly diagonally bounded on an infinite subset T of G with a bound $K \geq 1$. Dales and Dedania showed that there is a set V consisting of $\lfloor K \rfloor + 1$ points such that V is determining for the topological centre of $\ell^\infty(\omega^{-1})^*$, where $\lfloor K \rfloor$ denotes the integer part of K. More recently, Budak, Işık and Pym proved in their exquisite paper [3] that continuity at one point is enough to determine the topological centres of $LUC(G)^*$ and $L^1(G)^{**}$.

In this paper, we carry on our investigation on topological centres of convolution algebras started in [10]. The method proposed in [10] combined with elements from [7], [5] and [3] leads also to the topological centres in the weighted cases, assuming the weight is diagonally bounded. Moreover, in the σ-compact case, we find finite sets determining the topological centres (for the case of the second dual $L^1(\omega)^{**}$ of the weighted group algebra $L^1(\omega)$ we need that G is SIN).

Throughout the paper the weight ω will be diagonally bounded on $E \subseteq G$ with bound K, where E has the same compact covering number as G (precise definitions are given in Section 2). We shall consider the C*-algebras $LUC(\omega^{-1})$, $C_0(\omega^{-1})$, $L^\infty(\omega^{-1})$ and $L_0^\infty(\omega^{-1})$, defined as weighted function algebras. We prove in Section 3 that if G is σ-compact, then $\lfloor K \rfloor + 1$ many points in the spectrum Δ of $LUC(\omega^{-1})$ are enough to determine the topological centre of $LUC(\omega^{-1})^*$. In general, we show that the topological centre of $LUC(\omega^{-1})^*$ is the weighted measure algebra $M(\omega)$. Similarly, the topological centre of $C_0(\omega^{-1})^\perp$ is trivial for any locally compact group, and in the case of σ-compact groups, it is enough to check continuity at $\lfloor K \rfloor + 1$ many points.

The main technique behind the proofs is to use points x in the spectrum Δ of $LUC(\omega^{-1})$ that satisfy the bi-Lipschitz property

$$\frac{\|\mu\|}{K} \leq \|\mu \,\Box\, x\| \leq \|\mu\| \tag{1}$$

and that can be separated by slowly oscillating functions. Right isometries and slowly oscillating functions were the main tools in [10]. The use of points with the bi-Lipschitz property replaces the use of right isometries in the unweighted case. The first inequality in (1) is special for these points: it is due to two crucial facts, first, x is taken from the closure of a specially constructed set T giving x a factorisation property, and second, the weight ω is diagonally bounded on T. These techniques are then combined with methods from [7] and [5]. We should note that right cancellation, factorisation and right isometries are the three basic properties used by different authors to find the topological centres. In [11], they are shown to be basically the same, under certain assumptions.

We continue in Section 4 with $L^1(\omega)^{**}$. With the help of an argument based on [3], we prove that $\lfloor K \rfloor + 2$ many points are enough to determine the topological centres of $L^1(\omega)^{**}$ and $L_0^\infty(\omega^{-1})^\perp$ when G is a σ-compact SIN-group. Our method shows also that the topological centre of $L^1(\omega)^{**}$ is $L^1(\omega)$ and the topological centre of $L_0^\infty(\omega)^\perp$ is trivial for any non-compact locally compact SIN-group.

Our method applies also to weakly cancellative, right cancellative discrete semigroups, as presented in Section 6. We prove that $\lfloor K \rfloor + 1$ is again the number of points enough to determine the centres of $\ell^1(\omega)^{**}$ and $c_0(\omega^{-1})^\perp$ when S is countable; the result for $\ell^1(\omega)^{**}$ is thus similar to the result of Dales and Dedania [5, Theorem 5.6] but our assumptions on the semigroup and the weight are slightly different. Moreover, we also deduce that the topological centres of $\ell^1(\omega)^{**}$ and $c_0(\omega^{-1})^\perp$ are respectively $\ell^1(\omega)$ and $\{0\}$, without assuming that S is countable.

2. Preliminaries

2.1. Function algebras

In this paper, a *weight* on a locally compact group G means a continuous function $\omega\colon G \to (0,\infty)$ such that the value of ω at the identity $e \in G$ is 1 and ω is submultiplicative, that is,

$$\omega(st) \le \omega(s)\omega(t) \qquad \text{for every } s,t \in G.$$

For any function space $F(G)$, we let $F(\omega^{-1})$ denote the weighted analogue defined by

$$F(\omega^{-1}) = \{\, f\colon G \to \mathbb{C};\ \omega^{-1} f \in F(G)\,\}$$

with the norm that makes the map

$$f \mapsto \omega^{-1} f\colon F(\omega^{-1}) \to F(G) \tag{2}$$

an isometry. We shall apply this construction to the the C*-algebra $C_0(G)$ of all continuous functions vanishing at infinity, to the C*-algebra $LUC(G)$ of the bounded left uniformly continuous functions on G, to the C*-algebra $L^\infty(G)$ of essentially bounded locally measurable functions on G and to the C*-algebra $L_0^\infty(G)$. For $f \in L^\infty(G)$, we put $\|f\|_K = \operatorname{ess\,sup}\{\, |f(x)|;\ x \in K\,\}$, and define

$$L_0^\infty(G) = \{\, f \in L^\infty(G);\ \text{for } K \text{ compact } \|f\|_{G\setminus K} \to 0 \text{ as } K \to G\,\}.$$

The resulting weighted function spaces are C*-algebras under the weighted pointwise product

$$f \cdot_\omega g(s) = \frac{f(s)g(s)}{\omega(s)},$$

and the map $f \mapsto \omega^{-1} f\colon F(\omega^{-1}) \to F(G)$ is a $*$-isomorphism.

2.2. Convolution algebras

The space $M(\omega)$ consists of all Radon measures μ on G such that the weighted measure $\omega\mu$ is bounded. This space is normed such that the map $\mu \mapsto \omega\mu\colon M(\omega) \to M(G)$ is an isometry, where $M(G)$ is the usual measure algebra consisting of bounded Radon measures on G. The measure algebra $M(G)$ is the dual space of $C_0(G)$, and similarly $M(\omega)$ is the dual space of $C_0(\omega^{-1})$. Since ω is submultiplicative, $M(\omega)$ is a Banach algebra under the convolution product

$$\langle \mu * \nu, f\rangle = \iint f(st)\, d\mu(s)\, d\nu(t) \qquad (\mu, \nu \in M(\omega), f \in C_0(\omega^{-1})).$$

Note that although $M(\omega)$ is isometric to $M(G)$ their Banach algebra structures may be very different, depending on ω.

Let $L^1(G)$ be the group algebra of G, consisting of those measures in $M(G)$ that are absolutely continuous with respect to the left Haar measure. The weighted group algebra $L^1(\omega) \subseteq M(\omega)$ is defined via the isometry $\mu \mapsto \omega\mu$. Then $L^1(\omega)$ is a closed two-sided ideal in $M(\omega)$, similarly as $L^1(G)$ is in $M(G)$. When $\omega \geq 1$, we have $L^1(\omega) \subseteq L^1(G)$ (with different norms); in this well-studied case $L^1(\omega)$ is called a *Beurling algebra*.

Let $\pi\colon LUC(G)^* \to LUC(\omega^{-1})^*$ be the adjoint of the isometry given in (2) for $F(G) = LUC(G)$; that is,

$$\langle \pi(\mu), f\rangle = \langle \mu, \omega^{-1} f\rangle \qquad (\mu \in LUC(G)^*, f \in LUC(\omega^{-1})).$$

Since $f \mapsto \omega^{-1} f$ is a $*$-isomorphism, π maps the spectrum G^{LUC} of $LUC(G)$ onto the spectrum Δ of $LUC(\omega^{-1})$. Define $\epsilon\colon G \to \Delta$ by

$$\langle \epsilon(s), f\rangle = \frac{f(s)}{\omega(s)} \qquad (f \in LUC(\omega^{-1})).$$

Then ϵ is a homeomorphism, and in fact

$$\epsilon(s) = \pi(s),$$

where s in the right-hand side is considered as an element of G^{LUC} (the point evaluation at s). We shall always use the identification $G \subseteq G^{LUC}$. We may also identify G with its image in Δ (topologically), but keep writing $\epsilon(s)$ in the weighted case to deter confusion.

