

On the extreme non-Arens regularity of Banach algebras

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ABSTRACT

As is well-known, on an Arens regular Banach algebra all continuous functionals are weakly almost periodic. In this paper, we show that ℓ^1 -bases which approximate upper and lower triangles of products of elements in the algebra produce large sets of functionals that are not weakly almost periodic. This leads to criteria for extreme non-Arens regularity of Banach algebras in the sense of Granirer. We find in particular that bounded approximate identities (bai's) and bounded nets converging to invariance (TI-nets) both fall into this approach, suggesting that this is indeed the main tool behind most known constructions of non-Arens regular algebras.

These criteria can be applied to the main algebras in harmonic analysis such as the group algebra, the measure algebra, the semigroup algebra (with certain weights) and the Fourier algebra. In this paper, we apply our criteria to the Lebesgue-Fourier algebra, the 1-Segal Fourier algebra and the Figà-Talamanca Herz algebra.

1. Introduction

In [2], Richard Arens extended the product of a Banach algebra \mathcal{A} to its second dual \mathcal{A}^{**} in two different, but completely symmetric, ways, both making \mathcal{A}^{**} into a Banach algebra. The way these products are defined makes them one-sided $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ -continuous, but each of them on a different side. As it turned out, on some algebras these extensions yield the same product but on others two genuinely different multiplications are obtained. Banach algebras thus could be divided in those with a single Arens product, called *Arens regular*, and those with two different Arens products, called *non-Arens regular* or *Arens irregular*. All C^* -algebras can be counted among the former (in the bidual of a C^* -algebra both Arens products coincide with the operator multiplication of the enveloping von Neumann algebra, see [7, Theorem 7.1]). But Arens himself proved in [2] that the convolution semigroup algebra ℓ^1 is not Arens regular, and neither is the group algebra $L^1(G)$ for any infinite G , as proved by Young in [41] about twenty years later. With important examples to be found on either side, it is clear why the problem of finding conditions for these two products to be the same has attracted the attention of many researchers ever since.

The crucial fact, when dealing with Arens regularity, is that all the Banach algebras \mathcal{A} have in common the space $\mathcal{WAP}(\mathcal{A})$ of weakly almost periodic functionals. Due to Grothendieck's double limit criterion satisfied by these functionals, $\mathcal{WAP}(\mathcal{A})$ is precisely the subspace of \mathcal{A}^* where the two Arens-products agree, see [36]. So when $\mathcal{A}^* = \mathcal{WAP}(\mathcal{A})$, there is only one Arens product, which is therefore separately $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ -continuous.

This explains why it is natural to measure the degree of non-Arens regularity of an algebra \mathcal{A} by the relative size of $\mathcal{WAP}(\mathcal{A})$ with respect to \mathcal{A}^* , that is, the size of the quotient space

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$\mathcal{A}^*/\mathcal{W}\mathcal{A}\mathcal{P}(\mathcal{A})$. The extreme cases appear when this quotient is trivial, that is, the algebra is Arens regular, and when $\mathcal{A}^*/\mathcal{W}\mathcal{A}\mathcal{P}(\mathcal{A})$ is as large as \mathcal{A}^* . The algebras with this latter property were termed extremely non-Arens regular (enAr for short) by Granirer, see [25].

Non-Arens regularity may also be measured by the degree of defect of the separate $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ -continuity of Arens products. In the extreme situation (that is, when separate $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ -continuity of the product in \mathcal{A}^{**} happens only when one of the elements in the product is in \mathcal{A}), Dales and Lau called the algebra \mathcal{A} strongly Arens irregular (sAir for short), see [9]. It may be worthwhile to note that the properties enAr and sAir do not imply each other, examples distinguishing the two concepts can be found in [30]. Natural examples of algebras which are enAr but not sAir, can be obtained from Remark 5.8 and Corollary 5.9 at the end of the paper. In these examples, the Fourier algebra $A(G)$ even satisfies a reinforced version of extreme non-Arens regularity, as the quotient $A(G)^*/\mathcal{W}\mathcal{A}\mathcal{P}(A(G))$ contains an isometric copy of $A(G)^*$, see our forthcoming paper [14].

