Some inverse scattering problems for perturbations of the biharmonic operator

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Abstract

Some inverse scattering problems for the three-dimensional biharmonic operator are considered. The operator is perturbed by first and zero order perturbations, which may be complex-valued and singular. We show the existence of the scattering solutions in the Sobolev space $W^1_\infty(R^3)$. One of the main result of this paper is the proof of analogue of Saito's formula (in different form as known before), which can be used to prove a uniqueness theorem for the inverse scattering problem. Another main result is to obtain the estimates for the kernel of the resolvent of the direct operator in W^1_∞ and to prove the reconstruction formula for the unknown coefficients of this perturbation.

1 Introduction

We consider the following three-dimensional biharmonic operator

$$H_4 u(x) = \Delta^2 u(x) + \vec{W}(x) \cdot \nabla u(x) + V(x) u(x) = 0, \tag{1}$$

where Δ is the Laplacian and $\cdot$ denotes the dot-product in R^3 for complex-valued vectors in C^3. The bi-Laplacian is perturbed by the first and zero order perturbations, vector-valued function $\vec{W}$ and a scalar function V, that may be complex-valued and very singular. More precisely, we assume that $\vec{W}$ belongs to $L^\infty(R^3)$ and V belongs to the Kato space K_3, i.e.

$$\sup_{x \in R^3} \int\limits_{|x-y|\le 1} \frac{|V(y)|}{|x-y|}\, dy < \infty, \tag{2}$$

and both have special behaviour at the infinity

$$|\vec{W}(x)|, \quad |V(x)| \le \frac{C}{|x|^\mu}, \quad |x| \ge R, \quad \mu > 3, \tag{3}$$

where $C > 0$, and $R > 0$ is big enough.

The motivation to study operators of order four appears for example in the study of elasticity and the theory of vibration of beams. As a concrete example, the linear beam equation [2]

$$\partial_t^2 U(x,t) + \Delta^2 U(x,t) + m(x)U(x,t) = 0,$$

under time-harmonic assumptions $U(x,t) = u(x)e^{-i\omega t}$ leads to the equation

$$\Delta^2 u(x) + m(x)u(x) = \omega^2 u(x).$$

The wave parameter ω is fixed (in general) here, nevertheless we can consider it fixed but big enough. This allows to consider some scattering problems with high frequency for this potential equation. In particular, we can use some numerical methods in that case. For the scattering problems (including linear or nonlinear equations), see for example [8] and references therein. In terms of inverse problems for bi- and poly-harmonic operators it might be mentioned some solutions to inverse boundary value problems, see for example [6]. One can refer also to [13], where the fundamental result concerning the global uniqueness for an inverse boundary value problem was proved. For the operators with vector potential one can mention [14].

The present work is concerned with the following scattering problem for operator H_4 given by

$$H_4 u(x) = k^4 u(x), \quad u(x) = u_0(x) + u_{sc}(x), \quad u_0(x) = e^{ikx\cdot\theta}, \quad \theta \in S^2, \tag{4}$$

where scattered wave u_{sc} and its Laplacian Δu_{sc} are required to satisfy Sommerfeld radiation condition at the infinity

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial f(x)}{\partial r} - ikf(x) \right) = 0, \quad r = |x|, \quad f = u_{sc} \quad or \quad f = \Delta u_{sc}. \tag{5}$$

The author was originally motivated to start studying scattering for fourth order operators by the article [1] (see also [15]) , where the time-evolution of several scattering coefficients for the one-dimensional biharmonic operator were studied. In terms of inverse scattering problems for fourth order operator might be mentioned Iwasaki's results [4], [5]. In these works Iwasaki studied the scattering problem in one-dimensional case and considered the inverse problem as a Riemann-Hilbert boundary value problem with respect to the wave number k in the complex cone $\arg([0, \frac{\pi}{4}]) \setminus \{0\}$.

The main differences of present work is that all considered scattering problems are studying here in the usual Sobolev spaces (compare with the weighted Sobolev spaces in the previous publications) and that the possible local singularities of the unknown coefficients $\vec{W}$ and V are stronger than it was considered before. Another important difference is concerned to Theorem 2, where the inverse scattering problem in terms of the Green's functions is considered.

We are looking for the scattering solutions u_{sc} to the equation (4) in the Sobolev space $W^1_\infty(R^3)$. Under the Sommerfeld radiation conditions (5) the scattering solutions to equation (4) are the unique solutions of the integral Lippmann-Schwinger equation (see [15] for details)

$$u(x) = u_0(x) - \int\limits_{R^3} G_k^+(|x-y|)(\vec{W}(y)\cdot\nabla u(y) + V(y)u(y))\,dy, \tag{6}$$

where G_k^+ is the outgoing fundamental solution of the operator $(\Delta^2 - k^4)$ in R^3, i.e., the kernel of the integral operator $(\Delta^2 - k^4 - i0)^{-1}$. This function G_k^+ in R^3 has the following form

$$G_k^+(|x|) = \frac{e^{ik|x|} - e^{-k|x|}}{8\pi k^2|x|}, \quad k > 0. \tag{7}$$

Since u_0 is just a bounded function with the norm $\|u_0\|_{L^\infty(R^3)} = 1$ it is more convenient to study (in stead of (6)) the equivalent integral equation for the scattered wave, namely

