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ABSTRACT

We show that there exists $0 < \alpha_0 < 1$ (depending on the parameters) such that the fractal percolation is almost surely purely α -unrectifiable for all $\alpha > \alpha_0$.

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1. Introduction

Fractal percolation, also known as Mandelbrot percolation, is a classical random process introduced by Mandelbrot in 1974 for the purpose of modelling turbulence [13]. Mandelbrot called the model canonical curdling whereas the name fractal percolation was established for the process later. We begin by describing the model briefly and refer to Section 3 for more precise definitions.

Fix $0 \leq p \leq 1$, and let $N \in \mathbb{N} := \{0, 1, 2, \dots\}$ with $N \geq 2$. Letting $d \in \mathbb{N} \setminus \{0\}$, construct a random compact subset E of the unit cube $Q_0 := [0, 1]^d \subset \mathbb{R}^d$ in the following manner: Divide Q_0 into N^d subcubes of equal size. Independently of each other, each of them is chosen with probability p and deleted with probability $1 - p$, and the collection of the chosen subcubes is denoted by \mathcal{C}_1 . Continue by repeating the same process for each $Q \in \mathcal{C}_1$. The set of all chosen cubes at the second level is denoted by \mathcal{C}_2 . Iterating this process inductively gives the fractal percolation set E , defined as

$$E := \bigcap_{n=1}^{\infty} \bigcup_{Q \in \mathcal{C}_n} Q.$$

The probability space Ω is the space of all constructions and the natural probability measure on Ω induced by this procedure is denoted by \mathbb{P}_p .

We shortly describe some basic properties of fractal percolation set relevant to our purposes and refer to [5,10,11] for further information. It is clear that $E = \emptyset$ with positive probability if $p < 1$, since $\mathcal{C}_1 = \emptyset$ with probability $(1 - p)^{N^d}$. It follows from the theory of branching processes that $E = \emptyset$ almost surely if the expected number of chosen cubes is at most one, that is, $p \leq N^{-d}$. Kahane and Peyrière [12] proved that, in the opposite case $p > N^{-d}$, the Hausdorff dimension, \dim_{H} , of the limiting set is almost surely a constant conditioned on non-extinction, that is,

$$\dim_{\text{H}} E = \frac{\log(pN^d)}{\log N} \tag{1.1}$$

almost surely conditioned on the event $E \neq \emptyset$.

In [7], J.T. Chayes, L. Chayes and R. Durrett verified in the case $d = 2$ that there is a critical probability $0 < p_c < 1$ such that if $p < p_c$, then E is totally disconnected with probability one, whereas the opposing sides of Q_0 are connected by a connected component of E with positive probability provided that $p \geq p_c$. The latter phenomenon is commonly referred to as fractal percolation. The exact value of p_c is not known. From (1.1) it trivially follows that $p_c \geq N^{-1}$ if $d = 2$, and in [7] it is proved that $N^{-1} < p_c < 1$. Corresponding results are apparently valid also for $d > 2$ (see [8]).

Even though $p_c > N^{-1}$, the set E looks connected from outside as soon as its dimension is larger than one. Indeed, in [8], Falconer and Grimmett proved that in this case the coordinate projections of E contain an interval almost surely conditioned on non-extinction. Further, Rams and Simon [22,23] showed that almost surely all projections of E contain an interval simultaneously if $\dim_{\mathbb{H}} E > 1$. This result also follows from [21]. Finally, almost surely all visible parts of E are 1-dimensional (see [1]).

By the above mentioned result of [7], conditioned on non-extinction, E contains almost surely a non-trivial connected component as soon as $p \geq p_c$. It is natural to ask whether E contains a non-trivial path connected component in this case. This was answered positively by Meester in [18]. As far as the regularity of paths contained in E is concerned, Chayes showed in [6] that the lower box counting dimension of any path contained in E is strictly larger than 1 with a bound depending on the parameters p and N . Thus, E does not contain uniform α -Hölder curves for α close to 1. In particular, E does not contain any rectifiable curves. An explicit lower bound for the lower box counting dimension of the non-trivial curves contained in E was given by Orzechowski in [20]. In [19], he proved that E contains non-trivial curves whose upper box counting dimension is strictly less than 2. Again, there is an explicit expression for the upper bound.

Broman et al. [4] showed that, in the case $d = 2$ and $p \geq p_c$, the set E can be decomposed as $E = E^c \cup E^d$, where E^d is the totally disconnected part of E and E^c consists of non-trivial connected components of E . Moreover, $\dim_{\mathbb{H}} E^c < \dim_{\mathbb{H}} E^d = \dim_{\mathbb{H}} E$ and there exists $0 < \beta < 1$, depending on the parameters, such that E^c is an uncountable union of non-trivial β -Hölder curves.

In this paper, we supplement the result of Broman et al. We first define a concept of α -unrectifiability: given $0 < \alpha \leq 1$, a set $A \subset \mathbb{R}^d$ is purely α -unrectifiable if $\mathcal{H}^{\frac{1}{\alpha}}(A \cap \gamma([0, 1])) = 0$ for all α -Hölder curves $\gamma: [0, 1] \rightarrow \mathbb{R}^d$, where \mathcal{H}^s is the s -dimensional Hausdorff measure. We note that related fractional rectifiability questions have been studied for example by Martín and Mattila for deterministic sets in [14–16] and, quite recently, by Badger, Naples and Vellis for measures in [2,3]. In our main theorem (Theorem 6.13), we verify that, for every $0 \leq p < 1$ and $N, d \in \mathbb{N} \setminus \{0, 1\}$, there exists $\alpha_0 < 1$ such that almost surely the fractal percolation set E is purely α -unrectifiable for all $\alpha_0 < \alpha \leq 1$. Since the case $\alpha = 1$ corresponds to standard 1-unrectifiability, our result implies that E is almost surely purely 1-unrectifiable and, thus, purely k -unrectifiable for all $k \in \mathbb{N}$. In Section 4 (see Theorem 4.17), we give a simpler proof than that of our main theorem for 1-unrectifiability. The general case, requiring new tools, is consid-

ered in Section 6. We believe that these new tools turn out to be useful in many other problems related to the fractal percolation and other random geometric constructions.

The paper is organised as follows. In Section 2, we explain the heuristic idea of the proof of our main result. In Section 3, we define the fractal percolation model by introducing a slightly different viewpoint than the standard one described earlier in this section but leading to same probabilities. In Section 4, we introduce basic concepts, prove preliminary results and give a short proof for the 1-unrectifiability of the fractal percolation in Theorem 4.17 which, in turn, implies the k -unrectifiability, see Corollary 4.18. Section 5 is concerned with probability estimates guaranteeing the existence and abundance of hereditarily good cubes, one of the key concepts in our paper. In these cubes holes generated during the fractal percolation process are “sufficiently uniformly” located. In Section 6, we study α -unrectifiability and prove our main result: Theorem 6.13. The proof is based on Proposition 6.12, which is our main tool guaranteeing the length increase in hereditarily good cubes for broken line approximations of curves staying close to the fractal percolation set. The proof of Proposition 6.12 requires several quite technical definitions, algorithms and results. These are collected in three Appendices.

In Section 7 (Appendix A), we construct the special sequences used in Section 8 to define correct zoom levels. Appendix B, that is, Section 8, is dedicated to our new tool consisting of several algorithms utilised to construct special broken line approximations of curves staying close to the fractal percolation set. Finally, in Appendix C, that is, in Section 9, we verify growth estimates for the length of the broken line approximations defined in Section 8 and prove Proposition 6.12.

2. Idea of the proof

In this section, we describe the heuristic idea behind the proof of our main result concerning the α -unrectifiability of fractal percolation. We also describe some subtleties encountered while making this idea rigorous.

The length of a smooth curve can be calculated by constructing finer and finer broken line approximations of the curve and by taking the limit of their lengths. In particular, the lengths of the broken line approximations are uniformly bounded. If the curve is genuinely α -Hölder continuous, the lengths of the broken line approximations typically tend to infinity and the exponent α controls how fast this may happen. Indeed, let $\gamma: [a, b] \rightarrow \mathbb{R}^d$ be α -Hölder continuous. Suppose that, for some integer L and some number $r > 0$, one can find points $a =: a_1 < a_2 < \dots < a_L < a_{L+1} := b$ such that

$$(1+r)|\gamma(a) - \gamma(b)| \leq \sum_{i=1}^L |\gamma(a_i) - \gamma(a_{i+1})|. \quad (2.1)$$

Assuming that this process can be iterated q times, there are points $a =: a_1 < a_2 < \dots < a_{L^q} < a_{L^q+1} := b$ such that

$$(1 + r)^q |\gamma(a) - \gamma(b)| \leq \sum_{i=1}^{L^q} |\gamma(a_i) - \gamma(a_{i+1})|.$$

Using α -Hölder continuity and Jensen’s inequality on the right hand side, one obtains

$$(1 + r)^q |\gamma(a) - \gamma(b)| \leq C \sum_{i=1}^{L^q} |a_i - a_{i+1}|^\alpha \leq CL^q (L^{-q} |a - b|)^\alpha. \tag{2.2}$$

If this is true for large q , we have that $1 + r \leq L^{1-\alpha}$, that is, the exponential growth rate of the broken line approximation is controlled by the Hölder exponent. In particular, if the above inequality is valid for some fixed r and L , the exponent α cannot be very close to 1.

The idea for the existence of r is as follows: If the intersection of a curve with the fractal percolation set has positive measure, the density point theorem implies that, at small scales, there cannot be big gaps, that is, parts of the curve whose intersection with the set is empty. On the other hand, there are a lot of holes at many scales in the percolation set. Hence, in order to avoid gaps, the curve has to go around the holes, increasing the length compared to a straight line.

For the purpose of iterating Inequality (2.1), one needs to find the fixed relative increase of length at many successive scales and at various different places, that is, for all subcurves. There are two competing phenomena: the size of a hole and the probability of its existence. In order to have many successive scales with holes, high probability for the existence of a hole is needed. This is obtained by decreasing the size of a hole, which, in turn, makes r smaller.

There is also a trade-off between L and probabilities. The larger L , the easier to obtain a length increase of $1 + r$, but the harder to iterate, since there are more subcurves which should satisfy suitable conditions.

The choice of q is also delicate. Taking large q makes the scales very small, which implies that the holes and, therefore, gaps become very small. Therefore, extremely small scales are required when using the density point argument. It turns out that we need to choose the points a and b in (2.2) such that they depend on q . To overcome this difficulty, we introduce a concept called tightness (see Definition 6.6).

Finally, we note that the order of quantifiers is tricky. Due to the definition of unrectifiability, one has to show that, almost surely, the intersection of the fractal percolation set with all α -Hölder curves has zero measure.

3. Fractal percolation model

Letting $d \in \mathbb{N} \setminus \{0, 1\}$ (the case $d = 1$ is trivial for our purposes), we begin by describing the underlying probability space related to the fractal percolation set in \mathbb{R}^d . Fix $N \in \mathbb{N} \setminus \{0, 1\}$. Let \mathcal{T} be the rooted N^d -branching tree and set $\Omega := \{0, 1\}^{v(\mathcal{T})} = \{\omega \mid \omega : v(\mathcal{T}) \rightarrow \{0, 1\}\}$, where $v(\mathcal{T})$ is the set of vertices of \mathcal{T} . Let $J := \{1, \dots, N^d\}$.

The vertices of \mathcal{T} may be naturally encoded by finite words with letters in J , that is, by elements of $\bigcup_{n=0}^{\infty} J^n$, where the root corresponds to the empty word \emptyset and the vertices whose distance to the root is n are coded by the words $\mathbf{i} := i_1 \cdots i_n$ of length n , where $i_j \in J$ for all $j = 1, \dots, n$. We denote the length of a word \mathbf{i} by $|\mathbf{i}|$ and define a metric ρ on Ω by setting $\rho(\omega, \omega') := N^{-|\omega \wedge \omega'|}$, where

$$|\omega \wedge \omega'| := \min\{n \in \mathbb{N} \mid \text{there exists } \mathbf{i} \in J^n \text{ with } \omega(\mathbf{i}) \neq \omega'(\mathbf{i})\}.$$

For $0 \leq p \leq 1$, define a Borel probability measure \mathbb{P}_p on Ω by

$$\mathbb{P}_p := ((1-p)\delta_0 + p\delta_1)^{v(\mathcal{T})},$$

where δ_k is the Dirac measure at k .

We consider the probability space $(\Omega, \mathcal{B}, \mathbb{P}_p)$, where \mathcal{B} is the completion of the Borel σ -algebra. Every $\omega \in \Omega$ defines a fractal percolation set $E(\omega) \subset \mathbb{R}^d$ as follows: For $n \in \mathbb{N}$, let

$$\mathcal{Q}_n := \left\{ \prod_{i=1}^d [(l_i - 1)N^{-n}, l_i N^{-n}] \mid l_i = 1, \dots, N^n \text{ and } i = 1, \dots, d \right\}$$

be the collection of grid cubes of $Q_0 := [0, 1]^d$ with side length N^{-n} . *The level of a cube* $Q \in \mathcal{Q}_n$ is n . Enumerating the elements of \mathcal{Q}_1 by J and using the same enumeration for the subcubes of Q belonging to \mathcal{Q}_{n+1} for all $Q \in \mathcal{Q}_n$, we define a natural bijection between J^n and \mathcal{Q}_n . The image of $\mathbf{i} \in J^n$ under this bijection is denoted by $Q_{\mathbf{i}}$. Given $\omega \in \Omega$, a cube $Q_{\mathbf{i}} \in \mathcal{Q}_n$ is *chosen* if $\omega(\mathbf{i}) = 1$ and *deleted* if $\omega(\mathbf{i}) = 0$. The set of chosen cubes in \mathcal{Q}_n is denoted by $\mathcal{C}_n(\omega) := \{Q_{\mathbf{i}} \in \mathcal{Q}_n \mid \omega(\mathbf{i}) = 1\}$. For every $\omega \in \Omega$, we define the fractal percolation set $E(\omega)$ by

$$E(\omega) := \bigcap_{n=0}^{\infty} \bigcup_{Q \in \mathcal{C}_n(\omega)} Q.$$

Note that $E(\omega) \neq \emptyset$ if and only if there exists an infinite subtree $T \subset \mathcal{T}$ rooted at \emptyset such that $\omega(\mathbf{i}) = 1$ for all vertices \mathbf{i} of T . In particular, $E(\omega)$ may be identified with the infinite component (determined by the condition $\omega(\mathbf{i}) = 1$) of \mathcal{T} containing the root.

Remark 3.1. In this section, we have chosen a slightly different viewpoint while defining the probability space Ω than the standard one described in the introduction. Indeed, the set $\mathcal{C}_n(\omega)$ depends only on $\omega(\mathbf{i})$ with $|\mathbf{i}| = n$ and, therefore, the sequence $(\mathcal{C}_n)_{n \in \mathbb{N}}$ is not nested. This is merely a notational convention, which does not change the probabilities related to the fractal percolation sets $E(\omega)$, but turns out to be useful for our purposes.

4. Pure 1-unrectifiability

We begin with some notation used throughout the paper. For all $i \in \{1, \dots, d\}$, the orthogonal projection onto the i -th coordinate axis is denoted by Π_i . For all $x, y \in \mathbb{R}^d$, $L(x, y)$ is the line segment connecting x and y . The complement of a set $A \subset \mathbb{R}^d$ is denoted by A^c and $\text{Int } A$ refers to the interior of A . Finally, set

$$\mathcal{Q}_n(A) := \{Q \in \mathcal{Q}_n \mid Q \subset A\} \tag{4.1}$$

for all $A \subset \mathbb{R}^d$ and $n \in \mathbb{N}$.

Definition 4.1. We say that $i \in \{1, \dots, d\}$ is a *principle direction* for a line $\ell \subset \mathbb{R}^d$ if $j = i$ maximises the length of $\Pi_j(s)$ for subsegments s of ℓ . Observe that for certain lines ℓ the principle direction is not unique. For all $n \in \mathbb{N}$, we say that a line $\ell \subset \mathbb{R}^d$ *intersects a cube* $Q \in \mathcal{Q}_n$ *properly* if $\mathcal{H}^1(\Pi_i(\ell \cap Q)) \geq d^{-1}N^{-n}$, where i is a principle direction for ℓ . For any $n \in \mathbb{N}$ and $i \in \{1, \dots, d\}$, we say that a line $\ell \subset \mathbb{R}^d$ *intersects a cube* $Q \in \mathcal{Q}_n$ *very properly in direction* i if $\Pi_i(\ell \cap Q)$ contains an interval of the form $[kd^{-1}N^{-n}, (k + 1)d^{-1}N^{-n}]$ for some integer k .

A set $\mathcal{L} \subset \mathbb{R}^d$ is an *i -layer* if $\mathcal{L} = \Pi_i^{-1}([a, b])$ for some $a, b \in \mathbb{R}$ with $a \leq b$, and \mathcal{L} is a *layer* if it is an i -layer for some $i \in \{1, \dots, d\}$. A set $\mathcal{L} \subset \mathbb{R}^d$ is an *(n, i) -layer* if $\mathcal{L} = \Pi_i^{-1}([kN^{-n}, (k+1)N^{-n}])$ for some $k \in \{0, \dots, N^n - 1\}$ and it is an *(n, i) -double-layer* if $\mathcal{L} = \Pi_i^{-1}([kN^{-n}, (k + 2)N^{-n}])$.

Let $a, b \in \mathbb{R}$ with $a < b$. We say that a curve $\gamma: [a, b] \rightarrow \mathbb{R}^d$ *passes through a layer* \mathcal{L} , if $\gamma(a)$ and $\gamma(b)$ belong to different connected components of $(\text{Int } \mathcal{L})^c$. For $x, y \in \mathbb{R}^d$, the line segment $L(x, y)$ passes through \mathcal{L} , if $\gamma: [0, 1] \rightarrow \mathbb{R}^d$, $\gamma(t) := x + t(y - x)$, passes through \mathcal{L} .

We say that cubes $Q, Q' \in \mathcal{Q}_n$ are *neighbours* if $Q \cap Q' \neq \emptyset$. In this case, we use the notation $Q \sim Q'$. Cubes $Q, Q' \in \mathcal{Q}_n$ are *i -neighbours*, denoted by $Q \sim_i Q'$, if $Q \sim Q'$ and $Q, Q' \subset \mathcal{L}$ for some (n, i) -layer \mathcal{L} .

The above definition of intersecting properly is motivated by the following geometric observation.

Observation 4.2. Fix $i \in \{1, \dots, d\}$, $n \in \mathbb{N}$ and an (n, i) -layer \mathcal{L} . Let $x, y \in \mathbb{R}^d$ and assume that

- (a) the line segment $L(x, y)$ passes through \mathcal{L} and
- (b) for all $j \neq i$ there are (n, j) -layers \mathcal{L}_j^1 and \mathcal{L}_j^2 such that $L(x, y) \subset \mathcal{L}_j^1 \cup \mathcal{L}_j^2$.

Let ℓ be the line containing $L(x, y)$. Then there exists $Q \in \mathcal{Q}_n(\mathcal{L})$ with the following properties:

- (i) The line ℓ intersects Q very properly in direction i .

- (ii) The line ℓ intersects Q properly and $\ell \cap Q = L(x, y) \cap Q$.
- (iii) The cube Q is a neighbour to every cube belonging to $\mathcal{Q}_n(\mathcal{L})$ that is intersected by $L(x, y)$ in a set of positive length.

Proof. Define $\gamma: [0, 1] \rightarrow \mathbb{R}^d$ by $\gamma(t) := x' + t(y' - x')$, where $\{x', y'\} = L(x, y) \cap \partial\mathcal{L}$ and the boundary of a set A is denoted by ∂A . By (b), the line segment $L(x, y)$ intersects at most two different cubes in a set of positive length in each direction $j \neq i$. Note that along $L(x, y)$ one moves from one cube to another one at most once in each coordinate direction $j \neq i$. Therefore, there are points $0 < t_1 < \dots < t_m < 1$ with $m \leq d - 1$ satisfying the property that, for all $l = 0, \dots, m$, there is $Q \in \mathcal{Q}_n(\mathcal{L})$ such that $\gamma([t_l, t_{l+1}]) \subset Q$. Hence if we subdivide the (n, i) -layer \mathcal{L} into d parallel i -layers $\tilde{\mathcal{L}}_1, \dots, \tilde{\mathcal{L}}_d$ of equal width then there exist $l, k \in \{0, \dots, d - 1\}$ such that $\gamma|_{[t_l, t_{l+1}]}$ passes through $\tilde{\mathcal{L}}_k$, which implies (i). The first statement in (ii) is true since a principle direction maximises the length of the projection of a line segment, and the second claim follows from assumption (a). Finally, (iii) follows from (b). \square

The next definition deals with random concepts, that is, concepts which depend on $\omega \in \Omega$.

Definition 4.3. Let $m_0 \in \mathbb{N} \setminus \{0\}$, and fix $\omega \in \Omega$. For all $n \in \mathbb{N}$, a cube $Q' \in \mathcal{Q}_{n+m_0}$ is *strongly i -deleted* if $\tilde{Q} \in \mathcal{Q}_{n+m_0} \setminus \mathcal{C}_{n+m_0}(\omega)$ for all $\tilde{Q} \sim_i Q'$, see Fig. 4.1. A cube $Q' \in \mathcal{Q}_{n+m_0}$ is *weakly i -chosen* if it is not strongly i -deleted, that is, either Q' or at least one of its neighbours in the same $(n + m_0, i)$ -layer is chosen.

Let $\ell \subset \mathbb{R}^d$ be a line with a principle direction $i \in \{1, \dots, d\}$ and let $Q \in \mathcal{Q}_n$ be such that $\ell \cap Q$ has positive length. Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be the $(n + m_0, i)$ -layers (in the natural order) for which $\ell \cap Q \cap \mathcal{L}_j$ has positive length. We say that Q is *m_0 -good for the line ℓ* if there is a strongly i -deleted cube $Q' \in \mathcal{Q}_{n+m_0}(Q)$ so that

- (a) $Q' \not\subset \mathcal{L}_j$ for $j \in \{1, 2, k - 1, k\}$ and
- (b) ℓ intersects Q' properly.

A cube $Q \in \mathcal{Q}_n$ is *m_0 -bad for a line ℓ* if it is not m_0 -good for ℓ .

For a $(d - 1)$ -dimensional face F of $Q \in \mathcal{Q}_n$, we denote by $G_{m_0}(F)$ the grid points of the natural grid of side length N^{-n-m_0} in F . For all $Q \in \mathcal{Q}_n$, define a collection of lines

$$\Gamma(Q, m_0) := \bigcup_F \bigcup_v \Gamma_{F,v}(m_0),$$

where the first union is over all $(d - 1)$ -dimensional faces F of Q , the second one is over all vertices of Q not contained in F and $\Gamma_{F,v}(m_0)$ is the collection of all lines ℓ such that $v \in \ell$ and $\ell \cap G_{m_0}(F) \neq \emptyset$, see Fig. 4.1. Note that if F is perpendicular to the i -th coordinate axis, then i is a principle direction for every $\ell \in \Gamma_{F,v}(m_0)$. We call a cube

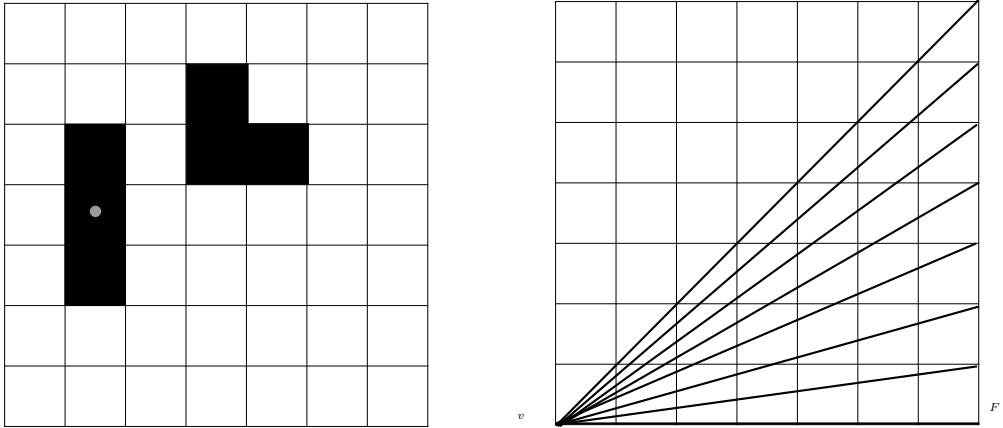


Fig. 4.1. On the left hand side, the black cubes are deleted, white ones are chosen and the black cube with a grey spot is strongly i -deleted, where $i = 1$. A collection $\Gamma_{F,v}(m_0)$ is illustrated on the right hand side.

$Q \in \mathcal{Q}_n$ m_0 -good if it is m_0 -good for all lines which intersect Q properly and are parallel to some line in $\Gamma(Q, m_0)$. Finally, a cube $Q \in \mathcal{Q}_n$ is m_0 -bad if it is not m_0 -good.

Remark 4.4. (a) Observe that all the concepts in Definition 4.3 depend only on $\mathcal{C}_{n+m_0}(\omega)$. If $Q, Q' \in \mathcal{Q}_n$ with $Q \approx Q'$, then the events “ Q is m_0 -good” and “ Q' is m_0 -good” are independent since only those cubes of $\mathcal{C}_{n+m_0}(\omega)$ play any role that intersect Q or Q' and since m_0 is at least 1, no cube intersects both Q and Q' . If $Q \sim Q'$ then these events are not independent due to the lines passing through Q or Q' close to their boundaries.

(b) If $Q \in \mathcal{Q}_n$ is m_0 -good for a line ℓ , then there is a line segment $I_\ell \subset Q \cap \ell \cap E(\omega)^c$ with $\mathcal{H}^1(I_\ell) \geq d^{-1}N^{-n-m_0}$.

We continue by introducing the notation utilised in Lemmas 4.5–4.8. Set

$$N_{m_0} := \lfloor d^{-1}N^{m_0} \rfloor - 4, \tag{4.2}$$

where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$. If $Q \in \mathcal{Q}_n$ and ℓ is a line such that $\ell \cap Q$ contains more than one point, let $\ell^a(Q)$ and $\ell^b(Q)$ be the end points of the line segment $\ell \cap Q$. Given $n \in \mathbb{N}$, $Q \in \mathcal{Q}_n$ and $i \in \{1, \dots, d\}$, let ℓ be a line with a principle direction i and intersecting Q properly. The number of $(n + m_0, i)$ -layers which the line segment $L(\ell^a(Q), \ell^b(Q))$ intersects in a set of positive length is at least $\lfloor d^{-1}N^{m_0} \rfloor$. Ignoring the first two and the last two of them (in the natural order, recall property (a) in Definition 4.3), we are left with at least N_{m_0} layers which $L(\ell^a(Q), \ell^b(Q))$ passes through. For each such $(n + m_0, i)$ -layer \mathcal{L}' , by choosing x and y to be the end points of the line segment $\mathcal{L}' \cap \ell$, the assumptions of Observation 4.2 are valid for i, x, y and \mathcal{L}' . Therefore, in each of these layers \mathcal{L}' , there is at least one cube $Q' \in \mathcal{Q}_{n+m_0}(\mathcal{L}')$ which ℓ intersects very properly in direction i . Select the first N_{m_0} of these layers. If there is

more than one very properly intersected cube inside some layer, order the cubes inside the layer in some systematic way and select the smallest very properly intersected cube with respect to this order. Let

$$\mathcal{K}_{m_0}^Q(\ell) := \{Q'_1, \dots, Q'_{N_{m_0}}\} \tag{4.3}$$

be the collection of cubes selected in this manner, see Fig. 4.2. If ℓ does not intersect Q properly, we set $\mathcal{K}_{m_0}^Q(\ell) := \emptyset$.