After a section with preliminaries, our first main theorem (Theorem 3.9) pinpoints quite precisely the natural conditions under which a general Banach algebra is enAr, namely the existence of an ℓ^1 -base that approximates upper and lower triangles. With this theorem, not only most (if not all) of the known cases of non-Arens regular Banach algebras \mathcal{A} are proved to be so, but in many cases the algebra is proved to be enAr. Once an ℓ^1 -base with cardinality η is constructed in \mathcal{A} , a copy of $\ell^\infty(\eta)$ will appear in the quotient space $\mathcal{A}^*/\mathcal{W}\mathcal{A}\mathcal{P}(\mathcal{A})$, and when η equals the density character $d(\mathcal{A})$ of \mathcal{A} , this leads to the algebra being enAr.

These ideas have certainly come up, at least in part, in previous works in special cases. The upper and lower triangles made of elements in discrete groups and semigroups are quite apparent particularly in [3, 8, 39].

The first applications of Theorem 3.9 come in Section 4 where we show how both bounded approximate identities (bai's for short) and bounded nets convergent to invariance (TI-nets) can be used to link the conditions in Theorem 3.9 and the non-Arens regularity of the algebra. It may be worthwhile to note that bai's and TI-nets are opposite to each other in the sense that while one measures non-Arens regularity locally, the other measures it at infinity. In the familiar case of $L^1(G)$, a bai could be used whenever G is non-discrete and a TI-net when G is not compact, while in the Fourier algebra $A(G)$, with G additionally amenable, the roles of bai and TI are reversed.

We next tackle the application of Theorem 3.9 to the algebras of harmonic analysis related to the Fourier algebra. The Fourier algebra $A(G)$, and more generally the Figà-Talamanca Herz algebras $A_p(G)$, and the Segal algebras defined on $L^1(G) \cap A(G)$, with norm $\|f\|_S = \|f\|_{L^1} + \|f\|_{A(G)}$ are studied.

On the linear space $(A(G) \cap L^1(G), \|\cdot\|_S)$, two different Segal algebra structures will be considered. Following [20], the Segal algebra over $A(G)$ that is obtained after equipping $A(G) \cap L^1(G)$ with pointwise product will be denoted by $S^1A(G)$ and called the 1-Segal Fourier algebra. The Segal algebra over $L^1(G)$ obtained when convolution is used as multiplication will be denoted by $LA(G)$ and referred to as the Lebesgue–Fourier algebra. Arens regularity of these Segal algebras was studied by Ghahramani and Lau in [21, 22]. In [22, Theorems 5.1 and 5.2], they proved that $S^1A(G)$ is Arens regular if and only if G is discrete. The necessity of discreteness shall follow immediately from our Theorem 4.6, and will be improved further by showing that the quotient space $S^1A(G)^*/\mathcal{W}\mathcal{A}\mathcal{P}(S^1A(G))$ is at least non-separable when G is non-discrete. In particular, this implies that $S^1A(G)$ is enAr when G is non-discrete and second countable, see Theorem 5.4.

Ghahramani and Lau proved further in [22, Theorem 5.1] that the convolution algebra $LA(G)$ of a unimodular group is Arens regular if and only if G is compact. Again using Theorem 3.9, this theorem shall be strengthened in Theorem 5.5 by disposing of the unimodularity condition on G and by showing that the quotient space $LA(G)^*/\mathcal{W}\mathcal{A}\mathcal{P}(LA(G))$ has $\ell^\infty(\kappa(G))$ as a quotient whenever G is non-compact. It is then immediate that $LA(G)$ is

enAr when in addition G satisfies $\kappa(G) \geq \chi(G)$, where $\kappa(G)$ is the compact covering of G and $\chi(G)$ is its local weight of G .

Concerning the Figà-Talamanca Herz algebra, Forrest proved in [18, Theorem 3.2] that $A_p(G)$ is non-Arens regular when G is not discrete and also when $A_p(G)$ is weakly sequentially complete (this is always true if $p = 2$) and G contains a discrete and amenable open subgroup. With the help of Theorem 3.9, a copy of ℓ^∞ is found in the quotient space $A_p(G)^*/\mathcal{WAP}(A_p(G))$ whenever G is either non-discrete, or discrete with an infinite amenable subgroup (assuming that $A_p(G)$ is weakly sequentially complete in this second case). We deduce in particular that $A_p(G)$ is enAr when G is either a non-discrete second countable group, or a discrete countable group containing an infinite amenable group and $p = 2$.