$$u_{sc}(x) = \tilde{u}_0(x) - \int\limits_{R^3} G_k^+(|x-y|)(\vec{W}(y)\cdot\nabla u_{sc}(y) + V(y)u_{sc}(y))\,dy =: \tilde{u}_0 + L_k(u_{sc}), \tag{8}$$

where $\tilde{u}_0 = L_k(u_0)$. As it is shown (see [15]) that a solution to the scattering problem (4), (5) also satisfies equation (6). This translates the study of the scattering problem to the study of integral equation (6) ((8)). It will be shown that this solution admits for fixed $k > 0$ asymptotic representation

$$u(x,k,\theta) = e^{ikx\cdot\theta} + C\frac{e^{ik|x|}}{k^2|x|}A(k,\theta',\theta) + o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

where $\theta, \theta' = \frac{x}{|x|} \in S^2$, C is known constant, and function $A(k,\theta',\theta)$ is called a scattering amplitude and defined by

$$A(k,\theta',\theta) := \int\limits_{R^3} e^{-ik\theta'\cdot y}(\vec{W}(y)\cdot\nabla u(y,k,\theta) + V(y)u(y,k,\theta))\,dy. \tag{9}$$

From the point of view of inverse problems one regards this scattering amplitude as one possible scattering data. For these purposes one requires the scattering amplitude to be known for all possible angles θ and θ' and all arbitrarily high frequencies ($k > 0$ large). Then Saito's formula is given by the following theorem.

Theorem 1 (Saito's formula) *Assume that $\vec{W}$ belongs to $L^\infty(R^3)$, V belongs to the Kato space K_3 and both satisfy conditions (3). Then the limit*

$$\lim_{k\to+\infty} k^2 \int\limits_{S^2\times S^2} e^{-ik(\theta-\theta')\cdot x}A(k,\theta',\theta)\,d\theta\,d\theta' =$$

$$= 4\pi^2 \nabla_x \int\limits_{R^3} \frac{\vec{W}(y)}{|x-y|^2}\, dy + 8\pi^2 \int\limits_{R^3} \frac{V(y)}{|x-y|^2}\, dy \tag{10}$$

in the sense of distributions.

The significance of Saito's formula for inverse problems is apparent from its corollary.

Corollary 1 (Uniqueness) *Let $\vec{W}_1, V_1$ and $\vec{W}_2, V_2$ be as in Theorem 1. Let, in addition, Fourier transform of $\vec{W}_1$ and $\vec{W}_2$ behaves as $o(|\xi|^{-1})$ at the infinity. If the corresponding scattering amplitudes for these coefficients coincide for some sequence $k_j \to +\infty$ then*

$$V_1(x) - \frac{1}{2}\nabla\delta \star \vec{W}_1(x) = V_2(x) - \frac{1}{2}\nabla\delta \star \vec{W}_2(x)$$

in the sense of tempered distributions, where δ is Dirac delta function and $\star$ denotes the convolution.

The limiting absorption principle can be applied for the operator H_4 to obtain the existence of the integral operator (see [15])

$$\hat{G}_p := \lim_{\epsilon \to +0} (H_4 - k^4 - i\epsilon)^{-1}$$

such that the kernel $G_p(x, y, k)$ for $k > 0$ large enough is the unique solution of the integral equation

$$G_p(x,y,k) = G_k^+(x,y,k) - \int\limits_{R^3} G_k^+(|x-z|)(\vec{W}(z)\cdot\nabla G_p(z,y,k) + V(z) G_p(y,z,k))\, dz, \tag{11}$$

From the point of view of inverse problems this kernel G_p can be considered as another possible scattering data. More precisely the following theorem holds.

Theorem 2 (Reconstruction) *Assume that $\vec{W}$ belongs to $L^\infty(R^3)$, V belongs to the Kato space K_3 and V satisfies conditions (3) and, in addition, $\vec{W}$ satisfies this condition with $\mu > 4$. Then for each fixed $\xi \in R^3$*

$$\mathcal{F}^{-1}(V)(\xi) - \frac{i\xi}{2} \cdot \mathcal{F}^{-1}(\vec{W})(\xi) =$$

$$= \lim_{k,|x|,|y| \to +\infty} 32\sqrt{2\pi} k^4 |x||y| e^{-ik(|x|+|y|)} (G_k^+(|x-y|) - G_p(x,y,k)) \tag{12}$$

such that $\xi = k\left(\frac{x}{|x|} + \frac{y}{|y|}\right)$.

Corollary 2 (Uniqueness-II) *Let $\vec{W}_1, V_1$ and $\vec{W}_2, V_2$ be as in Theorem 2. If the corresponding kernels $G_p^{(1)}(x,y,k)$ and $G_p^{(2)}(x,y,k)$ for these coefficients coincide for all x, y large enough and some sequence $k_j \to +\infty$ then*

$$V_1(x) - \frac{1}{2}\nabla\delta \star \vec{W}_1(x) = V_2(x) - \frac{1}{2}\nabla\delta \star \vec{W}_2(x)$$

in the sense of tempered distributions, where δ is Dirac delta function and $\star$ denotes the convolution.