Lemma 4.5. *Let $n \in \mathbb{N}$, $Q \in \mathcal{Q}_n$ and $i \in \{1, \dots, d\}$. Fix $m_0 \in \mathbb{N}$ such that $N_{m_0} > 1$. Assume that a line ℓ with a principle direction i intersects Q properly. For all $0 \leq p < 1$, there exists $0 \leq q = q(p, d) < 1$ such that*

$$\mathbb{P}_p(\{\omega \in \Omega \mid \text{every } Q' \in \mathcal{K}_{m_0}^Q(\ell) \text{ is weakly } i\text{-chosen}\}) \leq q^{N_{m_0}}.$$

In particular,

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is } m_0\text{-good for } \ell\}) \geq 1 - q^{N_{m_0}}.$$

Proof. By definition, for every $Q' \in \mathcal{Q}_{n+m_0}$,

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q' \text{ is strongly } i\text{-deleted}\}) \geq (1 - p)^{3^{d-1}},$$

giving

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q' \text{ is weakly } i\text{-chosen}\}) \leq 1 - (1 - p)^{3^{d-1}} =: q.$$

Since the cubes in $\mathcal{K}_{m_0}^Q(\ell)$ belong to different $(n + m_0, i)$ -layers, the events “ Q'_j is weakly i -chosen” and “ Q'_k is weakly i -chosen” are independent provided that $j \neq k$. Therefore,

$$\mathbb{P}_p(\{\omega \in \Omega \mid \text{every } Q' \in \mathcal{K}_{m_0}^Q(\ell) \text{ is weakly } i\text{-chosen}\}) \leq q^{N_{m_0}}.$$

Note that Q is m_0 -good for ℓ , if at least one of the cubes $Q' \in \mathcal{K}_{m_0}^Q(\ell)$ is strongly i -deleted. Thus,

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is } m_0\text{-good for } \ell\}) \geq 1 - q^{N_{m_0}}. \quad \square$$

Definition 4.6. Let $n \in \mathbb{N}$. If F is a face of a cube $Q \in \mathcal{Q}_n$, we denote by $-F$ the face of Q which is parallel to F and not equal to F . If ℓ is a line and a is a point in \mathbb{R}^d , we denote by ℓ_a the line parallel to ℓ and containing a .

Lemma 4.7. *Let $n, m_0 \in \mathbb{N}$ with $m_0 > 0$ and let $Q \in \mathcal{Q}_n$. Assume that ℓ intersects Q properly and is parallel to some $\ell' \in \Gamma(Q, m_0)$. Then there exist a face F and a vertex v of Q and a line $\hat{\ell} \in \Gamma_{F,v}(m_0)$ such that $\ell = \hat{\ell}_a$ for some $a \in -F$.*

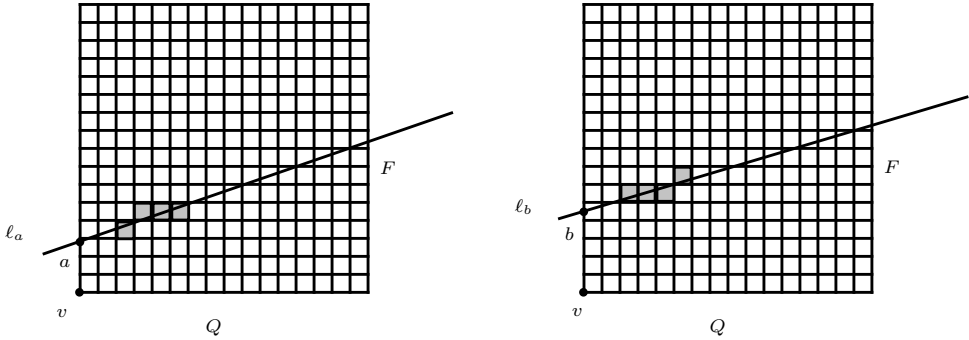


Fig. 4.2. Illustration of collections $\mathcal{K}_{m_0}^Q(\ell_a)$ and $\mathcal{K}_{m_0}^Q(\ell_b)$ in the case $d = 2$, $N_{m_0} = 4$ and $i = 1$. The cubes belonging to $\mathcal{K}_{m_0}^Q(\ell_a)$ and $\mathcal{K}_{m_0}^Q(\ell_b)$ are shaded.

Proof. Since $\ell' \in \Gamma(Q, m_0)$, we have $\ell' \in \Gamma_{F',v'}(m_0)$ for some face F' and vertex v' of Q . If $\ell \cap (-F') \neq \emptyset$, we may choose $\hat{\ell} = \ell'$ and $F = F'$. If $\ell \cap (-F') = \emptyset$, then ℓ intersects F' and there is $\hat{\ell} \in \Gamma_{-F',-v'}(m_0)$ parallel to ℓ such that $\ell = \hat{\ell}_a$ for some $a \in F'$, where $-v'$ is the opposite vertex to v' . \square

Next we estimate the number of essentially different translates of a line $\ell \in \Gamma_{F,v}(m_0)$. Denote by d_{\max} the maximum metric on \mathbb{R}^d .

Lemma 4.8. *Let $n \in \mathbb{N}$, $i \in \{1, \dots, d\}$ and $Q \in \mathcal{Q}_n$. Assume that $m_0 \in \mathbb{N}$ is such that $N_{m_0} \geq 1$. Fix a face F and a vertex v of Q such that $v \notin F$ and F is perpendicular to the i -th coordinate axis. Assume that $\ell \in \Gamma_{F,v}(m_0)$.*

- (a) *If $a, b \in -F$ with $d_{\max}(a, b) \geq N^{-n-m_0}$, then $\mathcal{K}_{m_0}^Q(\ell_a) \cap \mathcal{K}_{m_0}^Q(\ell_b) = \emptyset$.*
- (b) *We have*

$$\#\{\mathcal{K}_{m_0}^Q(\ell_b) \mid b \in -F\} \leq N^{2m_0(d-1)},$$

where the number of elements in a set B is denoted by $\#B$.

Proof. For an illustration of collections $\mathcal{K}_{m_0}^Q(\ell_a)$ and $\mathcal{K}_{m_0}^Q(\ell_b)$, see Fig. 4.2. Claim (a) follows directly from the definition of $\mathcal{K}_{m_0}^Q(\ell)$ and is obvious from Fig. 4.2.

To prove (b), by symmetry we can suppose that $i = 1$ and $-F$ is included in the hyperplane $(x_1 = 0) := \{x \in \mathbb{R}^d \mid x_1 = 0\}$. Let H be the union of the hyperplanes $(x_1 = jN^{-n-m_0}d^{-1})$ for $j = 2d, 2d + 1, \dots, (N_{m_0} + 2)d$, and denote by D the union of the hyperplanes spanned by the faces of the cubes of $\mathcal{Q}_{n+m_0}(Q)$ that are not parallel to F . Note that when b varies in $-F$, the set $\mathcal{K}_{m_0}^Q(\ell_b)$ can be changed only when ℓ_b intersects $H \cap D$. Thus, if we project $H \cap D$ in the direction of ℓ to the hyperplane $(x_1 = 0)$, we get a grid such that $\mathcal{K}_{m_0}^Q(\ell_b)$ is constant if b varies inside any cell of the grid. This grid has at most $(N_{m_0}d + 1)(N^{m_0} + 1)$ many $(d - 2)$ -dimensional walls in each

direction $j \neq 1$, so it has at most $((N_{m_0}d + 1)(N^{m_0} + 1) + 1)^{d-1}$ cells. Since by definition $N_{m_0}d \leq N^{m_0} - 4d \leq N^{m_0} - 2$, we have $((N_{m_0}d + 1)(N^{m_0} + 1) + 1)^{d-1} \leq N^{2m_0(d-1)}$, which completes the proof of (b). \square

According to the next proposition, which is one of our key preliminary results, the probability for being m_0 -good is large when m_0 is large.

Proposition 4.9. *For all $0 \leq p < 1$ and $m_0 \in \mathbb{N} \setminus \{0\}$, there exists a number $0 \leq p_g = p_g(m_0, p, d, N) \leq 1$ with $\lim_{m_0 \rightarrow \infty} p_g(m_0, p, d, N) = 1$ such that*

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is } m_0\text{-good}\}) \geq p_g$$

for all $n \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$.

Proof. If Q is not m_0 -good, there are $i \in \{1, \dots, d\}$, a face F of Q perpendicular to the i -th coordinate axis, a vertex v of Q not contained in F and a line ℓ with a principle direction i intersecting Q properly such that ℓ is parallel to some $\ell' \in \Gamma_{F,v}(m_0)$, ℓ intersects $-F$ and Q is m_0 -bad for ℓ . In particular, every $Q' \in \mathcal{K}_{m_0}^Q(\ell)$ is weakly i -chosen. By Lemma 4.8, the number of different collections $\mathcal{K}_{m_0}^Q(\ell_b)$ is at most $N^{2m_0(d-1)}$. Using the fact that F and v may be chosen in $2d$ and 2^{d-1} different ways, respectively, leads to

$$\#\Gamma(Q, m_0) \leq d2^d \#G_{m_0}(F) = d2^d(N^{m_0} + 1)^{d-1} \leq CN^{m_0(d-1)},$$

where C depends only on d . Using Lemma 4.5 and Lemma 4.8, we conclude that

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is } m_0\text{-bad}\}) \leq CN^{m_0(d-1)}N^{2m_0(d-1)}q^{N_{m_0}} = CN^{3m_0(d-1)}q^{N_{m_0}} =: s.$$

Since $N^{m_0} \leq 2dN_{m_0}$ when $N_{m_0} > 5$, the claim follows with $p_g := \max\{1 - s, 0\}$. \square

We need a few geometrical lemmas. We denote by

$$\text{dist}(x, A) := \inf\{|x - y| \mid y \in A\} \tag{4.4}$$

the Euclidean distance between a point $x \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$.

Lemma 4.10. *Let $x = (1, x_2, \dots, x_d)$ and $w = (0, 1, w_3, \dots, w_d)$ be points in \mathbb{R}^d such that $0 \leq x_j \leq 1$ for all $j = 2, \dots, d$ and $0 \leq w_j \leq 1$ for all $j = 3, \dots, d$. Let ℓ be the line spanned by x . Then*

$$\text{dist}(w, \ell) \geq \frac{1}{\sqrt{2}}.$$

Proof. A standard calculation shows that $\text{dist}(w, \ell)$ is minimised when $x_2 = 1$ and $|(x_3, \dots, x_d)|$ obtains an extreme value, that is, $(x_3, \dots, x_d) \in \{(0, \dots, 0), (1, \dots, 1)\}$. If $(x_3, \dots, x_d) = (0, \dots, 0)$, the claim easily follows. In the other case, a standard calculation shows that the minimum is obtained when $w_j = \frac{1}{2}$ for all $j = 3, \dots, d$ and then $\text{dist}(w, \ell) = \frac{1}{\sqrt{2}}$. \square

Lemma 4.11. *If $j \in \{1, \dots, d\}$ is a principle direction for a line $\ell \subset \mathbb{R}^d$ then, for all $w \in \mathbb{R}^d$, we have that*

$$d_{\max}(w, \ell \cap \Pi_j^{-1}(\Pi_j(w))) \leq \sqrt{2} \text{dist}(w, \ell). \tag{4.5}$$

Proof. By choosing the coordinate system properly, this follows directly from Lemma 4.10. \square

Lemma 4.12. *Let $n, m_0 \in \mathbb{N}$ with $m_0 > 0$ and $Q \in \mathcal{Q}_n$. Suppose that $j \in \{1, \dots, d\}$ is a principle direction for a line $\ell \subset \mathbb{R}^d$ and ℓ intersects Q properly. Then there exist $\ell' \in \Gamma(Q, m_0)$ and a line $\ell'' \subset \mathbb{R}^d$ such that*

- (i) *the lines ℓ' and ℓ'' are parallel and j is their principle direction,*
- (ii) $\Pi_j(\ell \cap Q) = \Pi_j(\ell'' \cap Q)$ *and*
- (iii) $d_{\max}(\ell \cap \Pi_j^{-1}(z), \ell'' \cap \Pi_j^{-1}(z)) \leq \frac{1}{2}N^{-n-m_0}$ *for every $z \in \Pi_j(Q)$.*

Proof. Fig. 4.3 is serving as an illustration for the proof of this lemma and also for its application in the proof of Lemma 4.13. Hence, apart from objects used in this proof, some other ones are also shown. The dashed line in the figure is $\Pi_j^{-1}(z)$.

Without loss of generality, we can suppose that $n = 0$, $Q = [0, 1]^d$ and $j = 1$. Let the direction vector of ℓ be $(1, a_2, \dots, a_d)$. Since 1 is a principle direction for ℓ , we have that $|a_i| \leq 1$ for all $i = 2, \dots, d$. By symmetry, we can suppose that $a_i \geq 0$ for every i . Let $v = (0, \dots, 0)$, $u = (1, a_2, \dots, a_d)$ and F be the face of Q in the hyperplane $(x_1 = 1)$. Then there is a point $u' = (1, a'_2, \dots, a'_d) \in G_{m_0}(F)$ such that $|a_i - a'_i| \leq \frac{1}{2}N^{-n-m_0}$ for all $i = 2, \dots, d$. The points v and u' define an element $\ell' \in \Gamma(Q, m_0)$. Let $\tilde{x} = (0, b_2, \dots, b_d)$ and $\tilde{y} = (1, c_2, \dots, c_d)$ be the intersections of ℓ with the hyperplanes $(x_1 = 0)$ and $(x_1 = 1)$, respectively. Clearly, $c_i - b_i = a_i$ for all $i = 2, \dots, d$.

Note that if we want to define ℓ'' as a line through $(0, b''_2, \dots, b''_d)$ and $(1, c''_2, \dots, c''_d)$, the requirements (i), (ii) and (iii) about ℓ'' are satisfied if and only if, for all $i = 2, \dots, d$, we have

- (1) $c''_i - b''_i = a'_i$,
- (2) $(\forall t \in [0, 1]) a_i t + b_i \in [0, 1] \Leftrightarrow a'_i t + b''_i \in [0, 1]$ and
- (3) $|b''_i - b_i| \leq \frac{1}{2}N^{-n-m_0}$ and $|c''_i - c_i| \leq \frac{1}{2}N^{-n-m_0}$.

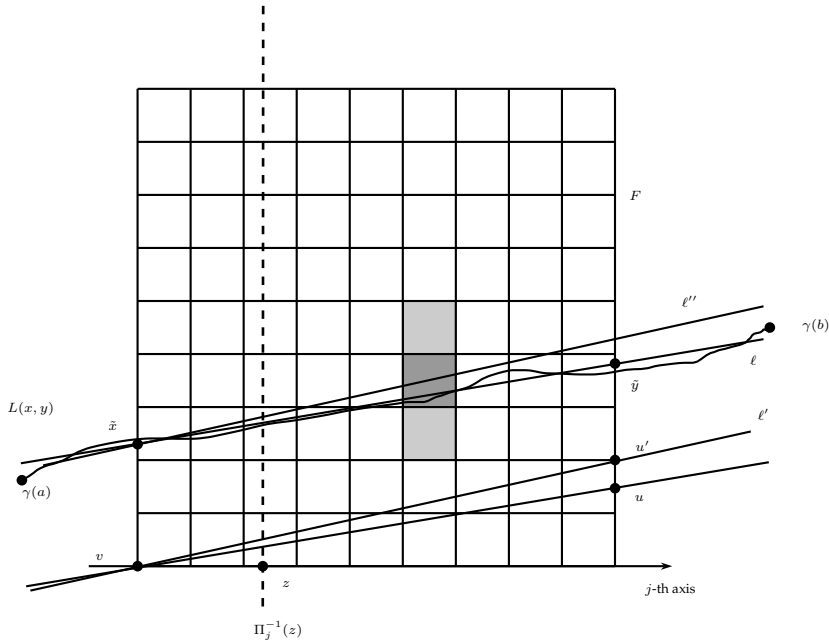


Fig. 4.3. An illustration for the proofs of Lemmas 4.12 and 4.13.

For different values of i these conditions are independent, and for any fixed i they are equivalent to the two-dimensional versions of the original (i), (ii) and (iii). Therefore, we can suppose that $d = 2$. Then either one or both endpoints of the segment $\ell \cap Q$ are on the vertical sides of the square Q . In the first case, it is easy to see that we can choose ℓ'' as the line parallel to ℓ' and going through the other endpoint of $\ell \cap Q$. In the latter case, there are two possibilities: either $\ell \cap Q$ intersects $\ell' \cap Q$ or not. If the intersection is non-empty, we may choose $\ell'' = \ell'$. In the opposite case, we can choose ℓ'' as the line parallel to ℓ' and going through the endpoint of $\ell \cap Q$ closer to ℓ' . This situation is shown in Fig. 4.3, where ℓ'' is going through the endpoint \tilde{x} . \square

We denote by d_H the Hausdorff distance between compact subsets of \mathbb{R}^d .

Lemma 4.13. *Let $n, m_0 \in \mathbb{N}$ with $N_{m_0} \geq 1$, $\omega \in \Omega$ and $i \in \{1, \dots, d\}$. Suppose that $x, y \in [0, 1]^d$ are such that $L(x, y)$ passes through an (n, i) -layer \mathcal{L} . Let $\gamma: [a, b] \rightarrow \mathbb{R}^d$ be a curve passing through \mathcal{L} such that*

$$d_H(\gamma([a, b]), L(x, y)) < \frac{1}{2\sqrt{2}} N^{-n-m_0}. \tag{4.6}$$

Finally, assume that all the cubes $Q \in \mathcal{Q}_n(\mathcal{L})$ intersecting $L(x, y)$ are m_0 -good. Then there are $\tilde{a}, \tilde{b} \in [a, b]$ such that $\gamma([\tilde{a}, \tilde{b}]) \cap E(\omega) = \emptyset$ and $|\gamma(\tilde{a}) - \gamma(\tilde{b})| \geq d^{-1} N^{-n-m_0}$.

Proof. If x, y and \mathcal{L} do not satisfy the assumptions of Observation 4.2, there is $j \in \{1, \dots, d\} \setminus \{i\}$ such that $L(x, y)$ intersects three successive (n, j) -layers, and one may pick points $x', y' \in L(x, y)$ and an (n, j) -layer \mathcal{L}' such that the line segment $L(x', y') \subset L(x, y)$ passes through \mathcal{L}' and x' and y' belong to the boundary of \mathcal{L}' . Iterating this argument, we end up with points \tilde{x} and \tilde{y} and an (n, \tilde{j}) -layer $\tilde{\mathcal{L}}$ satisfying assumptions of Observation 4.2 with i replaced by \tilde{j} .

Let ℓ be the line containing $L(x, y)$ and let $j \in \{1, \dots, d\}$ be its principal direction. Clearly, ℓ contains \tilde{x} and \tilde{y} . By Observation 4.2, there is $Q \in \mathcal{Q}_n(\tilde{\mathcal{L}})$ such that ℓ intersects Q properly and $\ell \cap Q = L(x, y) \cap Q$. By Lemma 4.12, there exist $\ell' \in \Gamma(Q, m_0)$ and a line $\ell'' \subset \mathbb{R}^d$ such that (i), (ii) and (iii) of Lemma 4.12 hold.

Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be the $(n + m_0, j)$ -layers such that $\ell'' \cap Q \cap \mathcal{L}_h$ has positive length (recall this type of notation in Definition 4.3). Since $N_{m_0} \geq 1$ and ℓ'' intersects Q properly, we have that $k \geq N_{m_0} + 4 \geq 5$. Since Q is m_0 -good, there is $h \in \{1, \dots, k\} \setminus \{1, 2, k - 1, k\}$ such that ℓ'' intersects properly a cube $Q' \in \mathcal{Q}_{n+m_0}(Q \cap \mathcal{L}_h)$ that is strongly j -deleted. By (4.6), there are points $c, d \in [a, b]$ such that $|\gamma(c) - x| < N^{-n-m_0}$ and $|\gamma(d) - y| < N^{-n-m_0}$. Applying property (ii) of Lemma 4.12 and the fact that $h \notin \{1, 2, k - 1, k\}$, we conclude that the curve $\gamma|_{[c,d]}$ passes through \mathcal{L}_h . Suppose that $\gamma(t) \in \mathcal{L}_h$. Applying (4.5) of Lemma 4.11 with $w = \gamma(t)$ and using (4.6) and property (iii) of Lemma 4.12, we conclude that $\gamma|_{[c,d]}$ passes through $\Pi_j^{-1}(\Pi_j(Q' \cap \ell''))$ inside $\bigcup_{Q'' \sim_j Q'} Q''$. Write $\partial(\Pi_j^{-1}(\Pi_j(Q' \cap \ell''))) =: B_c \cup B_d$, where B_c and B_d are the hyperplanes closer to $\gamma(c)$ and $\gamma(d)$, respectively. Now the points $\tilde{a} := \max\{t \in [c, d] \mid \gamma(t) \in B_c\}$ and $\tilde{b} := \min\{t \in [c, d] \mid \gamma(t) \in B_d\}$ satisfy the claim. \square

The next proposition will be used for studying the size of the set of points that belong to a good cube with good neighbour cubes at infinitely many scales. For our purposes, the critical value of dimension will be 1 (see the proof of Theorem 4.17 and Proposition 5.15) but, with the same effort, we prove a more general statement, since it might be useful in other connections related to dimensions of random constructions.

Proposition 4.14. *Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a sequence of independent sub- σ -algebras of \mathcal{B} on Ω . For all $n \in \mathbb{N}$, let $A_n: \Omega \times \mathcal{Q}_n \rightarrow \{0, 1\}$ be a function such that $A_n(\cdot, Q)$ is \mathcal{F}_n -measurable for all $Q \in \mathcal{Q}_n$. Let $0 \leq \varrho \leq 1$. Assume that, for all $n \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$,*

$$\mathbb{P}_p(\{\omega \in \Omega \mid A_n(\omega, Q) = 1\}) \leq \varrho. \tag{4.7}$$

Let $M_1 \in \mathbb{N}$ and assume that \simeq is a reflexive relation on $\bigcup_{n \in \mathbb{N}} \mathcal{Q}_n$ such that, for all $Q \in \mathcal{Q}_n$, there are at most M_1 cubes $Q' \in \mathcal{Q}_n$ with $Q' \simeq Q$, and $Q' \not\simeq Q$, if $Q' \in \mathcal{Q}_m$ with $m \neq n$. Given $\omega \in \Omega$, we say that a cube $Q \in \mathcal{Q}_n$ is selected, if $A_n(\omega, Q') = 1$ for some $Q' \simeq Q$. For all $n, k \in \mathbb{N}$, define

$$\tilde{E}_n(\omega) := \bigcup_{\substack{Q \in \mathcal{Q}_n \\ Q \text{ is selected}}} Q \quad \text{and} \quad \tilde{E}^k(\omega) := \bigcap_{n=k}^{\infty} \tilde{E}_n(\omega).$$

Then, for all $k \in \mathbb{N}$, we have

$$\dim_{\mathbb{H}} \tilde{E}^k(\omega) \leq \frac{\log(\max\{M_1 \varrho N^d, 1\})}{\log N}$$

for \mathbb{P}_p -almost all $\omega \in \Omega$. In particular, $\dim_{\mathbb{H}} \tilde{E}^k(\omega) < 1$ for all small enough ϱ .

Proof. Note that $\tilde{E}^k(\omega)$ is a finite union of sets satisfying similar assumptions as $\tilde{E}^1(\omega)$. Hence, it suffices to prove the claim for $\tilde{E}^1(\omega)$. For all $n \in \mathbb{N}$, let

$$\mathcal{S}_n(\omega) := \left\{ Q \in \mathcal{Q}_n \mid Q \subset \bigcap_{k=1}^n \tilde{E}_k(\omega) \right\},$$

that is, $\mathcal{S}_n(\omega)$ is the collection of selected cubes in \mathcal{Q}_n which are subsets of selected cubes in \mathcal{Q}_k for all $k = 1, \dots, n - 1$. Set $\tilde{\varrho} := \max\{M_1 \varrho N^d, 1\}$ and $s := \frac{\log \tilde{\varrho}}{\log N}$. Then

$$\tilde{\varrho} N^{-s} = 1. \tag{4.8}$$

For all $n \in \mathbb{N}$, define

$$Z_n(\omega) := (\sqrt{d})^s \sum_{Q \in \mathcal{S}_n(\omega)} N^{-ns}. \tag{4.9}$$

Note that the diameter of $Q \in \mathcal{Q}_n$ is $\text{diam } Q = \sqrt{d} N^{-n}$ and

$$Z_n(\omega) = (\sqrt{d})^s \sum_{Q \in \mathcal{S}_{n-1}(\omega)} N^{-(n-1)s} \sum_{\substack{Q' \in \mathcal{S}_n(\omega) \\ Q' \subset Q}} N^{-s}. \tag{4.10}$$

Let $\tilde{\mathcal{F}}_n := \bigvee_{i=1}^n \mathcal{F}_i$ be the σ -algebra generated by $\mathcal{F}_1, \dots, \mathcal{F}_n$. Let $Q \in \mathcal{S}_{n-1}(\omega)$. Note that every $Q' \in \mathcal{Q}_n(Q)$ has at most M_1 cubes $Q'' \in \mathcal{Q}_n(Q)$ with $Q'' \simeq Q'$ (including Q' itself), and there are N^d elements in $\mathcal{Q}_n(Q)$. Therefore, combining (4.10), (4.9), (4.7) and (4.8), we conclude that

$$\begin{aligned} \mathbb{E}(Z_n \mid \tilde{\mathcal{F}}_{n-1}) &= (\sqrt{d})^s \sum_{Q \in \mathcal{S}_{n-1}(\omega)} N^{-(n-1)s} \mathbb{E} \left(\sum_{\substack{Q' \in \mathcal{S}_n(\omega) \\ Q' \subset Q}} N^{-s} \right) \\ &\leq Z_{n-1} M_1 \varrho N^d N^{-s} \leq Z_{n-1}, \end{aligned}$$

implying that $(Z_n)_{n \in \mathbb{N}}$ is a supermartingale with respect to the filtration $(\tilde{\mathcal{F}}_n)_{n \in \mathbb{N}}$. By Doob's supermartingale convergence theorem (see, for example, [24, Section 11.5]), the limit $\lim_{n \rightarrow \infty} Z_n(\omega) = Z(\omega)$ exists and is finite for \mathbb{P}_p -almost all $\omega \in \Omega$. For all such $\omega \in \Omega$, we have for all $\delta > 0$ and for large enough $n \in \mathbb{N}$ that

$$\mathcal{H}_\delta^s(\tilde{E}^1(\omega)) \leq Z_n(\omega) \leq Z(\omega) + 1.$$

Therefore, $\mathcal{H}^s(\tilde{E}^1(\omega)) < \infty$, giving $\dim_{\mathbb{H}} \tilde{E}^1(\omega) \leq s$ and completing the proof. \square

Next we prove that the fractal percolation set is purely 1-unrectifiable almost surely. Even though this result is a special case of our main theorem (Theorem 6.13), we give here a simple alternative proof. Recall that a set $F \subset \mathbb{R}^d$ is purely k -unrectifiable if $\mathcal{H}^k(f(\mathbb{R}^k) \cap F) = 0$ for all Lipschitz maps $f: \mathbb{R}^k \rightarrow \mathbb{R}^d$. The following characterisation of pure k -unrectifiability will be utilised in the proof of Theorem 4.17.

Theorem 4.15. *Let $k \in \mathbb{N} \setminus \{0\}$. A set $F \subset \mathbb{R}^d$ is purely k -unrectifiable if and only if $\mathcal{H}^k(M \cap F) = 0$ for all k -dimensional C^1 -submanifolds $M \subset \mathbb{R}^d$.*

Proof. See [17, Theorem 15.21] or [9, Theorem 3.2.29]. \square

Corollary 4.16. *If $F \subset \mathbb{R}^d$ is a purely 1-unrectifiable Borel set, then it is also purely k -unrectifiable for any $k \in \mathbb{N} \setminus \{0\}$.*

Proof. Assume that F is not purely k -unrectifiable. Then there exists a k -dimensional C^1 -submanifold M such that $\mathcal{H}^k(M \cap F) > 0$. Fubini’s theorem, in turn, implies that there exists a 1-dimensional C^1 -submanifold L on M such that $\mathcal{H}^1(L \cap F) > 0$ and, therefore, $E(\omega)$ is not purely 1-unrectifiable. \square

Theorem 4.17. *For all $0 \leq p < 1$, the set $E(\omega)$ is purely 1-unrectifiable for \mathbb{P}_p -almost all $\omega \in \Omega$.*

Proof. Suppose that the claim is not true. Then

$$A := \{\omega \in \Omega \mid E(\omega) \text{ is not purely 1-unrectifiable}\}$$

has positive \mathbb{P}_p -measure. Consider $\omega \in A$. By Theorem 4.15, there exists a 1-dimensional C^1 -submanifold $M \subset \mathbb{R}^d$ such that $\mathcal{H}^1(M \cap E(\omega)) > 0$. Let $\mu := \mathcal{H}^1|_M$ be the restriction of \mathcal{H}^1 to M and denote by $D(\omega)$ the set of μ -density points of $E(\omega)$, that is,

$$D(\omega) = \left\{ x \in E(\omega) \mid \lim_{r \searrow 0} \frac{\mu(E(\omega) \cap B(x, r))}{\mu(B(x, r))} = 1 \right\}$$

where $B(x, r) \subset \mathbb{R}^d$ is the closed ball with radius $r > 0$ centred at $x \in \mathbb{R}^d$. By [17, Corollary 2.14], we have that $D(\omega) \subset M$ and $\mu(D(\omega)) = \mu(E(\omega))$.