Let A be a Banach algebra; in our interests A is either $L^1(\omega)$ or $\ell^1(\omega)$, the latter being the weighted semigroup algebra of a discrete semigroup S (see

Section 6 for more details). The first Arens product on the second dual of A is defined by the following formulas:

$$\begin{aligned} \langle fa, b\rangle &= \langle f, ab\rangle, && a, b \in A, f \in A^*, \\ \langle \nu f, a\rangle &= \langle \nu, fa\rangle, && \nu \in A^{**}, \\ \langle \mu \,\Box\, \nu, f\rangle &= \langle \mu, \nu f\rangle, && \mu \in A^{**}. \end{aligned} \tag{3}$$

This makes A^{**} a Banach algebra.

There is also the second Arens product on the second dual of any Banach algebra A; this is defined by the following formulas:

$$\begin{aligned} \langle af, b\rangle &= \langle f, ba\rangle, && a, b \in A, f \in A^*, \\ \langle f\mu, a\rangle &= \langle \mu, af\rangle, && \mu \in A^{**}, \\ \langle \mu \diamond \nu, f\rangle &= \langle \nu, f\mu\rangle, && \nu \in A^{**}. \end{aligned} \tag{4}$$

The product on $LUC(\omega^{-1})^*$ is defined by

$$\begin{aligned} L_s f(t) &= f(st) && s, t \in G, f \in LUC(\omega^{-1}), \\ \nu f(s) &= \langle \nu, L_s f\rangle, && \nu \in LUC(\omega^{-1})^*, \\ \langle \mu \,\Box\, \nu, f\rangle &= \langle \mu, \nu f\rangle, && \mu \in LUC(\omega^{-1})^*. \end{aligned}$$

Let $C_0(\omega^{-1})^\perp$ denote the functionals in $LUC(\omega^{-1})^*$ that annihilate $C_0(\omega^{-1}) \subseteq LUC(\omega^{-1})$. It is easily seen that $C_0(\omega^{-1})^\perp$ is a weak*-closed ideal in $LUC(\omega^{-1})^*$. Moreover,

$$LUC(\omega^{-1})^* \cong M(\omega) \oplus C_0(\omega^{-1})^\perp$$

as an ℓ^1-direct sum.

Let $\Phi\colon L^1(\omega)^{**} \to LUC(\omega^{-1})^*$ denote the natural quotient map, i.e., the adjoint of the inclusion $LUC(\omega^{-1}) \hookrightarrow L^\infty(\omega^{-1})$. Then Φ is a weak*-continuous homomorphism with respect to the first Arens product $\Box$ on $L^1(\omega)^{**}$ and the product $\Box$ on $LUC(\omega^{-1})^*$. Moreover, Φ maps $L_0^\infty(\omega^{-1})^*$ onto $M(\omega)$, $L_0^\infty(\omega^{-1})^\perp$ onto $C_0(\omega^{-1})^\perp$ and the spectrum Ω of $L^\infty(\omega^{-1})$ *onto* Δ (the last statement follows from [21, Lemma 4.1.7] for example).

It is a consequence of Cohen's factorisation theorem that $LUC(\omega^{-1}) = L^\infty(\omega^{-1})L^1(\omega)$, where the action of $L^1(\omega)$ on $L^\infty(\omega^{-1})$ defined by the first formula of (3) (see for example [13], there is a small gap in the proof of [13, Proposition 1.3], which is mended in [6, Proposition 7.15]). We therefore see that the Banach algebra $LUC(\omega^{-1})^*$ acts on $L^\infty(\omega^{-1})$ and on $L^1(\omega)^{**}$ by

$$\langle x \bullet f, \phi\rangle = \langle x, f\phi\rangle \quad \text{and} \quad \langle \mu \bullet x, f\rangle = \langle \mu, x \bullet f\rangle, \tag{5}$$

where $\mu \in L^\infty(\omega^{-1})^*$, $x \in LUC(\omega^{-1})^*$, $\phi \in L^1(\omega)$ and $f \in L^\infty(\omega^{-1})$. This leads immediately to the identity

$$\mu \,\Box\, \nu = \mu \bullet \Phi(\nu) \qquad (\mu, \nu \in L^1(\omega)^{**}), \tag{6}$$

which implies that the right shift by ν in $L^1(\omega)^{**}$ depends only on the restriction of ν to $LUC(\omega^{-1})$. Moreover, note that $x \bullet f = xf$ if $f \in LUC(\omega^{-1})$ and $x \in LUC(\omega^{-1})^*$.

The right translations on $(L^1(\omega)^{**}, \Box)$ and on $LUC(\omega^{-1})^*$ are weak*-continuous, but left translations need not be. On these algebras, the *topological centre* is the collection of all μ such that the left translation by μ, i.e. $\nu \mapsto \mu \,\Box\, \nu$, is weak*-continuous.

It is not difficult to show that $M(\omega)$, $L^1(\omega)$ and $\ell^1(\omega)$ are contained in the topological centres of $LUC(\omega^{-1})^*$, $L^1(\omega)^{**}$ and $\ell^1(\omega)^{**}$, respectively.

2.3. More definitions

For $E \subseteq G$, let $\kappa(E)$ denote the compact covering number of E, i.e., the minimal cardinality of a covering of E by compact subsets of G. Then the *height* of $x \in \Delta$ is defined to be

$$\rho(x) = \min\{\, \kappa(E);\, E \subseteq G,\, x \in \overline{\epsilon(E)}\,\},$$

where overline denotes the closure in Δ. Finally, let $\mathcal{U}_\Delta$ denote the points in Δ with height $\kappa(G)$.

The weight ω is *diagonally bounded* on $E \subseteq G$ with bound $K > 0$ if

$$\sup_{s \in E} \omega(s)\omega(s^{-1}) \leq K.$$

Necessarily, $K \geq 1$ (assuming $E \neq \emptyset$).

A function $f\colon G \to \mathbb{C}$ is said to be *slowly oscillating* if for every $\epsilon > 0$ and $s \in G$ there is $A \subseteq G$ with $\kappa(A) < \kappa(G)$ such that

$$|f(st) - f(t)| < \epsilon \quad \text{and} \quad |f(ts) - f(t)| < \epsilon \quad \text{for every} \quad t \in G \setminus A.$$

Slowly oscillating functions were constructed in [10, Lemma 1]. If $f \in LUC(G)$ is slowly oscillating, then

$$\langle y \,\Box\, s, f\rangle = \langle y, f\rangle = \langle x \,\Box\, y, f\rangle$$

whenever $s \in G$, $x \in G^{LUC}$ and the height of $y \in G^{LUC}$ is $\kappa(G)$.

3. Sets determining topological centres

The notion of sets *determining for the (left) topological centre* (dtc) of A^{**} was introduced by Dales, Lau, and Strauss [7]. A set $V \subset A^{**}$ is a *dtc set* if

$$\mu \,\Box\, y = \mu \diamond y \quad \text{for every} \quad y \in V \quad \text{implies that} \quad \mu \in A.$$

Note that since $\mu \,\Box\, a = \mu \diamond a$ for every $a \in A$, we have $\mu \,\Box\, y = \mu \diamond y$ if and only if $\mu \,\Box\, y = \lim_\alpha \mu \,\Box\, y_\alpha$ whenever (y_α) is a net in A such that $y_\alpha \to y$ in A^{**}.

Recall that G is an *SIN group* if it has a neighbourhood base at the identity consisting of symmetric, invariant neighbourhoods. When G is an SIN group, the left and right uniformities coincide and $LUC(G)$ is the C*-algebra of all bounded uniformly continuous functions. So in this case, a second product may be defined on $LUC(\omega^{-1})^*$ as follows:

$$R_s f(t) = f(ts), \quad f\nu(s) = \langle \nu, R_s f\rangle, \quad \langle \mu \diamond \nu, f\rangle = \langle \mu, f\nu\rangle,$$

where $s, t \in G$, $f \in LUC(\omega^{-1})$, and $\mu, \nu \in LUC(\omega^{-1})^*$. In this case, we say that a subset V of Δ is a dtc set of $LUC(\omega^{-1})^*$ if $\mu \in M(\omega)$ whenever

$$\mu \,\Box\, x = \mu \diamond x \quad \text{for every} \quad x \in V. \tag{7}$$

Equivalently, $\mu = 0$ is the only element in $C_0(\omega^{-1})^\perp$ with property (7). We remark at this point that when G is a SIN group, the map $\Phi\colon L^1(\omega)^{**} \to LUC(\omega^{-1})^*$ is a homomorphism also with respect to the second products.