This paper is organised as follows. In Section 2 some notations and estimates for G_k^+ are recalled. Then it will be proved the existence and the uniqueness of the solutions to (6) and (11) together with asymptotic behaviour of the scattering solution u and the kernel G_p. Several estimates for u and G_p are also given. Finally in Section 3 it will be given the proof of Theorem 1 and 2.

2 Solvability of direct scattering problems

We use the usual definitions of the Sobolev space W^1_∞ and the Lebesgue spaces $L^p, 1 \le p \le \infty$. We use also the following definitions for three-dimensional Fourier transform $\mathcal{F}$ and inverse Fourier transform $\mathcal{F}^{-1}$:

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{(2\pi)^3}} \int\limits_{R^3} f(x) e^{ix\cdot\xi}\, dx, \quad \mathcal{F}^{-1}(f)(x) = \frac{1}{\sqrt{(2\pi)^3}} \int\limits_{R^3} f(x) e^{-ix\cdot\xi}\, d\xi.$$

Next, taking into account the definition (7) of $G_k^+(|x|)$ in R^3 we obtain

$$|G_k^+(|x|)| \le \frac{1}{4\pi k^2 |x|}, \quad |\nabla G_k^+(|x|)| \le \frac{1}{2\pi k |x|}, \quad k > 0, \quad x \in R^3. \tag{13}$$

We now proceed to prove some estimates for the operator L_k.

Proposition 1 *Let $\vec{W}$ belongs $L^\infty(R^3)$, V belongs to the Kato space K_3 and both satisfy conditions (3). Then the following properties are satisfied.*

1. The function $\tilde{u}_0$ belongs to $W^1_\infty(R^3)$ with the estimates

$$\|\tilde{u}_0\|_{L^\infty(R^3)} \le \frac{c_0}{k}, \quad \|\nabla \tilde{u}_0\|_{L^\infty(R^3)} \le c_0, \quad k \ge 1,$$

where constant c_0 is equal to

$$c_0 = \frac{1}{2\pi}\left(\|\vec{W}\|_{L^\infty(R^3)} C_R + C_V + 2CC_\mu\right), \quad C_R := \sup_{x\in R^3} \int\limits_{|y|\le R} \frac{1}{|x-y|}\, dy,$$

$$C_V := \sup_{x\in R^3} \int\limits_{|y|\le R} \frac{|V(y)|}{|x-y|}\, dy, \quad C_\mu := \sup_{x\in R^3} \int\limits_{|y|\ge R} \frac{1}{|x-y||y|^\mu}\, dy \tag{14}$$

with C, R, μ are as in (3).

2. The operator $L_k : W^1_\infty(R^3) \to W^1_\infty(R^3)$ is bounded and satisfies for $k \ge 1$ the norm estimates

$$\|L_k f\|_{L^\infty(R^3)} \le \frac{c_0}{k^2} \|f\|_{W^1_\infty(R^3)}, \quad \|\nabla L_k f\|_{L^\infty(R^3)} \le \frac{c_0}{k} \|f\|_{W^1_\infty(R^3)}. \tag{15}$$

Proof Applying (13) one can obtain (see (14))

$$|\tilde{u}_0(x)| \leq \frac{1}{4\pi k^2}\left(\int\limits_{|y|\leq R} \frac{k|\vec{W}(y)|+|V(y)|}{|x-y|}\,dy + C\int\limits_{|y|\geq R} \frac{k+1}{|x-y||y|^{\mu}}\,dy\right) \leq$$

$$\leq \frac{1}{4\pi k}\left(\|\vec{W}\|_{L^\infty}C_R + C_V + 2CC_\mu\right) \leq \frac{c_0}{k}, \quad k\geq 1.$$

By similar method it can be proved that

$$\|\nabla\tilde{u}_0\|_{L^\infty(R^3)} \leq c_0, \quad k\geq 1.$$

This proves 1. Suppose now that $f\in W^1_\infty(R^3)$. Then

$$|L_k f(x)| \leq \frac{1}{4\pi k^2}\left(\int\limits_{|y|\leq R} \frac{|\nabla f(y)||\vec{W}(y)|+|V(y)||f(y)|}{|x-y|}\,dy + CC_\mu\|f\|_{W^1_\infty}\right) \leq$$

$$\leq \frac{1}{4\pi k^2}\left(\|\nabla f\|_{L^\infty}\int\limits_{|y|\leq R} \frac{|\vec{W}(y)}{|x-y|}\,dy + \|f\|_{L^\infty}\int\limits_{|y|\leq R}\frac{|V(y)|}{|x-y|}\,dy + CC_\mu\|f\|_{W^1_\infty}\right) \leq$$

$$\leq \frac{1}{4\pi k^2}\left(\|\nabla f\|_{L^\infty}\|\vec{W}\|_{L^\infty}C_R + \|f\|_{L^\infty}C_V + CC_\mu\|f\|_{W^1_\infty}\right).$$

This proves first inequality from (15). The second inequality from (15) can be proved using (13) similarly. Thus, Proposition 1 is completely proved.

Proposition 2 *Under the same assumptions for $\vec{W}$ and V as in Proposition 1 there is a constant $k_0>1$ such that the function $u_{sc}(x,k,\theta)$ defined by the series*

$$u_{sc}(x,k,\theta) = \sum_{j=0}^{\infty} L_k^j(\tilde{u}_0)(x,k,\theta) \tag{16}$$

solves integral equation (8) ((6)) uniquely in $W^1_\infty(R^3)$, when $k\geq k_0$. Moreover,

$$\|u_{sc}\|_{L^\infty(R^3)} \leq \frac{2c_0}{k}, \quad \|\nabla u_{sc}\|_{L^\infty(R^3)} \leq 2c_0 \tag{17}$$

uniformly in $\theta\in S^2$, when $k\geq k_0$.