For all $n \in \mathbb{N}$, define a function $A_n: \Omega \times \mathcal{Q}_n \rightarrow \{0, 1\}$ by setting $A_n(\eta, Q) := 1$, if and only if Q is m_0 -bad. Let \simeq be the relation \sim from Definition 4.1 defining neighbouring cubes. Then \simeq satisfies the assumptions of the relation in Proposition 4.14 with $M_1 = 3^d$. By means of Proposition 4.14, we will show that

$$D(\omega) \subset \bigcup_k \tilde{E}^k(\omega), \tag{4.11}$$

where $\tilde{E}^k(\omega)$ is as in Proposition 4.14. Note that, by Remark 4.4, A_n satisfies the measurability assumption of Proposition 4.14 with \mathcal{F}_n being the σ -algebra generated by \mathcal{C}_{n+m_0} and, moreover, Proposition 4.9 implies that assumption (4.7) is valid with $\varrho = 1 - p_g$. For verifying (4.11), let $x \in D(\omega)$ and let ℓ be the tangent space of M at x . Since M is a C^1 -submanifold, for all $m_0 \in \mathbb{N}$, there exist $r_0 > 0$ such that

$$d_{\mathbb{H}}(M \cap B(x, r), \ell \cap B(x, r)) < \frac{1}{2\sqrt{2d}} N^{-m_0} r$$

for all $0 < r \leq r_0$. Further, there is a constant $c_1 \geq 1$ such that $r \leq \mu(B(x, r)) \leq c_1 r$ for all $0 < r \leq r_0$. Combining this with the fact that x is a μ -density point of $E(\omega)$ implies the existence of $r_1 > 0$ such that $\mu(E(\omega)^c \cap B(x, r)) < d^{-1} N^{-m_0} r$ for all $0 < r \leq r_1$. Applying Lemma 4.13 with the line segment $\ell \cap B(x, \sqrt{d}N^{-n})$ and the curve $M \cap B(x, \sqrt{d}N^{-n})$, we conclude that, for all large $n \in \mathbb{N}$, either $Q_n(x)$ or one of its neighbour cubes is m_0 -bad, where $Q_n(x)$ is the cube in \mathcal{Q}_n whose half open counterpart contains x . Hence, $x \in \tilde{E}^k(\omega)$ for some $k \in \mathbb{N}$, completing the proof of (4.11).

Since $\lim_{m_0 \rightarrow \infty} p_g = 1$, we deduce from Proposition 4.14 that, for large enough $m_0 \in \mathbb{N}$, we have $\dim_{\mathbb{H}}(\bigcup_k \tilde{E}^k(\omega)) < 1$ for \mathbb{P}_p -almost all $\omega \in A$. This leads to a contradiction with (4.11), since $\dim_{\mathbb{H}}(D(\omega)) = 1$. \square

Combining Corollary 4.16 and Theorem 4.17, we obtain pure k -unrectifiability of typical fractal percolation sets for all $k \in \mathbb{N} \setminus \{0\}$.

Corollary 4.18. *Let $k \in \mathbb{N} \setminus \{0\}$. For all $0 \leq p < 1$, the set $E(\omega)$ is purely k -unrectifiable for \mathbb{P}_p -almost all $\omega \in \Omega$.*

5. Existence of hereditarily good cubes

In the previous section, dealing with 1-unrectifiability, one of our key concepts is the m_0 -good cube. While we zoom in, differentiable curves start to look almost like lines and, therefore, the little cubes in m_0 -good cubes removed during the fractal percolation process guarantee that differentiable curves cannot intersect the fractal percolation set in a set of positive linear measure. If $\alpha < 1$ then α -Hölder curves are more flexible and can “go around” holes. The price one needs to pay for it is a tiny bit of length increase compared to a straight line segment. In order to prove that curves staying close to the fractal percolation set cannot be α -Hölder for α close to 1, we need that m_0 -good cubes at different levels are occurring sufficiently uniformly. To be more precise, as we zoom out at certain zoom levels, with high probability, some m_0 -bad cubes show up. However, with high probability, in one “large” cube we do not have too many m_0 -bad cubes. The “large” cubes are considered “good” if this is the case. If we zoom out further, we might find some “much larger” cubes which, with high probability, contain some “large” cubes

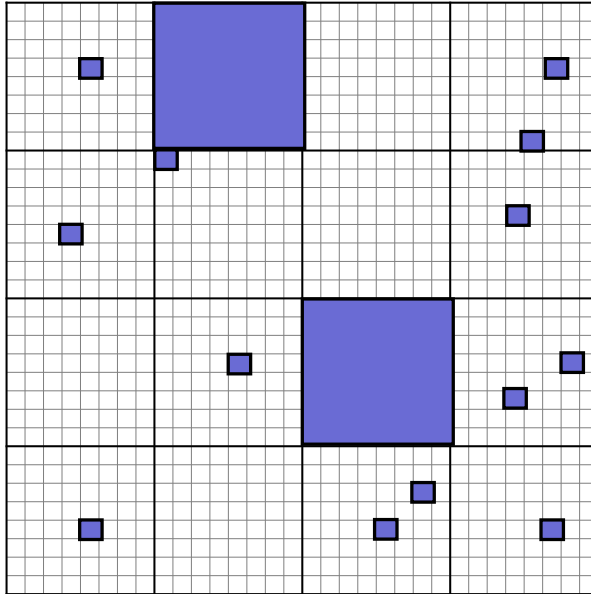


Fig. 5.1. A (k, \mathbf{L}, c, m_0) -good cube containing $(k - 1, \mathbf{L}, c, m_0)$ -bad and $(k - 2, \mathbf{L}, c, m_0)$ -bad cubes. Here $c = 2$. (In black and white printed version of the paper, the phrase “blue squares” refers to the shaded squares in the figure.)

which are “bad”. However, with high probability, in one “much larger” cube we do not have too many “large” cubes which are “bad”. The “much larger” cubes are considered “good” if this is the case. Clearly, this procedure can be iterated. To have an idea about this procedure, one may look at Fig. 5.1. One can think of the small blue squares (two dimensional cubes) as the m_0 -bad cubes. Then the larger blue cubes are bad “large” cubes (they contain too many m_0 -bad cubes) and one “good” much larger level cube is shown in the figure.

To determine the different zoom levels, at which we look at good and bad cubes, is very delicate. If we zoom out too fast, we might end up with too many bad cubes and, on the other hand, if we zoom out too slowly, the probability estimates about the number of good cubes do not work. The complicated and quite technical task of finding the proper zoom levels is postponed to Section 7.

In this section, our main aim is to define hereditarily good cubes in Definition 5.12, that is, cubes where the above procedure can be repeated for several different sequences of scales, and to prove Proposition 5.15 stating that, with probability one, the Hausdorff dimension of the set of points, which are not included in infinitely many “highly hereditarily good” cubes, is less than one.

Definition 5.1. Fix $\omega \in \Omega$ and $n \in \mathbb{N}$. Let $c, m_0 \in \mathbb{N} \setminus \{0\}$. Assume that $\mathbf{L} := (L_j)_{j=0}^k$ is a strictly decreasing finite sequence of nonnegative integers such that $L_k = 0$. Let $Q \in \mathcal{Q}_n$. We say that Q is (\mathbf{L}, c, m_0) -good if it is (k, \mathbf{L}, c, m_0) -good, where the concept

of being $(k, \mathbf{L}, \mathbf{c}, m_0)$ -good is defined inductively below. If Q is not $(\mathbf{L}, \mathbf{c}, m_0)$ -good, it is $(\mathbf{L}, \mathbf{c}, m_0)$ -bad.

- If $k = 0$, we say that Q is $(0, \mathbf{L}, \mathbf{c}, m_0)$ -good if it is m_0 -good. Otherwise, Q is $(0, \mathbf{L}, \mathbf{c}, m_0)$ -bad.

- Assume that we have defined the concepts of being $(\tilde{k}, \mathbf{L}, \mathbf{c}, m_0)$ -good and $(\tilde{k}, \mathbf{L}, \mathbf{c}, m_0)$ -bad for all strictly decreasing sequences $\mathbf{L} = (L_j)_{j=0}^{k-1}$ of natural numbers with $L_{k-1} = 0$ and $\tilde{k} = 0, \dots, k - 1$.

- Let $\mathbf{L} = (L_j)_{j=0}^k$ with $L_k = 0$. For $\tilde{k} \leq k - 1$, define a sequence $\tilde{\mathbf{L}} = (\tilde{L}_j)_{j=0}^{\tilde{k}}$ by setting $\tilde{L}_j := L_j - L_{\tilde{k}}$ for all $j = 0, \dots, \tilde{k}$. We say that Q is $(\tilde{k}, \mathbf{L}, \mathbf{c}, m_0)$ -good if it is $(\tilde{k}, \tilde{\mathbf{L}}, \mathbf{c}, m_0)$ -good. Otherwise, Q is $(\tilde{k}, \mathbf{L}, \mathbf{c}, m_0)$ -bad. Finally, we say that Q is $(k, \mathbf{L}, \mathbf{c}, m_0)$ -good if there are at most \mathbf{c} cubes $Q' \in \mathcal{Q}_{n+L_{k-1}}(Q)$ (recall (4.1)) which are $(k - 1, \mathbf{L}, \mathbf{c}, m_0)$ -bad and, otherwise, Q is $(k, \mathbf{L}, \mathbf{c}, m_0)$ -bad.

Remark 5.2. If $Q \in \mathcal{Q}_n$ is $(k, \mathbf{L}, \mathbf{c}, m_0)$ -good, then all but \mathbf{c} cubes $Q' \in \mathcal{Q}_{n+L_{k-1}}(Q)$ are $(k - 1, \mathbf{L}, \mathbf{c}, m_0)$ -good. This, in turn, means that all except \mathbf{c} cubes $Q'' \in \mathcal{Q}_{n+L_{k-2}}(Q')$ are $(k - 2, \mathbf{L}, \mathbf{c}, m_0)$ -good. Iterating this, we conclude that, for $Q \in \mathcal{Q}_n$, the property of being $(\mathbf{L}, \mathbf{c}, m_0)$ -good is determined by m_0 -good cubes $Q^{(k)} \in \mathcal{Q}_{n+L_0}(Q)$. On the other hand, whether a cube $Q^{(k)} \in \mathcal{Q}_{n+L_0}(Q)$ is m_0 -good or not depends only on $\mathcal{C}_{n+L_0+m_0}(\omega)$.

As observed in Remark 4.4.(a), the events “ Q is m_0 -good” and “ Q' is m_0 -good” are not independent if $Q \sim Q'$. To overcome this difficulty, we introduce a notion of independently good cubes in Definition 5.4. We begin by defining a class of shifted grids. Write

$$S := \{-1, 1\}^d \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ is the origin of \mathbb{R}^d .

Definition 5.3. Let $n', m_0 \in \mathbb{N}$ and $s \in S$. For all $n \in \mathbb{N}$, define

$$\mathcal{Q}_n^{s, n', m_0} := \left\{ \prod_{i=1}^d [(l_i - 1)N^{-n}, l_i N^{-n}] + \sum_{i=1}^d s_i N^{-n-n'-m_0} e_i \mid l_i = 1, \dots, N^n, \right. \\ \left. i = 1, \dots, d \right\},$$

where $\{e_1, \dots, e_d\}$ is the standard basis of \mathbb{R}^d .

When $s = \mathbf{0}$, we have that $\mathcal{Q}_n^{\mathbf{0}, n', m_0} = \mathcal{Q}_n$ for all $n, n', m_0 \in \mathbb{N}$. For all $n, n', m_0 \in \mathbb{N}$ and for all $Q \in \bigcup_{s \in S} \mathcal{Q}_n^{s, n', m_0}$, set

$$\text{Int}_{m_0} Q := (1 - 2N^{-m_0})Q,$$

where, for a cube \tilde{Q} and a positive constant C , we denote by $C\tilde{Q}$ the cube having the same centre as \tilde{Q} and the side length C times that of \tilde{Q} .

Definition 5.4. Let $n, m_0 \in \mathbb{N}$ with $m_0 > 0$, and fix $\omega \in \Omega$. Let $\ell \subset \mathbb{R}^d$ be a line with a principle direction $i \in \{1, \dots, d\}$, and let $Q \in \bigcup_{s \in S} \mathcal{Q}_n^{s,0,m_0}$ be such that $\ell \cap \text{Int}_{m_0} Q$ has positive length. Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be the $(n + m_0, i)$ -layers (in the natural order) for which $\ell \cap \text{Int}_{m_0} Q \cap \mathcal{L}_j$ has positive length. We say that Q is *independently m_0 -good for the line ℓ* if there is a strongly i -deleted cube $Q' \in \mathcal{Q}_{n+m_0}(\text{Int}_{m_0} Q)$ so that

- (a) $Q' \not\subset \mathcal{L}_j$ for $j \in \{1, 2, k - 1, k\}$ and
- (b) ℓ intersects Q' properly.

A cube $Q \in \bigcup_{s \in S} \mathcal{Q}_n^{s,0,m_0}$ is *independently m_0 -good* if it is simultaneously independently m_0 -good for all lines which intersect $\text{Int}_{m_0} Q$ properly and are parallel to some line in $\Gamma(Q, m_0)$, where (with a slight abuse of definition since the side length of $\text{Int}_{m_0} Q$ is not equal to N^{-n}) a line ℓ intersects the cube CQ properly in direction i if $\mathcal{H}^1(\Pi_i(\ell \cap CQ)) \geq d^{-1}N^{-n}$. Finally, a cube $Q \in \mathcal{Q}_n$ is *independently m_0 -bad* if it is not independently m_0 -good.

Remark 5.5. Clearly, if $Q, \tilde{Q} \in \mathcal{Q}_n^{s,0,m_0}$ with $Q \neq \tilde{Q}$, the events “ Q is independently m_0 -good” and “ \tilde{Q} is independently m_0 -good” are independent. However, if Q is independently m_0 -good, we have no information about those lines which intersect Q properly but do not intersect $\text{Int}_{m_0} Q$ properly. This is the reason why shifted grids are needed.

Next lemma describes the usefulness of shifted grids.

Lemma 5.6. Let $n, m_0 \in \mathbb{N}$ such that $m_0 > 0$ and let $Q \in \mathcal{Q}_n$. Suppose that all cubes $\tilde{Q} \in \bigcup_{s \in S} \mathcal{Q}_n^{s,0,m_0}$ intersecting the interior of Q are independently m_0 -good. Then Q is m_0 -good.

Proof. If a line ℓ intersects Q properly, there exists $\tilde{Q} \in \bigcup_{s \in S} \mathcal{Q}_n^{s,0,m_0}$ such that ℓ intersects $\text{Int}_{m_0} \tilde{Q}$ properly, which gives the claim since $\text{Int}_{m_0} \tilde{Q} \subset Q$. \square

Now we state a modified version of Definition 5.1 valid for shifted and non-shifted cubes.

Definition 5.7. Fix $\omega \in \Omega$ and $m_0, \mathfrak{c} \in \mathbb{N} \setminus \{0\}$. Let $k \in \mathbb{N}$ and let $\mathbf{L} = (L_j)_{j=0}^k$ be a strictly decreasing sequence of integers such that $L_k = 0$. For all $n \in \mathbb{N}$, the concept of $Q \in \bigcup_{s \in S} \mathcal{Q}_n^{s,L_0,m_0}$ being *independently $(\mathbf{L}, \mathfrak{c}, m_0)$ -good or bad* is defined by replacing goodness with independently goodness in Definition 5.1.

Remark 5.8. Whether a cube $Q \in \bigcup_{s \in S} \mathcal{Q}_n^{s,L_0,m_0}$ is independently $(\mathbf{L}, \mathfrak{c}, m_0)$ -good or not is determined by the independently m_0 -good cubes $Q' \in \mathcal{Q}_{n+L_0}^{s,0,m_0}(Q)$. Furthermore, whether a cube $Q' \in \mathcal{Q}_{n+L_0}^{s,0,m_0}(Q)$ is independently m_0 -good or not depends only on $\mathcal{C}_{n+L_0+m_0}(\omega)|_{Q'}$. Therefore, the events “ Q is independently $(\mathbf{L}, \mathfrak{c}, m_0)$ -good” and “ \tilde{Q} is independently $(\mathbf{L}, \mathfrak{c}, m_0)$ -good” are independent provided $Q, \tilde{Q} \in \mathcal{Q}_n^{s,L_0,m_0}$ with $Q \neq \tilde{Q}$.

Next lemma may be regarded as an extension of Lemma 5.6. It determines the value of the constant \mathfrak{c} we will utilise later.

Lemma 5.9. *Let $k, m_0 \in \mathbb{N}$ with $m_0 > 0$ and let $\mathbf{L} = (L_j)_{j=0}^k$ be a strictly decreasing sequence of integers with $L_k = 0$. Let $n \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$. Suppose that all cubes $\tilde{Q} \in \bigcup_{s \in S} \mathcal{Q}_n^{s, L_0, m_0}$ which intersect the interior of Q are independently $(\mathbf{L}, 1, m_0)$ -good. Then Q is $(\mathbf{L}, \mathfrak{c}, m_0)$ -good, where $\mathfrak{c} := 4^d + 1$.*

Proof. We prove the claim by induction on k . The case $k = 0$ is verified in Lemma 5.6. Suppose that the claim is true for $k - 1$ and fix $s \in S$. Since every cube $\tilde{Q} \in \mathcal{Q}_n^{s, L_0, m_0}$, intersecting the interior of Q , is independently $(k, \mathbf{L}, 1, m_0)$ -good, there is at most one $Q' \in \mathcal{Q}_{n+L_{k-1}}^{s, L_0-L_{k-1}, m_0}(\tilde{Q})$ which is independently $(k - 1, \mathbf{L}, 1, m_0)$ -bad. If $s \neq \mathbf{0}$, Q' is not in the standard grid, and we colour in green all cubes in $\mathcal{Q}_{n+L_{k-1}}(Q)$ intersecting Q' . On the other hand, if $s = \mathbf{0}$, Q' is in the standard grid, and we colour it in green. Since $\#(S \setminus \{\mathbf{0}\}) = 2^d$ and every shifted cube intersects 2^d standard cubes of the same size, the number of green cubes is at most \mathfrak{c} .

Consider a cube $Q' \in \mathcal{Q}_{n+L_{k-1}}(Q)$ which is not green. Then Q' is independently $(k - 1, \mathbf{L}, 1, m_0)$ -good and, moreover, every $\hat{Q} \in \bigcup_{s \in S \setminus \{\mathbf{0}\}} \mathcal{Q}_{n+L_{k-1}}^{s, L_0-L_{k-1}, m_0}$ intersecting Q' is independently $(k - 1, \mathbf{L}, 1, m_0)$ -good, since otherwise Q' would be green. By the induction hypothesis, Q' is $(k - 1, \mathbf{L}, \mathfrak{c}, m_0)$ -good. Since this applies to all cubes $Q' \in \mathcal{Q}_{n+L_{k-1}}(Q)$ which are not green, and the number of green cubes in Q is bounded by \mathfrak{c} , we obtain that Q is $(k, L, \mathfrak{c}, m_0)$ -good. \square

Observe that if $Q \in \mathcal{Q}_n$ is m_0 -good, it is not necessarily independently m_0 -good. This follows from the fact that there may be a line ℓ intersecting $\text{Int}_{m_0} Q$ properly such that ℓ intersects only one strongly i -deleted $Q' \in \mathcal{Q}_{n+m_0}(Q)$ and $Q' \not\subset \text{Int}_{m_0} Q$. However, analogously to Lemma 4.5 and Proposition 4.9, one can prove the following proposition.

Proposition 5.10. *For all $0 \leq p < 1$ and $m_0 \in \mathbb{N} \setminus \{0\}$, there exists a number $0 \leq p_{ig} = p_{ig}(m_0, p, d, N) \leq 1$ with $\lim_{m_0 \rightarrow \infty} p_{ig}(m_0, p, d, N) = 1$ such that*

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is independently } m_0\text{-good}\}) \geq p_{ig}$$

for all $n \in \mathbb{N}$ and $Q \in \bigcup_{s \in S} \mathcal{Q}_n^{s, 0, m_0}$.

In our proof, we will use a very specific choice of sequences \mathbf{L} depending on parameters $k_0, m_1 \in \mathbb{N} \setminus \{0\}$ to be fixed later. For every $i \in \mathbb{N}$, we will define in Section 7 inductively a finite decreasing sequence $\mathbf{L}^i(m_1, k_0) := (L^i(m_1, k_0)_j)_{j=0}^{k_i}$ of nonnegative integers. Here we list a few properties we use in the sequel. These properties are verified in Section 7. Set

$$\Delta_j^i := L^i(m_1, k_0)_{j-1} - L^i(m_1, k_0)_j. \tag{5.1}$$

We will suppose that we have (see (7.13))

$$\Delta_j^i \leq 5\sqrt{2}^j m_1 \text{ for all } i \in \mathbb{N} \text{ and } j = 1, \dots, k_i. \tag{5.2}$$

We also assume that (see (7.1), (7.2) and (7.4) in Definition 7.1)

$$L^l(m_1, k_0)_0 = (l + 1)l_0 \text{ for all } l \in \mathbb{N}, \tag{5.3}$$

where $l_0 := m_1(1 + 2 + \dots + k_0)$. An index i_0 will be determined such that $m_1 k_{i_0} = l_0$, see (7.6). In Lemma 7.6, we show that

$$k_i \geq i + 3 \text{ for all } i \leq i_0 + 2 \text{ and} \tag{5.4}$$

$$k_{i_0+l} \geq 2 \log_2(i_0 + l + 1) \text{ for all } l \geq 3. \tag{5.5}$$

For the rest of this section, we use these special sequences $\mathbf{L}^l(m_1, k_0)$. For all $n, l \in \mathbb{N}$, $m_0, k_0, m_1 \in \mathbb{N} \setminus \{0\}$, $k \in \{0, \dots, k_l\}$ and $0 \leq p < 1$, define

$$q_{k,l} = q_{k,l}(m_1, k_0, m_0, p, d, N) := \mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is independently } (k, \mathbf{L}^l(m_1, k_0), 1, m_0)\text{-bad}\}),$$

where $Q \in \bigcup_{s \in S} \mathcal{Q}_n^{s, L^l(m_1, k_0)_0 - L^l(m_1, k_0)_k, m_0}([N^{-n}, 1 - N^{-n}]^d)$. Note that the value of $q_{k,l}$ does not depend on the choice of n, s or Q . Moreover, $q_{0,l} \leq 1 - p_{ig}$ does not depend on l, m_1 or k_0 .

Lemma 5.11. *For every $l \in \mathbb{N}$ and $k \in \{1, \dots, k_l\}$,*

$$q_{k,l} \leq (q_{0,l} N^{25dm_1})^{2^k}.$$

Proof. Since the claim does not depend on the choice of $s \in S$, for notational simplicity, we assume that $s = \mathbf{0}$. Let $k \in \{1, \dots, k_l\}$. If $Q \in \mathcal{Q}_n$ is independently $(k, \mathbf{L}^l(m_1, k_0), 1, m_0)$ -good, it contains at most one independently $(k - 1, \mathbf{L}^l(m_1, k_0), 1, m_0)$ -bad cube $Q' \in \mathcal{Q}_{n+\Delta_k^l}(Q)$. By independence (recall Remark 5.8), we have that

$$\begin{aligned} 1 - q_{k,l} &= (1 - q_{k-1,l})^{N^{d\Delta_k^l}} + N^{d\Delta_k^l} q_{k-1,l} (1 - q_{k-1,l})^{N^{d\Delta_k^l} - 1} \\ &= (1 - q_{k-1,l})^{N^{d\Delta_k^l} - 1} (1 - q_{k-1,l} + N^{d\Delta_k^l} q_{k-1,l}) \\ &\geq (1 - (N^{d\Delta_k^l} - 1)q_{k-1,l})(1 + (N^{d\Delta_k^l} - 1)q_{k-1,l}) \geq 1 - (N^{d\Delta_k^l} q_{k-1,l})^2, \end{aligned}$$

implying that

$$q_{k,l} \leq (N^{d\Delta_k^l} q_{k-1,l})^2. \tag{5.6}$$

Iterating Inequality (5.6) and recalling Inequality (5.2), we conclude that

$$\begin{aligned}
 q_{k,l} &\leq N^{d \sum_{j=0}^{k-1} 2^{j+1} \Delta_k^l} \cdot q_{0,l}^{2^k} = q_{0,l}^{2^k} N^{2d2^k \sum_{j=1}^k 2^{-j} \Delta_j^l} \\
 &\leq q_{0,l}^{2^k} N^{10d2^k m_1 \sum_{j=1}^k \sqrt{2}^{-j}} < (q_{0,l} N^{25dm_1})^{2^k}. \quad \square
 \end{aligned}$$

We proceed by defining the concept of hereditarily good cubes, which will enable us to apply our length gain estimates repeatedly.

Definition 5.12. Fix $\omega \in \Omega$ and $m_0, \mathfrak{c}, k_0, m_1 \in \mathbb{N} \setminus \{0\}$. Let $n, q \in \mathbb{N}$. A cube $Q \in \mathcal{Q}_n$ is $(q, m_1, k_0, \mathfrak{c}, m_0)$ -hereditarily good if Q is $(\mathbf{L}^l(m_1, k_0), \mathfrak{c}, m_0)$ -good for all $l = 0, \dots, q$.

Remark 5.13. (a) Whether a cube $Q \in \mathcal{Q}_n$ is $(q, m_1, k_0, \mathfrak{c}, m_0)$ -hereditarily good or not depends only on

$$\bigcup_{l=0}^q \mathcal{C}_{n+L^l(m_1, k_0)_0+m_0}(\omega) = \bigcup_{l=0}^q \mathcal{C}_{n+(l+1)l_0+m_0}(\omega)$$

by Remark 5.2 and (5.3).

(b) The choice of the parameters m_0, \mathfrak{c}, k_0 and m_1 will be crucial in our proof. In the proof of Theorem 6.13, we explain how they are selected. In Sections 5–9 some restrictions on them are given, see in particular Lemmas 5.9, 9.1 and 9.6, Propositions 5.14, 5.15 and 9.3 and Construction 8.9.

According to the next proposition, the probability that a cube is hereditarily good is large when m_0 is large.

Proposition 5.14. Let $0 \leq p < 1$ and $k_0, m_1 \in \mathbb{N} \setminus \{0\}$ with $k_0 \geq 3$. For every $\varepsilon > 0$, there exists $m_0 = m_0(m_1, p, d, N, \varepsilon) \in \mathbb{N}$ such that

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is } (q, m_1, k_0, \mathfrak{c}, m_0)\text{-hereditarily good}\}) \geq 1 - \varepsilon$$

for every $n, q \in \mathbb{N}$ and $Q \in \mathcal{Q}_n([N^{-n}, 1 - N^{-n}]^d)$, where \mathfrak{c} is as in Lemma 5.9.

Proof. From Lemma 5.9 we obtain that

$$\mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is } (\mathbf{L}^l(m_1, k_0), \mathfrak{c}, m_0)\text{-good}\}) \geq 1 - (2^d + 1)q_{k_l, l}$$

for all $l = 0, \dots, q$. Further, by Remark 5.2, the events “ Q is $(\mathbf{L}^l(m_1, k_0), \mathfrak{c}, m_0)$ -good” and “ Q is $(\mathbf{L}^t(m_1, k_0), \mathfrak{c}, m_0)$ -good” are independent for $l \neq t$ and, by Proposition 5.10, we have $\lim_{m_0 \rightarrow \infty} q_{0,l} = 0$. Recall that $q_{0,l}$ does not depend on l, m_1 or k_0 . Thus, choosing sufficiently large $m_0 \in \mathbb{N}$, we can make $q_{0,l} N^{25dm_1}$ as small as we wish. Combining (5.4) and (5.5) with Lemma 5.11 leads to

$$\begin{aligned} & \mathbb{P}_p(\{\omega \in \Omega \mid Q \text{ is } (q, m_1, k_0, \mathbf{c}, m_0)\text{-hereditarily good}\}) \\ & \geq \prod_{l=0}^q (1 - (2^d + 1)(q_{0,l} N^{25dm_1})^{2^{k_l}}) \geq \prod_{l=0}^q (1 - (2^d + 1)(q_{0,l} N^{25dm_1})^{(l+1)^2}) \geq 1 - \varepsilon. \quad \square \end{aligned}$$

Recalling Definition 4.1, set

$$k_Q := \bigcup_{Q' \sim Q} Q' \text{ and } K_Q := \bigcup_{Q' \sim Q} k_{Q'} \tag{5.7}$$

for all $n \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$, that is, k_Q and K_Q are cubes with same centre as Q having three and five times the side length of Q , respectively. We conclude this section with a proposition which states that the hereditarily good cubes are abundant.