Lemma 1. *Suppose that G is a locally compact SIN group. Let $\mu \in LUC(\omega^{-1})^*$ and $x \in \Delta$. Then $\mu \,\Box\, x = \mu \diamond x$ if and only if $\mu \,\Box\, x = \lim_\alpha \mu \,\Box\, \epsilon(s_\alpha)$ whenever (s_α) is a net in G such that $\epsilon(s_\alpha) \to x$ in Δ.*

Proof. Let (s_α) be a net in G such that $\epsilon(s_\alpha) \to x$ in Δ. As $\mu \,\Box\, \epsilon(s) = \mu \diamond \epsilon(s)$ for every $s \in G$, we have

$$\lim_\alpha \mu \,\Box\, \epsilon(s_\alpha) = \lim_\alpha \mu \diamond \epsilon(s_\alpha) = \mu \diamond x.$$

$\Box$

The preceding lemma justifies the following definition which applies to all locally compact groups. A subset V of Δ is *determining for the topological centre of* $LUC(\omega^{-1})^*$ (a *dtc set*) if $\mu \in M(\omega)$ whenever

$$\mu \,\Box\, x = \lim_\alpha \mu \,\Box\, \epsilon(s_\alpha)$$

for every $x \in V$ and for every net (s_α) in G such that $\epsilon(s_\alpha) \to x$.

The approach in this paper shall lead to a subset Y of Δ of cardinality $2^{\mathfrak{c}}$ such that any $\lfloor K \rfloor + 1$ points from Y form a dtc set for the algebra $LUC(\omega^{-1})^*$ when G is σ-compact. Moreover, it will then be enough to check the continuity of the left translations at $\lfloor K \rfloor + 1$ points to deduce that the topological centre of $C_0(\omega^{-1})^\perp$ is $\{0\}$.

When G is in addition an SIN group, we find a subset Y in the spectrum Ω of cardinality $2^{\mathfrak{c}}$ such that any $\lfloor K \rfloor + 1$ points from Y together with any right identity in Ω form a dtc set for the algebra $L^1(\omega)^{**}$. It will also be enough then to check the continuity of the left translations at $\lfloor K \rfloor + 2$ points to deduce that the topological centre of $L_0^\infty(\omega^{-1})^\perp$ is $\{0\}$.

In the general case when G is not necessarily σ-compact, the methods still lead to the topological centres of any of these algebras.

The dtc sets considered in [3] (in the unweighted case) are different than the dtc sets defined above. Budak, Işık and Pym find one point x together with a net (x_α) both in the remainder $G^{LUC} \setminus G$ such that $x_\alpha \to x$, and

$\mu \,\Box\, x_\alpha \to \mu \,\Box\, x$ happens only for $\mu = 0$ in $C_0(\omega^{-1})^\perp$. So their dtc set is a singleton in $G^{LUC} \setminus G$ but the whole net of points in $G^{LUC} \setminus G$ is also needed in the process, while our dtc set consists simply of two points in $G^{LUC} \setminus G$. In fact, with our definition, 2 is the smallest cardinality dtc sets may have in general: take G abelian, $\omega = 1$, $x \in G^{LUC} \setminus G$ and (s_α) any net in G converging to x. Then $x \,\Box\, x = \lim_\alpha s_\alpha \,\Box\, x = \lim_\alpha x \,\Box\, s_\alpha$, and so a dtc set (even for G^{LUC}) must have more than one point.

4. Dtc sets for $LUC(\omega^{-1})^*$

This section forms the core of the paper. Throughout the section we assume that ω is a weight on a non-compact locally compact group G, which is diagonally bounded, with bound $K \geq 1$, on a subset $E \subseteq G$ with $\kappa(E) = \kappa(G)$.

The proof of the following lemma builds upon elements from [10] and [5, Theorem 5.6] (the latter of which is adapted from [7]).

Recall that $\mathcal{U}_\Delta$ denotes the set of uniform points in Δ, i.e., points with height $\kappa(G)$.

Lemma 2. *There exist $n = \lfloor K \rfloor + 1$ points $x_1, x_2, \ldots, x_n$ in Δ such that $\mu = 0$ is the only element in $LUC(\omega^{-1})^*$ with $\operatorname{supp} \mu \subseteq \mathcal{U}_\Delta$ having the property that $\lim_\alpha \mu \,\Box\, \epsilon(s_\alpha) = \mu \,\Box\, x_k$ whenever $k = 1, \ldots, n$ and (s_α) is a net in G with $\lim_\alpha \epsilon(s_\alpha) = x_k$ in Δ.*

Proof. We start by constructing points $x_1, x_2, \ldots, x_n$ in $\mathcal{U}_\Delta$ that can be separated by slowly oscillating functions and that have the following factorisation property (obtained with the help of [20]): for any $f \in LUC(\omega^{-1})$ there is $g \in LUC(\omega^{-1})$ such that $f = x_k g$.

Let $T \subseteq E$ be a set as constructed in Lemma 1 of [10]. Let $f \in LUC(\omega^{-1})$ and $\epsilon > 0$ be arbitrary. Since $L^1(\omega)$ has a bounded approximate identity with bound 1 (as $\omega(e) = 1$), it follows from the Cohen factorisation theorem [15, Theorem 32.22] that $f = h\varphi$ where $h \in L^\infty(\omega^{-1})$ and $\varphi \in L^1(\omega)$ are such that $\|h - f\|_\omega < \epsilon$ and $\|\varphi\|_\omega \leq 1$. Comparing the construction of our set T in Lemma 1 of [10] to the construction leading to Lemma 8 of [9], we see that the set T satisfies the properties needed for the latter result. Therefore, any x in the closure of $\epsilon(T)$ with height $\kappa(G)$ has the factorisation property that any $h \in L^\infty(\omega^{-1})$ can be written as $h = x \bullet g'$ for some $g' \in L^\infty(\omega^{-1})$. (This factorisation result goes back to Neufang [20].) Put $g = g'\varphi \in LUC(\omega^{-1})$ so that

$$xg = x(g'\varphi) = (x \bullet g')\varphi = h\varphi = f,$$

as required. Since ω is diagonally bounded on E with bound K, an inspection of the argument in [9] shows that $\|g'\|_\omega \leq K\|h\|_\omega$ and so

$$\|g\|_\omega \leq \|g'\|_\omega \|\varphi\|_\omega \leq K\|h\|_\omega \leq K(\|f\|_\omega + \epsilon).$$

Therefore, for every $\mu \in LUC(\omega^{-1})^*$, $f \in LUC(\omega^{-1})$ and $\epsilon > 0$, writing $f = xg$ as above, we have

$$|\langle \mu, f \rangle| = |\langle \mu \,\Box\, x, g \rangle| \leq K\|\mu \,\Box\, x\|(\|f\|_\omega + \epsilon).$$

It follows that

$$\frac{\|\mu\|}{K} \leq \|\mu \,\square\, x\| \tag{8}$$

for every μ in $LUC(\omega^{-1})^*$.

Now let $x_1, x_2, \ldots, x_n$ be n distinct points in $\overline{\epsilon(T)}$ with height $\kappa(G)$, and pick n distinct points $y_1, y_2, \ldots, y_n$ in $\overline{T} \subseteq G^{LUC}$ (necessarily with height $\kappa(G)$) such that $x_k = \pi(y_k)$ for $1 \leq k \leq n$. The construction in [10, Lemma 1] gives slowly oscillating functions $f_j \in LUC(G)$, $1 \leq j \leq n$, such that $0 \leq f_j \leq 1$, $\langle y_k, f_j \rangle = \delta_{k,j}$ and the supports of f_j's are disjoint (regarded as continuous functions on G^{LUC}). For every $1 \leq j \leq n$, let $g_j = \omega f_j$ and note that g_j is in the unit ball of $LUC(\omega^{-1})$ and $\langle x_k, g_j \rangle = \delta_{k,j}$.