2. Preliminaries and definitions

Let \mathcal{A} be a Banach algebra and let \mathcal{A}^* and \mathcal{A}^{**} be its first and second Banach duals, respectively. In his paper [2], Arens defined two multiplications in \mathcal{A}^{**} . To define these multiplications, consider first the following actions of \mathcal{A} and \mathcal{A}^{**} on \mathcal{A}^* . For every $\mu, \nu \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$, and $\varphi, \psi \in \mathcal{A}$,

$$\begin{aligned} \langle f\varphi, \psi \rangle &= \langle f, \varphi\psi \rangle & \langle \varphi.f, \psi \rangle &= \langle f, \psi\varphi \rangle & (\text{actions of } \mathcal{A} \text{ on } \mathcal{A}^*) \\ \langle \nu f, \varphi \rangle &= \langle \nu, f\varphi \rangle & \langle f.\mu, \varphi \rangle &= \langle \mu, \varphi.f \rangle & (\text{actions of } \mathcal{A}^{**} \text{ on } \mathcal{A}^*). \end{aligned}$$

The multiplications $\mu\nu$ and $\mu.\nu$ are then defined as

$$\langle \mu\nu, f \rangle = \langle \mu, \nu f \rangle \quad \langle \mu.\nu, f \rangle = \langle \nu, f.\mu \rangle.$$

Arens [2] proved that these products make \mathcal{A}^{**} into a Banach algebra that contains \mathcal{A} as a subalgebra. When these two products coincide, the Banach algebra \mathcal{A} is said to be *Arens regular*.

In [36], Pym considered the space $\mathcal{WAP}(\mathcal{A})$ of weakly almost periodic functionals on \mathcal{A} , this is the space of all elements $f \in \mathcal{A}^*$ such that the left orbit

$$L(f) = \{f\varphi : \varphi \in \mathcal{A}, \|\varphi\| \leq 1\}$$

or, equivalently, the right orbit

$$R(f) = \{\varphi.f : \varphi \in \mathcal{A}, \|\varphi\| \leq 1\}$$

is relatively weakly compact in \mathcal{A}^* . He proved that \mathcal{A} is Arens regular if and only if $\mathcal{WAP}(\mathcal{A}) = \mathcal{A}^*$, that is, when the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ is trivial. For more information, the reader is directed to [9, 16].

To describe the other end, Granirer [25] introduced the concept of extreme non-Arens regularity.

DEFINITION 2.1. A Banach algebra \mathcal{A} is *extremely non-Arens regular* (enAr for short) when $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ contains a closed subspace which has \mathcal{A}^* as a quotient.

So far, the Banach algebras known to be enAr are: the group algebra $L^1(G)$ for any infinite locally compact group [13] (some particular, but important, cases were previously proved in [4, 25]), $\ell^1(S)$ for any infinite discrete cancellative semigroup [17] (the proof in [4] is given for locally compact groups but can also be applied also in this case) and the Fourier algebra $A(G)$ for any locally compact group satisfying $\chi(G) \geq \kappa(G)$, where $\chi(G)$ is the smallest cardinality of an open basis of the identity of G and $\kappa(G)$ is the smallest cardinality of a compact covering of G ([25, 29] when $\chi(G) = \omega$).

To fix our terminology, we need to recall a few elementary facts relating Banach spaces and operators between them. If E_1 and E_2 are Banach spaces, we say that E_1 has E_2 as a quotient if there is a surjective bounded linear map $T: E_1 \rightarrow E_2$. We say that E_2 contains an isomorphic copy of E_1 when there is a linear isomorphism of E_1 into E_2 , that is, when there is a linear mapping $T: E_1 \rightarrow E_2$ such that for some positive constants K_1 and K_2 ,

$$K_1\|x\| \leq \|Tx\| \leq K_2\|x\| \quad \text{for all } x \in E_1.$$

DEFINITION 2.2. $\ell^1(\eta)$ -BASE: Let E be a normed space and η be an infinite cardinal. A bounded set $B = \{a_\alpha : \alpha < \eta\}$ of cardinality η is an $\ell^1(\eta)$ -base in E , with constant $K > 0$, when the inequality

$$\sum_{n=1}^p |z_n| \leq K \left\| \sum_{n=1}^p z_n a_{\alpha_n} \right\|$$

holds for all $p \in \mathbb{N}$ and for every possible choice of scalars z_1, \dots, z_p and elements $a_{\alpha_1}, \dots, a_{\alpha_p}$ in B .