Proposition 5.15. *Let $0 \leq p < 1$ and $k_0, m_1 \in \mathbb{N} \setminus \{0\}$ with $k_0 \geq 3$. Then there exists a positive integer $m_0 = m_0(m_1, p, d, N)$ such that, for \mathbb{P}_p -almost all $\omega \in \Omega$,*

$$\begin{aligned} & \dim_{\mathbb{H}} \left\{ x \in [0, 1]^d \mid \#\{n \in \mathbb{N} \mid Q' \text{ is } (q, m_1, k_0, \mathbf{c}, m_0)\text{-hereditarily good} \right. \\ & \left. \text{for all } Q' \in \mathcal{Q}_n(K_{Q_n(x)})\} < \infty \right\} < 1 \end{aligned}$$

for all $q \in \mathbb{N}$, where \mathbf{c} is defined in Lemma 5.9.

Proof. Combining Proposition 5.14 with Proposition 4.14 gives the claim (recall the proof of Theorem 4.17). \square

6. Pure α -unrectifiability

In Section 4, we proved that a typical realisation of the fractal percolation is purely 1-unrectifiable, that is, all Lipschitz curves intersect the fractal percolation in a set of zero 1-dimensional Hausdorff measure. In view of [4] (see Section 1), it is natural to ask whether Lipschitz curves can be replaced by α -Hölder curves, and \mathcal{H}^1 by $\mathcal{H}^{\frac{1}{\alpha}}$, for some $\alpha < 1$. We define a concept of α -rectifiability for the purpose of answering this question positively. Throughout the section, $I \subset \mathbb{R}$ is a generic closed and bounded interval.

Definition 6.1. Let $0 < \alpha \leq 1$ and $H \geq 0$. A curve $\gamma: I \rightarrow \mathbb{R}^d$ is (H, α) -Hölder (continuous) at $a \in I$, if it satisfies the condition

$$|\gamma(a) - \gamma(b)| \leq H|a - b|^\alpha$$

for every $b \in I$. A curve γ is (H, α) -Hölder, if it is (H, α) -Hölder at every $a \in I$. Finally, a curve γ is α -Hölder, if for every $a \in I$ there is $H_a \geq 0$ such that γ is (H_a, α) -Hölder at $a \in I$.

The following well-known lemma is an immediate corollary of definitions.

Lemma 6.2. *Let $\gamma: I \rightarrow \mathbb{R}^d$ be an (H, α) -Hölder curve for some $0 < \alpha \leq 1$ and $H \geq 0$. Assume that $A \subset I$. Then $\mathcal{H}^{\frac{1}{\alpha}}(\gamma(A)) \leq H^{\frac{1}{\alpha}} \mathcal{H}^1(A)$.*

According to the next lemma, every α -Hölder curve can be covered by a countable collection of (H, α) -Hölder curves.

Lemma 6.3. *Let $\gamma: I \rightarrow \mathbb{R}^d$ be an α -Hölder curve for some $0 < \alpha \leq 1$. Then there is a countable collection of curves $\gamma_i: I \rightarrow \mathbb{R}^d, i \in \mathbb{N}$, such that γ_i is (i, α) -Hölder and*

$$\gamma(I) \subset \bigcup_{i=1}^{\infty} \gamma_i(I).$$

Proof. For all $i \in \mathbb{N}$, let $F_i \subset I$ be the set of points where γ is (i, α) -Hölder. By definition, $I = \bigcup_{i=1}^{\infty} F_i$ and, therefore, it is enough to show that $\gamma(F_i)$ can be covered by an (i, α) -Hölder curve $\gamma_i: I \rightarrow \mathbb{R}^d$.

Fix $i \in \mathbb{N}$. Suppose that $(t_n)_{n \in \mathbb{N}}$ is a sequence in F_i converging to $t \in I$. For all $u \in I$ and $n \in \mathbb{N}$, we have

$$|\gamma(t) - \gamma(u)| \leq |\gamma(t) - \gamma(t_n)| + |\gamma(t_n) - \gamma(u)| \leq i|t - t_n|^\alpha + i|t_n - u|^\alpha \xrightarrow{n \rightarrow \infty} i|t - u|^\alpha,$$

implying that $t \in F_i$. Thus F_i is closed for all $i \in \mathbb{N}$. Since the complement of F_i is open, we can write it as $F_i^c = \bigcup_{j=1}^{\infty}]a_j^i, b_j^i[$, where the intervals are disjoint. (The case of finite union is included by adding infinitely many empty sets.) Define $\gamma_i: I \rightarrow \mathbb{R}^d$ as $\gamma_i(t) := \gamma(t)$ for $t \in F_i$, and on each interval $]a_j^i, b_j^i[$ with $j \in \mathbb{N}$, define γ_i as the affine map connecting $\gamma(a_j^i)$ and $\gamma(b_j^i)$.

We verify that γ_i is (i, α) -Hölder. Letting $t, u \in I$ with $t < u$, we need to prove that $|\gamma_i(t) - \gamma_i(u)| \leq i|t - u|^\alpha$. This is trivial if $t, u \in F_i$. Assume that $t \in F_i$ and $u \notin F_i$. Then $u \in]a_j^i, b_j^i[$ for some $j \in \mathbb{N}$. Considering the functions $g_t, f_t: [t, b_j^i] \rightarrow [0, \infty[$, $g_t(s) = i|t - s|^\alpha$ and $f_t(s) = |\gamma_i(t) - \gamma_i(s)|$, it suffices to show that $f_t(s) \leq g_t(s)$ for all $s \in]a_j^i, b_j^i[$. This follows from the concavity of g_t , since $f_t(a_j^i) \leq g_t(a_j^i)$, $f_t(b_j^i) \leq g_t(b_j^i)$ and f_t is affine on $]a_j^i, b_j^i[$. By symmetry, $f_t(u) \leq g_t(u)$ when $u \in F_i$ and $t \notin F_i$. Finally, let $t \in]a_k^i, b_k^i[$ and $u \in]a_j^i, b_j^i[$ for some $k, j \in \mathbb{N}$. Since $a_j^i, b_j^i \in F_i$, we have that $f_t(a_j^i) \leq g_t(a_j^i)$ and $f_t(b_j^i) \leq g_t(b_j^i)$. Thus, concavity of g_t and affinity of f_t on $]a_j^i, b_j^i[$ imply that $f_t(u) \leq g_t(u)$, completing the proof. \square

In view of Lemmas 6.2 and 6.3, the following definition is natural.

Definition 6.4. Let $0 < \alpha \leq 1$. A set $A \subset \mathbb{R}^d$ is α -rectifiable, if there exist α -Hölder curves $\gamma_i: I \rightarrow \mathbb{R}^d, i \in \mathbb{N}$, such that $\mathcal{H}^{\frac{1}{\alpha}}(A \setminus (\bigcup_{i=1}^{\infty} \gamma_i(I))) = 0$. A set $A \subset \mathbb{R}^d$ is purely α -unrectifiable, if $\mathcal{H}^{\frac{1}{\alpha}}(A \cap \gamma(I)) = 0$ for all α -Hölder curves $\gamma: I \rightarrow \mathbb{R}^d$.

Remark 6.5. When $\alpha = 1$, the above definition agrees with the standard definition of 1-rectifiability and pure 1-unrectifiability.

According to Lemma 6.2, the images of (H, α) -Hölder curves have finite $\mathcal{H}^{\frac{1}{\alpha}}$ -measure. One may address the question whether the images have always positive $\mathcal{H}^{\frac{1}{\alpha}}$ -measure. The answer is negative, since any β -Hölder curve is α -Hölder for all $\alpha < \beta$ and $\mathcal{H}^{\frac{1}{\beta}}(\gamma(I)) < \infty$ implies $\mathcal{H}^{\frac{1}{\alpha}}(\gamma(I)) = 0$. To avoid problems caused by this fact, we give the following definition.

Definition 6.6. Let $\eta, R > 0$ and $0 < \alpha \leq 1$. A curve $\gamma: I \rightarrow \mathbb{R}^d$ is (α, η, R) -tight at $t \in I$ if

$$\frac{\text{diam } \gamma([t - r, t + r])}{(2r)^\alpha} \geq \eta$$

for all $r \leq R$. We say that γ is (α, η, R) -tight if it is (α, η, R) -tight at every $t \in I$. A curve γ is (α, η) -tight at $t \in I$ if it is (α, η, R) -tight at $t \in I$ for some $R > 0$. Finally, a curve γ is (α, η) -tight if it is (α, η) -tight at every $t \in I$.

Remark 6.7. There exist tight Hölder curves. For example, the natural parametrisation $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ of the von Koch curve is $(\frac{\log 3}{\log 4}, \frac{1}{\sqrt{3}}, 1)$ -tight since, for all $t \in [0, 1]$, $n \in \mathbb{N}$ and $4^{-n} \leq r < 4^{-n+1}$, the set $\gamma([t - r, t + r])$ contains points whose distance is 3^{-n} . It is also $(1, \frac{\log 3}{\log 4})$ -Hölder continuous. Modifying this example, one can construct $(\alpha, \eta, 1)$ -tight (H, α) -Hölder curves for any $0 < \alpha, \eta \leq 1$ and $H > 0$.

Next lemma states that a Hölder curve is tight at most points.

Lemma 6.8. Letting $\gamma: I \rightarrow \mathbb{R}^d$ be an (H, α) -Hölder curve, define

$$A_0 := \bigcap_{\eta > 0} \{t \in I \mid \gamma \text{ is not } (\alpha, \eta)\text{-tight at } t\}.$$

Then $\mathcal{H}^{\frac{1}{\alpha}}(\gamma(A_0)) = 0$.

Proof. For all $i \in \mathbb{N}$, set

$$A_i := \left\{ t \in I \mid \liminf_{r \searrow 0} \frac{\text{diam } \gamma([t - r, t + r])}{(2r)^\alpha} < \frac{1}{i} \right\}.$$

Then $A_{i+1} \subset A_i$ for all $i \in \mathbb{N}$ and $A_0 = \bigcap_{i=1}^\infty A_i$. So it is enough to show that $\lim_{i \rightarrow \infty} \mathcal{H}^{\frac{1}{\alpha}}(\gamma(A_i)) = 0$. Fixing $i \in \mathbb{N}$ and using Vitali's covering theorem [17, Theorem 2.8], we find, for all $\delta > 0$, disjoint balls $B_j \subset I$, $j \in \mathbb{N}$, such that $\text{diam } B_j \leq \delta$, $\mathcal{H}^1(A_i \setminus \bigcup_{j=1}^\infty B_j) = 0$ and $\text{diam } \gamma(B_j) \leq \frac{1}{i}(\text{diam } B_j)^\alpha$. By Lemma 6.2,

$\mathcal{H}^{\frac{1}{\alpha}}(\gamma(A_i \setminus \bigcup_{j=1}^{\infty} B_j)) = 0$ which implies $\mathcal{H}_{\varepsilon}^{\frac{1}{\alpha}}(\gamma(A_i \setminus \bigcup_{j=1}^{\infty} B_j)) = 0$ for all $\varepsilon > 0$. Therefore,

$$\mathcal{H}_{\frac{1}{i} \delta}^{\frac{1}{\alpha}}(\gamma(A_i)) \leq \left(\frac{1}{i}\right)^{\frac{1}{\alpha}} \sum_{j=1}^{\infty} \text{diam } B_j \leq \left(\frac{1}{i}\right)^{\frac{1}{\alpha}} \text{diam } I,$$

from which the claim follows. \square

Next lemma guarantees that α -Hölder curves can be essentially covered by tight curves.

Lemma 6.9. *Let $\gamma: I \rightarrow \mathbb{R}^d$ be an (H, α) -Hölder curve. Then there exist sequences $(H_i)_{i \in \mathbb{N}}$, $(\eta_i)_{i \in \mathbb{N}}$ and $(R_i)_{i \in \mathbb{N}}$ of positive real numbers and (α, η_i, R_i) -tight (H_i, α) -Hölder curves $\gamma_i: I \rightarrow \mathbb{R}^d$, $i \in \mathbb{N}$, such that*

$$\mathcal{H}^{\frac{1}{\alpha}}(\gamma(I) \setminus \bigcup_{i=1}^{\infty} \gamma_i(I)) = 0.$$

Proof. For $\eta, R > 0$, set

$$F_{\eta, R} := \{t \in I \mid \gamma \text{ is } (\alpha, \eta, R)\text{-tight at } t\}.$$

By Lemma 6.8,

$$\mathcal{H}^{\frac{1}{\alpha}}(\gamma(I \setminus \bigcup_{i,j=1}^{\infty} F_{\frac{1}{i}, \frac{1}{j}})) = 0.$$

Therefore, it suffices to show that, for all $\eta, R > 0$, the set $\gamma(F_{\eta, R})$ can be covered by an (α, η', R) -tight (H', α) -Hölder curve for some $\eta', H' > 0$. To that end, let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $F_{\eta, R}$ tending to $t \in I$. Let $0 < r \leq R$ and define $r_n := r - |t - t_n|$ for all $n \in \mathbb{N}$. Then

$$\text{diam } \gamma([t - r, t + r]) \geq \lim_{n \rightarrow \infty} \text{diam } \gamma([t_n - r_n, t_n + r_n]) \geq \eta(2r)^\alpha.$$

Therefore, $F_{\eta, R}$ is closed. Write $F_{\eta, R}^c = \bigcup_{n \in \mathbb{N}}]a_n, b_n[$, where the union is disjoint. Define a curve $\gamma_\eta: I \rightarrow \mathbb{R}^d$ by setting $\gamma_\eta(t) := \gamma(t)$ for all $t \in F_{\eta, R}$ and, on each interval $]a_n, b_n[$ with $n \in \mathbb{N}$, define γ_η as an $(\alpha, \eta, \frac{1}{2}|a_n - b_n|)$ -tight (H, α) -Hölder curve connecting $\gamma(a_n)$ and $\gamma(b_n)$ (recall Remark 6.7).

Let $t \in I$ and $r \leq R$. Choose a constant $K > 1$ (to be fixed later). If

$$|a_{n_0} - b_{n_0}| := \max\{|a_n - b_n| \mid [a_n, b_n] \subset [t - r, t + r]\} \geq \frac{r}{K},$$

then, by the $(\alpha, \eta, \frac{1}{2}|a_n - b_n|)$ -tightness of $\gamma|_{[a_n, b_n]}$, we have that

$$\text{diam } \gamma_\eta([t - r, t + r]) \geq \text{diam } \gamma_\eta([a_{n_0}, b_{n_0}]) \geq \eta \left(\frac{2r}{2K} \right)^\alpha =: \eta'(2r)^\alpha.$$

Otherwise, for every $u \in [t - r, t + r]$, there is $s_u \in [t - r, t + r] \cap F_{\eta,R}$ with $|s_u - u| < \frac{r}{K}$. By Hölder continuity, $|\gamma(u) - \gamma_\eta(s_u)| = |\gamma(u) - \gamma(s_u)| \leq H \left(\frac{r}{K} \right)^\alpha$, so

$$\begin{aligned} \text{diam } \gamma_\eta([t - r, t + r]) &\geq \text{diam } \gamma([s_t - (1 - \frac{1}{K})r, s_t + (1 - \frac{1}{K})r]) - 2H \left(\frac{r}{K} \right)^\alpha \\ &\geq \eta(2(1 - \frac{1}{K})r)^\alpha - 2H \left(\frac{r}{K} \right)^\alpha =: \eta''(2r)^\alpha. \end{aligned}$$

Solving K from the equation

$$\frac{\eta}{(2K)^\alpha} = \eta' = \eta'' = \eta \left(\left(1 - \frac{1}{K}\right)^\alpha - \frac{2H}{\eta(2K)^\alpha} \right),$$

we obtain that $K = 1 + \frac{1}{2} \left(1 + \frac{2H}{\eta}\right)^{\frac{1}{\alpha}}$, which leads to $\eta' = \eta'' = \frac{\eta}{(2+(1+\frac{2H}{\eta})^{\frac{1}{\alpha}})^\alpha}$. Thus, γ_η is (α, η', R) -tight at t .

Finally, we prove that γ_η is $(3H, \alpha)$ -Hölder continuous. Let $t, s \in I$ with $t < s$. If $t, s \in [a_n, b_n]$ for some $n \in \mathbb{N}$, then $|\gamma_\eta(t) - \gamma_\eta(s)| \leq H|t - s|^\alpha$ by construction. Otherwise, set $t' := \min\{u \in F_{\eta,R} \mid u \geq t\}$ and $s' := \max\{u \in F_{\eta,R} \mid u \leq s\}$. Then

$$\begin{aligned} |\gamma_\eta(t) - \gamma_\eta(s)| &\leq |\gamma_\eta(t) - \gamma_\eta(t')| + |\gamma(t') - \gamma(s')| + |\gamma_\eta(s') - \gamma_\eta(s)| \\ &\leq 3H|t - s|^\alpha. \end{aligned}$$

Thus, γ_η is $(3H, \alpha)$ -Hölder continuous. \square

Next we define a concept of an ε -gap which depends on $\omega \in \Omega$.

Definition 6.10. Let $\gamma: I \rightarrow \mathbb{R}^d$ be a curve and let $\varepsilon > 0$. Fix $\omega \in \Omega$. For $r > 0$, we say that γ has an ε -gap of scale r at $t \in I$ if there are $\tilde{a}, \tilde{b} \in [t - r, t + r]$ such that $\gamma([\tilde{a}, \tilde{b}]) \cap E(\omega) = \emptyset$ and $|\gamma(\tilde{a}) - \gamma(\tilde{b})| \geq \varepsilon \text{diam } \gamma([t - r, t + r])$.

The following lemma shows that around typical points of tight Hölder curves there are no ε -gaps of small scale.

Lemma 6.11. Let $\gamma: I \rightarrow \mathbb{R}^d$ be an (H, α) -Hölder and (α, η) -tight curve. Fix $\varepsilon > 0$ and $\omega \in \Omega$. Denote by A the set of $t \in I$ such that γ has ε -gaps of arbitrarily small scales at t . Then $\mathcal{H}^1(A \cap \gamma^{-1}(E(\omega))) = 0$ and $\mathcal{H}^{\frac{1}{\alpha}}(\gamma(A) \cap E(\omega)) = 0$.

Proof. Fix $t \in A$. There exist a sequence $(r_i)_{i \in \mathbb{N}}$ of positive real numbers tending to zero and points $\tilde{a}_i, \tilde{b}_i \in [t - r_i, t + r_i]$ for which $\gamma([\tilde{a}_i, \tilde{b}_i]) \cap E(\omega) = \emptyset$ and $|\gamma(\tilde{a}_i) - \gamma(\tilde{b}_i)| \geq \varepsilon \text{diam } \gamma([t - r_i, t + r_i])$. Since γ is (H, α) -Hölder and (α, η) -tight, we have, for all large enough $i \in \mathbb{N}$, that

$$|\tilde{a}_i - \tilde{b}_i| \geq \left(\frac{1}{H} |\gamma(\tilde{a}_i) - \gamma(\tilde{b}_i)| \right)^{\frac{1}{\alpha}} \geq \left(\frac{\varepsilon}{H} \right)^{\frac{1}{\alpha}} (\text{diam } \gamma([t - r_i, t + r_i]))^{\frac{1}{\alpha}} \geq \left(\frac{\varepsilon}{H} \right)^{\frac{1}{\alpha}} \eta^{\frac{1}{\alpha}} 2r_i.$$

Combining this with the fact that $[\tilde{a}_i, \tilde{b}_i] \subset I \setminus \gamma^{-1}(E(\omega))$ implies that t cannot be a density point of $\gamma^{-1}(E(\omega))$. This is true for all $t \in A$, so $\mathcal{H}^1(A \cap \gamma^{-1}(E(\omega))) = 0$. Since $\gamma(A) \cap E(\omega) = \gamma(A \cap \gamma^{-1}(E(\omega)))$, we have that

$$\mathcal{H}^{\frac{1}{\alpha}}(\gamma(A) \cap E(\omega)) \leq H^{\frac{1}{\alpha}} \mathcal{H}^1(A \cap \gamma^{-1}(E(\omega))) = 0$$

by Lemma 6.2. \square

The next proposition is the main result about the length gain of our broken line approximations to Hölder curves. Its proof, which we postpone until the end of Section 9, is quite technical and depends on several lemmas which are in the Appendices. Recall the notation $l_0 = m_1(1 + 2 + \dots + k_0)$ from (5.3) or Definition 7.1.

Proposition 6.12. *Fix $\omega \in \Omega$ and $m_0, \mathfrak{c}, q \in \mathbb{N} \setminus \{0\}$. Suppose that $k_0, m_1 \in \mathbb{N} \setminus \{0\}$ are large enough so that the assumptions in Lemma 9.6 concerning them are satisfied. Let $n \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$. Assume that every $Q' \in \mathcal{Q}_n(K_Q)$ is $(q, m_1, k_0, \mathfrak{c}, m_0)$ -hereditarily good. Further, suppose that $\gamma: [a, b] \rightarrow K_Q$ is a curve passing through an (n, i) -layer for some $i \in \{1, \dots, d\}$. Assume that, for all $h = 0, \dots, q$, there are no points $a^h, b^h \in [a, b]$ and $\tilde{a}^h, \tilde{b}^h \in [a^h, b^h]$ such that $\text{diam } \gamma([a^h, b^h]) \leq 5\sqrt{d}N^{-hl_0-n}$, $\gamma([\tilde{a}^h, \tilde{b}^h]) \cap E(\omega) = \emptyset$ and $|\gamma(\tilde{a}^h) - \gamma(\tilde{b}^h)| \geq d^{-1}N^{-m_0-(h+1)l_0-n}$. Then there exist points $a \leq b_1 < d_1 \leq \dots \leq b_{2M} < d_{2M} \leq b$ such that*

$$\sum_{j=1}^{2M} |\gamma(b_j) - \gamma(d_j)| \geq (1 + C_3N^{-2m_0})^{q+1} |\gamma(a) - \gamma(b)|, \tag{6.1}$$

where $M < (C_0N^{l_0})^{q+1}$.

We are now ready to prove our main theorem.

Theorem 6.13. *For all $0 \leq p < 1$, there exists $\alpha_0 = \alpha_0(p, d, N) < 1$ such that, for \mathbb{P}_p -almost all $\omega \in \Omega$, the set $E(\omega)$ is purely α -unrectifiable for all $\alpha_0 < \alpha \leq 1$.*

Proof. By Lemmas 6.3 and 6.9, it is enough to show that, for \mathbb{P}_p -almost all $\omega \in \Omega$, all (η, α, R) -tight (H, α) -Hölder curves $\gamma: I \rightarrow \mathbb{R}^d$ with $\alpha_0 < \alpha \leq 1$ satisfy

$$\mathcal{H}^{\frac{1}{\alpha}}(\gamma(I) \cap E(\omega)) = 0$$

for all $H, \eta, R > 0$.

Clearly, we may assume that $\eta \leq 1 \leq H$. Choose $\mathfrak{c} = \mathfrak{c}(d)$ as in Lemma 5.9 and $m_1 = m_1(\mathfrak{c}, d, N)$ as in Lemma 9.6. Select $m_0 = m_0(m_1, p, d, N)$ such that Proposition 5.15 is valid. Finally, let $k_0 = k_0(m_1, m_0, p, d, N)$ be as in Lemma 9.6. Then the parameters

\mathfrak{c} , m_1 , m_0 and k_0 satisfy the assumptions of Proposition 6.12. Fix $H, \eta, R > 0$. Let $0 < \alpha \leq 1$ and assume that $\gamma: I \rightarrow \mathbb{R}^d$ is (η, α, R) -tight (H, α) -Hölder curve. Consider $\omega \in \Omega$ satisfying the conclusion of Proposition 5.15. We show that the assumption

$$\mathcal{H}^{\frac{1}{\alpha}}(\gamma(I) \cap E(\omega)) > 0 \tag{6.2}$$

implies that α is bounded away from 1.

To that end, fix $q \in \mathbb{N}$ and set $\varepsilon := (5\sqrt{d}d)^{-1}N^{-m_0-l_0}$. Let G be the set whose dimension is proved to be less than 1 in Proposition 5.15. By assumption (6.2) and Lemma 6.2, $\mathcal{H}^1(\gamma^{-1}(E(\omega) \setminus G)) > 0$. Set

$$B_\varrho := \{t \in I \mid \forall r \leq \varrho, \gamma \text{ has no } \varepsilon\text{-gaps of scale } r \text{ at } t\}.$$

By Lemma 6.11, there exists $\varrho_0 > 0$ such that $\mathcal{H}^1(B_{\varrho_0} \cap \gamma^{-1}(E(\omega) \setminus G)) > 0$. Let $t_0 \in D := B_{\varrho_0} \cap \gamma^{-1}(E(\omega) \setminus G)$ be a density point of D . For all $0 < \delta < 1$, there exists $0 < r_\delta \leq \min\{R, \varrho_0\}$ such that, for all $r \leq r_\delta$ and $s \in [t_0 - r, t_0 + r]$, we have $[s - \delta r, s + \delta r] \cap D \neq \emptyset$. Fix $0 < \alpha_1 < \alpha$ such that $\alpha_1 < \alpha_0$, where α_0 is defined later. Choose $\delta := ((20\sqrt{d}H^2)^{-1}\eta^2N^{-q l_0})^{\frac{1}{\alpha_1}}$. Let $r \leq r_\delta$ and assume that $\text{diam } \gamma([t_0 - r, t_0 + r]) \leq 5\sqrt{d}N^{-n}$. If $]c, d[\subset [t_0 - r, t_0 + r] \setminus D$, then $]c, d[\subset [s - \delta r, s + \delta r]$ for some $s \in [t_0 - r, t_0 + r]$, and

$$\begin{aligned} \text{diam } \gamma([c, d]) &\leq \text{diam } \gamma([s - \delta r, s + \delta r]) \leq H\delta^\alpha(2r)^\alpha \leq \frac{H}{\eta}\delta^\alpha \text{diam } \gamma([t_0 - r, t_0 + r]) \\ &\leq \frac{H}{\eta}((20\sqrt{d}H^2)^{-1}\eta^2N^{-q l_0})^{\frac{\alpha}{\alpha_1}}5\sqrt{d}N^{-n} < \frac{\eta}{4H}N^{-q l_0 - n}. \end{aligned}$$

In particular, for all $c, d \in [t_0 - r, t_0 + r]$, we have that

$$\text{diam } \gamma([c, d]) \geq \frac{\eta}{4H}N^{-q l_0 - n} \implies [c, d] \cap D \neq \emptyset. \tag{6.3}$$

By Proposition 5.15, there exist $0 < r_1 \leq r_\delta$ and $n \in \mathbb{N}$ such that every $Q' \in \mathcal{Q}_n(K_{Q_m(\gamma(t_0))})$ is $(q, m_1, k_0, \mathfrak{c}, m_0)$ -hereditarily good, $\gamma([t_0 - r_1, t_0 + r_1]) \subset K_{Q_m(\gamma(t_0))}$, $\gamma|_{[t_0 - r_1, t_0 + r_1]}$ passes through an (n, i) -layer for some $i \in \{1, \dots, d\}$ and $\gamma|_{[t_0 - r_1, t_0 + r_1]}$ is an (η, α, r_1) -tight (H, α) -Hölder curve. Suppose that there are $t \in [t_0 - r_1, t_0 + r_1]$ and $r < r_1$ such that $N^{-hl_0 - n} \leq \text{diam } \gamma([t - r, t + r]) \leq 5\sqrt{d}N^{-hl_0 - n}$ for some $h \in \{0, \dots, q\}$. Since $N^{-hl_0 - n} \leq \text{diam } \gamma([t - r, t + r]) \leq H(2r)^\alpha$ and $h \leq q$, we have

$$\text{diam } \gamma([\tilde{t}, \tilde{t} + \frac{r}{2}]) \geq \eta(\frac{r}{2})^\alpha \geq \frac{\eta}{4^\alpha H}N^{-q l_0 - n} \geq \frac{\eta}{4H}N^{-q l_0 - n}$$

for all $\tilde{t} \in [t - r, t + r]$. By (6.3), we conclude that there are $\tilde{t}_1 \in [t - \frac{r}{2}, t] \cap D$ and $\tilde{t}_2 \in [t, t + \frac{r}{2}] \cap D$. Since $\tilde{t}_1 \in B_{\varrho_0}$, we have, for all $\tilde{a}, \tilde{b} \in [t - r, \tilde{t}_1 + (\tilde{t}_1 - (t - r))]$ with

$\gamma([\tilde{a}, \tilde{b}]) \cap E(\omega) = \emptyset$, that

$$\begin{aligned} |\gamma(\tilde{a}) - \gamma(\tilde{b})| &< \varepsilon \operatorname{diam} \gamma([\tilde{t}_1 - (\tilde{t}_1 - (t - r)), \tilde{t}_1 + (\tilde{t}_1 - (t - r))]) \\ &\leq \varepsilon 5\sqrt{d} N^{-hl_0-n} \leq \frac{5\sqrt{d}}{5\sqrt{dd}} N^{-m_0-l_0} N^{-hl_0-n}, \end{aligned}$$

and similarly for $\tilde{a}, \tilde{b} \in [\tilde{t}_2 - (t + r - \tilde{t}_2), t + r]$. Since

$$[t - r, t + r] \subset [t - r, \tilde{t}_1 + (\tilde{t}_1 - (t - r))] \cup [\tilde{t}_2 - (t + r - \tilde{t}_2), t + r],$$

$\gamma|_{[t_0-r_1, t_0+r_1]}$ satisfies the assumptions of Proposition 6.12.