Choose $\epsilon > 0$ such that $\frac{1}{K} - \epsilon \geq \frac{1}{n}$. Let $z \in \mathcal{U}_\Delta$ and let h be an arbitrary function in the unit ball of $LUC(\omega^{-1})$. Now for every $1 \leq k \leq n$ and $s, t \in G$,

$$(\epsilon(s)(h \cdot_\omega g_k))(t) = \langle \epsilon(s), L_t(h \cdot_\omega g_k) \rangle = \frac{(h \cdot_\omega g_k)(ts)}{\omega(s)} = \frac{h(ts) g_k(ts)}{\omega(s)\omega(ts)}.$$

Then

$$\begin{aligned} |\langle z \,\square\, \epsilon(s), h \cdot_\omega g_k \rangle| &= |\langle z, \epsilon(s)(h \cdot_\omega g_k) \rangle| = \lim_{\epsilon(t) \to z} \left| \frac{h(ts) g_k(ts)}{\omega(t)\omega(s)\omega(ts)} \right| \\ &\leq \lim_{\epsilon(t) \to z} \left| \frac{g_k(ts)}{\omega(ts)} \right| = \lim_{t \to \pi^{-1}(z)} |f_k(ts)| \\ &\overset{(*)}{=} \lim_{t \to \pi^{-1}(z)} |f_k(t)| = |\langle \pi^{-1}(z), \omega^{-1} g_k \rangle| = |\langle z, g_k \rangle|, \end{aligned} \tag{9}$$

where the right slow oscillation of f_k is used at $(*)$. Since the supports of the functions g_k (regarded as continuous functions on Δ) are disjoint, there is at most one k such that $\langle z, g_k \rangle$ is non-zero. It follows that if μ is a finite sum of the form $\sum_{i \in I} \alpha_i z_i$ such that $z_i \in \mathcal{U}_\Delta$ and $\alpha_i \in \mathbb{R}$ with $\sum_{i \in I} |\alpha_i| \leq 1$, then for some $1 \leq k \leq n$

$$|\langle \mu \,\square\, \epsilon(s), h \cdot_\omega g_k \rangle| \leq \frac{1}{n} \leq \frac{1}{K} - \epsilon. \tag{10}$$

Indeed, if $I_k = \{ i \in I;\, \langle z_i, g_k \rangle \neq 0 \}$, then the sets I_k are disjoint, and by (9),

$$\begin{aligned} |\langle \mu \,\square\, \epsilon(s), h \cdot_\omega g_k \rangle| &\leq \sum_{i \in I} |\alpha_i| \, |\langle z_i \,\square\, \epsilon(s), h \cdot_\omega g_k \rangle| \leq \sum_{i \in I} |\alpha_i| \, |\langle z_i, g_k \rangle| \\ &= \sum_{i \in I_k} |\alpha_i| \, |\langle z_i, g_k \rangle| \leq \sum_{i \in I_k} |\alpha_i|. \end{aligned} \tag{11}$$

Since

$$\sum_{k=1}^{n} \sum_{i \in I_k} |\alpha_i| \leq 1,$$

inquality (10) is clear.

By weak*-approximation, (10) holds for any μ in the unit ball of $LUC(\omega^{-1})^*$ that is supported by $\mathcal{U}_\Delta$ (as a measure on Δ).

Suppose that $\|\mu\| = 1$. Now we show that for every $1 \le k \le n$, we can find h_k in the unit ball of $LUC(\omega^{-1})$ such that

$$|\langle \mu \,\square\, x_k, h_k \cdot_\omega g_k\rangle| > \frac{1}{K} - \epsilon. \tag{12}$$

Since $\|\mu \,\square\, x_k\| \ge \|\mu\|/K = 1/K$ by (8), we can already pick h_k in the unit ball of $LUC(\omega^{-1})$ such that

$$|\langle \mu \,\square\, x_k, h_k\rangle| > \frac{1}{K} - \epsilon. \tag{13}$$

Next, for each $1 \le k \le n$, we let

$$F_k = \{\, z \in \Delta;\ \langle z, g_k\rangle = 1\,\}$$

and show that $\mathrm{supp}(\mu \,\square\, x_k) \subseteq F_k$ (recall that Δ is usually not closed under the multiplication of $LUC(\omega^{-1})^*$). We first claim that for every $z \in \Delta$, the support of $z \,\square\, x_k$ is contained in F_k. For every $s, t \in G$,

$$\epsilon(s) \,\square\, \epsilon(t) = \frac{\omega(st)}{\omega(s)\omega(t)}\epsilon(st).$$

Noting that $\omega(st)/\omega(s)\omega(t)$ is bounded by 1, and taking first $\epsilon(t) \to x_k$ then $\epsilon(s) \to z$, the preceding identity leads to

$$z \,\square\, x_k = \lambda u$$

where $\lambda \in (0, 1]$ and $u \in \Delta$. Since

$$\begin{aligned}\langle \lambda u, g_k\rangle &= \lim_{\epsilon(s)\to z}\lim_{\epsilon(t)\to x_k} \frac{\omega(st)}{\omega(s)\omega(t)}\frac{g_k(st)}{\omega(st)}\\ &= \lim_{s\to\pi^{-1}(z)}\lim_{t\to\pi^{-1}(x_k)} \frac{\omega(st)}{\omega(s)\omega(t)} f_k(st) \overset{(**)}{=} \lambda\langle x_k, g_k\rangle = \lambda,\end{aligned}$$

where as in (9), the left slow oscillation of f_k is used at $(**)$. We have therefore $\langle u, g_k\rangle = 1$, and so $\mathrm{supp}(z \,\square\, x_k) = \{u\} \subseteq F_k$. Taking linear combinations of elements of the form $z \,\square\, x_k$, $z \in \Delta$, and then weak*-limits, we see that $\mathrm{supp}(\mu \,\square\, x_k) \subseteq F_k$ (as $F_k \subseteq \Delta$ is closed). Now, since $\widetilde{g}_k = 1$ on F_k, it is easy to check that $h_k \cdot_\omega g_k = h_k$ on F_k. Accordingly, inequality (12) follows from inequality (13).

Suppose now that $\mu \in LUC(\omega^{-1})^*$ is supported by $\mathcal{U}_\Delta$ and has norm 1. Since both (10) and (12) hold for μ, we see that

$$\lim_\alpha \mu \,\square\, \epsilon(s_\alpha) \ne \mu \,\square\, x_k$$

even if (s_α) is a net in G such that $\lim_\alpha \epsilon(s_\alpha) = x_k$. This proves the claim. $\square$

Theorem 3. *Suppose that G is σ-compact. Then Δ contains a dtc set for $LUC(\omega^{-1})^*$ of cardinality $\lfloor K \rfloor + 1$. In other words, there exist $n = \lfloor K \rfloor + 1$ points $x_1, x_2, \ldots, x_n$ in Δ such that if $\mu \in LUC(\omega^{-1})^*$ and $\lim_\alpha \mu \,\Box\, \epsilon(s_\alpha) = \mu \,\Box\, x_k$ whenever $k = 1, \ldots, n$ and (s_α) is a net in G with $\lim_\alpha \epsilon(s_\alpha) = x_k$ in Δ, then $\mu \in M(\omega)$.*

In particular, the topological centre of $LUC(\omega^{-1})^*$ is $M(\omega)$.

Proof. Let $x_1, x_2, \ldots, x_n$ be the points in $\overline{\epsilon(T)}$ as given by Lemma 2. Let $\mu \in LUC(\omega^{-1})^*$ and write it as $\mu = \mu_0 + \mu_1$ where $\mu_0 \in M(\omega)$ and $\mu_1 \in C_0(\omega^{-1})^\perp$. Suppose that μ has the property that $\lim_\alpha \mu \,\Box\, \epsilon(s_\alpha) = \mu \,\Box\, x_k$ whenever $k = 1$, $\ldots$, n and (s_α) is a net in G with $\lim_\alpha \epsilon(s_\alpha) = x_k$ in Δ. Since $M(\omega)$ is contained in the topological centre, it is clear that μ_0 satisfies also the property, and so does μ_1. But as G is σ-compact, μ_1 is supported by $\Delta \setminus \epsilon(G) = \mathcal{U}_\Delta$. It then follows from Lemma 2 that $\mu_1 = 0$. $\Box$

Next we shall show that the topological centre of $LUC(\omega^{-1})^*$ is $M(\omega)$ for any locally compact group G. Let H be an open subgroup of G. Restricting ω to H gives a weight on H and we denote the associated weighted LUC-space by $LUC(H, \omega^{-1})$. For every f in $LUC(H, \omega^{-1})$, let $\hat{f}$ denote extension of f to G by 0 and note that $\hat{f} \in LUC(\omega^{-1})$. Then if $\mu \in LUC(\omega^{-1})^*$, let $\check{\mu}$ denote the functional in $LUC(H, \omega^{-1})^*$ defined by

$$\langle \check{\mu}, f \rangle = \langle \mu, \hat{f} \rangle \qquad (f \in LUC(H, \omega^{-1})).$$

Note that the spectrum $\Delta(H)$ of $LUC(H, \omega^{-1})$ may be identified with the closure of $\epsilon(H)$ in Δ. Moreover, denote the set of points in $\overline{\epsilon(H)} \subseteq \Delta$ with height $\kappa(H)$ by $\mathcal{U}_\Delta(H)$.