Proof The estimates (15) imply that the norm estimate for operator L_k for $k\geq 1$ is

$$\|L_k\|_{W^1_\infty\to W^1_\infty} \leq \frac{2c_0}{k}.$$

Since $\tilde{u}_0$ belongs to $W^1_\infty(R^3)$ this estimate in turn implies that

$$\|u_{sc}\|_{W^1_\infty} \leq \sum_{j=0}^{\infty}\left(\frac{2c_0}{k}\right)^j \|\tilde{u}_0\|_{W^1_\infty}.$$

We may choose any $k_0 > max\{1, 2c_0\}$ to conclude that the series (16) converges in $W^1_\infty(R^3)$. Because the operator L_k is linear and maps continuously in $W^1_\infty(R^3)$ the series (16) solves (8). Choosing now $k_0 > max\{1, 4c_0\}$ one can easily obtain (17). Uniqueness of solution follows from the contraction condition of L_k. Proposition 2 is completely proved.

Concerning the kernel G_p (see integral equation (11)) of the integral operator $(H_4 - k^4 - i0)^{-1}$ one can prove the following result.

Proposition 3 *Under the same assumptions for $\vec{W}$ and V as in Proposition 1 there is a constant $k_0 > 1$ such that the function $G_p(x, y, k)$ can be defined by the series*

$$G_p(x, y, k) = \sum_{j=0}^{\infty} G^{(j)}(x, y, k), \quad G^{(0)} = G_k^+,$$

$$G^{(j)}(x, y, k) := -\int_{R^3} G_k^+(|x-z|)(\vec{W}(z)\cdot\nabla G^{(j-1)}(z, y, k)+V(z)G^{(j-1)}(y, z, k))\, dz \tag{18}$$

which solves integral equation (11) uniquely, when $k \geq k_0$. Moreover,

$$|G_p(x, y, k) - G_k^+(x, y, k)| \leq \frac{\tilde{c}_0}{4\pi^2 k^3 |x-y|},$$

$$|\nabla G_p(x, y, k) - \nabla G_k^+(x, y, k)| \leq \frac{\tilde{c}_0}{2\pi^2 k^2 |x-y|}, \tag{19}$$

where $\tilde{c}_0 = 2\|\vec{W}\|_{L^\infty} C_R + C_V + 3CC_\mu$ with constants C_R, C_V, C and C_μ are as in (14).

Proof To prove (18)-(19) it is needed first to estimate $G^{(1)}$. Indeed, using (13) one can obtain ($k \geq 1$)

$$|G^{(1)}(x, y, k)| \leq \frac{1}{8\pi^2 k^3}\left(\int_{|z|\leq R} \frac{|\vec{W}(z)| + \frac{1}{2k}|V(z)|}{|x-z||z-y|}\, dz + \int_{|z|\geq R} \frac{C + \frac{C}{2k}}{|x-z||z-y||z|^\mu}\, dz\right)$$

with constant C from the condition (3). Considering now two cases: $|x-z| \leq |z-y|$ and $|x-z| \geq |z-y|$ and taking into account conditions (2), one can obtain ($k \geq 1$)

$$|G^{(1)}(x, y, k)| \leq \frac{1}{8\pi^2 k^3}\left(\frac{2\|\vec{W}\|_{L^\infty} C_R}{|x-y|} + \frac{C_V}{|x-y|} + \frac{3CC_\mu}{|x-y|}\right) \leq$$

$$\leq \frac{\tilde{c}_0}{8\pi^2 k^3 |x-y|}, \tag{20}$$

where the constant $\tilde{c}_0$ is as above. Similarly one can obtain

$$|\nabla_x G^{(1)}(x,y,k)| \leq \frac{\tilde{c}_0}{4\pi^2 k^2 |x-y|}. \tag{21}$$

We show now that for $k \geq 1$

$$|G^{(j)}(x,y,k)| \leq \frac{\tilde{c}_0^j}{2k(2\pi k)^{j+1}} \frac{1}{|x-y|}, \quad j = 1,2,..., \tag{22}$$

and

$$|\nabla_x G^{(j)}(x,y,k)| \leq \frac{\tilde{c}_0^j}{(2\pi k)^{j+1}} \frac{1}{|x-y|}, \quad j = 1,2,.... \tag{23}$$

By (20) and (21) the claim holds for $j = 1$. Suppose that the claim is proved for $j \geq 1$. The induction hypothesis leads to

$$|G^{(j+1)}(x,y,k)| \leq \frac{\tilde{c}_0^j}{4\pi k^2 (2\pi k)^{j+1}} \left(\int\limits_{|z|\leq R} \frac{|\vec{W}(z)| + \frac{1}{2k}|V(z)|}{|x-z|} \frac{1}{|z-y|} \, dz + \right.$$

$$\left. + \int\limits_{|z|\geq R} \frac{C + \frac{C}{2k}}{|x-z||z|^{\mu}} \frac{1}{|z-y|} \, dz \right) \leq \frac{\tilde{c}_0^j}{4\pi k^2 (2\pi k)^{j+1}} \left(\frac{2\|\vec{W}\|_{L^\infty} C_R}{|x-y|} + \right.$$

$$\left. + \frac{C_V}{|x-y|} + \frac{3CC_\mu}{|x-y|} \right) = \frac{\tilde{c}_0^j}{4\pi k^2 (2\pi k)^{j+1}} \frac{\tilde{c}_0}{|x-y|}.$$

This finishes the proof of (22) by induction. The estimate (23) can be obtained similarly (by induction). Choosing now $k_0 > max\{1, \frac{\tilde{c}_0}{\pi}\}$ we obtain the estimates (19). Thus, Proposition 3 is proved.