Set $L := C_0 N^{l_0}$, and denote by $a_i, i = 1, \dots, \widetilde{M} + 1$, the increasing sequence of division points given by Proposition 6.12 including the points $a_1 = t_0 - r_1$ and $a_{\widetilde{M}+1} = t_0 + r_1$. If $\widetilde{M} < 3L^{q+1}$, add extra division points to obtain points $a_1, \dots, a_f = t_0 + r_1$ with $f - 1 = 3L^{q+1}$. We remind that the points a_1 and a_f depend on q via the definition of δ . By Proposition 6.12, (H, α) -Hölder continuity, Jensen’s inequality and (η, α, r_1) -tightness, we obtain that

$$\begin{aligned} (1 + C_3 N^{-2m_0})^{q+1} |\gamma(a_1) - \gamma(a_f)| &\leq \sum_{i=1}^{3L^{q+1}} |\gamma(a_i) - \gamma(a_{i+1})| \leq H \sum_{i=1}^{3L^{q+1}} |a_i - a_{i+1}|^\alpha \\ &\leq H 3L^{q+1} (3^{-1} L^{-q-1} |a_1 - a_f|)^\alpha = H 3^{1-\alpha} L^{(1-\alpha)(q+1)} |a_1 - a_f|^\alpha \\ &\leq \eta^{-1} H 3^{1-\alpha} L^{(1-\alpha)(q+1)} \operatorname{diam}(\gamma([a_1, a_f])) \leq \eta^{-1} H 3L^{(1-\alpha)(q+1)} 5\sqrt{d} |\gamma(a_1) - \gamma(a_f)|. \end{aligned}$$

Hence $(1 + C_3 N^{-2m_0})^{q+1} \leq C(H, \eta, d) L^{(1-\alpha)q+1}$, which is a contradiction for large q provided that $(C_0 N^{l_0})^{1-\alpha} = L^{1-\alpha} < 1 + C_3 N^{-2m_0}$. Therefore, α_0 can be chosen to be the solution of the equation

$$(C_0 N^{l_0})^{1-\alpha} = 1 + C_3 N^{-2m_0}. \quad \square$$

To conclude, we pose a natural open question related to the results of Broman et al. [4] described in the Introduction.

Question 6.14. Is it possible to have $\alpha_0 = \beta$, where β is the constant obtained by Broman et al. in [4]? That is, is it true that $\mathcal{H}^{\frac{1}{\alpha}}(E \cap \gamma(I)) = 0$ for all α -Hölder curves $\gamma: I \rightarrow \mathbb{R}^d$ with $\beta < \alpha \leq 1$?

7. Appendix A: definition of zoom levels

In this appendix, we construct the sequences $\mathbf{L}^i := \mathbf{L}^i(m_1, k_0), i \in \mathbb{N}$, used in Sections 5–9 to determine the appropriate zoom levels, and prove their basic properties.

The explicit construction of the sequences \mathbf{L}^i is given in Definition 7.1 below, see also Fig. 7.1. Note that, according to Definition 5.1, the sequence \mathbf{L} determines the levels with information about the distribution of bad cubes and $\Delta_j := L_{j-1} - L_j$ is the number of levels between a $(j - 1, \mathbf{L}, \mathbf{c}, m_0)$ -bad cube and its $(j, \mathbf{L}, \mathbf{c}, m_0)$ -good parent cube. In Section 5, we saw that the larger the step size Δ_j , the smaller the probability that a given cube at level L_{j-1} is $(j - 1, \mathbf{L}, \mathbf{c}, m_0)$ -bad. Hence, we will consider sequences with increasing Δ_j . However, increasing Δ_j increases the total number of subcubes at level L_{j-1} which, in turn, increases the probability that some cubes at level L_{j-1} are $(j - 1, \mathbf{L}, \mathbf{c}, m_0)$ -bad and, therefore, a balance between these two competing phenomena will be needed.

In addition to probability estimates, the sequences \mathbf{L}^i are used to determine appropriate scales for broken line approximations of curves. In Section 8, we construct broken line approximations for curves using information of the distribution of good cubes given by hereditarily good cubes (see Definition 5.12). In order to do that, it is essential that the zooming levels determined by different sequences \mathbf{L}^i are “synchronised”, see Fig. 7.1. Since we are zooming in as we go down into the fractal set, the level zero is on the top of the figure and the levels are going downwards. In order to simplify the notation, we are not adding negative signs to these levels. The basic idea is as follows: We start with the first basic block of l_0 successive levels, which will be covered by a decreasing sequence \mathbf{L}^0 with linearly increasing step size. The basic block is used to obtain a macroscopic increase in length for a broken line approximation. For the purpose of obtaining an exponential increase of length, basic blocks are utilised iteratively as follows: We proceed by adding the second basic block below the first one and by defining \mathbf{L}^1 by means of the same step sizes Δ_j as for \mathbf{L}^0 but starting from the level $2l_0$ instead of level l_0 . Once the level l_0 is reached, we continue increasing the step size linearly but round up the sizes so that we end up using a subset of the levels utilised for \mathbf{L}^0 . Continue inductively by defining the sequence \mathbf{L}^i similarly starting from level $(i + 1)l_0$ and using the levels utilised for \mathbf{L}^{i-1} . For an illustration, see Example 7.2 and Fig. 7.1 below. Once the step size l_0 is reached, we change our strategy by allowing several steps of the same size followed by an exponential increase in the step size. This is explained in Construction 7.4 and represented pictorially in Fig. 7.2.

Definition 7.1. Fix $k_0, m_1 \in \mathbb{N} \setminus \{0\}$ and set $l_0 := m_1(1 + 2 + \dots + k_0)$. For every $i \in \mathbb{N}$, define inductively a finite decreasing sequence $\mathbf{L}^i(m_1, k_0) := (L^i(m_1, k_0)_j)_{j=0}^{k_i}$ of nonnegative integers as follows: Set

$$L^0(m_1, k_0)_j := l_0 - m_1(0 + 1 + 2 + \dots + j) \text{ for } j = 0, 1, \dots, k_0. \tag{7.1}$$

Assume that the sequence $\mathbf{L}^{i-1}(m_1, k_0)$ is defined for some $i \in \mathbb{N} \setminus \{0\}$. In order to define $\mathbf{L}^i(m_1, k_0)$, we distinguish two cases.

Case 1: $m_1 k_{i-1} < l_0$. In this case, let

$$L^i(m_1, k_0)_j := l_0 + L^{i-1}(m_1, k_0)_j \text{ for } j = 0, \dots, k_{i-1}. \tag{7.2}$$

When $j > k_{i-1}$, we define $L^i(m_1, k_0)_j$ inductively. If $L^i(m_1, k_0)_{j-1}$ is defined, let

$$L^i(m_1, k_0)_j := \max\{L^{i-1}(m_1, k_0)_l \mid L^{i-1}(m_1, k_0)_l \leq L^i(m_1, k_0)_{j-1} - m_1 j, \\ l \in \{0, \dots, k_{i-1}\}\} \tag{7.3}$$

provided we have $L^i(m_1, k_0)_j \geq m_1(j+1)$. Otherwise, set $k_i := j$ and let $L^i(m_1, k_0)_j := 0$ be the last element of the sequence $\mathbf{L}^i(m_1, k_0)$.

Case 2: $m_1 k_{i-1} \geq l_0$. In this case, let

$$L^i(m_1, k_0)_j := l_0 + L^{i-1}(m_1, k_0)_j \text{ for } j = 0, \dots, \frac{l_0}{m_1} - 1. \tag{7.4}$$

When $j \geq \frac{l_0}{m_1}$, we define $L^i(m_1, k_0)_j$ inductively by setting

$$L^i(m_1, k_0)_j := L^i(m_1, k_0)_{j-1} - \Delta_j^i, \tag{7.5}$$

where the values of Δ_j^i will be specified later in Construction 7.4 and are illustrated in Fig. 7.2.

Example 7.2. We calculate the sequences $\mathbf{L}^0(m_1, k_0)$, $\mathbf{L}^1(m_1, k_0)$ and $\mathbf{L}^2(m_1, k_0)$ when $k_0 = 8$ and illustrate them in Fig. 7.1. In this case, $L^0(m_1, k_0)_0 = L^0(m_1, 8)_0 = l_0 = 36 \cdot m_1$, $L^0(m_1, 8)_1 = 35 \cdot m_1$, $L^0(m_1, 8)_2 = 33 \cdot m_1$, $L^0(m_1, 8)_3 = 30 \cdot m_1$, $L^0(m_1, 8)_4 = 26 \cdot m_1$, $L^0(m_1, 8)_5 = 21 \cdot m_1$, $L^0(m_1, 8)_6 = 15 \cdot m_1$, $L^0(m_1, 8)_7 = 8 \cdot m_1$ and $L^0(m_1, 8)_8 = 0 \cdot m_1$. These numbers define the levels in the first column in Fig. 7.1. For $j = 0, \dots, 8$, we have that $L^1(m_1, 8)_j = 36 \cdot m_1 + L^0(m_1, 8)_j$. Since $36 \cdot m_1 - 9 \cdot m_1 = 27 \cdot m_1$ and the largest number in the sequence $L^0(m_1, 8)_j$ not exceeding $27 \cdot m_1$ is $26 \cdot m_1$, we conclude that $L^1(m_1, 8)_9 = 26 \cdot m_1$. Further, $26 \cdot m_1 - 10 \cdot m_1 = 16 \cdot m_1$ giving $L^1(m_1, 8)_{10} = 15 \cdot m_1$, and $15 \cdot m_1 - 11 \cdot m_1 = 4 \cdot m_1$ yielding $L^1(m_1, 8)_{11} = 0 \cdot m_1$. So $k_1 = 11$. This gives the second column in Fig. 7.1. For the third column in Fig. 7.1, we calculate $36 \cdot m_1 - 12 \cdot m_1 = 24 \cdot m_1$ and take the largest element in the middle column not exceeding it (keeping in mind that we are zooming into our fractal, so in Fig. 7.1 levels increase downwards). In this way, we obtain $L^2(m_1, k_0)_{k_1+1} = L^2(m_1, k_0)_{12} = 15 \cdot m_1$. Finally, $15 \cdot m_1 - 13 \cdot m_1 = 2 \cdot m_1$ and hence we need to take $L^2(m_1, k_0)_{13} = 0 \cdot m_1$ and $k_2 = 13$.

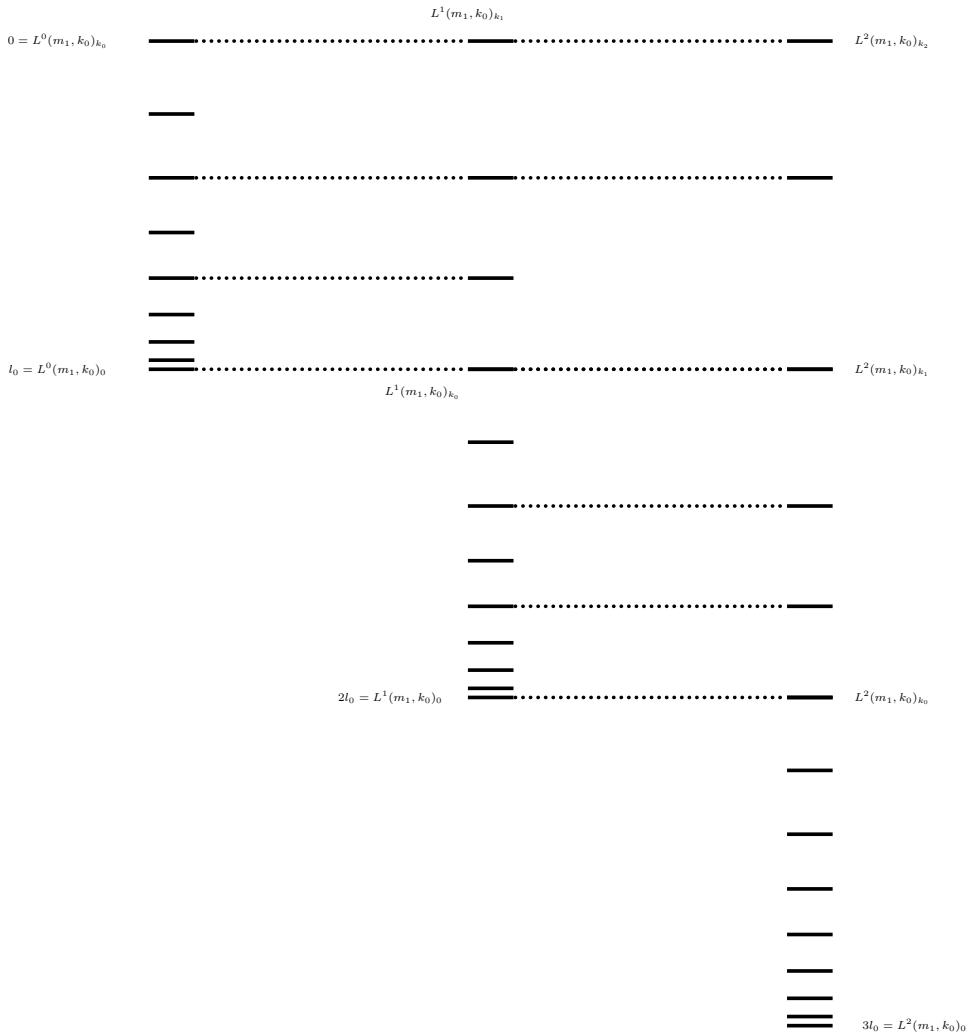


Fig. 7.1. Definition of $\mathbf{L}^i(m_1, k_0)$ for $i = 0, 1$ and 2 , when $k_0 = 8$.

Before we specify the values of Δ_j^i when $k_{i-1} \geq m_1 l_0$ and $j \geq \frac{l_0}{m_1}$, we introduce the notation Δ_j^i for all i and j in harmony with (7.5).

Definition 7.3. Let

$$\Delta_j^i := L^i(m_1, k_0)_{j-1} - L^i(m_1, k_0)_j \text{ for all } i \in \mathbb{N} \text{ and } j = 1, \dots, k_i.$$

Note that in Definition 7.1 (depending on the parameters k_0 and m_1) we already specified the values of Δ_j^i when $m_1 k_{i-1} < l_0$ or $j < \frac{l_0}{m_1}$ and, in order to complete the

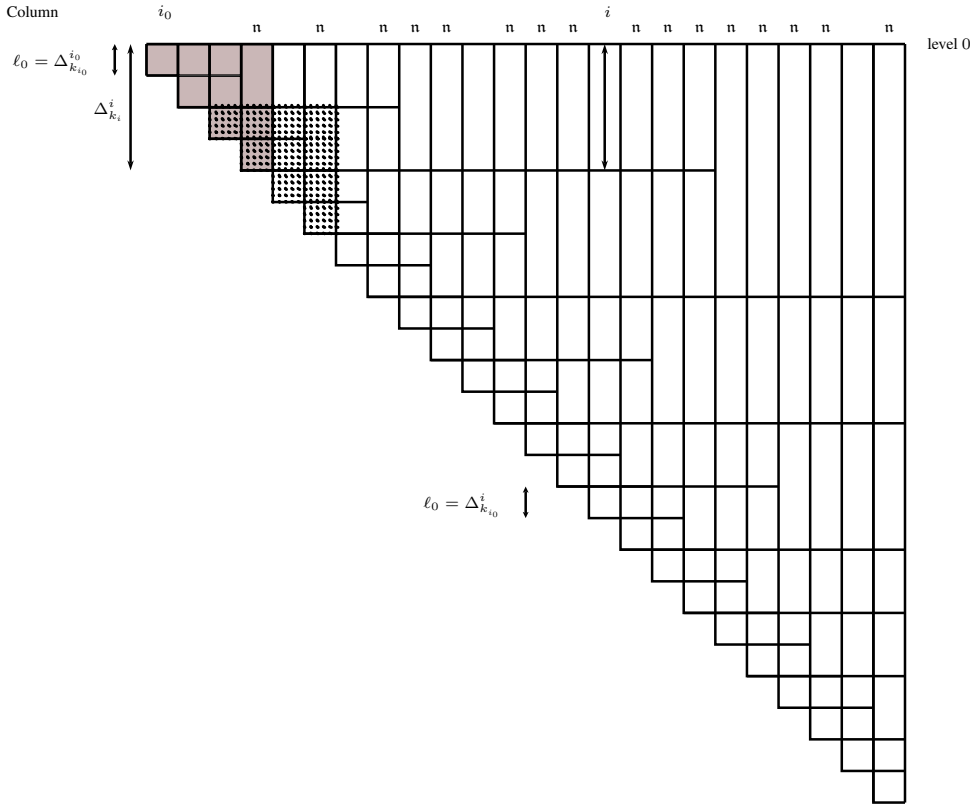


Fig. 7.2. Illustration of Construction 7.4.

definition of the sequences $\mathbf{L}^i(m_1, k_0)$, we need to specify the values of Δ_j^i for the rest of the cases.

Construction 7.4. Each column in Fig. 7.2 consists of a finite number of rectangles called blocks. The columns are labelled by $i_0, i_0 + 1, \dots$ referring to the sequences defined in Definition 7.1, where i_0 is determined via the formula

$$m_1 k_{i_0} = l_0. \tag{7.6}$$

From (7.3) we obtain for all $i = 0, \dots, i_0$ that

$$\Delta_j^i \geq m_1 j \text{ for all } j = k_{i-1} + 1, \dots, k_i, \tag{7.7}$$

where we have used the convention $k_{-1} := 0$.

In the i -th column the heights of the blocks determine Δ_j^i with $j = k_{i_0}, \dots, k_i$, the lowest and the highest ones being $\Delta_{k_{i_0}}^i$ and $\Delta_{k_i}^i$, respectively (see Fig. 7.2). In every column, each block has height of the form $2^n l_0$ for some $n \in \mathbb{N}$, the lowest block having

height l_0 , and the heights form a non-decreasing sequence while moving upwards. We enumerate the blocks such that the lowest one is the first, the second lowest one is the second etc. Note that the i -th column starts from level $(i - i_0 + 1)l_0$ and ends at level 0.

For all $i \in \mathbb{N}$ with $i \geq i_0$, let p_i be the number of blocks in column i . We choose the numbers p_i and the heights of the blocks such that the following properties hold:

$$\text{Either } p_{i+1} = p_i \text{ or } p_{i+1} = p_i + 1 \text{ for all } i. \tag{7.8}$$

$$\text{In each column the block height is increased by factor 2 from } 2^n l_0 \text{ to } 2^{n+1} l_0 \text{ when the inequality } p_i \leq 2(n + 1) + 1 \text{ would otherwise be violated.} \tag{7.9}$$

For a detailed explanation see below. Note that (7.9) guarantees that in each column the height of the j -th block is approximately $\sqrt{2}^j l_0$.

Fig. 7.2 is constructed as follows: For simplicity, the common factor l_0 is omitted from the notation. Clearly, the column i_0 consists of one block of height $1 = 2^0$, and the column $i_0 + 1$ consists of two blocks of height $1 = 2^0$, giving $p_{i_0} = 1$ and $p_{i_0+1} = 2$. In column $i_0 + 2$ we may still utilise blocks of height $1 = 2^0$, giving $p_{i_0+2} = 3$, since $p_{i_0+2} = 3 \leq 2(0 + 1) + 1$ satisfying (7.9). However, the column $i_0 + 3$ cannot consist of four blocks of height 1 since $4 > 2(0 + 1) + 1$. Instead, we use two blocks of height 1 and one block of height 2 in accordance with (7.8) and (7.9), giving $p_{i_0+3} = 3$. In Fig. 7.2 we indicate the fact that $p_{i_0+3} = p_{i_0+2}$ by the symbol **n** on top of the column $i_0 + 3$. Note that in the column $i_0 + 3$ two blocks of height 1 from the column $i_0 + 2$ are “merged” into one block of height 2.

We proceed by translating the pattern of the four columns obtained so far by two steps along the diagonal, that is, the block of height 1 in the column i_0 is moved to the lowest block in column $i_0 + 2$. In Fig. 7.2 the grey shaded part is translated to the dotted one. The translated columns $i_0 + 2$ and $i_0 + 3$ form a part of the columns $i_0 + 4$ and $i_0 + 5$ which are completed by adding a block of height 2 as the topmost block according to (7.8) and (7.9). Then $p_{i_0+4} = 4 = p_{i_0+5}$, as indicated by the symbol **n** above column $i_0 + 5$ in Fig. 7.2.

Next we apply the same translation to the pattern of the six columns constructed above and complete the columns $i_0 + 6$ and $i_0 + 7$ by adding a block of height 2 as the topmost block in accordance with (7.8) and (7.9). This leads to $p_{i_0+6} = 5 = p_{i_0+7}$, as indicated by **n** above column $i_0 + 7$ in Fig. 7.2.

Next we need to modify the translation we are utilising. This is due to the fact that shifting the column $i_0 + 6$ onto the lower part of the column $i_0 + 8$ would imply $p_{i_0+8} = 6$ violating Inequality (7.9) since $6 > 2 \cdot (1 + 1) + 1$. Therefore, we translate the pattern of the first eight rows by four steps along the diagonal, that is, the column i_0 is moved onto the lowest block in column $i_0 + 4$, and complete the incomplete columns by adding a block of height 2^2 to the top of each of them in accordance with (7.8) and (7.9).

We complete the construction inductively by translating the initial pattern by 2^{n+1} steps along the diagonal, whenever utilising the shift of 2^n steps leads to a contradiction

with (7.9), and by placing blocks of height 2^{n+1} to the top of each incomplete block. The symbol \mathbf{n} is used above every column where p_i is not increased. These columns are called \mathbf{n} -columns. \square

Remark 7.5. (a) Observe that $l_0 = \frac{m_1}{2}k_0(1 + k_0) \approx \frac{m_1}{2}k_0^2$. Without the round up process (7.3), by simply defining $L^i(m_1, k_0)_j$ as $L^i(m_1, k_0)_{j-1} - m_1j$ for $j > k_{i-1}$ provided $L^i(m_1, k_0)_j \geq m_1(j + 1)$, we would have $l_0 = \frac{m_1}{2}(k_1 - k_0)(k_0 + 1 + k_1) \approx \frac{m_1}{2}(k_1^2 - k_0^2)$, leading to $k_1 \approx \sqrt{2}k_0$ and, more precisely, $k_1 \leq \frac{3}{2}k_0$. Since the round up process (7.3) reduces the number of steps needed to reach the value 0 for $L^1(m_1, k_0)_j$, the number k_1 is, in fact, smaller than the above calculation indicates. In particular, $k_1 \leq \frac{3}{2}k_0$. Similarly, we obtain that $k_2 \approx \sqrt{3}k_0 < 2k_0$.

(b) We claim that, for all $i \in \mathbb{N}$,

$$\Delta_j^i = m_1j \text{ for } j = 1, \dots, k_0 \text{ and} \tag{7.10}$$

$$\Delta_j^i \leq (i + 1)m_1j \leq j^2m_1 \text{ for } j = k_{i-1} + 1, \dots, k_i < k_{i_0}. \tag{7.11}$$

Indeed, (7.10) follows directly from the definition. The first inequality in (7.11) follows easily by induction since $\Delta_j^i \leq m_1j + \Delta_l^{i-1}$ for some $l < j$ by (7.3), see also Fig. 7.1. Since $k_{i'} > k_{i'-1}$ for all $i' \leq i_0$ and $k_0 \geq 1$, we have that $k_i \geq i + 1$. This implies the second inequality in (7.11). Further, for $j = k_{i_0} + l$, (7.9) and (7.6) imply that

$$\Delta_{k_{i_0}+l}^i \leq \sqrt{2}^l l_0 = \sqrt{2}^l m_1 k_{i_0} \leq 2\sqrt{2}^{k_{i_0}+l} m_1. \tag{7.12}$$

Combining Inequalities (7.10), (7.11) and (7.12), we conclude that

$$\Delta_j^i \leq 5\sqrt{2}^j m_1 \text{ for all } i \in \mathbb{N} \text{ and } j = 1, \dots, k_i, \tag{7.13}$$

that is, (5.2) holds.

Next we estimate the number of blocks of height $2^n l_0$ ending at a given level and derive lower bounds for p_i and k_i .

Lemma 7.6. Consider Construction 7.4 illustrated in Fig. 7.2. For every $n, j \in \mathbb{N}$, let y_j^n be the number of blocks of height $2^n l_0$ having upper side at level jl_0 . Then

$$y_j^n \leq 3 \cdot 2^n. \tag{7.14}$$

Further,

$$p_i \geq \max\{2 \log_2(i - i_0 - 3) - 3, 3\} \text{ for all } i \geq i_0 + 3. \tag{7.15}$$

Finally, provided that $k_0 \geq 3$, we have that

$$k_i \geq i + 3 \text{ for all } i \leq i_0 + 2 \text{ and} \tag{7.16}$$

$$k_{i_0+l} \geq 2 \log_2(i_0 + l + 1) \text{ for all } l \geq 3. \tag{7.17}$$

Proof. By the self-repeating structure of Fig. 7.2, we have $y_j^n \leq y_0^n$ for all $j \in \mathbb{N}$. Thus, it is enough to prove the case $j = 0$. As in Construction 7.4, for notational simplicity, the common factor l_0 of the heights is omitted in what follows.

Clearly, the claim is true when $n = 0$, since $y_0^0 = 3$. In the case $n \geq 1$, the proof is based on counting the number of \mathbf{n} -columns. Let i_n be the column where a block of height 2^n appears for the first time. Note that $y_0^n = i_{n+1} - i_n$. By construction, the column i_n is an \mathbf{n} -column. Let x_n be the number of columns in the maximal chain of successive \mathbf{n} -columns including the column i_n . Then $x_1 = 1$ by Fig. 7.2.

Since the columns $i_0 + 3$ and $i_0 + 5$ are \mathbf{n} -columns (see Fig. 7.2), the self-repeating structure implies that each column $i_0 + 2i + 1$ is an \mathbf{n} -column when $i \geq 1$, that is, the number of blocks is not increased between the columns $i_0 + 2i$ and $i_0 + 2i + 1$ for $i \geq 1$. In particular, the height of the topmost block is never increased by factor two in columns $i_0 + 2i + 1$ for $i \geq 2$. (This only happens in column $i_0 + 3$.) Furthermore, by Construction 7.4, introducing a block of height 2^2 in column i_2 produces three successive \mathbf{n} -columns $i_2 - 1$, i_2 and $i_2 + 1$, that is, $x_2 = 3$.

Again, by the self-repeating structure of Fig. 7.2, there will be three successive \mathbf{n} -columns $i_0 + 4i - 1$, $i_0 + 4i$ and $i_0 + 4i + 1$ for all $i \geq 2$. In general, the self-repeating structure implies that there are x_{n-1} successive \mathbf{n} -columns before and after the column i_n , and moreover, the column i_n is an \mathbf{n} -column. Therefore, $x_n = 2x_{n-1} + 1$. Since $x_1 = 1$, we conclude that $x_n = 2^n - 1$ for all $n \geq 1$.

From (7.9) it follows that $p_{i_n} = 2n + 1$ and, thus, $p_{i_{n+1}} = p_{i_n} + 2$. Utilising the self-repeating structure of Fig. 7.2, we deduce that after the column i_n there are x_{n-1} successive \mathbf{n} -columns followed by a column which is not an \mathbf{n} -column, then x_n successive \mathbf{n} -columns followed by a column which is not an \mathbf{n} -column and after that again x_n successive \mathbf{n} -columns. Since $p_{i_{n+1}} = p_{i_n} + 2$, the next one is the column i_{n+1} . This implies that

$$y_0^n = 1 + x_{n-1} + 1 + x_n + 1 + x_n = 3 + 2^{n-1} - 1 + 2(2^n - 1) = \frac{5}{2} \cdot 2^n,$$

completing the proof of (7.14).