Lemma 4. *Let H be an open subgroup of G. Let $y \in \overline{\epsilon(H)} \cong \Delta(H)$ where the closure is taken in Δ. If the map*

$$L_\mu \colon x \mapsto \mu \,\Box\, x \colon \Delta \to LUC(\omega^{-1})^*$$

is continuous at y, then also the map

$$L_{\check{\mu}} \colon x \mapsto \check{\mu} \,\Box\, x \colon \Delta(H) \to LUC(H, \omega^{-1})^*$$

is continuous at y.

Proof. We begin by checking that $(xf)\hat{} = x\hat{f}$ whenever $x \in \overline{\epsilon(H)}$ and $f \in LUC(H, \omega^{-1})$. For every $s \in G$,

$$x\hat{f}(s) = \langle x, L_s \hat{f} \rangle = \lim_{\epsilon(h) \to x} \frac{\hat{f}(sh)}{\omega(h)} = \begin{cases} xf(s) & \text{if } s \in H \\ 0 & \text{if } s \notin H, \end{cases}$$

and so $(xf)\hat{} = x\hat{f}$. Now if (y_α) is a net in $\overline{\epsilon(H)}$ converging to y, then

$$\langle L_{\check{\mu}}(y_\alpha), f \rangle = \langle \check{\mu}, y_\alpha f \rangle = \langle \mu, (y_\alpha f)\hat{} \rangle = \langle \mu, y_\alpha \hat{f} \rangle \to \langle \mu \,\Box\, y, \hat{f} \rangle = \langle L_{\check{\mu}}(y), f \rangle$$

because L_μ is continuous at y. Hence $L_{\check{\mu}}$ is continuous at y. $\Box$

The following result is known from [20] and [6], but our methods are very different.

Theorem 5. *The topological centre of* $LUC(\omega^{-1})^*$ *is* $M(\omega)$*.*

Proof. Suppose that μ is in the topological centre of $LUC(\omega^{-1})^*$. Since $M(\omega)$ is contained in the topological centre, we may assume without loss of generality that $\mu \in C_0(\omega^{-1})^\perp$. Suppose that $\mu \neq 0$. Pick ξ from $\operatorname{supp}\mu \subseteq \Delta$ with minimal height and construct an open subgroup H of G such that $\xi \in \mathcal{U}_\Delta(H)$ and $\kappa(\overline{E \cap H}) = \kappa(H)$ (this is obtained by taking first a subset A of G such that $\xi \in \overline{\epsilon(A)}$ and $\kappa(A)$ equals the height of ξ, then any subset B of E with $\kappa(B) = \kappa(A)$ and a compact neighbourhood V of the identity, and finally defining H as the subgroup generated by $A \cup B \cup V$). Then $\check{\mu} \neq 0$ and $\operatorname{supp}\check{\mu} \subseteq \mathcal{U}_\Delta(H)$ due to the minimality of the height of ξ. Since μ is in the topological centre of $LUC(\omega^{-1})^*$, the map

$$L_{\check{\mu}} \colon x \mapsto \check{\mu} \,\square\, x \colon \Delta(H) \to LUC(H, \omega^{-1})^*$$

is continuous by Lemma 4. This contradicts Lemma 2 when applied to H. □

Theorem 6. *The topological centre of* $C_0(\omega^{-1})^\perp$ *is* $\{0\}$*. If* G *is* σ*-compact, then it is enough to check the continuity of left translations at* $\lfloor K \rfloor + 1$ *points.*

Proof. The argument is similar to the proof of [10, Theorem 17]. For the first statement, it is enough to show that any element μ in the topological centre of $C_0(\omega^{-1})^\perp$ is in the topological centre of $LUC(\omega^{-1})^*$. To this end, fix a right cancellable point x in $\overline{C_0}(\omega^{-1})^\perp$. For the definition of right cancellable, see section 6. Any point in $\overline{\epsilon(T)}$, where T is the set used throughout the paper starting from Lemma 2, is right cancellable, see [9, Theorem 10]. If (ν_α) is a bounded net in $LUC(\omega^{-1})^*$ converging to ν in the weak*-topology of $LUC(\omega^{-1})^*$, then $\nu_\alpha \square x \to \nu \square x$ in $C_0(\omega^{-1})^\perp$ with respect to the relative weak*-topology. Hence $\mu \,\square\, \nu_\alpha \,\square\, x \to \mu \,\square\, \nu \,\square\, x$. On the other hand, the net $(\mu \,\square\, \nu_\alpha)$ clusters at some $\eta \in LUC(\omega^{-1})^*$ due to boundedness. But then $\eta \,\square\, x = \mu \,\square\, \nu \,\square\, x$ and since x is right cancellable, we have $\eta = \mu \,\square\, \nu$. Therefore $\mu \,\square\, \nu$ is the unique cluster point of $(\mu \,\square\, \nu_\alpha)$, and so $\mu \,\square\, \nu_\alpha \to \mu \,\square\, \nu$. This shows that the left translation by μ is weak*-continuous on bounded sets of $LUC(\omega^{-1})^*$. Since bounded nets (in fact just nets from the group) were enough to deduce the topological centre in the argument of Theorem 5, the left translation by μ is weak*-continuous on all of $LUC(\omega^{-1})^*$. (It should be mentioned that this passage from bounded nets to general nets was wrongly argued in the proof of [10, Theorem 17].)

For the second statement, note that we only need to check the continuity of the left translation by μ at the points $x_1 \,\square\, x, \ldots, x_n \,\square\, x$, where $x_1, \ldots, x_n$ are as in Theorem 3. □

5. Dtc sets for $L^1(\omega)^{**}$

Again we assume that ω is a weight on a non-compact locally compact group G and that ω is diagonally bounded, with bound $K \geq 1$, on a subset $E \subseteq G$ with $\kappa(E) = \kappa(G)$.

Recall that $L_0^\infty(\omega^{-1})$ is the closure of the compactly supported functions in $L^\infty(\omega^{-1})$, and that $L^1(\omega)^{**}$ has an ℓ^1-direct sum decomposition

$$L^1(\omega)^{**} = L_0^\infty(\omega^{-1})^* \oplus L_0^\infty(\omega^{-1})^\perp \tag{14}$$

(see [9, 18]). Note that $L_0^\infty(\omega^{-1})^\perp$ is a weak*-closed ideal in $L^1(\omega)^{**}$ consisting of the functionals annihilating $L_0^\infty(\omega^{-1})$.

The main part of the section is inspired by the work of Budak, Işık and Pym [3]. We shall first show that if μ is in the topological centre of $L^1(\omega)^{**}$ and $\mu = \mu_0 + \mu_1$ is the decomposition of μ according to (14), then μ_0 is in $L^1(\omega)$. To this end, we say that $\mu \in L^1(\omega)^{**}$ is *singular* if for every $f \in L^1(\omega)$ we have $\mu \perp f$ as measures on Ω. Since $L^1(G)$ is a band in $L^1(G)^{**}$ (by Lemma 3.5 of [3]) and the isometry $L^1(G)^{**} \to L^1(\omega)^{**}$ is a lattice isomorphism, also $L^1(\omega)$ is a band in $L^1(\omega)^{**}$. Hence $L_0^\infty(\omega^{-1})^*$ has an orthogonal decomposition $L_0^\infty(\omega^{-1})^* = L^1(\omega) \oplus L_0^\infty(\omega^{-1})_s^*$ where $L_0^\infty(\omega^{-1})_s^*$ denotes the singular elements in $L_0^\infty(\omega^{-1})^*$ (as argued in [3]; for more details on Banach lattices, see [19], in particular Theorem 1.2.9).