Next we study the asymptotic behaviour of the scattering solutions u that provides the scattering data fo the inverse scattering problems in the foregoing section.

Proposition 4 *Assume that $\vec{W}$ is bounded, V belongs to the Kato space K_3 and both satisfy condition (3). Then for fixed $k \geq k_0$ the solution $u(x,k,\theta)$ to (6) ((8)) admits the representation*

$$u(x,k,\theta) = e^{ikx\cdot\theta} - \frac{1}{8\pi} \frac{e^{ik|x|}}{k^2|x|} A(k,\theta',\theta) + o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

The function $A(k,\theta,\theta')$ is called the scattering amplitude and is defined by equation (9).

Proof Since

$$u_{sc}(y,k,\theta) = -\int\limits_{R^3} G_k^+(|x-y|)(\vec{W}(y)\cdot\nabla u(y) + V(y)u(y)) \, dy =$$

$$= -\int\limits_{|y|\le|x|^a} G_k^+(|x-y|)(\vec{W}(y)\cdot\nabla u(y)+V(y)u(y))\,dy-$$

$$-\int\limits_{|y|\ge|x|^a} G_k^+(|x-y|)(\vec{W}(y)\cdot\nabla u(y)+V(y)u(y))\,dy =: I_1+I_2,$$

where parameter a is chosen such that $0<a<\frac{1}{2}$. For the integral I_1 we use the following asymptotic

$$G_k^+(|x-y|)=\frac{1}{8\pi k^2}\left(e^{ik|x|}e^{-ik\theta'\cdot y}-e^{-k|x|}e^{k\theta'\cdot y}\right)+O(|x|^{2a-2}).$$

This implies that

$$I_1=-\frac{1}{8\pi k^2}\int\limits_{|y|\le|x|^a}\left(e^{ik|x|}e^{-ik\theta'\cdot y}-e^{-k|x|}e^{k\theta'\cdot y}\right)\left(\vec{W}\cdot\nabla u+Vu\right)\,dy+O(|x|^{2a-2}).$$

And this in turn leads as $|x|\to+\infty$ to

$$I_1=-\frac{e^{ik|x|}}{8\pi k^2}\int\limits_{R^3} e^{-ik\theta'\cdot y}\left(\vec{W}\cdot\nabla u+Vu\right)\,dy+o(|x|^{-1}). \tag{24}$$

Next we consider the integral I_2 and split the region of integration as

$$|I_2|\le\int\limits_{|y|\ge|x|^a}|G_k^+(|x-y|)||\vec{W}(y)\cdot\nabla u(y)+V(y)u(y)|\,dy=$$

$$=\int\limits_{|x|^a\le|y|\le\frac{|x|}{2}}|G_k^+(|x-y|)||\vec{W}(y)\cdot\nabla u(y)+V(y)u(y)|\,dy+$$

$$+\int\limits_{|y|\ge\frac{|x|}{2}}|G_k^+(|x-y|)||\vec{W}(y)\cdot\nabla u(y)+V(y)u(y)|\,dy =: J_1+J_2.$$

In the case J_1 we have $|x-y|\ge|x|-|y|\ge\frac{|x|}{2}$. Thus, as $|x|\to+\infty$, one can have

$$J_1\le\frac{1}{4\pi k^2|x|}\int\limits_{|x|^a\le|y|\le\frac{|x|}{2}}|\vec{W}(y)\cdot\nabla u(y)+V(y)u(y)|\,dy=o(|x|^{-1})$$

due to conditions (2), (3) and Proposition 1. To estimate the integral J_2 we use condition (3) and Proposition 1 and obtain

$$J_2\le\frac{c}{k^2}\int\limits_{|y|\ge\frac{|x|}{2}}\frac{1}{|x-y||y|^{\mu}}\,dy\le\frac{c}{k^2|x|^{\epsilon}}\int\limits_{|y|\ge\frac{|x|}{2}}\frac{1}{|x-y||y|^{\mu-\epsilon}}\,dy,$$

where positive ϵ is chosen such that $2 < \mu - \epsilon < 3$ and $c > 0$. Apply now the estimate for the convolution of weak singularities (see, for example, [12], Lemma 34.3) one can finally obtain that

$$J_2 \le \frac{c}{k^2 |x|^{\mu-2}} = o(|x|^{-1})$$

since $\mu > 3$. The latter estimates and (24) finish the proof of Proposition 4.

3 Proof of the main results

This Section is devoted to the proof of the main results and their consequences.