We observed above that $p_{i_n+x_{n-1}} = 2n + 1$ and $p_{i_n+x_{n-1}+1} = 2n + 2$. Moreover,

$$i_n = i_{n-1} + y_0^{n-1} = i_1 + \sum_{j=1}^{n-1} y_0^j = i_0 + 3 + \frac{5}{2} \sum_{j=1}^{n-1} 2^j = i_0 + 3 + 5(2^{n-1} - 1),$$

giving $i_n + x_{n-1} = i_0 + 3 + 6(2^{n-1} - 1)$. Writing $i = i_0 + 3 + 6(2^{n-1} - 1)$, we conclude that

$$p_i \geq 2 \log_2 \left(\frac{1}{6} (i - i_0 - 3) + 1 \right) + 3 \geq 2 \log_2 (i - i_0 - 3) - 3.$$

This implies (7.15) since $p_{i_0+3} = 3$.

According to Definition 7.1 (see the line after (7.3)), we have that $k_i \geq k_{i-1} + 1$ for all $i \leq i_0$. This is true also for $i = i_0 + 1$ and $i = i_0 + 2$ by Fig. 7.2. Therefore, $k_i \geq i + 3$ for all $i \leq i_0 + 2$ provided that $k_0 \geq 3$, completing the proof of (7.16). To prove (7.17), observe that, by the definition of p_i , we get $k_{i_0+l} = k_{i_0} - 1 + p_{i_0+l}$ for all $l \in \mathbb{N}$. Thus, combining (7.15) and (7.16), we obtain

$$k_{i_0+l} \geq i_0 + 2 + \max\{2 \log_2(l - 3) - 3, 3\} \text{ for all } l \geq 3.$$

Inequality (7.17) follows from this by elementary calculations using the fact that condition $k_0 \geq 3$ implies that $i_0 \geq 3$, which can be easily checked from Definition 7.1 and (7.6). \square

8. Appendix B: broken line approximations

In this section, we derive one of our main tools – a special algorithm for constructing broken line approximations, having exponentially increasing arc lengths at different scaling levels, for curves γ which are close to the fractal percolation set. Since the algorithm is quite technical, we first try to give a heuristic outline to prepare the reader to what is coming up in the next two sections.

Our final goal is to prove Proposition 6.12. Definitions 5.1, 5.12 and 7.1 are utilised for the purpose of defining the appropriate scaling levels. The reason behind the length gain is quite simple: If γ passes through an m_0 -good cube (recall Definition 4.3) and is close to the fractal percolation set, it has to go around a strongly deleted cube. This increases the arc length slightly compared to a straight line. The macroscopic increase in length is a consequence of the fact that m_0 -good cubes are abundant and uniformly spread. To achieve this, we need to define several concepts depending on $\omega \in \Omega$ and based on the definitions of m_0 -good and m_0 -bad cubes. If we had m_0 -good cubes everywhere, the proof would be straightforward. However, as illustrated on Fig. 5.1, there might be blue cubes without sufficiently many, sufficiently uniformly distributed m_0 -good cubes to guarantee that the lengths of the broken line approximations increase at an exponential rate. In addition to this, there might also occur blue and red cubes coming from various scaling levels. We will paint our curve to mark on it the parts where we have some “bad” sections. The parts staying white will be good, the blue sections are bad, but not too bad, and the red sections are very bad.

The construction of the broken line approximation is carried through using five algorithms: colouring cubes (Algorithm 8.1), priming curves with colour (Algorithm 8.2), primed curve modification (Algorithm 8.4), layer division (Algorithm 8.6) and painting curves (Algorithm 8.8). Roughly speaking, as a result of a repeated application of these algorithms, we will end up with a modified painted curve where sections with various

properties are distinguished by different colours. We also obtain some division points plus some c -points on our curves at different zoom levels. The broken line approximation to our curve will be determined by the division and the c -points. The c -points will be responsible for the length gain of our broken line approximation as we move to finer and finer broken line approximations. The existence of these c -points is due to the fact that there are gaps in the fractal percolation set $E(\omega)$ and we work with curves which cannot do large jumps over gaps in $E(\omega)$. The increase of length will be achieved in white sections, allowing us to iterate our construction. The blue colour also indicates that the construction may be iterated in the corresponding sections even though there is neither length gain, nor length loss. In red sections we are unable to iterate the construction. These sections will be disregarded when estimating the length of the broken line approximation. In Proposition 9.3 below, we will verify that the white and blue sections make up most of our broken line approximation. The probability estimates were given in Section 5.

Next we describe a process of colouring cubes that will be used as a tool for constructing broken line approximations. As the result of the colouring process, we will have blue cubes and i -red cubes for different values of $i \geq 1$. One can think of i -red colours as different shades of red.

Algorithm 8.1. (Colouring cubes). Fix $\omega \in \Omega$ and $m_0, c, k_0, m_1 \in \mathbb{N} \setminus \{0\}$. Let $n, q \in \mathbb{N}$ and let $\mathbf{L}^i(m_1, k_0)$ and Δ_j^i be as in Definitions 7.1 and 7.3 for all $i = 0, \dots, q$ and $j = 1, \dots, k_i$. Suppose that $I \subset \{0, \dots, q\}$ and $\tilde{k}_i \in \{1, \dots, k_i\}$ for all $i \in I$. Assume that $Q \in \mathcal{Q}_n$ is $(\tilde{k}_i, \mathbf{L}^i(m_1, k_0), c, m_0)$ -good for all $i \in I$.

If $0 \in I$, then Q is $(\tilde{k}_0, \mathbf{L}^0(m_1, k_0), c, m_0)$ -good and there are at most c cubes $Q' \in \mathcal{Q}_{n+\Delta_{\tilde{k}_0}^0}(Q)$ that are $(\tilde{k}_0 - 1, \mathbf{L}^0(m_1, k_0), c, m_0)$ -bad. If such Q' exist, colour them in blue and attach a blue label n to them to denote the level of the parent cube Q of Q' . In this case, Q is called *the blue-labelled parent of Q'* and it is denoted by $\text{LP}_b(Q')$.

Similarly, for every $i \in I$ with $i \geq 1$, there are at most c cubes $Q^{(i+1)} \in \mathcal{Q}_{n+\Delta_{\tilde{k}_i}^i}(Q)$ that are $(\tilde{k}_i - 1, \mathbf{L}^i(m_1, k_0), c, m_0)$ -bad. Colour them in i -red and attach an i -red label n to them to denote the level of the parent cube Q of $Q^{(i+1)}$. As above, Q is called *the i -red-labelled parent of $Q^{(i+1)}$* and it is denoted by $\text{LP}_{i-r}(Q^{(i+1)})$. \square

We proceed by giving algorithms for the purpose of constructing a painted broken line approximation of a curve contained in hereditarily good cubes. The next priming procedure is taking care of the problems which might arise if our curve and the approximating broken line is going in and out of coloured “bad” cubes too many times.

Algorithm 8.2. (Priming curves with colour). Let $l, n, m \in \mathbb{N}$ with $m > n$. Let $Q \in \mathcal{Q}_n$ and assume that the cubes $Q_i \in \mathcal{Q}_m(K_Q)$ for $i = 1, \dots, l$ are coloured in the same colour. Let $\gamma: [a, b] \rightarrow K_Q$ be a curve. Fix $c \in]a, b[$.

Step 1: We denote by K the element of the collection $\{K_{Q_i}\}_{i=1}^l$ that γ enters first. If K does not exist, Step 1 terminates. If K exists, let

$$t_1 := \min\{t \in [a, b] \mid \gamma(t) \in K\}.$$

If $c \leq t_1$ or $c \geq \max\{t \in [a, b] \mid \gamma(t) \in K\}$ or $\text{dist}(\gamma(c), K) \leq N^{-m}$, define

$$t_2 := \max\{t \in [a, b] \mid \gamma(t) \in K\}.$$

Otherwise, set

$$t_2 := \max\{t \in [t_1, c] \mid \gamma(t) \in K\}.$$

Then t_1 is the first entrance time to K whereas t_2 is the last departure time from K (or the last departure time from K before c). We call t_1 and t_2 priming division points.

Step 2: We proceed by applying Step 1 with γ replaced by $\gamma|_{[t_2, b]}$. This results in priming division points t_3 and t_4 . Continue in this way until $\gamma([t_{2p}, b])$ does not hit any K_{Q_i} or $t_{2p} = b$. As a result, we obtain priming division points t_1, \dots, t_{2p} .

Step 3: We prime $\gamma([t_{2j-1}, t_{2j}])$ with the common colour of cubes Q_i for all $j = 1, \dots, p$ and call γ a primed curve with priming division points $\{t_i\}_{i=1}^{2p}$. \square

Remark 8.3. Algorithm 8.2 will be applied using different priming colours. Note that an element of the collection $\{K_{Q_i}\}_{i=1}^l$ is chosen at most twice – at most once before c and after c . Clearly, Algorithm 8.2 may be applied (in a simpler manner) also in the case where no point $c \in]a, b[$ is fixed.

Algorithm 8.4. (Primed curve modification). Assume that $\gamma: [a, b] \rightarrow \mathbb{R}^d$ is a primed curve with priming division points $\{t_i\}_{i=1}^{2p}$. Modify γ such that each part $\gamma|_{[t_{2j-1}, t_{2j}]}$, $j = 1, \dots, p$, is replaced by the line segment connecting $\gamma(t_{2j-1})$ to $\gamma(t_{2j})$ parameterised by the interval $[t_{2j-1}, t_{2j}]$. These line segments inherit the prime colour from the corresponding parts of γ . The modified curve is called $\tilde{\gamma}: [a, b] \rightarrow \mathbb{R}^d$. \square

The above modification “simplifies” γ by replacing certain parts of it, corresponding to going in and out of coloured “bad” cubes, by line segments.

Remark 8.5. If $\gamma([a, b]) \subset K_Q$ for some $Q \in \mathcal{Q}_n$, then $\tilde{\gamma}([a, b]) \subset K_Q$.

Next we introduce an algorithm that divides a curve $\gamma: [a, b] \rightarrow K_Q$ into a collection of subcurves $\gamma: [a_j, a_{j+1}] \rightarrow \mathbb{R}^d$, $j = 0, \dots, p$ for some $p \in \mathbb{N}$, such that

$$\begin{aligned} \gamma|_{[a_j, a_{j+1}]} &\text{ passes through an } (m, i)\text{-layer for some } i \in \{1, \dots, d\} \text{ and} \\ \gamma([a_j, a_{j+1}]) &\subset K_{Q_j} \text{ for some } Q_j \in \mathcal{Q}_m \text{ with } \gamma([a_j, a_{j+1}]) \cap Q_j \neq \emptyset. \end{aligned} \tag{8.1}$$

The points $\{a_j\}_{j=0}^{p+1}$ are called *layer division points*. Algorithm 8.6 consists of 3 steps, the last of which is the most complicated one. It is needed to guarantee that the last subcurve $\gamma: [a_p, a_{p+1}] \rightarrow \mathbb{R}^d$ satisfies (8.1). This algorithm corresponds to zooming from grid level

N^{-n} to grid level N^{-m} . It also defines a part of our approximating broken line division points as intersection points of our curve with the boundaries of certain N^{-m} -net cubes. In addition to these division points, other c -points will be selected later. These c -points will be responsible for the length gain of our broken line approximation at this level.

Algorithm 8.6. (Layer division). Let $n, m \in \mathbb{N}$ with $m > n + \log 5 / \log N$. Assume that $Q \in \mathcal{Q}_n$ and $\gamma: [a, b] \rightarrow K_Q$ is a curve which passes through an (n, i_1) -layer for some $i_1 \in \{1, \dots, d\}$.

Step 1: Set $a_0 := a$ and define

$$a_1 := \min\{t \in [a_0, b] \mid \gamma|_{[a_0, t]} \text{ passes through an } (m, i)\text{-double-layer for some } i \in \{1, \dots, d\}\}.$$

Note that a_1 exists since $m > n$ and $\gamma|_{[a, b]}$ passes through an (n, i_1) -layer. Clearly, $\gamma|_{[a_0, a_1]}$ satisfies (8.1) since $\gamma([a_0, a_1]) \subset K_{Q_m(\gamma(a_0))}$ (recall the notation $Q_n(x)$ from the proof of Theorem 4.17).

Step 2: Apply Step 1 to $\gamma|_{[a_1, b]}$ in order to define a point a_2 . Proceed by applying Step 1 recursively and defining points a_0, a_1, \dots, a_p until there is no $t \in [a_p, b]$ such that $\gamma|_{[a_p, t]}$ passes through an (m, i) -double-layer for some $i \in \{1, \dots, d\}$. If $a_p = b$, the algorithm terminates.

Step 3: The first standing assumption at this step is that there is no $t \in [a_p, b]$ such that $\gamma|_{[a_p, t]}$ passes through an (m, i) -double-layer for some $i \in \{1, \dots, d\}$. In particular, $\gamma([a_p, b]) \subset K_{Q_m(\gamma(a_p))}$. According to the second standing assumption, due to the construction, $\gamma([a_{p-1}, a_p]) \subset K_{Q_m(\gamma(a_{p-1}))}$.

- (1) If $\gamma|_{[a_p, b]}$ passes through an (m, i) -layer, setting $a_{p+1} := b$ terminates the algorithm. Obviously (8.1) is valid for $\gamma|_{[a_p, a_{p+1}]}$.
- (2) If there is $Q' \in \mathcal{Q}_m$ such that $\gamma([a_{p-1}, b]) \subset K_{Q'}$ and $\gamma([a_{p-1}, b]) \cap Q' \neq \emptyset$, redefining $a_p := b$ terminates the algorithm. Let \tilde{a}_p be the value of a_p before we redefined it. Note that $\gamma|_{[a_{p-1}, b]}$ passes through an (m, i) -layer, since $\gamma|_{[a_{p-1}, \tilde{a}_p]}$ passes through an (m, i) -double-layer and $\gamma|_{[\tilde{a}_p, b]}$ does not pass through any (m, i) -layer, since the algorithm did not terminate at point (1). So (8.1) is satisfied.
- (3) If the algorithm has not terminated, we have $\gamma(a_p) \in k_{Q_m(\gamma(b))}$, since $\gamma|_{[a_p, b]}$ does not pass through any (m, i) -layer. If $\gamma([a_p, b]) \not\subset K_{Q_m(\gamma(b))}$, defining $a_{p+1} := \sup\{t \in [a_p, b] \mid \gamma(t) \notin K_{Q_m(\gamma(b))}\}$ and $a_{p+2} := b$ terminates the algorithm. The fact that $\gamma(a_p) \in k_{Q_m(\gamma(b))}$ implies that $\gamma|_{[a_p, a_{p+1}]}$ passes through an (m, i) -layer and $\gamma([a_p, a_{p+1}]) \subset K_{Q_m(\gamma(a_p))}$ by the first standing assumption. Obviously, $\gamma|_{[a_{p+1}, a_{p+2}]}$ satisfies (8.1).
- (4) If $\gamma(a_{p-1}) \in k_{Q_m(\gamma(b))}$, then

$$\tilde{s} := \sup\{t \in [a_{p-1}, b] \mid \gamma(t) \notin K_{Q_m(\gamma(b))}\}$$

exists (since the algorithm did not terminate at point (2)) and is less than a_p (since the algorithm did not terminate at point (3)). Redefining $a_p := \tilde{s}$ and setting $a_{p+1} := b$ terminates the algorithm. Then $\gamma|_{[a_{p-1}, a_p]}$ passes through an (m, i) -layer and $\gamma([a_{p-1}, a_p]) \subset K_{Q_m(\gamma(a_{p-1}))}$ (since \tilde{s} is less than the original value of a_p). Clearly $\gamma|_{[a_p, a_{p+1}]}$ satisfies (8.1).

- (5) If the algorithm has not terminated, then $\gamma(a_{p-1}) \notin k_{Q_m(\gamma(b))}$ and there exists

$$s := \sup\{t \in [a_{p-1}, a_p] \mid \gamma(t) \notin k_{Q_m(\gamma(b))}\}.$$

- (6) If $\gamma(s) \notin \text{Int}(k_{Q_m(\gamma(a_{p-1}))})$, redefining $a_p := s$ and setting $a_{p+1} := b$ terminates the algorithm. Note that $\gamma([a_{p-1}, a_p]) \subset K_{Q_m(\gamma(a_{p-1}))}$ due to the second standing assumption, since the redefined value of a_p is at most its original value. Thus (8.1) is valid for $\gamma|_{[a_{p-1}, a_p]}$. Since the algorithm did not terminate at point (3), we have that $\gamma([a_p, a_{p+1}]) \subset K_{Q_m(\gamma(a_{p+1}))}$ and, thus, also $\gamma|_{[a_p, a_{p+1}]}$ satisfies (8.1) due to the definition of s .

- (7) Recalling that $\gamma(s) \in \partial k_{Q_m(\gamma(b))}$, we may choose $Q_{\gamma(s)} \in \mathcal{Q}_m$ in such a way that $\gamma(s) \in Q_{\gamma(s)}$ and $Q_{\gamma(s)} \subset k_{Q_m(\gamma(b))}$. In particular, $k_{Q_m(\gamma(b))} \subset K_{Q_{\gamma(s)}}$. Then there is

$$u := \sup\{t \in [a_{p-1}, b] \mid \gamma(t) \notin K_{Q_{\gamma(s)}}\},$$

since otherwise the algorithm would have terminated at point (2).

- (8) If $u < s$, redefine $a_p := u$. Setting $a_{p+1} := s$ and $a_{p+2} := b$ terminates the algorithm. Now the second standing assumption implies that $\gamma([a_{p-1}, a_p]) \subset K_{Q_m(\gamma(a_{p-1}))}$. Moreover, since the algorithm did not terminate at point (6), we have that $\gamma(s) \in \text{Int}(k_{Q_m(\gamma(a_{p-1}))})$ which, in turn, implies that $\gamma(a_{p-1}) \in k_{Q_{\gamma(s)}}$. Therefore, $\gamma|_{[a_{p-1}, a_p]}$ passes through an (m, i) -layer. Clearly $\gamma([a_p, a_{p+1}]) \subset K_{Q_{\gamma(s)}}$, and (8.1) is satisfied since $\gamma(u) \in \partial K_{Q_{\gamma(s)}}$. Finally, $\gamma([a_{p+1}, a_{p+2}]) \subset K_{Q_m(\gamma(b))}$ due to the definition of s and $\gamma|_{[a_{p+1}, a_{p+2}]}$ satisfies (8.1) since $\gamma(s) \in \partial k_{Q_m(\gamma(b))}$.

- (9) If the algorithm has not terminated, we have $u \geq s$, implying $u > a_p$ since $k_{Q_m(\gamma(b))} \subset K_{Q_{\gamma(s)}}$ (see point (7)). Define

$$w := \sup\{t \in [a_{p-1}, s] \mid \gamma(t) \notin K_{Q_{\gamma(s)}}\}.$$

- (10) If w exists, redefining $a_p := w$ and setting $a_{p+1} := s$, $a_{p+2} := u$ and $a_{p+3} := b$ terminates the algorithm. Then $\gamma([a_{p-1}, a_p]) \subset K_{Q_m(\gamma(a_{p-1}))}$ by the second standing assumption and the fact that s , and thus w , are at most the original value of a_p . Since $\gamma(s) \in \text{Int}(k_{Q_m(\gamma(a_{p-1}))})$ (the algorithm did not terminate at point (6)), we have that $\gamma(a_{p-1}) \in k_{Q_{\gamma(s)}}$, implying that the first property of (8.1) is true for $\gamma|_{[a_{p-1}, a_p]}$. Evidently, $\gamma|_{[a_p, a_{p+1}]}$ satisfies (8.1). Since the algorithm did not terminate at point (3), we obtain that $\gamma([\tilde{a}_p, b]) \subset K_{Q_m(\gamma(b))}$, where \tilde{a}_p is the original value of a_p . Therefore, $\gamma([s, u]), \gamma([u, b]) \subset K_{Q_m(\gamma(b))}$, where also the definition of s is used.

The first property in (8.1) is valid for $\gamma|_{[a_{p+1}, a_{p+2}]}$, since $\gamma(u) \in \partial K_{Q_{\gamma(s)}}$. It is also valid for $\gamma|_{[a_{p+2}, a_{p+3}]}$ since $k_{Q_m(\gamma(b))} \subset K_{Q_{\gamma(s)}}$.

(11) If w does not exist, then

$$\tilde{w} := \inf\{t \in [s, b] \mid \gamma(t) \notin K_{Q_{\gamma(s)}}\} > a_p$$

exists since the algorithm did not terminate at point (2). The inequality $\tilde{w} > a_p$ follows from the definition of s and the fact $k_{Q_m(\gamma(b))} \subset K_{Q_{\gamma(s)}}$. Redefine $a_p := \tilde{w}$, set $a_{p+1} := b$ and terminate the algorithm. Then $\gamma([a_{p-1}, a_p]) \subset K_{Q_{\gamma(s)}}$ since w does not exist, and (8.1) is true since $\gamma(a_{p-1}) \in k_{Q_{\gamma(s)}}$. As $k_{Q_m(\gamma(b))} \subset K_{Q_{\gamma(s)}}$, the subcurve $\gamma|_{[\tilde{w}, b]}$ passes through an (m, i) -layer. The definition of s and the fact that the algorithm did not terminate at point (3) imply that $\gamma([\tilde{w}, b]) \subset K_{Q_m(\gamma(b))}$ and, therefore, $\gamma|_{[a_p, a_{p+1}]}$ satisfies (8.1). \square

Remark 8.7. If $\gamma(a) \in \partial Q'$ and $\gamma(b) \in \partial Q''$ for some $Q', Q'' \in \mathcal{Q}_m$ while applying Algorithm 8.6, then, for all $j = 0, \dots, p + 1$, there is $Q'_j \in \mathcal{Q}_m$ such that $\gamma(a_j) \in \partial Q'_j$. If $p = 0$, meaning that Algorithm 8.6 does not give any proper subcurves, then $\gamma([a, b]) \subset K_{Q'}$ for some $Q' \in \mathcal{Q}_m$. Since γ passes through an (n, i_1) -layer, this implies that $m \leq n + \log 5 / \log N$, which is a contradiction due to the choice of m .

Algorithm 8.8. (Painting curves). Let $\gamma: [a, b] \rightarrow \mathbb{R}^d$ be a primed curve with layer division points $\{a_j\}_{j=0}^{p+1}$. Suppose that a colour is given. Let $j \in \{0, \dots, p\}$. If a part of $\gamma([a_j, a_{j+1}])$ is primed with the given colour, paint the corresponding closed set $\gamma([a_j, a_{j+1}])$ with the given colour. \square

One of our main tools, Construction 8.10 along with its special case, Construction 8.9, will be applied to curves contained in hereditarily good cubes, resulting in a painted modification of the curve with a collection of layer division points. We will use white, blue and i -red paints for $i = 1, 2, \dots$. Recall that white and blue parts will later enable us to iterate the construction – the difference between them being that only the white colour indicates an increase of length. Red parts will be disregarded in length estimations of broken line approximations.

To illustrate the main ideas behind the construction, we begin with the simplest case of $(0, m_1, k_0, \mathfrak{c}, m_0)$ -hereditarily good cubes. This construction will also be utilised in the general case discussed in Construction 8.10. In Construction 8.9 only white and blue colours are used and the curve is not modified. It corresponds to zooming from grid level N^{-n_0} to grid level $N^{-n_0-l_0} = N^{-n_0-L^0(m_1, k_0)_0}$ without the information that the cubes are also $(q, m_1, k_0, \mathfrak{c}, m_0)$ -hereditarily good for some $q > 0$, which is available in Construction 8.10. We remind that, by Definition 7.3 and Inequality (7.7), $\Delta_j^0 = L^0(m_1, k_0)_{j-1} - L^0(m_1, k_0)_j \geq m_1$. Therefore, the lower bound imposed on m_1 in Construction 8.9 guarantees that Algorithm 8.6 may be applied.

Construction 8.9. Fix $\omega \in \Omega$ and $m_0, \mathbf{c}, k_0, m_1 \in \mathbb{N} \setminus \{0\}$ with $m_1 > \log 5 / \log N$. Let $n_0 \in \mathbb{N}$ and $Q \in \mathcal{Q}_{n_0}$. Assume that every $Q' \in \mathcal{Q}_{n_0}(K_Q)$ is $(0, m_1, k_0, \mathbf{c}, m_0)$ -hereditarily good. Let $\gamma: [a, b] \rightarrow K_Q$ be a curve passing through an (n_0, j) -layer for some $j \in \{1, \dots, d\}$. Fix $c \in]a, b[$. We paint γ and define layer division points by applying the following steps.

Step 1: By assumption, all cubes $Q' \in \mathcal{Q}_{n_0}(K_Q)$ are $(k_0, \mathbf{L}^0(m_1, k_0), \mathbf{c}, m_0)$ -good. Apply Algorithm 8.1 to all of them with $I = \{0\}$ and $\tilde{k}_0 = k_0$. As a result, some cubes in $\mathcal{Q}_{n_0+L^0(m_1, k_0)_{k_0-1}}(K_Q)$ are coloured in blue. Recall that $\Delta_{k_0}^0 = L^0(m_1, k_0)_{k_0-1}$.

Step 2: Let $Q_1, \dots, Q_l \in \mathcal{Q}_{n_0+L^0(m_1, k_0)_{k_0-1}}(K_Q)$ be the blue cubes obtained at Step 1. Note that their blue-labelled parents belong to the set $\mathcal{Q}_{n_0}(K_Q)$. Apply Algorithm 8.2 with $n = n_0$ and $m = n_0 + L^0(m_1, k_0)_{k_0-1}$ using blue primer. Proceed by applying Algorithm 8.6 to γ with $n = n_0 + L^0(m_1, k_0)_{k_0}$ and $m = n_0 + L^0(m_1, k_0)_{k_0-1}$ and denote the resulting layer division points by $\{a_{j_{k_0}}\}_{j_{k_0}=0}^{p+1}$. Paint γ with blue by means of Algorithm 8.8 and, finally, paint with white those parts $\gamma([a_{j_{k_0}}, a_{j_{k_0}+1}])$ that are not painted with blue.

Step 3: For all $j_{k_0} = 0, \dots, p$, consider the curve $\gamma: [a_{j_{k_0}}, a_{j_{k_0}+1}] \rightarrow K_{\tilde{Q}_{j_{k_0}}}$, where $\tilde{Q}_{j_{k_0}} \in \mathcal{Q}_{n_0+L^0(m_1, k_0)_{k_0-1}}$ (recall (8.1)).

- If $\gamma([a_{j_{k_0}}, a_{j_{k_0}+1}])$ is painted blue, apply Algorithm 8.6 with $n = n_0 + L^0(m_1, k_0)_{k_0-1}$ and $m = n_0 + L^0(m_1, k_0)_{k_0-2}$ and denote the resulting layer division points by $\{a_{j_{k_0}, j_{k_0-1}}\}_{j_{k_0-1}=0}^{p_{j_{k_0}}+1}$. Prime with blue all the sets $\gamma([a_{j_{k_0}, j_{k_0-1}}, a_{j_{k_0}, j_{k_0-1}+1}])$. Go to Step 4.
- If the curve $\gamma([a_{j_{k_0}}, a_{j_{k_0}+1}])$ is white, then $\gamma([a_{j_{k_0}}, a_{j_{k_0}+1}]) \cap K_{Q_i} = \emptyset$ for all $i = 1, \dots, l$ by Algorithm 8.2. Further, since $\gamma([a_{j_{k_0}}, a_{j_{k_0}+1}]) \cap \tilde{Q}_{j_{k_0}} \neq \emptyset$ by (8.1), none of the cubes $Q' \in \mathcal{Q}_{n_0+L^0(m_1, k_0)_{k_0-1}}(K_{\tilde{Q}_{j_{k_0}}})$ is blue, that is, they all are $(k_0 - 1, \mathbf{L}^0(m_1, k_0), \mathbf{c}, m_0)$ -good. Apply Algorithm 8.1 to all of them with $I = \{0\}$ and $\tilde{k}_0 = k_0 - 1$. Let $Q'_1, \dots, Q'_l \in \mathcal{Q}_{n_0+L^0(m_1, k_0)_{k_0-2}}(K_{\tilde{Q}_{j_{k_0}}})$ be the resulting blue cubes whose blue-labelled parents belong to the set $\mathcal{Q}_{n_0+L^0(m_1, k_0)_{k_0-1}}(K_{\tilde{Q}_{j_{k_0}}})$. Apply Algorithm 8.2 to $\gamma|_{[a_{j_{k_0}}, a_{j_{k_0}+1}]}$ with $n = n_0 + L^0(m_1, k_0)_{k_0-1}$ and $m = n_0 + L^0(m_1, k_0)_{k_0-2}$ using blue primer. Note that there is only one j_{k_0} such that $c \in]a_{j_{k_0}}, a_{j_{k_0}+1}[$. Proceed by applying Algorithm 8.6 with the same n and m and denote the resulting layer division points by $\{a_{j_{k_0}, j_{k_0-1}}\}_{j_{k_0-1}=0}^{p_{j_{k_0}}+1}$. Go to Step 4.