REMARK 2.3. If B is an $\ell^1(\eta)$ -base, then clearly every $\mathbf{c} = (c_\alpha)_{\alpha < \eta} \in \ell^\infty(\eta)$, defines a continuous linear functional $\phi_{\mathbf{c}}: \langle B \rangle \rightarrow \mathbb{C}$. This is obtained simply by letting

$$\left\langle \phi_{\mathbf{c}}, \sum_{n=1}^p z_n a_{\alpha_n} \right\rangle = \sum_{n=1}^p z_n c_{\alpha_n}, \quad \text{for each } \sum_{n=1}^p z_n a_{\alpha_n} \in \langle B \rangle,$$

and noting that

$$\left| \left\langle \phi_{\mathbf{c}}, \sum_{n=1}^p z_n a_{\alpha_n} \right\rangle \right| = \left| \sum_{n=1}^p z_n c_{\alpha_n} \right| \leq \|\mathbf{c}\|_\infty \sum_{n=1}^p |z_n| \leq \|\mathbf{c}\|_\infty K \left\| \sum_{n=1}^p z_n a_{\alpha_n} \right\|.$$

Since this linear functional is bounded on $\langle B \rangle$, it may be extended to an element of E^* by Hahn–Banach Theorem.

Note that $\|a_\alpha\| \geq \frac{1}{K} > 0$ for every element $a_\alpha \in B$.

We finally record a standard fact from [29]. Recall that the density character of a normed space \mathcal{A} , denoted by $d(\mathcal{A})$, is the cardinality of the smallest norm-dense subset of \mathcal{A} .

LEMMA 2.4. If E is a normed space with density character $d(E) = \eta$, then there is a linear isometry of E^* into $\ell^\infty(\eta)$.

Proof. To see this, let $\{x_\alpha : \alpha < \eta\}$ be a norm-dense subset in the unit ball of E . Define for each $\psi \in E^*$, the function v_ψ in $\ell^\infty(\eta)$ by $v_\psi(\alpha) = \langle \psi, x_\alpha \rangle$. Then

$$\mathcal{I}: E^* \rightarrow \ell^\infty(\eta), \quad \mathcal{I}(\psi) = v_\psi,$$

is a linear isometry of \mathcal{A}^* into $\ell^\infty(\eta)$. □

COROLLARY 2.5. Let \mathcal{A} be a Banach algebra and η be an infinite cardinal number. If $\mathcal{A}^*/\mathcal{W}\mathcal{A}\mathcal{P}(\mathcal{A})$ has $\ell^\infty(\eta)$ as a quotient and $\ell^\infty(\eta)$ contains an isomorphic copy of \mathcal{A}^* (for instance, when $\eta = d(\mathcal{A})$), then \mathcal{A} is enAr.

3. *Triangles and weakly almost periodic functionals*

Our objective in this section is to set natural conditions under which a general Banach algebra is enAr. The concept of *triangle* in a directed set, adapted from [6, Definition 2.1], will be essential in this process.

By a directed set, it is always meant a set Λ together with a preorder \preceq with the additional property that every pair of elements has an upper bound. We will use a single letter, Λ usually, to denote a directed set, the existence of \preceq is implicitly assumed.

DEFINITION 3.1. Let (Λ, \preceq) be a directed set and let Λ_1, Λ_2 be two cofinal subsets of Λ . If \mathcal{U} is a subset of $\Lambda_1 \times \Lambda_2$, we say that:

- (i) \mathcal{U} is *vertically cofinal*, when for every $\alpha \in \Lambda_1$, there exists $\beta(\alpha) \in \Lambda_2$ such that $(\alpha, \beta) \in \mathcal{U}$ for every $\beta \in \Lambda_2, \beta \succeq \beta(\alpha)$;
- (ii) \mathcal{U} is *horizontally cofinal*, when for every $\beta \in \Lambda_2$, there exists $\alpha(\beta) \in \Lambda_1$ such that $(\alpha, \beta) \in \mathcal{U}$ for every $\alpha \in \Lambda_1, \alpha \succeq \alpha(\beta)$.