Proof We prove Theorem 1. Indeed, denoting by I the integral

$$I := k^2 \int\limits_{S^2 \times S^2} e^{-ik(\theta-\theta')\cdot x} A(k,\theta,\theta')\, d\theta\, d\theta'$$

one can write (because $u = u_0 + u_{sc}$)

$$I := k^2 \int\limits_{S^2 \times S^2} e^{-ik(\theta-\theta')\cdot x} \int\limits_{R^3} e^{-ik\theta'\cdot y} [ik\theta \cdot \vec{W}(y) + V(y)]\, dy\, d\theta\, d\theta' +$$

$$+k^2 \int\limits_{S^2 \times S^2} e^{-ik(\theta-\theta')\cdot x} \int\limits_{R^3} e^{-ik\theta'\cdot y} [\vec{W}(y) \cdot \nabla u_{sc}(y) + V(y) u_{sc}(y)]\, dy\, d\theta\, d\theta' =$$

$$=: I_1 + I_2.$$

Since $\vec{W}$ and V belong to $L^1(R^3)$ (see conditions (2) and (3)) then I_1 can be rewritten as

$$I_1 = k^2 \int\limits_{R^3} \vec{W}(y)\, dy \cdot \int\limits_{S^2} ik\theta e^{-ik\theta\cdot(x-y)}\, d\theta \int\limits_{S^2} e^{ik\theta'\cdot(x-y)}\, d\theta' +$$

$$+k^2 \int\limits_{R^3} V(y)\, dy \int\limits_{S^2} e^{-ik\theta\cdot(x-y)}\, d\theta \int\limits_{S^2} e^{ik\theta'\cdot(x-y)}\, d\theta'.$$

It is very well known that (see, for example, [7])

$$\int\limits_{S^2} e^{-ik\theta\cdot(x-y)}\, d\theta = \sqrt{(2\pi)^3} \frac{J_{\frac{1}{2}}(k|x-y|)}{\sqrt{k|x-y|}} = 4\pi \frac{\sin(k|x-y|)}{k|x-y|}. \tag{25}$$

Hence, I_1 is equal to

$$I_1 = 8\pi^2 \int\limits_{R^3} \vec{W}(y) \cdot \nabla_x \left(\frac{\sin^2(k|x-y|)}{|x-y|^2} \right) dy + 16\pi^2 \int\limits_{R^3} V(y) \frac{\sin^2(k|x-y|)}{|x-y|^2}\, dy.$$

If now $\varphi \in C_0^\infty(R^3)$ then in the sense of distributions

$$< I_1, \varphi >= 8\pi^2 \int\limits_{R^3}\int\limits_{R^3} \vec{W}(y) \cdot \nabla_x \left(\frac{\sin^2(k|x-y|)}{|x-y|^2} \right) \varphi(x)\, dy\, dx +$$

$$+16\pi^2 \int\limits_{R^3}\int\limits_{R^3} V(y) \frac{\sin^2(k|x-y|)}{|x-y|^2} \varphi(x)\, dy\, dx.$$

Using the smoothness of φ and compactness of its support and integrating by parts we obtain

$$< I_1, \varphi >= -8\pi^2 \int\limits_{R^3} \vec{W}(y)\, dy \int\limits_{R^3} \frac{\sin^2(k|x-y|)}{|x-y|^2} \nabla_x \varphi(x)\, dx +$$

$$+16\pi^2 \int\limits_{R^3} V(y)\, dy \int\limits_{R^3} \frac{\sin^2(k|x-y|)}{|x-y|^2} \varphi(x)\, dx =$$

$$= -4\pi^2 \int\limits_{R^3} \vec{W}(y)\, dy \int\limits_{R^3} \frac{1}{|x-y|^2} \nabla_x \varphi(x)\, dx +$$

$$+4\pi^2 \int\limits_{R^3} \vec{W}(y)\, dy \int\limits_{R^3} \frac{\cos(2k|x-y|)}{|x-y|^2} \nabla_x \varphi(x)\, dx +$$

$$+8\pi^2 \int\limits_{R^3} V(y)\, dy \int\limits_{R^3} \frac{1}{|x-y|^2} \varphi(x)\, dx -$$

$$-8\pi^2 \int\limits_{R^3} V(y)\, dy \int\limits_{R^3} \frac{\cos(2k|x-y|)}{|x-y|^2} \varphi(x)\, dx.$$

Then application of Fubini theorem and Riemann - Lebesgue lemma lead to the equality

$$\lim_{k\to+\infty} < I_1, \varphi >= 4\pi^2 < \nabla_x \int\limits_{R^3} \frac{\vec{W}(y)}{|x-y|^2}\, dy, \varphi > +8\pi^2 < \int\limits_{R^3} \frac{V(y)}{|x-y|^2}\, dy, \varphi > .$$

To estimate I_2 one can first rewrite it (using again (25)) as

$$I_2 = 4\pi k \int\limits_{R^3} \frac{\sin(k|x-y|)}{|x-y|} \vec{W}(y)\, dy \cdot \int\limits_{S^2} e^{-ik\theta\cdot x} \nabla u_{sc}(k, y, \theta)\, d\theta +$$

$$+4\pi k \int\limits_{R^3} \frac{\sin(k|x-y|)}{|x-y|} V(y)\, dy \int\limits_{S^2} e^{-ik\theta\cdot x} u_{sc}(k, y, \theta)\, d\theta.$$