The following lemma is a weighted version of [3, Lemma 5.3]. The result can be deduced from [3], because the proof there relies only on the lattice structure of $L^1(G)^{**}$, which is the same as that of $L^1(\omega)^{**}$. For completeness, here is another proof directly for $L^1(\omega)^{**}$. For compact $F \subseteq G$, define

$$\omega_F^{-1}(s) = \begin{cases} \omega(s)^{-1} & \text{if } s \in F \\ 0 & \text{otherwise,} \end{cases}$$

and note that $\omega_F^{-1} \in L^1(\omega)$.

For $\mu \in L^\infty(\omega^{-1})^*$ and compact set $K \subseteq G$, let $\mu|_K$ denote the functional defined by $\langle \mu|_K, f\rangle = \langle \mu, 1_K f\rangle$ for $f \in L^\infty(\omega^{-1})$ (note that $1_K f$ denotes the pointwise product).

Lemma 7. *Let $\mu \in L_0^\infty(\omega^{-1})^*$. Suppose that $\{K_n\}$ is a sequence of increasing compact sets such that $\|\mu|_{K_n}\| \to \|\mu\|$ as $n \to \infty$ (such a sequence always exists). Then there is a sequence of functions $(f_n) \subseteq L^\infty(\omega^{-1})$ such that*

1. $\|f_n\|_\omega \leq 1$,
2. $f_n = 0$ *off* K_n,
3. $\lim_{n\to\infty} \langle \phi, f_n\rangle = 0$ *whenever* $\phi \in L^1(\omega)$,
4. $\lim_{n\to\infty} \langle \mu, f_n\rangle = \|\mu_s\|$.

Proof. Write $\mu = \mu_{ab} + \mu_s$ where $\mu_{ab} \in L^1(\omega)$ and μ_s is singular. Fix a natural number n. Identify elements of $L^1(\omega)^{**}$ with bounded Radon measures on the spectrum Ω of $L^\infty(\omega^{-1})$; note that this identification preserves the lattice structure, and so $\mu_s \perp \omega_{K_n}^{-1}$ as measures on Ω. To simplify notation, write $\nu_n = \mu_s|_{K_n}$. Since $\nu_n \perp \omega_{K_n}^{-1}$, there exists a Borel measurable set $A \subseteq \Omega$ such that $|\nu_n|(A) = \|\nu_n\|$ and $\omega_{K_n}^{-1}(A) = 0$. By regularity, there is $g \in L^\infty(\omega^{-1})$ with $0 \leq g \leq \omega$ such that $\langle |\nu_n|, g\rangle \geq \|\nu_n\| - 1/n$ and $\langle \omega_{K_n}^{-1}, g\rangle \leq 1/n$. Choose $h \in$

$L^\infty(\omega^{-1})$ such that $|h| \le g$ and $\langle \nu_n, h\rangle \ge \langle |\nu_n|, g\rangle - 1/n$. Define $f_n \in L^\infty(\omega^{-1})$ by putting $f_n = h$ on K_n and $f_n = 0$ off K_n. The first two statements are then immediate.

To see that the third statement holds, note that for $\psi \in C_c(G)$ (the compactly supported continuous functions on G)

$$\begin{aligned}\left|\int_G \psi(s) f_n(s)\, ds\right| &= \left|\int_{K_n} \psi(s) f_n(s)\, ds\right| \\ &\le \sup_{s\in G} |\psi(s)\omega(s)| \int_{K_n} \omega^{-1}(s)|f_n(s)|\, ds \\ &\le \sup_{s\in G} |\psi(s)\omega(s)|\, \langle \omega^{-1}_{K_n}, g\rangle \le \frac{\sup_{s\in G}|\psi(s)\omega(s)|}{n}.\end{aligned}$$

So $\int_G \psi(s) f_n(s)\, ds \to 0$ for every $\psi \in C_c(G)$ and it follows that $\langle \phi, f_n\rangle \to 0$ whenever $\phi \in L^1(\omega)$.

As for the fourth statement, note that

$$|\langle \mu, f_n\rangle| \ge |\langle \nu_n, f_n\rangle| - |\langle \mu_{ab}, f_n\rangle| \ge \|\nu_n\| - 2/n - |\langle \mu_{ab}, f_n\rangle| \to \|\mu_s\|$$

as $n \to \infty$. □

Lemma 8. *If $\mu \in L_0^\infty(\omega^{-1})^*$ is in the topological centre of $L^1(\omega)^{**}$, then μ is in $L^1(\omega)$.*

Proof. Let $\{K_n\}$ be a sequence of increasing compact sets such that $\mu|_{K_n} \to \mu$ in norm, and let by Lemma 7, $(f_n) \subseteq L^\infty(\omega^{-1})$ be the sequence of functions obtained for μ and $\{K_n\}$. Then pick a sequence $(y_n) \subseteq E$ such that

$$K_n y_n \cap K_m y_m = \emptyset$$

whenever $n \ne m$. Then the function

$$h = \sum_{n=1}^{\infty} \omega(y_n) R_{y_n^{-1}} f_n$$

is in $L^\infty(\omega^{-1})$. For every $\phi \in L^1(\omega)$ supported by K_m, we have

$$\begin{aligned}\langle \phi, \epsilon(y_m) \bullet h\rangle = \frac{h\phi(y_m)}{\omega(y_m)} &= \sum_{n=1}^{\infty} \frac{\omega(y_n)}{\omega(y_m)} \int f_n(s y_m y_n^{-1}) \phi(s)\, ds \\ &= \langle \phi, f_m\rangle,\end{aligned} \tag{15}$$

as ϕ is supported by K_m and $K_n y_n \cap K_m y_m = \emptyset$ for $n \ne m$. It then follows that

$$\langle \mu|_{K_n}, \epsilon(y_m) \bullet h\rangle = \langle \mu|_{K_n}, f_m\rangle$$

for $m \ge n$. Now

$$\begin{aligned}|\langle \mu, \epsilon(y_m) \bullet h\rangle - \langle \mu, f_m\rangle| &\le 2\|\mu - \mu|_{K_n}\| + |\langle \mu|_{K_n}, \epsilon(y_m) \bullet h - f_m\rangle| \\ &= 2\|\mu - \mu|_{K_n}\|\end{aligned}$$

for $m \geq n$, and so

$$\lim_{m\to\infty} \langle \mu, \epsilon(y_m) \bullet h \rangle = \lim_{m\to\infty} \langle \mu, f_m \rangle = \|\mu_s\| \tag{16}$$

by Lemma 7.

Let y be a cluster point of the sequence $(\epsilon(y_n))_{n=1}^{\infty}$ in Δ. Since μ is in the topological centre of $L^1(\omega)^{**}$, it follows from (16) that

$$\langle \mu \bullet y, h \rangle = \|\mu_s\|. \tag{17}$$

On the other hand, it follows from (15) and Lemma 7 that

$$\langle \phi \bullet y, h \rangle = \lim_{m\to\infty} \langle \phi, f_m \rangle = 0$$

for every $\phi \in L^1(\omega)$ supported by any K_n. As $\mu|_{K_n} \to \mu$, we may take ϕ to μ in the weak*-topology, and so

$$\langle \mu \bullet y, h \rangle = 0.$$

Combining this with (17) we see that $\mu_s = 0$, and so $\mu \in L^1(G)$. □

Recall that $\Phi\colon L^1(\omega)^{**} \to LUC(\omega^{-1})^*$ is the natural quotient map, which maps $L_0^\infty(\omega^{-1})^*$ onto $M(\omega)$ and $L_0^\infty(\omega^{-1})^\perp$ onto $C_0(\omega^{-1})^\perp$.

Lemma 9. *Suppose that G is a locally compact SIN group and that V is a dtc set for $LUC(\omega^{-1})^*$. For every $y \in V$, let $\tilde{y} \in L^1(\omega)^{**}$ such that $\Phi(\tilde{y}) = y$. If $\mu \in L^1(\omega)^{**}$ and*

$$\mu \,\Box\, \tilde{y} = \mu \diamond \tilde{y}$$

for every $y \in V$, then $\Phi(\mu) \in M(\omega)$.