Step 4: Using blue colour, paint $\gamma|_{[a_{j_{k_0}}, a_{j_{k_0}+1}]}$ by means of Algorithm 8.8. Finally, paint with white those sets $\gamma([a_{j_{k_0}, j_{k_0-1}}, a_{j_{k_0}, j_{k_0-1}+1}])$ that are not painted blue.

Step 5: Iterate Step 3 utilising curves determined by the layer division points obtained in the previous iteration step, and selecting $n = n_0 + L^0(m_1, k_0)_k$ and $m = n_0 + L^0(m_1, k_0)_{k-1}$ for $k = k_0 - 2, \dots, 1$. As a result, we obtain a curve with layer division points $\{a_{j_{k_0}, \dots, j_1}\}$, $j_k = 0, \dots, p_{j_{k_0}, \dots, j_{k+1}} + 1$ for $k = k_0, \dots, 1$, such that the sets $\gamma([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_k+1}])$ are painted with either blue or white. \square

Now we are ready to present a general construction leading to a modification of a curve γ having layer division points that determine parts which are painted white, blue or i -red for $i = 1, \dots, q$. First we will define inductively curves $\gamma_1, \dots, \gamma_q$, making use of Algorithm 8.2 with i -red primers, respectively, and Algorithm 8.4. In particular, $\gamma_1, \dots, \gamma_{q-1}$ are auxiliary curves that will be utilised when defining γ_q . Next we apply Construction 8.9 to γ_q in order to identify some layer division points and painted curve segments. The final outcome is obtained as a result of an iteration process. In this construction, we are zooming again from grid level N^{-n_0} to grid level $N^{-n_0-l_0}$. However, later (see Remark 8.12.(b)) we will zoom in to grid level $N^{-n_0-(q+1)l_0} = N^{-n_0-L^q(m_1, k_0)_0}$ and, therefore, we need to take into consideration i -red cubes coming up from deeper zoom levels of our construction. In Example 8.11, we illustrate Construction 8.10 in the special case depicted in Fig. 7.1. In Construction 8.10, we are taking advantage of the property that $L^i(m_1, k_0)_j$ for $j \geq k_0$ are defined using the levels determined by $L^{i-1}(m_1, k_0)$, see Definition 7.1 and Fig. 7.1.

Construction 8.10. Fix $\omega \in \Omega$ and $m_0, c, k_0, m_1 \in \mathbb{N} \setminus \{0\}$ with $m_1 > \log 5 / \log N$. Let $n_0, q \in \mathbb{N}$ and $Q \in \mathcal{Q}_{n_0}$. Assume that every $Q' \in \mathcal{Q}_{n_0}(K_Q)$ is (q, m_1, k_0, c, m_0) -hereditarily good. Let $\gamma: [a, b] \rightarrow K_Q$ be a curve passing through an (n_0, j) -layer for some $j \in \{1, \dots, d\}$. Fix $c \in]a, b[$. We define a painted modification of γ with layer division points by applying the following steps. During the first three steps, we identify some “bad parts” of γ and modify it at these bad zones. Set $\hat{q} := q, I := \{0, \dots, \hat{q}\}$ and let $\hat{k}_i := k_i$ for all $i \in I$.

Step 1: By assumption, all cubes $Q' \in \mathcal{Q}_{n_0}(K_Q)$ are $(\hat{k}_i, L^i(m_1, k_0), c, m_0)$ -good for all $i \in I$. Apply Algorithm 8.1 to all cubes $Q' \in \mathcal{Q}_{n_0}(K_Q)$ to colour some of their subcubes in blue corresponding to the case $i = 0$ or i -red for $i \in I \setminus \{0\}$.

Step 2: Set $n = n_0$ and $m = n_0 + L^1(m_1, k_0)_{\hat{k}_1-1}$ and let $Q_1, \dots, Q_l \in \mathcal{Q}_m(K_Q)$ be the 1-red cubes. (Recall that their labelled parents belong to the set $\mathcal{Q}_{n_0}(K_Q)$.) Apply Algorithm 8.2 to γ using 1-red primer. Next apply Algorithm 8.4 and denote the modified curve by γ_1 .

Step 3: Set $n = n_0$ and $m = n_0 + L^2(m_1, k_0)_{\hat{k}_2-1}$ and let $Q'_1, \dots, Q'_i \in \mathcal{Q}_m(K_Q)$ be the 2-red cubes. Apply Algorithm 8.2 to γ_1 using 2-red primer. Next apply Algorithm 8.4 to γ_1 and denote the modified curve by γ_2 . When $n = n_0$ and $m = n_0 + L^i(m_1, k_0)_{\hat{k}_i-1}$, with $i = 3, \dots, \hat{q}$, continue iteratively until the curve $\gamma_{\hat{q}}$ is defined. Note that parts of $\gamma_{\hat{q}}([a, b])$ are primed with i -red primer for $i \in I$, and some parts may be primed with several i -red primers.

Next we start to introduce the level division points used in our broken line approximation at different levels.

Step 4: Apply Steps 2–5 of Construction 8.9 to the curve $\gamma_{\hat{q}}$ until the curve $\gamma_{\hat{q}}|_{[a^{j_{k_0}}, \dots, j_{k+1}, a^{j_{k_0}}, \dots, j_{k+1}+1]}$ is considered, where

$$m = n_0 + L^0(m_1, k_0)_{k-1} = n_0 + L^1(m_1, k_0)_{\hat{k}_1-1} = \dots = n_0 + L^{\hat{q}}(m_1, k_0)_{\hat{k}_{\hat{q}}-1}$$

for some $q' \in \{1, \dots, q\}$, that is, until the size of blue cubes is same as the size of i -red cubes for $i = 1, \dots, q'$. Now apply Steps 2 and 3 of Construction 8.9. Instead of applying Step 4 of Construction 8.9, proceed by painting as follows: Apply q' times Algorithm 8.8 to $\gamma_{\hat{q}}|_{[a_{j_{k_0}, \dots, j_{k+1}}, a_{j_{k_0}, \dots, j_{k+1}+1}]}$ using i -red paint for $i = 1, \dots, q'$.

- If $\gamma_{\hat{q}}|_{[a_{j_{k_0}, \dots, j_{k+1}}, a_{j_{k_0}, \dots, j_{k+1}+1}]}$ is blue, those sets $\gamma_{\hat{q}}([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}])$ which are not painted i -red for any $i \in \{1, \dots, q'\}$ inherit the blue paint. Go to Step 5.
- If $\gamma_{\hat{q}}|_{[a_{j_{k_0}, \dots, j_{k+1}}, a_{j_{k_0}, \dots, j_{k+1}+1}]}$ is white, apply Algorithm 8.8 to the curve $\gamma_{\hat{q}}|_{[a_{j_{k_0}, \dots, j_{k+1}}, a_{j_{k_0}, \dots, j_{k+1}+1}]}$ using blue paint, ignoring those curve segments $\gamma_{\hat{q}}([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}])$ which are painted i -red for some $i \in \{1, \dots, q'\}$, that is, if $\gamma_{\hat{q}}([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}])$ is painted i -red, do not paint it blue even though a part of it is primed with a blue primer. Finally, paint white those sets $\gamma_{\hat{q}}([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}])$ that are not painted i -red or blue. Go to Step 5.

Step 5: For all j_{k_0}, \dots, j_k , consider $\gamma_{\hat{q}}: [a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}] \rightarrow K_{Q_{j_{k_0}, \dots, j_k}}$.

- If $\gamma_{\hat{q}}([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}])$ is i -red for some $i \in \{1, \dots, q'\}$, the construction terminates.
- In the case that $\gamma_{\hat{q}}([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}])$ is blue, all the cubes in the collection $\mathcal{Q}_{n_0+L^0(m_1, k_0)_{k-1}}(K_{Q_{j_{k_0}, \dots, j_k}})$ are $(\hat{k}_i - 1, \mathbf{L}^i(m_1, k_0), \mathbf{c}, m_0)$ -good for all $i = 1, \dots, q'$, since otherwise $\gamma_{\hat{q}}([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}])$ had received i -red paint for some $i \in \{1, \dots, q'\}$ (recall the argument from the second bullet of Step 3 in Construction 8.9). Repeat the construction from Step 1 utilising the curve $\gamma_{\hat{q}}|_{[a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}]}$ with $I = \{1, \dots, q'\}$, replacing \hat{k}_i by $\hat{k}_i - 1$ for $i = 1, \dots, q'$ and keeping the value of \hat{k}_i for $i = q' + 1, \dots, q$.
- If $\gamma_{\hat{q}}([a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}])$ is white, all cubes in $\mathcal{Q}_{n_0+L^0(m_1, k_0)_{k-1}}(K_{Q_{j_{k_0}, \dots, j_k}})$ are $(k-1, \mathbf{L}^0(m_1, k_0), \mathbf{c}, m_0)$ -good and $(\hat{k}_i - 1, \mathbf{L}^i(m_1, k_0), \mathbf{c}, m_0)$ -good for all $i = 1, \dots, q'$. Repeat the construction from Step 1 using the curve $\gamma_{\hat{q}}|_{[a_{j_{k_0}, \dots, j_k}, a_{j_{k_0}, \dots, j_{k+1}}]}$ with $I = \{0, \dots, q'\}$, replacing \hat{k}_i by $\hat{k}_i - 1$ for $i = 0, \dots, q'$ and letting \hat{k}_i be as they are for $i = q' + 1, \dots, q$.

While iterating the Steps 1–5, the curve $\gamma_{\hat{q}}$ is further modified. For $i = q' + 1, \dots, q$, the i -red parts are taken into account once their levels are reached in the construction at Step 4.

Step 6: The construction is complete once the level $n_0 + L^0(m_1, k_0)_0$ is reached, the modified painted curve $\gamma_{\tilde{q}}$ for some $\tilde{q} \geq q$ and the layer division points $a_{j_{k_0}, \dots, j_1}$ are defined and the final curve segments $\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}])$ are painted. \square

Example 8.11. We demonstrate Construction 8.10 in the special case $k_0 = 8$ and $q = 2$ as shown in Fig. 7.1. During the Steps 1–3 a curve γ_2 , containing parts with 1-red and 2-red primer, is defined. Recall from Example 7.2 that $k_1 = 11$, $k_2 = 13$ and

$$L^0(m_1, 8)_6 = L^1(m_1, 8)_{10} = L^2(m_1, 8)_{12} = 15 \cdot m_1. \tag{8.2}$$

At Step 4 we apply Step 2 of Construction 8.9 once with $m = n_0 + L^0(m_1, 8)_7 = n_0 + 8 \cdot m_1$ and define curves $\gamma_2|_{[a_{j_8}, a_{j_8+1}]}$. While considering these curves, we have already reached the level, where the size of blue cubes corresponding to level $n_0 + L^0(m_1, 8)_6 = n_0 + 15 \cdot m_1$ is the same as the size of 1-red and 2-red cubes by (8.2), so $q' = 2$. We continue with Step 4 and define curves $\gamma_2|_{[a_{j_8, j_7}, a_{j_8, j_7+1}]}$. For blue and white curves we proceed from Step 5 and go back to Step 1, where we find new 1-red cubes at level $n_0 + 26 \cdot m_1$ and 2-red cubes at level $n_0 + 36 \cdot m_1$. After Step 3 we have defined the modified curve γ_4 . At Step 4 we have again only one “blue” step and after that we have reached the level where $m = n_0 + L^0(m_1, 8)_4 = n_0 + L^1(m_1, 8)_9 = n_0 + 26 \cdot m_1$. So in this case $q' = 1$. After Step 4 we have constructed the curves $\gamma_4|_{[a_{j_8, j_7, j_6, j_5}, a_{j_8, j_7, j_6, j_5+1}]}$. For blue and white curves we continue from Step 5 and go back to Step 1, where we produce new 1-red cubes at level $n_0 + 36 \cdot m_1$. After Step 2 we have defined the modified curve γ_5 and go to Step 4. (There is no Step 3 since $q' = 1$.) Next we have three “blue” iteration steps at levels $n_0 + 30 \cdot m_1$, $n_0 + 33 \cdot m_1$ and $n_0 + 35 \cdot m_1$, and we end up with curves $\gamma_5|_{[a_{j_8, \dots, j_2}, a_{j_8, \dots, j_2+1}]}$. Now we have reached the final level $n_0 + 36 \cdot m_1$, where the sizes of blue, 1-red and 2-red cubes are the same. We complete Step 4 in order to define the final curves $\gamma_5|_{[a_{j_8, \dots, j_1}, a_{j_8, \dots, j_1+1}]}$.

Remark 8.12. (a) Denote by $b_j, j = 0, \dots, p$, the layer division points $a_{j_{k_0}, \dots, j_1}$ obtained in Construction 8.10. Then the family of curves $\gamma_{\bar{q}}: [b_j, b_{j+1}] \rightarrow K_{Q_j}, j = 0, \dots, p$, satisfies (8.1) with $m = n_0 + l_0$. If $\gamma_{\bar{q}}([b_j, b_{j+1}])$ is not red and $q \leq i_0 + 2$ (recall (7.6)), all the cubes $Q' \in \mathcal{Q}_{n_0+l_0}(K_{Q_j})$ are $(q - 1, m_1, k_0, c, m_0)$ -hereditarily good due to the iterative construction of sequences $\mathbf{L}^i(m_1, k_0)$ (see Definition 7.1), $\gamma_{\bar{q}}(b_j) = \gamma(b_j)$ and $\gamma_{\bar{q}}(b_{j+1}) = \gamma(b_{j+1})$. On the other hand, if $\gamma_{\bar{q}}([b_j, b_{j+1}])$ is red, it may happen that $\gamma_{\bar{q}}(b_j) \neq \gamma(b_j)$ or $\gamma_{\bar{q}}(b_{j+1}) \neq \gamma(b_{j+1})$. In this case, $\gamma_{\bar{q}}(b_j)$, respectively $\gamma_{\bar{q}}(b_{j+1})$, is on a line segment produced by Algorithm 8.4 and, therefore, $\gamma_{\bar{q}}([b_{j-1}, b_j])$, respectively $\gamma_{\bar{q}}([b_{j+1}, b_{j+2}])$, is red.

(b) If $q > i_0 + 2$, some parts of $\gamma_{\bar{q}}$ may be primed with i -red for $i > i_0 + 2$. These parts are not painted with i -red in Construction 8.10, since the corresponding i -red cubes are at higher levels than $n_0 + L^0(m_1, k_0)_0$, where the construction terminates. For later purposes (see Proposition 6.12), we emphasise that, due to the iterative construction of sequences $\mathbf{L}^i(m_1, k_0)$, Construction 8.10 may be continued until the level $n_0 + L^q(m_1, k_0)_0$ is reached. In this case, there will be no primed parts that are not painted.

(c) Construction 8.10 may be applied also in the case when no $c \in]a, b[$ is fixed.

(d) Let $j_{k_0} = i_c$ be an index such that $c \in [a_{i_c}, a_{i_c+1}]$. If $\gamma_{\bar{q}}([a_{i_c}, a_{i_c+1}])$ does not contain parts which are painted or primed with red, we have

$$|\gamma_{\bar{q}}(a_{i_c}) - \gamma(c)| = |\gamma(a_{i_c}) - \gamma(c)| \leq 5\sqrt{d}N^{-n_0 - L^0(m_1, k_0)_{k_0-1}}.$$

Otherwise, it may happen that $\gamma_{\bar{q}}(c) \neq \gamma(c)$. By construction, $\gamma_{\bar{q}}([a_{i_c}, a_{i_c+1}]) \subset K_{Q'}$ for some $Q' \in \mathcal{Q}_{n_0+L^0(m_1, k_0)_{k_0-1}}$, where γ_q is the modified curve obtained after Step 3 in Construction 8.10. By Algorithms 8.2 and 8.4 and Definition 7.1, we deduce that

$$|\gamma(c) - \gamma_q(c)| \leq 6\sqrt{d}N^{-n_0} \sum_{i=1}^q N^{-L^i(m_1, k_0)_{k_i-1}} \leq 6\sqrt{d}N^{-n_0-L^0(m_1, k_0)_{k_0-1}}.$$

Furthermore, according to Remark 8.5, $\gamma_{\tilde{q}}([a_{i_c}, a_{i_c+1}]) \subset K_{Q'}$, where $\gamma_{\tilde{q}}$ is the final modified curve in Construction 8.10. Therefore,

$$|\gamma_{\tilde{q}}(a_{i_c}) - \gamma(c)| \leq 11\sqrt{d}N^{-n_0-L^0(m_1, k_0)_{k_0-1}}.$$

In both cases, recalling that $L^0(m_1, k_0)_{k_0-1} = m_1 k_0$ (see Definition 7.1) and $|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)| \geq N^{-n_0}$, we conclude that

$$\sum_{j_{k_0}=0}^p |\gamma_{\tilde{q}}(a_{j_{k_0}}) - \gamma_{\tilde{q}}(a_{j_{k_0}+1})| \geq (1 - CN^{-m_1 k_0})(|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|),$$

where $C := 22\sqrt{d}$.

(e) By definition, the modified curve $\gamma_{\tilde{q}}$ enters every expanded red cube $K_{Q'}$ at most twice – at most once before c and at most once after c .

9. Appendix C: increase of length

In this section, we estimate the length of white and non-red parts of $\gamma_{\tilde{q}}$ obtained in Construction 8.10. We begin by verifying a lemma concerning diameters of blue and red cubes inside white or blue cubes. Recall from Definition 7.1 and Equation (7.6) that $l_0 = m_1(1 + 2 + \dots + k_0)$ and $m_1 k_{i_0} = l_0$.

Lemma 9.1. *Fix $\omega \in \Omega$ and $m_0, \mathbf{c}, k_0, m_1 \in \mathbb{N} \setminus \{0\}$ with $k_0 \geq 2$. Let $n, q \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$. Assume that Q is $(q, m_1, k_0, \mathbf{c}, m_0)$ -hereditarily good. Apply Algorithm 8.1 iteratively as in Construction 8.10 until the level $n + L^q(m_1, k_0)_0 = n + (q + 1)l_0$ is reached (recall Remark 8.12.(b)). Set $m := n + \tilde{q}l_0 + L^0(m_1, k_0)_{j_0}$, where $\tilde{q} \in \{0, \dots, q\}$ and $j_0 \in \{1, \dots, k_0\}$. Assume that $\tilde{Q} \in \mathcal{Q}_m(Q)$ is white. Then*

$$\sum_{Q', LP_b(Q')=\tilde{Q}} \text{diam } Q' \leq \mathbf{c}N^{-m_1 j_0} \text{diam } \tilde{Q}.$$

Suppose that, for some $q' \leq q$, there exists $j_i \in \{k_{i-1} + 1, \dots, k_i\}$ for $i = 1, \dots, q'$ such that $m = n + \tilde{q}l_0 + L^i(m_1, k_0)_{j_i}$. If $\hat{Q} \in \mathcal{Q}_m(Q)$ is white or blue, then

$$\sum_{Q', LP_{1-r}(Q')=\hat{Q}} \text{diam } Q' \leq \mathbf{c}N^{-m_1 j_1} \text{diam } \hat{Q}$$

and

$$\sum_{i=1}^{q'} \sum_{Q', LP_{i-r}(Q')=\hat{Q}} \text{diam } Q' \leq 6\mathbf{c}N^{-m_1 j_1} \text{diam } \hat{Q}.$$

Further, if $q' \geq 2$ and $k_0 \geq 3$, then

$$\sum_{i=2}^{q'} \sum_{Q', LP_{i-r}(Q')=\widehat{Q}} \text{diam } Q' \leq 6cN^{-m_1j_2} \text{diam } \widehat{Q}.$$

Proof. Since $\widetilde{Q} \in \mathcal{Q}_m(Q)$ is white, it is $(j_0, \mathbf{L}^0(m_1, k_0), \mathbf{c}, m_0)$ -good and, therefore, the blue-labelled parent of at most \mathbf{c} blue cubes at level $m + \Delta_{j_0}^0 = m + m_1j_0$. This implies the first claim. The second claim follows similarly.

Since $\widehat{Q} \in \mathcal{Q}_m(Q)$ is white or blue, it is $(j_i, \mathbf{L}^i(m_1, k_0), \mathbf{c}, m_0)$ -good for $i = 1, \dots, q'$, where $q' \leq q$ and $j_1 < j_2 < \dots < j_{i_0} \leq j_{i_0+1} \leq \dots \leq j_{q'}$. Therefore, the number of i -red cubes at level $m + \Delta_{j_i}^i$, having i -red-labelled parent \widehat{Q} , is at most \mathbf{c} for all $i = 1, \dots, q'$. By Inequality (7.7), $\Delta_{j_i}^i \geq m_1j_i$ for all $i = 1, \dots, i_0$. According to Inequality (7.14) in Lemma 7.6, the number of i 's with $i > i_0$ and $\Delta_{j_i}^i = 2^k l_0$ is at most $3 \cdot 2^k$. Combining the above facts, we conclude that

$$\begin{aligned} \sum_{i=1}^{q'} \sum_{\substack{Q' \\ LP_{i-r}(Q')=\widehat{Q}}} \text{diam } Q' &\leq \mathbf{c} \text{diam } \widehat{Q} \left(\sum_{j=j_1}^{i_0} N^{-m_1j} + \sum_{k=0}^{\infty} 3 \cdot 2^k N^{-2^k l_0} \right) \\ &\leq \mathbf{c} \text{diam } \widehat{Q} \left(\frac{1}{1-N^{-m_1}} N^{-m_1j_1} + 4N^{-l_0} \right) \leq 6cN^{-m_1j_1} \text{diam } \widehat{Q}, \end{aligned}$$

where we used the fact that, by Remark 7.5.(a), for $k_0 \geq 2$,

$$j_1 \leq k_1 \leq \frac{3}{2}k_0 \leq \frac{1}{2}k_0 + \frac{1}{2}k_0^2 = \frac{l_0}{m_1}.$$

The last claim follows in a similar manner for $k_0 \geq 3$. \square

Remark 9.2. We have quite a few parameters in our construction and, as mentioned in Remark 5.13.(b), the order in which they are selected is very delicate. It is done in the second paragraph of the proof of Theorem 6.13, but in this Section we impose some restrictions on them. The parameter m_1 controls the contribution of blue curve segments (see (9.1) below). We can make that contribution small, but we have to fix m_1 before we can choose the parameter m_0 , which is used to tune the length gain in our broken line approximation (see Lemma 9.5 and Propositions 5.14 and 5.15). The role of parameter k_0 is to make the contribution of red curve segments arbitrarily small (see (9.2)).

The following proposition is the key result of this section. It is essential in the proof of Lemma 9.6 (see (9.12)) which, in turn, is the basis of Proposition 6.12, whose proof is the main goal of this section. A curve is called red, if it is i -red for some $i \in \mathbb{N} \setminus \{0\}$.

Proposition 9.3. Fix $\omega \in \Omega$ and $m_0, \mathbf{c}, k_0 \in \mathbb{N} \setminus \{0\}$ with $k_0 \geq 3$. Let $m_1, n, q \in \mathbb{N}$ with $m_1 > \log 5 / \log N$ and $Q \in \mathcal{Q}_n$. Assume that every $Q' \in \mathcal{Q}_n(K_Q)$ is $(q, m_1, k_0, \mathbf{c}, m_0)$ -hereditarily good. Let $\gamma: [a, b] \rightarrow K_Q$ be a curve passing through an (n, i) -layer for some

$i \in \{1, \dots, d\}$ and assume that $\gamma(x) \in \partial Q_x$ for some $Q_x \in \mathcal{Q}_n$, where $x \in \{a, b\}$. Fix $c \in]a, b[$. Applying Construction 8.10 to γ and denoting the modified painted curve by $\gamma_{\bar{q}}$, there exist positive constants C_1 and M_1 depending only on d and \mathfrak{c} such that, for all $m_1 \geq M_1$,

$$\sum_{\substack{j_{k_0}, \dots, j_1 \\ \gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]) \text{ is white}}} |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_1+1})| \geq (1 - C_1 N^{-m_1})(|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|) \tag{9.1}$$

and

$$\sum_{\substack{j_{k_0}, \dots, j_1 \\ \gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]) \text{ is not red}}} |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_1+1})| \geq (1 - C_1 N^{-m_1 k_0})(|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|). \tag{9.2}$$

Proof. Let $l \in \{k_0, \dots, 1\}$. Here we use the convention $[a_{j_{k_0+1}}, a_{j_{k_0+1}+1}] := [a, b]$. Consider $\gamma_{\bar{q}}: [a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}] \rightarrow \mathbb{R}^d$. By Remarks 8.7 and 8.5 and Condition (8.1), there exist cubes $Q_{j_{k_0}, \dots, j_{l+1}, \eta} \in \mathcal{Q}_{n+L^0(m_1, k_0)_l}$, $\eta \in \{\alpha, \beta\}$, such that

$$\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}]) \subset K_{Q_{j_{k_0}, \dots, j_{l+1}, \eta}}, \tag{9.3}$$

$$\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{l+1}}) \in \partial Q_{j_{k_0}, \dots, j_{l+1}, \alpha} \quad \text{and} \quad \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{l+1}+1}) \in \partial Q_{j_{k_0}, \dots, j_{l+1}, \beta} \quad \text{and} \tag{9.4}$$

$$|\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{l+1}}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{l+1}+1})| \geq N^{-n-L^0(m_1, k_0)_l} = \frac{1}{\sqrt{d}} \text{diam } Q_{j_{k_0}, \dots, j_{l+1}}. \tag{9.5}$$

If $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}])$ is red, the interval $[a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}]$ does not contain any further layer division points and, in particular, it does not contain any white or blue curve segments. If $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}])$ is blue, it does not contain any white curve segments of the form $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_l+1}])$ but may contain blue and red ones. Finally, if $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}])$ is white, it may contain white, blue and red curve segments $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_l+1}])$.

Suppose that $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}])$ is white. We will first estimate the contribution of blue curve segments $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_l+1}])$ it contains. If the segment $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_l+1}])$ is blue, it intersects $K_{Q'}$ for some blue cube $Q' \in \mathcal{Q}_{n+L^0(m_1, k_0)_{l-1}}(K_{Q_{j_{k_0}, \dots, j_{l+1}}})$. Applying (9.3) with l replaced by $l - 1$, we conclude that $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_l+1}]) \subset 15Q'$. Since $\gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}])$ is white, all cubes in the set $\mathcal{Q}_{n+L^0(m_1, k_0)_l}(K_{Q_{j_{k_0}, \dots, j_{l+1}}})$ are white according to Step 3 of Construction 8.9. Combining Inequality (9.5) and the first claim in Lemma 9.1 gives

$$\begin{aligned}
 & \sum_{j_l} |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_l}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_{l+1}})| \\
 & \gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_{l+1}}]) \text{ is not blue} \\
 & \geq |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_{l+1}}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_{l+1}+1})| \\
 & - \sum_{\tilde{Q} \in \mathcal{Q}_{n+L^0(m_1, k_0)_l}(K_{Q_{j_{k_0}, \dots, j_{l+1}}})} \sum_{\text{LP}_b(Q') = \tilde{Q}}^{Q'} 15 \text{ diam } Q' \\
 & \geq (1 - 5^d \mathfrak{c} 15 \sqrt{d} N^{-m_1 l}) |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_{l+1}}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_{l+1}+1})|.
 \end{aligned} \tag{9.6}$$

Next we estimate the contribution of red parts which are contained in the curve $\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}])$. If $\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_{l+1}}])$ is red, it intersects $K_{Q'}$ for some i -red cube $Q' \in \mathcal{Q}_{n+L^0(m_1, k_0)_{l-1}}(K_{Q_{j_{k_0}, \dots, j_{l+1}}})$ and, as above, we obtain that $\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_{l+1}}]) \subset 15Q'$. Now $\text{LP}_{i-r}(Q') \in \mathcal{Q}_{n+L^0(m_1, k_0)_{r(i, l)}}$ for $r(i, l) \geq l + 1$. Suppose first that $c \notin [a_{j_{k_0}, \dots, j_{r(i, l)+1}}, a_{j_{k_0}, \dots, j_{r(i, l)+1}+1}]$. Let

$$t_1 := \min\{t \in [a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}] \mid \gamma_{\tilde{q}}(t) \in K_{Q'}\} \text{ and}$$

$$t_2 := \max\{t \in [a_{j_{k_0}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{l+1}+1}] \mid \gamma_{\tilde{q}}(t) \in K_{Q'}\}.$$

By Algorithm 8.4, the curve $\gamma_{\tilde{q}}([t_1, t_2])$ is a straight line inside $K_{Q'}$. Assuming that $K_{Q'} \subset \tilde{Q} \in \mathcal{Q}_{n+L^0(m_1, k_0)_l}$, a repeated application of condition (9.4) with l replaced by $l + 1, \dots, r(i, l)$ implies the existence of a unique sequence $j_{r(i, l)}, \dots, j_{l+1}$ such that the curve $\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_{r(i, l)}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{r(i, l)}, \dots, j_{l+1}+1}])$ intersects $K_{Q'}$. If there exists an index $\tilde{l} \in \{l, \dots, r(i, l)\}$ such that $K_{Q'}$ intersects the interiors of at least two cubes in $\mathcal{Q}_{n+L^0(m_1, k_0)_{\tilde{l}}}$, there are at most two sequences $j_{r(i, l)}, \dots, j_{l+1}$ with the property that the curve $\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_{r(i, l)}, \dots, j_{l+1}}, a_{j_{k_0}, \dots, j_{r(i, l)}, \dots, j_{l+1}+1}])$ intersects $K_{Q'}$, and these sequences are next to each other with respect to the natural order of sequences given by the layer division points they are labelling. We pick up the first one of these sequences. If $c \in [a_{j_{k_0}, \dots, j_{r(i, l)+1}}, a_{j_{k_0}, \dots, j_{r(i, l)+1}+1}]$, by Remark 8.12.(e), there may be two sequences $j_{r(i, l)}, \dots, j_{l+1}$ of this type – one before c and one after c .