Next, we use the following equalities (see, for example, [15] or [12])

$$u_{sc}(y) = -\hat{G}_p\left(\vec{W}(z)\cdot\nabla u_0(z) + V(z)u_0(z)\right)(y),$$

$$\nabla_y u_{sc}(y) = -\nabla_y\hat{G}_p\left(\vec{W}(z)\cdot\nabla u_0(z) + V(z)u_0(z)\right)(y),$$

where $\hat{G}_p$ denotes the integral operator with kernel $G_p(x,y,k)$ (see Proposition 3). These equalities and (25) allow to obtain for I_2 the following representation

$$-16\pi^2\int\limits_{R^3}\frac{\sin(k|x-y|)}{|x-y|}\vec{W}(y)\cdot\nabla_y\hat{G}_p\left(\frac{\sin(k|x-z|)}{|x-z|}V(z)+\vec{W}(z)\nabla_x\frac{\sin(k|x-z|)}{|x-z|}\right)dy-$$

$$-16\pi^2\int\limits_{R^3}\frac{\sin(k|x-y|)}{|x-y|}V(y)\hat{G}_p\left(\frac{\sin(k|x-z|)}{|x-z|}V(z)+\vec{W}(z)\nabla_x\frac{\sin(k|x-z|)}{|x-z|}\right)dy.$$

The estimates (13) and (19) for $G_p(x,y,k)$ and the same technique as for I_1 allow easily to obtain that for any $\varphi\in C_0^\infty(R^3)$

$$\lim_{k\to+\infty} < I_2,\phi >= 0.$$

This finishes the proof of Theorem 1.

Remark If we assume in addition that $\vec{W}\in W_p^1(R^3)$ and $V\in L^p(R^3)$ with some $3<p\le\infty$ and with the same behaviour at the infinity then it can be proved that the limit in Saito's formula holds uniformly in $x\in R^3$ (see [15], [9]).

Proof We prove Corollary 1 (Uniqueness). We have only to show that the homogeneous equation

$$\Psi(x) := \frac{1}{2}\nabla_x\int\limits_{R^3}\frac{\vec{W}(y)}{|x-y|^2}\,dy + \int\limits_{R^3}\frac{V(y)}{|x-y|^2}\,dy = 0$$

has a unique solution such that $\frac{1}{2}\nabla\delta\star\vec{W}-V=0$. Indeed, following [11] and [10] consider the space $S_0(R^3)$ of all functions from the Schwarz space which vanish in some neighborhood of the origin. Then for every $\varphi(\xi)\in S_0(R^3)$ it follows that

$$0 =< \mathcal{F}(\Psi(x))(\xi),\varphi(\xi) >= 2\pi^2 < -\frac{i\xi}{2|\xi|}\cdot\mathcal{F}(\vec{W})(\xi)+\frac{\mathcal{F}(V)(\xi)}{|\xi|},\varphi(\xi) >=$$

$$= 2\pi^2 < -\frac{i\xi}{2}\cdot\mathcal{F}(\vec{W})(\xi)+\mathcal{F}(V)(\xi),\frac{\varphi(\xi)}{|\xi|} >,$$

where $\mathcal{F}$ is usual Fourier transform in R^3. Since $\varphi(\xi)\in S_0(R^3)$ then $\frac{\varphi(\xi)}{|\xi|}\in S_0(R^3)$ also. Hence, for every $h\in S_0(R^3)$, one can see that

$$< \frac{i\xi}{2}\cdot\mathcal{F}(\vec{W})(\xi)-\mathcal{F}(V)(\xi),h(\xi) >= 0.$$

This means that the support of the function $\frac{i\xi}{2}\cdot\mathcal{F}(\vec{W})(\xi)-\mathcal{F}(V)(\xi)$ is at most in the origin, and therefore it can be represented as follows (with some integer m)

$$\frac{i\xi}{2}\cdot\mathcal{F}(\vec{W})(\xi)-\mathcal{F}(V)(\xi)=\sum_{|\alpha|\le m} C_\alpha \partial^\alpha \delta(\xi),$$

where $\delta(\xi)$ is Dirac $\delta-$function and C_α are constants. The conditions for $\vec{W}$ and V imply that all these constants are equal to 0. Thus,

$$\mathcal{F}^{-1}\left(\frac{i\xi}{2}\cdot\mathcal{F}(\vec{W})(\xi)-\mathcal{F}(V)(\xi)\right)=0, \quad i.e. \quad \frac{1}{2}\nabla\delta\star\vec{W}(x)-V(x)=0.$$

This finishes the proof of Corollary 1.