Proof. Since G is SIN, there are two well-defined products $\Box$ and $\diamond$ on $LUC(\omega^{-1})^*$. Suppose that

$$\mu \,\Box\, \tilde{y} = \mu \diamond \tilde{y}$$

for every $y \in V$. Since Φ is a homomorphism with respect to the first products $\Box$ as well as the second products $\diamond$, we have that

$$\Phi(\mu) \,\Box\, y = \Phi(\mu) \diamond y$$

for every $y \in V$. Since V is a dtc set for $LUC(\omega^{-1})^*$, we have that $\Phi(\mu) \in M(\omega)$ (via Lemma 1). □

The first part of the following result is known to be true for any locally compact group; see [20] and [6].

Theorem 10. *Suppose that G is a locally compact SIN group. The topological centre of $L^1(\omega)^{**}$ is $L^1(\omega)$. When G is also σ-compact, there exists a dtc set of $n = \lfloor K \rfloor + 2$ points in Ω.*

Proof. Let μ be in the topological centre of $L^1(\omega)^{**}$, and decompose μ as in (14) with μ_0 as the local component and μ_1 as the component at infinity. It follows from Lemma 9 that $\Phi(\mu) \in M(\omega)$. Since $\Phi(\mu_0) \in M(\omega)$, we have $\Phi(\mu_1) \in M(\omega)$. But since $\mu_1 \in L_0^\infty(\omega^{-1})^\perp$, we must have $\Phi(\mu_1) = 0$. This means that $\mu_1 = 0$ on $LUC(\omega^{-1})$. Since G is SIN, there is a central bounded approximate identity (e_α) in $L^1(\omega)$, with the supports contained in some common compact set. Note that each e_α is also in the algebraic centre of $L^1(\omega)^{**}$. In particular, for every α and for every $f \in L^\infty(\omega^{-1})$,

$$\langle \mu_1 \Box e_\alpha, f \rangle = \langle e_\alpha \Box \mu_1, f \rangle = \langle e_\alpha, \mu_1 f \rangle = \langle \mu_1, f e_\alpha \rangle = 0.$$

So, if ν is a weak* cluster point of (e_α), then ν is a right identity in $L^1(\omega)^{**}$ and

$$\begin{aligned} \mu = \mu_0 + \mu_1 &= (\mu_0 + \mu_1) \Box \nu = \text{w*-}\lim_\alpha (\mu_0 + \mu_1) \Box e_\alpha \\ &= \text{w*-}\lim_\alpha (\mu_0 \Box e_\alpha + \mu_1 \Box e_\alpha) = \text{w*-}\lim_\alpha \mu_0 \Box e_\alpha \\ &= \text{w*-}\lim_\alpha e_\alpha \Box \mu_0 = \nu \Box \mu_0. \end{aligned}$$

Now μ_0 is the norm limit of a compactly supported functionals $\mu_{0,n}$ in $L_0^\infty(\omega^{-1})^*$, $n = 1, 2, \dots$ For every n, there is a compact set K_n such that the support of $e_\alpha \Box \mu_{0,n}$ is contained in K_n for every α. Hence $\nu \Box \mu_{0,n} = \text{w*-}\lim_\alpha e_\alpha \Box \mu_{0,n}$ is also supported by K_n. Therefore μ, as the norm limit of the sequence $(\nu \Box \mu_{0,n})_{n=1}^\infty \subseteq L_0^\infty(\omega^{-1})^*$, is in $L_0^\infty(\omega^{-1})^*$, and so $\mu = \mu_0$. Consequently, μ_0 is in the topological centre of $L^1(\omega)^{**}$, and by Lemma 8, $\mu = \mu_0 \in L^1(\omega)$, as required.

Now suppose that G is also σ-compact. Let $x_1, x_2, \dots, x_{n-1}$ be the points in Δ given by Theorem 3. For every $k = 1, 2, \dots, n-1$, pick $y_k \in \Omega$ such that $\Phi(y_k) = x_k$ (that this is possible, recall that Φ maps Ω onto Δ by [21, Lemma 4.1.7]). If

$$\mu \Box y_k = \mu \diamond y_k \quad \text{for every} \quad k = 1, 2, \dots, n-1,$$

then by Lemma 9 $\Phi(\mu) \in M(\omega)$ and hence $\Phi(\mu_1) = 0$. To deduce that $\mu_1 = 0$, we need as above one right identity in Ω, taking the number of necessary points to $\lfloor K \rfloor + 2$. Note that any element in $\Phi^{-1}(\delta_e)$ is a right identity, and $\Phi^{-1}(\delta_e) \cap \Omega$ is non-empty as $\delta_e \in \Delta$. Finally, to deduce that $\mu_0 \in L^1(\omega)$, we need to apply Lemma 8. This will not add to the number of necessary points as the element y used in Lemma 8 can be one of the x_k's (so effectively y_k). ☐

The same argument, which gives Theorem 6 (using [9, Theorem 10]), proves the following.

Theorem 11. *Suppose that G is SIN. The topological centre of $L_0^\infty(\omega^{-1})^\perp$ is $\{0\}$. If G is also σ-compact, it is enough to check the continuity at $\lfloor K \rfloor + 2$ points.*

6. Dtc sets for $\ell^1(\omega)^{**}$

Let S be a discrete semigroup and consider the weighted semigroup algebra $\ell^1(\omega)$ where $\omega\colon S \to (0, \infty)$ is a submultiplicative weight function. We want to show that the topological centre of $\ell^1(\omega)^{**} \cong \ell^\infty(\omega^{-1})^*$ is $\ell^1(\omega)$ under some conditions on S and ω. This case is very similar to the case of $LUC(\omega^{-1})^*$ considered in section 4.

The spaces $\ell^1(\omega)$ and $\ell^\infty(\omega^{-1})$ are defined via isometries

$$f \mapsto \omega f \colon \ell^1(\omega) \to \ell^1(S)$$

and

$$f \mapsto \omega^{-1} f \colon \ell^\infty(\omega^{-1}) \to \ell^\infty(S).$$

Then $\ell^1(\omega)$ is a Banach algebra with respect to the convolution product and $\ell^\infty(\omega^{-1})$ is a C*-algebra with respect to the weighted pointwise product. We let $\pi\colon \ell^1(S)^{**} \to \ell^1(\omega)^{**}$ be the adjoint of the $*$-isomorphism $f \mapsto w^{-1}f$. Similarly to the previous case, we denote the spectrum of $\ell^\infty(\omega^{-1})$ by Δ, and let $\epsilon\colon S \to \Delta$ be the map

$$\langle \epsilon(s), f\rangle = \frac{f(s)}{\omega(s)} \qquad (f \in \ell^\infty(\omega^{-1})).$$

Note that the spectrum of $\ell^\infty(S)$ is the Stone–Čech compactification βS of S, and so (e, Δ) is a realisation of the Stone–Čech compactification of S. We define the height of points in Δ similarly as before, and denote by $\mathcal{U}_\Delta$ the set of points with the maximal height $|S|$. We define also slowly oscillating functions as in the group case.

We say that ω is *diagonally bounded* on $E \subseteq S$ with bound $K > 0$ if

$$\omega(s)\omega(t) \leq K\omega(st) \quad \text{for every } s \text{ in } S \text{ and } t \text{ in } E.$$

(In the case when S is a group this is equivalent to the previous definition.)

We say that an element s in a semigroup S is *right cancellable* if $t_1 s = t_2 s$ implies $t_1 = t_2$ whenever $t_1, t_2 \in S$; left cancellable elements are defined analogously. A semigroup S is *right cancellative* if every element in S is right cancellable, and S is *weakly left cancellative* if for every fixed $s, u \in S$, the equation $st = u$ has finitely many solutions $t \in S$. A *weakly cancellative* semigroup is both weakly left and weakly right cancellative.

Throughout this last section, S is an infinite discrete, right cancellative, weakly cancellative semigroup and ω is a weight on S that is diagonally bounded, with bound K, on $E \subseteq S$ with $|E| = |S|$.