Let $\tilde{r}(i, l)$ be such that $L^0(m_1, k_0)_{r(i, l)} = L^i(m_1, k_0)_{\tilde{r}(i, l)}$. For all $j_{k_0}, \dots, j_{r(i, l)+1}$, define a function $\chi_{i, l}^{j_{k_0}, \dots, j_{r(i, l)+1}}$ by setting $\chi_{i, l}^{j_{k_0}, \dots, j_{r(i, l)+1}}(j_{r(i, l)}, \dots, j_{l+1}) = 1$ provided that $j_{r(i, l)}, \dots, j_{l+1}$ is a sequence determined by some i -red cube Q' as above and, otherwise, $\chi_{i, l}^{j_{k_0}, \dots, j_{r(i, l)+1}}(j_{r(i, l)}, \dots, j_{l+1}) = 0$. In particular, if i and l are such that there are no i -red cubes at level $n + L^0(m_1, k_0)_{l-1}$, then $\chi_{i, l}^{j_{k_0}, \dots, j_{r(i, l)+1}} \equiv 0$. Note that the function χ depends on ω but, for simplicity, we suppress it from the notation. Set $\tilde{C} := 5^d \mathfrak{c} 360 \sqrt{d}$ (the factor 360 instead of 15 will be needed at later stages of the proof). Combining the above information with Inequality (9.6) and multiplying the contribution of the red cubes by an extra factor 2, to be utilised in the proof of Lemma 9.6, leads to

$$\begin{aligned}
 & \sum_{\substack{j_l \\ \gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_l}, a_{j_{k_0}, \dots, j_l+1}]) \text{ is white}}} |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_l}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_l+1})| \\
 & \geq (1 - \tilde{C}N^{-m_1 l})(|\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_l+1}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_l+1+1})|) \\
 & \quad - \sum_{i=1}^q \chi_{i,l}^{j_{k_0}, \dots, j_{r(i,l)+1}} 30\sqrt{d}N^{-\Delta_{\bar{r}(i,l)}^i} N^{-n-L^i(m_1, k_0)_{\bar{r}(i,l)}}.
 \end{aligned} \tag{9.7}$$

By Definition 7.1, the level $n + l_0$, corresponding to the sum over j_1 , is such that there may be i -red cubes for all $i = 1, \dots, i_0 + 2$ and $\chi_{i,l}^{j_{k_0}, \dots, j_{r(i,l)+1}} \equiv 0$ for $i > i_0 + 2$. Recalling that only white curves contain white curve segments, we conclude from (9.7) that

$$\begin{aligned}
 & \sum_{j_{k_0}, \dots, j_2} \sum_{\substack{j_1 \\ \gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]) \text{ is white}}} |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_1+1})| \\
 & \geq \sum_{j_{k_0}, \dots, j_3} \sum_{\substack{j_2 \\ \gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_2}, a_{j_{k_0}, \dots, j_2+1}]) \text{ is white}}} (1 - \tilde{C}N^{-m_1})(|\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_2}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_2+1})|) \\
 & \quad - \sum_{i=1}^q \sum_{j_{k_0}, \dots, j_{r(i,1)+1}} 30\sqrt{d}N^{-\Delta_{\bar{r}(i,1)}^i} N^{-n-L^i(m_1, k_0)_{\bar{r}(i,1)}} \\
 & \quad \times \sum_{j_{r(i,1)}, \dots, j_2} \chi_{i,1}^{j_{k_0}, \dots, j_{r(i,1)+1}}(j_{r(i,1)}, \dots, j_2) =: A.
 \end{aligned}$$

While summing over $j_2, \dots, j_{r(1,1)-1}$, we only need to subtract the contribution of blue cubes, since there are 1-red cubes next time at level $n + L^1(m_1, k_0)_{\bar{r}(1,1)} = n + L^0(m_1, k_0)_{r(1,1)}$ corresponding the summing index $j_{r(1,1)+1}$. Further, the contribution of $\chi_{1,1}^{j_{k_0}, \dots, j_{r(1,1)+1}}(j_{r(1,1)}, \dots, j_2)$ cannot be estimated before we reach the summing index $j_{r(1,1)}$. By construction, every i -red cube $Q' \in \mathcal{Q}_{n+L^0(m_1, k_0)_{l-1}}$ defines at most two sequences $j_{r(i,l)}, \dots, j_{l+1}$ such that $\chi_{i,l}^{j_{k_0}, \dots, j_{r(i,l)+1}}(j_{r(i,l)}, \dots, j_{l+1}) \neq 0$. By (9.3) with l replaced by $r(i, l)$, there are at most 5^d cubes which are i -red-labelled parents for some Q' related to a fixed $\chi_{i,l}^{j_{k_0}, \dots, j_{r(i,l)+1}}$. Thus, by the second claim of Lemma 9.1 and Inequality (9.5), we have that

$$\begin{aligned}
 & 30\sqrt{d}N^{-\Delta_{\bar{r}(1,1)}^1} N^{-n-L^1(m_1, k_0)_{\bar{r}(1,1)}} \sum_{j_{r(1,1)}, \dots, j_2} \chi_{1,1}^{j_{k_0}, \dots, j_{r(1,1)+1}}(j_{r(1,1)}, \dots, j_2) \\
 & \leq 5^d c60\sqrt{d}N^{-\Delta_{\bar{r}(1,1)}^1} |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1}}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1+1})|.
 \end{aligned} \tag{9.8}$$

Combining the above facts, we obtain that

$$A \geq \sum_{\substack{j_{k_0}, \dots, j_{r(1,1)+1} \\ \gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_{r(1,1)+1}}, a_{j_{k_0}, \dots, j_{r(1,1)+1+1}]) \text{ is white}}} \left[\prod_{r=1}^{r(1,1)} (1 - \tilde{C}N^{-m_1 r}) \right]$$

$$\begin{aligned}
 & \times |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1}}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1+1})| \\
 & - \tilde{C}N^{-\Delta_{\bar{r}(1,1)}^1} |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1}}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1+1})| \Big] \\
 & - \sum_{i=2}^q \sum_{j_{k_0}, \dots, j_{r(i,1)+1}} 30\sqrt{d}N^{-\Delta_{\bar{r}(i,1)}^i} N^{-n-L^i(m_1, k_0)_{\bar{r}(i,1)}} \\
 & \times \sum_{j_{r(i,1)}, \dots, j_2} \chi_{i,1}^{j_{k_0}, \dots, j_{r(i,1)+1}}(j_{r(i,1)}, \dots, j_2) =: B.
 \end{aligned}$$

There exists M_1 , depending only on d and \mathfrak{c} , such that, for all $m_1 \geq M_1$, we have that $\prod_{r=1}^\infty (1 - 2\tilde{C}N^{-m_1 r}) > \frac{1}{2}$ (the factor 2 appearing in front of \tilde{C} is only needed at later stages of the proof), which implies that

$$\begin{aligned}
 & \left(\prod_{r=1}^{r(1,1)} (1 - \tilde{C}N^{-m_1 r}) \right) |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1}}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1+1})| \\
 & - \tilde{C}N^{-\Delta_{\bar{r}(1,1)}^1} |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1}}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1+1})| \\
 & \geq (1 - 2\tilde{C}N^{-\Delta_{\bar{r}(1,1)}^1}) \left(\prod_{r=1}^{r(1,1)} (1 - \tilde{C}N^{-m_1 r}) \right) |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1}}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+1+1})|.
 \end{aligned}$$

Therefore, using Inequality (9.7) with $l = r(1, 1) + 1$ and recalling that $i = 1$ gives the only non-zero contribution to the sum in (9.7), we deduce that

$$\begin{aligned}
 B & \geq \sum_{j_{k_0}, \dots, j_{r(1,1)+2}} (1 - 2\tilde{C}N^{-\Delta_{\bar{r}(1,1)}^1}) \\
 & \gamma_{\bar{q}}([a_{j_{k_0}, \dots, j_{r(1,1)+2}, a_{j_{k_0}, \dots, j_{r(1,1)+2+1}}]) \text{ is white} \\
 & \times \left(\prod_{r=1}^{r(1,1)+1} (1 - \tilde{C}N^{-m_1 r}) \right) |\gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+2}}) - \gamma_{\bar{q}}(a_{j_{k_0}, \dots, j_{r(1,1)+2+1})| \\
 & - \sum_{j_{k_0}, \dots, j_{r(1, r(1,1)+1)+1}} 30\sqrt{d}N^{-\Delta_{\bar{r}(1, r(1,1)+1)}^1} N^{-n-L^1(m_1, k_0)_{\bar{r}(1, r(1,1)+1)}} \\
 & \times \sum_{j_{r(1, r(1,1)+1)}, \dots, j_{r(1,1)+2}} \chi_{1, r(1,1)+1}^{j_{k_0}, \dots, j_{r(1, r(1,1)+1)+1}}(j_{r(1, r(1,1)+1)}, \dots, j_{r(1,1)+2}) \\
 & - \sum_{i=2}^q \sum_{j_{k_0}, \dots, j_{r(i,1)+1}} 30\sqrt{d}N^{-\Delta_{\bar{r}(i,1)}^i} N^{-n-L^i(m_1, k_0)_{\bar{r}(i,1)}} \\
 & \times \sum_{j_{r(i,1)}, \dots, j_2} \chi_{i,1}^{j_{k_0}, \dots, j_{r(i,1)+1}}(j_{r(i,1)}, \dots, j_2) =: D.
 \end{aligned}$$

We proceed by estimating D in a similar manner. When computing the contribution of i -red cubes for $i = 1, \dots, q'$, where $q' \leq q$, we apply the third claim of Lemma 9.1, that is, instead of Inequality (9.8), we have

$$\begin{aligned}
 & 30\sqrt{d} \sum_{i=1}^{q'} \sum_{j_{r(i,l_i)}, \dots, j_{l_i+1}} N^{-\Delta_{\tilde{r}(i,l_i)}^i} N^{-n-L^i(m_1, k_0)_{\tilde{r}(i,l_i)}} \chi_{i,l_i}^{j_{k_0}, \dots, j_{r(i,l_i)+1}}(j_{r(i,l_i)}, \dots, j_{l_i+1}) \\
 & \leq 5^d \cdot 6\mathfrak{c}60\sqrt{d} N^{-\Delta_{\tilde{r}(1,l_1)}^1} |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_{r(1,l_1)+1}}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_{r(1,l_1)+1+1})|,
 \end{aligned}$$

explaining the factor 360 in the definition of \tilde{C} . Proceeding in this way, recalling Remark 8.12.(d) while summing over j_{k_0} and recalling (7.7), we end up with the estimate

$$\begin{aligned}
 D & \geq (1 - CN^{-m_1 k_0})(|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|) \left(\prod_{r=1}^{k_0} (1 - \tilde{C}N^{-m_1 r}) \right) \\
 & \quad \times \prod_{s=k_0+1}^{k_1} (1 - 2\tilde{C}N^{-\Delta_s^1}) \\
 & \geq (1 - \tilde{C}_1 N^{-m_1})(|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|),
 \end{aligned}$$

where \tilde{C}_1 depends only on d and \mathfrak{c} .

The proof of the second claim is similar. The role of blue cubes is taken by 1-red cubes and the last claim of Lemma 9.1 is utilised. Instead of the factor

$$\left(\prod_{r=1}^{k_0} (1 - \tilde{C}N^{-m_1 r}) \right) \prod_{s=k_0+1}^{k_1} (1 - 2\tilde{C}N^{-\Delta_s^1})$$

the computation results, by using (7.7) for $i = 2$, in the factor

$$\left(\prod_{r=k_0+1}^{k_1} (1 - \tilde{C}N^{-m_1 r}) \right) \prod_{s=k_1+1}^{k_2} (1 - 2\tilde{C}N^{-\Delta_s^2})$$

leading to a constant different from \tilde{C}_1 , denoted by C_1 . \square

Remark 9.4. (a) By construction (recall Remark 8.12.(a)), we have that

$$|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)| \leq 10\sqrt{d}N^{-n} \text{ and } |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1+1})| \geq N^{-l_0-n}$$

for all (j_{k_0}, \dots, j_1) . Therefore, one may choose a subcollection $J \subset \{(j_{k_0}, \dots, j_1) \mid j_l = 0, \dots, p_l + 1, l = k_0, \dots, 1\}$ with $\#J \leq \lfloor 10\sqrt{d}N^{l_0} \rfloor + 1 < C_0 N^{l_0}$, where $C_0 := 11\sqrt{d}$, such that Proposition 9.3 is valid when the sum is restricted to the indices in J .

(b) If $q > i_0 + 2$ in Proposition 9.3, there may be curve segments which are primed with i -red primer but not painted with i -red paint for $i > i_0 + 2$. These curve segments play no role in Proposition 9.3 (recall Remark 8.12.(b)). However, in the proof of Proposition 9.3 we used Lemma 9.1, which takes into account also the contribution of these i -red segments. We will use this fact in the proof of Proposition 6.12 later.

Next we estimate the increase rate of the length of the broken line approximation provided the curve stays close to the fractal percolation set $E(\omega)$. Next lemma is a key observation guaranteeing length gain.

Lemma 9.5. *Fix $\omega \in \Omega$ and $m_0 \in \mathbb{N} \setminus \{0\}$. Let $n \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$. Suppose that Q' is m_0 -good for all $Q' \in \mathcal{Q}_n(K_Q)$. Let $\gamma: [a, b] \rightarrow K_Q$ be a curve passing through an (n, i) -layer for some $i \in \{1, \dots, d\}$. Suppose further that there are no points $\tilde{a}, \tilde{b} \in [a, b]$ such that $\gamma([\tilde{a}, \tilde{b}]) \cap E(\omega) = \emptyset$ and $|\gamma(\tilde{a}) - \gamma(\tilde{b})| \geq d^{-1}N^{-m_0-n}$. Then there is $c \in]a, b[$ such that*

$$|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)| \geq (1 + C_2N^{-2m_0})|\gamma(a) - \gamma(b)|, \tag{9.9}$$

where C_2 depends only on d .

Proof. Observe that $|\gamma(a) - \gamma(b)| \leq 5\sqrt{d}N^{-n}$. If γ passes through an (n, j) -layer for several $j \in \{1, \dots, d\}$, we consider the index j which maximises the length of $\Pi_j(L(\gamma(a), \gamma(b)))$. By the contrapositive form of Lemma 4.13, there exists $c \in]a, b[$ such that

$$\text{dist}(\gamma(c), L(\gamma(a), \gamma(b))) \geq \frac{1}{2\sqrt{2}}N^{-n-m_0} \geq (10\sqrt{2d})^{-1}N^{-m_0}|\gamma(a) - \gamma(b)|.$$

Under this condition, the minimum of $|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|$ is attained when $\gamma(c)$ is in the hyperplane consisting of the points which are equally far away from $\gamma(a)$ and $\gamma(b)$. Combining this with the fact that $\sqrt{1+x} \geq 1 + \frac{x}{2\sqrt{2}}$ for $0 < x < 1$ leads to

$$\begin{aligned} |\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)| &\geq 2\sqrt{\left(\frac{1}{2}\right)^2 + (10\sqrt{2d})^{-2}N^{-2m_0}}|\gamma(a) - \gamma(b)| \\ &\geq (1 + C_2N^{-2m_0})|\gamma(a) - \gamma(b)|, \end{aligned}$$

where $C_2 := (100\sqrt{2d})^{-1}$. \square

Lemma 9.5 guarantees that m_0 -good cubes produce a relative length gain of order N^{-2m_0} provided a curve has no gaps of relative order N^{-m_0} in $E(\omega)$ (see Definition 6.10). According to the next lemma, a length gain of the same relative order is also produced by $(\mathbf{L}^0(m_1, k_0), \mathbf{c}, m_0)$ -good cubes. Note that in Lemma 9.5 the diameter of the curve is of order N^{-n} and the gap size is N^{-n-m_0} . In the next lemma, the diameter of the curve is still of order N^{-n} but the gap size is $N^{-n-l_0-m_0}$. The reason why we nevertheless obtain the length gain of relative order N^{-2m_0} is that the painted curve $\gamma_{\tilde{q}}$ has at least N^{l_0} white curve segments with diameter of order N^{-n-l_0} , each of them producing a length gain $N^{-2m_0-n-l_0}$ by Lemma 9.5. The point $c \in]a, b[$ in the next lemma is needed later in the proof of Proposition 6.12 when we iterate our construction.

Lemma 9.6. Fix $\omega \in \Omega$ and $m_0, \mathbf{c} \in \mathbb{N} \setminus \{0\}$. Let $k_0, m_1 \in \mathbb{N} \setminus \{0\}$ be such that $C_1 N^{-m_1} \leq \frac{1}{4}$, Proposition 9.3 is valid and $C_1 N^{-m_1 k_0} \leq \frac{1}{4} C_2 N^{-2m_0}$, where C_1 is as in Proposition 9.3 and C_2 as in Lemma 9.5. Let $q, n \in \mathbb{N}$ and $Q \in \mathcal{Q}_n$. Assume that every $Q' \in \mathcal{Q}_n(K_Q)$ is $(q, m_1, k_0, \mathbf{c}, m_0)$ -hereditarily good. Further, suppose that $\gamma: [a, b] \rightarrow K_Q$ is a curve passing through an (n, i) -layer for some $i \in \{1, \dots, d\}$. Fix $c \in]a, b[$. Assume that there are no points $\tilde{a}, \tilde{b} \in [a, b]$ such that $\gamma([\tilde{a}, \tilde{b}]) \cap E(\omega) = \emptyset$ and $|\gamma(\tilde{a}) - \gamma(\tilde{b})| \geq (4d)^{-1} N^{-m_0 - l_0 - n}$. Let $\gamma_{\tilde{q}}$ be the painted curve obtained when applying Construction 8.10 to γ . Then there exist $C_3 > 0$, depending only on d , and a sequence of points $a \leq b_1 < d_1 \leq \dots \leq b_{2M} < d_{2M} \leq b$ such that

$$\sum_{\substack{j=1 \\ \gamma_{\tilde{q}}([b_j, d_j]) \text{ is not red}}}^{2M} |\gamma_{\tilde{q}}(b_j) - \gamma_{\tilde{q}}(d_j)| \geq (1 + C_3 N^{-2m_0})(|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|), \tag{9.10}$$

where $M < C_0 N^{l_0}$.

Proof. Let $a_{j_{k_0}, \dots, j_1}$ be the layer division points defined in Construction 8.10. If $q \leq i_0 + 2$, there are no segments in $\gamma_{\tilde{q}}$ which are primed but not painted (recall Remark 8.12.(b)). Therefore, if $\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}])$ is white, then the curve $\gamma_{\tilde{q}}: [a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}] \rightarrow K_{Q_{j_{k_0}, \dots, j_1}}$ satisfies the assumptions of Lemma 9.5, since $\gamma_{\tilde{q}}|_{[a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]} = \gamma|_{[a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]}$. Let $c_{j_{k_0}, \dots, j_1} \in [a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]$ be the point given by Lemma 9.5. We get

$$\begin{aligned} & |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(c_{j_{k_0}, \dots, j_1})| + |\gamma_{\tilde{q}}(c_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1+1})| \\ & \geq (1 + C_2 N^{-2m_0}) |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1+1})|. \end{aligned} \tag{9.11}$$

If $q > i_0 + 2$, $\gamma_{\tilde{q}}$ and γ may differ even on white segments if they contain parts which are primed with i -red primer but not painted with i -red paint for some $i > i_0 + 2$. In this case, $\gamma_{\tilde{q}}$ contains a line segment included in $K_{Q'}$ for some i -red cube Q' (recall Algorithm 8.4). If the sum of the side lengths of all such cubes $K_{Q'}$ is at least $(2d)^{-1} N^{-m_0 - l_0 - n}$, we can use the extra factor 2 introduced in (9.7) to obtain the length gain (9.11), recalling that $C_2 N^{-2m_0} < (2d)^{-1} N^{-m_0}$. If the above sum is less than $(2d)^{-1} N^{-m_0 - l_0 - n}$ and if the conclusion of Lemma 9.5 is not valid for $\gamma_{\tilde{q}}|_{[a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]}$, there is $j \in \{1, \dots, d\}$ and a j -layer \mathcal{L} of width $d^{-1} N^{-m_0 - l_0 - n}$ such that $\gamma_{\tilde{q}}|_{[a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]}$ passes through \mathcal{L} without intersecting $E(\omega)$ (recall the proof of Lemma 4.13). Since the total length of the modified part of $\gamma_{\tilde{q}}|_{[a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]}$ is less than $(2d)^{-1} N^{-m_0 - l_0 - n}$, there is $t \in [a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]$ such that $\gamma_{\tilde{q}}(t) = \gamma(t)$ and $\text{dist}(\gamma(t), \mathcal{L}^c) \geq (4d)^{-1} N^{-m_0 - l_0 - n}$, leading to a contradiction with assumptions on γ . Therefore, we obtain (9.11) in all cases.

If $\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}])$ is blue, we will not have any length gain but, for notational simplicity, we set $c_{j_{k_0}, \dots, j_1} := \frac{1}{2}(a_{j_{k_0}, \dots, j_1} + a_{j_{k_0}, \dots, j_1+1})$. By Remark 9.4, we may choose a set J of indices (j_{k_0}, \dots, j_1) such that $\#J < C_0 N^{l_0}$ and Proposition 9.3 remains valid when summing over J . Define points b_i and d_i by setting

$$\begin{aligned} \{b_{2p-1} \mid p = 1, \dots, M\} &= \{a_{j_{k_0}, \dots, j_1} \mid (j_{k_0}, \dots, j_1) \in J\}, \\ \{d_{2p-1} \mid p = 1, \dots, M\} &= \{c_{j_{k_0}, \dots, j_1} \mid (j_{k_0}, \dots, j_1) \in J\} = \{b_{2p} \mid p = 1, \dots, M\} \text{ and} \\ \{d_{2p} \mid p = 1, \dots, M\} &= \{a_{j_{k_0}, \dots, j_1+1} \mid (j_{k_0}, \dots, j_1) \in J\}. \end{aligned}$$

Using (9.11), Proposition 9.3 and the choices of m_1 and k_0 , we obtain that

$$\begin{aligned} &\sum_{\substack{j=1 \\ \gamma_{\tilde{q}}([b_j, d_j]) \text{ is not red}}}^{2M} |\gamma_{\tilde{q}}(b_j) - \gamma_{\tilde{q}}(d_j)| \\ &= \sum_{\substack{(j_{k_0}, \dots, j_1) \in J \\ \gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]) \text{ is white}}} (|\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(c_{j_{k_0}, \dots, j_1})| \\ &\quad + |\gamma_{\tilde{q}}(c_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1+1})|) \\ &+ \sum_{\substack{(j_{k_0}, \dots, j_1) \in J \\ \gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]) \text{ is blue}}} (|\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(c_{j_{k_0}, \dots, j_1})| \\ &\quad + |\gamma_{\tilde{q}}(c_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1+1})|) \\ &\geq \sum_{\substack{(j_{k_0}, \dots, j_1) \in J \\ \gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]) \text{ is white}}} (1 + C_2 N^{-2m_0}) |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1+1})| \\ &+ \sum_{\substack{(j_{k_0}, \dots, j_1) \in J \\ \gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]) \text{ is blue}}} |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1+1})| \\ &\geq \sum_{\substack{(j_{k_0}, \dots, j_1) \in J \\ \gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}]) \text{ is not red}}} |\gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1}) - \gamma_{\tilde{q}}(a_{j_{k_0}, \dots, j_1+1})| \\ &+ C_2 N^{-2m_0} (1 - C_1 N^{-m_1}) (|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|) \\ &\geq (1 - C_1 N^{-m_1 k_0} + \frac{3}{4} C_2 N^{-2m_0}) (|\gamma(a) - \gamma(c)| + |\gamma(c) - \gamma(b)|), \tag{9.12} \end{aligned}$$

which gives the claim with $C_3 := \frac{1}{2} C_2$. \square

Next we utilise Lemma 9.6 iteratively to prove Proposition 6.12.

Proof of Proposition 6.12. Apply Construction 8.10 to γ without fixing any $c \in]a, b[$. Choosing $h = 0$ in the assumptions, we see that the assumptions of Lemma 9.6 are satisfied. Applying Lemma 9.6 to γ , we obtain points \tilde{b}_j and \tilde{d}_j , $j = 1, \dots, \tilde{M}$, with $\tilde{M} < C_0 N^{l_0}$. For every $p \in \{1, \dots, \tilde{M}\}$, there is (j_{k_0}, \dots, j_1) such that $\tilde{b}_{2p-1} = a_{j_{k_0}, \dots, j_1}$, $\tilde{d}_{2p} = a_{j_{k_0}, \dots, j_1+1}$, $\tilde{d}_{2p-1} = \tilde{b}_{2p} = c_{j_{k_0}, \dots, j_1} \in]a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}[$ and

$\gamma_{\tilde{q}}([a_{j_{k_0}, \dots, j_1}, a_{j_{k_0}, \dots, j_1+1}])$ is either white or blue. By Remark 8.12.(a), the conditions of Construction 8.10 are valid for $\gamma_{\tilde{q}}|_{[\tilde{b}_{2p-1}, \tilde{d}_{2p}]}$ with q replaced by $q-1$. If $\gamma_{\tilde{q}}([\tilde{b}_{2p-1}, \tilde{d}_{2p}])$ is white, apply Construction 8.10 to it with $c = \tilde{b}_{2p}$ recalling that, if $q > i_0 + 2$, a part of $\gamma_{\tilde{q}}([\tilde{b}_{2p-1}, \tilde{d}_{2p}])$ may already be primed with an i -red primer for $i > i_0$ (recall Remark 8.12.(b)). If $\gamma_{\tilde{q}}([\tilde{b}_{2p-1}, \tilde{d}_{2p}])$ is blue, apply Construction 8.10 to it without fixing any c and interpreting that it is white, that is, the blue paint of $\gamma_{\tilde{q}}([\tilde{b}_{2p-1}, \tilde{d}_{2p}])$ is not inherited by its subcurves. Choosing $h = 1$ in the assumptions, we see that $\gamma|_{[\tilde{b}_{2p-1}, \tilde{d}_{2p}]}$ satisfies the assumptions of Lemma 9.6 for all $p \in \{1, \dots, \widetilde{M}\}$. By means of Lemma 9.6, we find new points \tilde{b}_j and \tilde{d}_j as above. Repeat this process $q+1$ times until the level $n + L^q(m_1, k_0)_0$ is reached and the final points $a \leq b_1 < d_1 \leq \dots \leq b_{2M} < d_{2M} \leq b$ are defined. Note that at this level all primed parts are painted which implies that $\gamma_{\tilde{q}}$ and γ agree on white and blue segments. Therefore, $\gamma_{\tilde{q}}(b_i) = \gamma(b_i)$ and $\gamma_{\tilde{q}}(d_i) = \gamma(d_i)$ for all $i = 1, \dots, 2M$. By Lemma 9.6, at every step the number of chosen subcurves is less than $C_0 N^{l_0}$, so $M < (C_0 N^{l_0})^{q+1}$. Recalling Remark 9.4.(b), the claim follows by using recursively the estimate given by Lemma 9.6 starting from the lowest level $n + L^q(m_1, k_0)_0$ and finishing at level n . \square

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