DEFINITION 3.2. Let \mathcal{U} and X be two sets. We say that:

- (i) X is *indexed* by \mathcal{U} , when there exists a surjective map $x : \mathcal{U} \rightarrow X$. When $\mathcal{U} \subset \Lambda \times \Lambda$, for some other set Λ , we say that X is *double-indexed* by \mathcal{U} and write $X = \{x_{\alpha\beta} : (\alpha, \beta) \in \mathcal{U}\}$, where $x_{\alpha\beta} = x(\alpha, \beta)$;
- (ii) if X is double-indexed by \mathcal{U} , we say it is *vertically injective* if $x_{\alpha\beta} = x_{\alpha'\beta'}$ implies $\beta = \beta'$ for every $(\alpha, \beta) \in \mathcal{U}$. If $x_{\alpha\beta} = x_{\alpha'\beta'}$ implies $\alpha = \alpha'$ for every $(\alpha, \beta) \in \mathcal{U}$, we say that X is *horizontally injective*.

DEFINITION 3.3. Let \mathcal{A} be a Banach algebra, (Λ, \preceq) be a directed set and Λ_1, Λ_2 be two cofinal subsets of Λ . Consider two subsets, A and B , of \mathcal{A} indexed, respectively, by Λ_1 and Λ_2 , that is,

$$A = \{a_\alpha : \alpha \in \Lambda_1\} \quad \text{and} \quad B = \{b_\alpha : \alpha \in \Lambda_2\}.$$

- (i) The sets

$$T_{AB}^u = \{a_\alpha b_\beta : (\alpha, \beta) \in \Lambda_1 \times \Lambda_2, \alpha \prec \beta\} \quad \text{and}$$

$$T_{AB}^l = \{a_\alpha b_\beta : (\alpha, \beta) \in \Lambda_1 \times \Lambda_2, \beta \prec \alpha\}$$

are called, respectively, the *upper* and *lower triangles defined by A and B*.

- (ii) A set $X \subseteq \mathcal{A}$ is said to *approximate segments in T_{AB}^u* , if there exists a vertically cofinal set \mathcal{U} in $\Lambda_1 \times \Lambda_2$ so that X is double-indexed as $X = \{x_{\alpha\beta} : (\alpha, \beta) \in \mathcal{U}\}$, and for each $\alpha \in \Lambda_1$,

$$\lim_{\beta \succeq \beta(\alpha)} \|x_{\alpha\beta} - a_\alpha b_\beta\| = 0.$$

Note that, by considering an appropriate subset of X we can assume that $(\alpha, \beta) \in \mathcal{U}$ implies $\beta \succ \alpha$.

- (iii) A set $X \subseteq \mathcal{A}$ is said to *approximate segments in T_{AB}^l* , if there exists a horizontally cofinal set \mathcal{U} in $\Lambda_1 \times \Lambda_2$ so that X is double-indexed as $X = \{x_{\alpha\beta} : (\alpha, \beta) \in \mathcal{U}\}$, and for each $\beta \in \Lambda_2$,

$$\lim_{\alpha \succeq \alpha(\beta)} \|x_{\alpha\beta} - a_\alpha b_\beta\| = 0.$$

As before, we can assume here that $(\alpha, \beta) \in \mathcal{U}$ implies $\alpha \succ \beta$.

In Theorem 3.9, the key theorem of the paper, we need to be able to partition a directed set of cardinality η into as many cofinal subsets. This is easily done when the cardinality is countable but may not be possible for general directed sets. We will use the following definition and theorem, due to van Douwen to delimitate the pathological situations that will not be pertinent to our applications.

DEFINITION 3.4. Let Λ be a directed set and for every $\xi \in \Lambda$, define $\xi^+ = \{\alpha \in \Lambda : \xi \prec \alpha\}$. We define the *true cardinality* of Λ as $\text{tr}|\Lambda| = \min_{\xi \in \Lambda} |\xi^+|$.

LEMMA 3.5 (Lemma of [12]). If Λ is a directed set, then Λ admits a pairwise disjoint collection of $\text{tr}|\Lambda|$ -many cofinal subsets of Λ each having true cardinality $|\Lambda|$.