Proof We prove now Theorem 2 (Reconstruction). Based on Proposition 3, one can represent (see (11) and (18))

$$G_k^+(|x-y|)-G_p(x,y,k)=\int_{R^3} G_k^+(|x-z|)[\vec{W}(z)\cdot\nabla_z G_p(z,y,k)+V(z)G_p(z,y,k)]\,dz=$$

$$=\int_{R^3} G_k^+(|x-z|)[\vec{W}(z)\cdot\nabla_z G_k^+(|z-y|)+V(z)G_k^+(|z-y|)]\,dz+$$

$$+\int_{R^3} G_k^+(|x-z|)[\vec{W}(z)\cdot\nabla_z \sum_{j=1}^{\infty} G^{(j)}(z,y,k)+V(z)\sum_{j=1}^{\infty} G^{(j)}(z,y,k)]\,dz:=J_1+J_2,$$

where $k\ge k_0$. We first consider J_1 and divide it into to parts J_1' and J_1'' w.r.t. $|z|<k$ and $|z|>k$, respectively for fixed k big enough. Using conditions (3) and estimates (13) the value J_1'' can be estimated as

$$|J_1''|\le \frac{c}{k^3}\int_{|z|>k}\frac{1}{|x-z||z-y||z|^\mu}\,dz, \tag{26}$$

where constant $c>0$ is independent on x,y,k. Since $\mu>4$ then the latter integral is $o\left(\frac{1}{k^4|x||y|}\right)$ uniformly in k,x,y big enough. For estimation of J_1' we assume that $|x|>k^{4+s}, |y|>k^{4+s}$ with $s>0$ and with k is big enough. In this case

$$ik(z-y)=-|y|ik\theta'+ikz, \quad ik|x-z|=ik|x|-ik\theta\cdot z+O(k^{-1-s}),$$

where $\theta=\frac{x}{|x|}, \theta'=\frac{y}{|y|}$. These representations lead to the following asymptotic behaviour

$$G_k^+(|x-z|)=\frac{e^{ik|x|}e^{-ik\theta\cdot z}}{8\pi k^2|x|}\left(1+\frac{O(1)}{k^{1+s}}\right), \quad |z|<k,$$

$$\nabla_z G_k^+(|y-z|) = \frac{e^{ik|y|}e^{-ik\theta'\cdot z}}{8\pi k^2|y|}\left(1+\frac{O(1)}{k^{1+s}}\right)\left(-ik\theta'+\frac{ikz}{|y|}\right), \quad |z|<k.$$

This allows to obtain that

$$J_1' = -\frac{e^{ik(|x|+|y|)}}{64\pi^2 k^4|x||y|}\int\limits_{|z|<k}(ik\vec{W}(z)\cdot\theta' - V(z))e^{-ik(\theta+\theta')\cdot z}\,dz + O\left(\frac{1}{k^{4+s}|x||y|}\right). \tag{27}$$

The next step is: symmetric (w.r.t. θ and θ') form of the latter integral

$$\int\limits_{|z|<k} ik\vec{W}(z)\cdot\theta' e^{-ik(\theta+\theta')\cdot z}\,dz$$

leads to the fact that

$$\int\limits_{|z|<k} ik\vec{W}(z)\cdot\theta' e^{-ik(\theta+\theta')\cdot z}\,dz = \frac{1}{2}\int\limits_{|z|<k} ik\vec{W}(z)\cdot(\theta'+\theta')e^{-ik(\theta+\theta')\cdot z}\,dz.$$

Hence, the first term of the sum (27) can be represented as

$$-\frac{e^{ik(|x|+|y|)}}{64\pi^2 k^4|x||y|}\int\limits_{|z|<k}\left(\frac{ik}{2}\vec{W}(z)\cdot(\theta+\theta') - V(z)\right)e^{-ik(\theta+\theta')\cdot z}\,dz. \tag{28}$$

For estimation of the term J_2 one can use the estimates (19), (22), (23) and conditions (2) and (3). Then using quite similar technique (with more careful considerations of $G^{(2)}$) as for the proof of the estimates (27) and (28) one can obtain that

$$J_2 = o\left(\frac{1}{k^4|x||y|}\right), \quad k, |x|, |y| \to +\infty. \tag{29}$$

Letting now $k, |x|, |y| \to +\infty$ such that fixed $\xi = k\left(\frac{x}{|x|}+\frac{y}{|y|}\right)$ and combining now (26) - (29) we obtain formula (12). Thus, Theorem 2 is completely proved.

Corollary 2 can be proved similarly as the proof of Corollary 1.

Conclusions

The direct scattering problems for a first order perturbation of the three-dimensional biharmonic operator with singular coefficients ($\vec{W}$ belongs to L^∞_{loc} and V belongs to the Kato space $K_{3,loc}$) was studied. It was shown that a solution to scattering problem with certain radiation conditions satisfies the integral Lippmann-Schwinger equation. The same was shown for the Green's function of the perturbed biharmonic operator with these coefficients. These integral equations have the unique solutions in the usual Sobolev spaces $W^1_\infty(R^3)$. The asymptotic behaviour of the scattering solutions for fixed $k > 0$ as $|x| \to +\infty$

was studied and a formula for the scattering amplitude was obtained. Similar asymptotic is obtained for the Green's function when $k, |x|, |y| \to +\infty$.

The main results of this paper, Saito's formula and formula for the Green's functions of the bi-Laplacian and the perturbed bi-Laplacian, were proved under very general assumptions on the coefficients. The proof of these formulas itself was based on explicit calculations starting from the formula for scattering amplitude and from the integral equation for the Green's functions. Some consequences of these results were discussed w.r.t. inverse scattering problems. Namely, the scattering amplitude uniquely determines a combination of the coefficients for the direct problem and in turn gives a uniqueness result for the inverse problem. And the behaviour of the Green's function of the perturbed bi-Laplacian for x, y, k large enough uniquely determines the same combination of the unknown coefficients.

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