Lemma 12. *There is a subset $T \subseteq E$, with $|T| = |E|$, such that*

1. *the points in $\overline{T} \subseteq \beta S$ can be separated by slowly oscillating functions;*
2. *for every $x \in \overline{\epsilon(T)} \cap \mathcal{U}_\Delta$ and $\nu \in \ell^1(\omega)^{**}$*

$$\frac{\|\mu\|}{K} \leq \|\mu \,\Box\, x\|.$$

Proof. Let $\{S_\alpha\}_{\alpha<|S|}$ be an increasing cover of S as constructed in [10, Lemma 7]; in particular, $|S_\alpha|$ is finite when $|\alpha|$ is finite and $|S_\alpha| = |\alpha|$ otherwise. We may assume without loss of generality that S has an identity element e and $e \in S_0$. By transfinite induction, there is a subset $T = \{t_\alpha\}_{\alpha<|S|}$ of E such that $S_\alpha t_\alpha S_\alpha \cap S_\beta t_\beta S_\beta = \emptyset$ when $\alpha \neq \beta$ (note that weak cancellation is needed at this point). Then the points in the closure of T in βS can be separated by slowly oscillating functions, as constructed in [10, Lemma 7], so the first statement holds.

To prove the second statement, it suffices in fact that the set T satisfies that $S_\alpha t_\alpha \cap S_\beta t_\beta = \emptyset$ whenever $\alpha \neq \beta$. We show that each function $f \in \ell^\infty(\omega^{-1})$ factorises as $f = xg$, where $x \in \overline{\epsilon(T)} \cap \mathcal{U}_\Delta$ and $g \in \ell^\infty(\omega^{-1})$. To see this, consider for a given $f \in \ell^\infty(\omega^{-1})$, the function

$$g(s) = \sum_\alpha \omega(t_\alpha) 1_{S_\alpha t_\alpha}(s) f_\alpha(s) \qquad (s \in S)$$

where $1_{S_\alpha t_\alpha}$ is the characteristic function of $S_\alpha t_\alpha$ and

$$f_\alpha(s) = \begin{cases} f(u) & \text{if } s = ut_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Note that f_α is well defined since S is right cancellative and so is g since $\{S_\alpha t_\alpha\}$ is a disjoint family. Since ω is diagonally bounded on T with bound K, we have $\omega(t_\alpha)\omega(ut_\alpha)^{-1} \leq K\omega(u)^{-1}$ and so

$$|g(s)\omega(s)^{-1}| \leq K \sum_\alpha 1_{S_\alpha t_\alpha}(s) \|f\|_\omega \leq K\|f\|_\omega \tag{18}$$

for every $s \in S$.

Let now x be any point in $\overline{\epsilon(T)} \cap \mathcal{U}_\Delta$ and let $\epsilon(t_{\alpha_\gamma}) \to x$. Let $s \in S$ and pick β such that $s \in S_\beta$. We may suppose that $\alpha_\gamma \geq \beta$ for every γ. Then

$$\begin{aligned} xg(s) = \langle x, L_s g\rangle &= \lim_\gamma \omega(t_{\alpha_\gamma})^{-1} g(st_{\alpha_\gamma}) \\ &= \lim_\gamma \sum_\alpha \omega(t_{\alpha_\gamma})^{-1} \omega(t_\alpha) 1_{S_\alpha t_\alpha}(st_{\alpha_\gamma}) f_\alpha(st_{\alpha_\gamma}). \end{aligned}$$

Note that $1_{S_\alpha t_\alpha}(st_{\alpha_\gamma}) = 0$ if $\alpha \neq \alpha_\gamma$ and that $f_\alpha(st_{\alpha_\gamma}) = f(s)$ if $\alpha = \alpha_\gamma$. It follows that $xg(s) = f(s)$, and so we have our wanted factorisation.

If now $\mu \in \ell^\infty(\omega^{-1})^*$ is non-zero and $\epsilon > 0$ is given, pick f from the unit ball of $\ell^\infty(\omega^{-1})$ such that $|\langle \mu, f\rangle| > \|\mu\| - \epsilon$. If g is as above, then

$$|\langle \mu \,\Box\, x, g\rangle| = |\langle \mu, f\rangle| \geq \|\mu\| - \epsilon.$$

As $\|g\|_\omega \leq K$ by (18), we have

$$\|\mu \,\Box\, x\| \geq \frac{\|\mu\|}{K}.$$

$\square$

After choosing T as in the preceding lemma, repeating the proof of Lemma 2 gives the following result.

Lemma 13. *There exist $n = \lfloor K \rfloor + 1$ points $x_1, x_2, \ldots, x_n$ in Δ such that $\mu = 0$ is the only element in $\ell^1(\omega)^{**}$ with $\operatorname{supp}\mu \subseteq \mathcal{U}_\Delta$ having the property that $\lim_\alpha \mu \,\Box\, \epsilon(s_\alpha) = \mu \,\Box\, x_k$ whenever $k = 1, \ldots, n$ and (s_α) is a net in S with $\lim_\alpha \epsilon(s_\alpha) = x_k$ in Δ.*

The next result follows immediately from the preceding lemma. Note that Dales and Dedania [5, Theorem 5.6] proved a similar result under the assumptions that S is both left and right cancellative, countable semigroup and that the weight ω is weakly diagonally bounded on an infinite $E \subseteq S$.

Theorem 14. *Suppose in addition that S is countable. Then Δ contains a dtc set for $\ell^1(\omega)^{**}$ of cardinality $\lfloor K \rfloor + 1$. In other words, there exist $n = \lfloor K \rfloor + 1$ points $x_1, x_2, \ldots, x_n$ in Δ such that if $\mu \in \ell^1(\omega)^{**}$ and $\lim_\alpha \mu \Box \epsilon(s_\alpha) = \mu \Box x_k$ whenever $k = 1, \ldots, n$ and (s_α) is a net in S with $\lim_\alpha \epsilon(s_\alpha) = x_k$ in Δ, then $\mu \in \ell^1(\omega)$.*

Theorem 15. *The topological centre of $\ell^1(\omega)^{**}$ is $\ell^1(\omega)$.*

Proof. Let μ be in the topological centre of $\ell^1(\omega)^{**}$. Suppose that $\mu \in c_0(\omega^{-1})^\perp$ so that it is enough to show that $\mu = 0$. Assume towards contradiction that $\mu \neq 0$ and pick an element ξ from the support of μ in Δ such that the height of ξ is minimal (but note that the height of ξ is infinite). There is a subtlety in the construction of a subsemigroup S_0 such that an analogue of Lemma 4 holds for S_0. Let $A \subseteq S$ such that $|A| = |A \cap E|$ is equal to the height of ξ and ξ is in the closure of $\epsilon(A)$. Put $A_0 = A$ and define inductively $A_{n+1} = A_n \cup A_n^2 \cup A_n A_n^{-1}$ (where $A_n A_n^{-1}$ denotes those $t \in S$ such that $ts \in A_n$ for some $s \in A_n$). Since S is right cancellative, $|A_n| = |A|$ for every n and so $S_0 := \bigcup_{n=0}^\infty A_n$ is a subsemigroup of S with $|S_0|$ equal to the height of ξ. Moreover, Lemma 4 applies when G and H are replaced by S and S_0, respectively (that $x\hat{f} = (xf)\widehat{\ }$ requires that $S_0 S_0^{-1} \subseteq S_0$, which is guaranteed by the construction of S_0). Identifying the closure of $\epsilon(S_0)$ in Δ with the spectrum of $\ell^\infty(S_0, \omega^{-1})$, we have $\xi \in \mathcal{U}_\Delta(S_0)$ and ω is diagonally bounded on the set $E_0 := E \cap S_0$ of cardinality $|S_0|$. Therefore we may apply Lemma 13 to see that the restriction of μ to $\overline{\epsilon(S_0)}$ is 0. This contradicts the fact that $\xi \in \operatorname{supp}\mu$. $\square$

The following result is proved similarly as Theorem 6.

Theorem 16. *The topological centre of $c_0(\omega^{-1})^\perp$ is trivial. If S is in addition countable, then it is enough to check the continuity at $\lfloor K \rfloor + 1$ points.*

Remark 17. In Theorems 3 and 14, the dtc sets are picked from $\overline{\epsilon(T)} \cap \mathcal{U}_\Delta$. Since T is right uniformly discrete and countably infinite, this set has the same cardinality as the set of points in $\overline{T} \setminus T$ (the closure in G^{LUC}) and $\overline{T}$ may be identified with the Stone–Čech compactification βT of T. Thus $\overline{\epsilon(T)} \cap \mathcal{U}_\Delta$ has cardinality $2^{\mathfrak{c}}$. Any $\lfloor K \rfloor + 1$ points from this set form a dtc set in the above-mentioned results. In Theorem 10, we further need one right identity.

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