REMARKS 3.6. In [12], Lemma 3.5 only states that each of the disjoint cofinal sets have cardinality $|\Lambda|$ but its proof also shows that they actually have true cardinality $|\Lambda|$.

The case when the set Λ is countable does not need the generality of van Douwen’s lemma. One only needs to know that in this case the set may be partitioned into infinitely many infinite sets. This will be the case in Theorems 4.4, 5.4, 5.6 and 5.7.

In our application to the algebras in harmonic analysis, our set Λ will be the initial ordinal associated to the cardinal $\chi(G)$, the smallest cardinality of an open basis of the identity e of G , or $\kappa(G)$, the smallest cardinality of a compact covering of G , see, for example, Theorem 5.5. In both situations, $|\xi^+| = \Lambda$ for each $\xi \in \Lambda$ and so $\text{tr}|\Lambda| = |\Lambda|$.

This way of partitioning has also come up in some of our previous papers (see, for example, [4, 15]). In these works, the directed set Λ consisted of a compact cover $\{K_\alpha : \alpha < \kappa(G)\}$ of G with cardinality $\kappa(G)$ or of a neighbourhood base at the identity $\{U_\alpha : \alpha < \chi(G)\}$ with cardinality $\chi(G)$, both directed by set inclusion with $K_\alpha \preceq K_\beta$ if and only if $K_\alpha \subseteq K_\beta$, and $U_\alpha \preceq U_\beta$ if and only if $U_\alpha \supseteq U_\beta$. In each case, the minimality of $\chi(G)$ and $\kappa(G)$ proves that $\text{tr}|\Lambda| = \Lambda$.

Before proving our main theorem, we recall a definition and a fact obtained in [13] that help to simplify the remaining proofs.

DEFINITION 3.7. Let $T: E_1 \rightarrow E_2$ be a linear map between Banach spaces. If F is a closed subspace of E_2 , we say that T is preserved by F when there is $c > 0$ such that the following property holds:

$$\|T\xi - \phi\| \geq c\|\xi\|, \quad \text{for all } \phi \in F \text{ and } \xi \in E_1.$$

The following elementary fact is proved in [13, Lemma 2.2] for isometries. Its proof is easily adapted to this case.

LEMMA 3.8. Let $T: E_1 \rightarrow E_2$ be a linear isomorphism of the Banach spaces E_1 into E_2 and let D, F be closed linear subspaces of E_2 with $D \subseteq F$. Denote by $Q: E_2 \rightarrow E_2/D$ the quotient map. If T is preserved by F , then the map $Q \circ T: E_1 \rightarrow E_2/D$ is a linear isomorphism.

THEOREM 3.9. Let \mathcal{A} be a Banach algebra and η an infinite cardinal number. Suppose that \mathcal{A} contains two bounded subsets A and B indexed, respectively, by two cofinal subsets Λ_1 and Λ_2 of a common directed set (Λ, \preceq) with $\text{tr}|\Lambda_1| = \text{tr}|\Lambda_2| = \eta$. Suppose as well that \mathcal{A} contains two other disjoint sets X_1 and X_2 with the following properties.

- (i) $X = X_1 \cup X_2$ is an $\ell^1(\eta)$ -base in \mathcal{A} with constant $K > 0$ contained in the ball of radius $M > 0$.
- (ii) X_1 and X_2 approximate segments in T_{AB}^u and T_{AB}^l , respectively.
- (iii) Either X_1 is vertically injective and X_2 is horizontally injective or vice versa.

Then there is a linear isomorphism $\mathcal{J}: \ell^\infty(\eta) \rightarrow \frac{\langle X \rangle^*}{\mathcal{WAP}(\mathcal{A})|_{\langle X \rangle}}$. In particular, \mathcal{A} is non-Arens regular.

Proof. Put $A = \{a_\alpha: \alpha \in \Lambda_1\}$ and $B = \{b_\beta: \beta \in \Lambda_2\}$. Let in addition

$$X_1 = \{x_{\alpha\beta}: (\alpha, \beta) \in \mathcal{U}_1\} \text{ and } X_2 = \{x_{\alpha\beta}: (\alpha, \beta) \in \mathcal{U}_2\}$$

be the enumerations of X_1 and X_2 satisfying, respectively, Conditions (ii) and (iii) in Definition 3.3, where $\mathcal{U}_1, \mathcal{U}_2$ are, respectively, vertically cofinal and horizontally cofinal subsets of $\Lambda_1 \times \Lambda_2$ such that $(\alpha, \beta) \in \mathcal{U}_1$ implies $\beta \succ \alpha$ and $(\alpha, \beta) \in \mathcal{U}_2$ implies $\alpha \succ \beta$. We assume that X_1 is vertically injective and X_2 is horizontally injective. The proof in the other case proceeds with straightforward modifications.

We first use Lemma 3.5 to partition $\Lambda_j = \bigcup_{\lambda < \eta} I_{\lambda,j}$, $j = 1, 2$ in η -many cofinal subsets, with $\text{tr}|I_{\lambda,j}| = \eta$, for every $\lambda < \eta$ and $j = 1, 2$.

Now, for each $\mathbf{c} \in \ell^\infty(\eta)$, we define $\psi_{\mathbf{c}}: X_1 \cup X_2 \rightarrow \mathbb{C}$ by

$$\langle \psi_{\mathbf{c}}, x_{\alpha\beta} \rangle = \begin{cases} \mathbf{c}(\lambda), & \text{if } (\alpha, \beta) \in \mathcal{U}_1 \text{ and } \beta \in I_{\lambda,2}, \\ -\mathbf{c}(\lambda), & \text{if } (\alpha, \beta) \in \mathcal{U}_2 \text{ and } \alpha \in I_{\lambda,1}. \end{cases}$$

We first observe that $\psi_{\mathbf{c}}$ is a well-defined bounded function. It could happen that $x_{\alpha\beta} = x_{\alpha'\beta'}$ but, in case $x_{\alpha\beta} \in X_1$, vertical injectivity would then imply that $\beta = \beta'$ and thus that $\psi_{\mathbf{c}}(x_{\alpha\beta}) = \psi_{\mathbf{c}}(x_{\alpha'\beta'})$. The same argument applies when $x_{\alpha\beta} \in X_2$.

Since $X_1 \cup X_2$ is an $\ell^1(\eta)$ -base, this map can be extended by linearity to a bounded linear functional $\psi_{\mathbf{c}} \in \langle X \rangle^*$.

We next check that the linear map $\mathcal{I}: \ell^\infty(\eta) \rightarrow \langle X \rangle^*$ given by $\mathcal{I}(\mathbf{c}) = \psi_{\mathbf{c}}$ is a linear isomorphism preserved (in the sense of Definition 3.7) by $\mathcal{WAP}(\mathcal{A})|_{\langle X \rangle}$. We first observe that

$$\|\mathcal{I}(\mathbf{c})\|_{\langle X \rangle^*} \leq K \|\mathbf{c}\|_\infty. \tag{3.1}$$

Let now $\mathbf{c} \in \ell^\infty(\eta)$ and $f \in \mathcal{WAP}(\mathcal{A})$ be given.

Fix $\lambda < \eta$ and $\varepsilon > 0$.

Since A and B are bounded, we may find (using that $f \in \mathcal{WAP}(\mathcal{A})$ and that bounded sets of \mathcal{A} are relatively weak*-compact in \mathcal{A}^{**}) directed sets Λ'_1 and Λ'_2 and monotonic maps $\xi_i: \Lambda'_i \rightarrow I_{\lambda,i}$ with $\xi_i(\Lambda'_i)$ cofinal in $I_{\lambda,i}$ for $i = 1, 2$, such that

$$\lim_{\alpha \in \Lambda'_1} \lim_{\beta \in \Lambda'_2} f(a_{\xi_1(\alpha)} b_{\xi_2(\beta)}) = \lim_{\beta \in \Lambda'_2} \lim_{\alpha \in \Lambda'_1} f(a_{\xi_1(\alpha)} b_{\xi_2(\beta)}). \tag{3.2}$$

Since X_2 approximates T_{AB}^l , we have that, for each $\beta \in \Lambda'_2$, $\lim_{\gamma \in \Lambda_1} \|x_{\gamma\xi_2(\beta)} - a_\gamma b_{\xi_2(\beta)}\| = 0$. Using that ξ_1 is monotonic and that $\xi_1(\Lambda'_1)$ is cofinal in $I_{\lambda,1}$ (and hence in Λ_1), we can find $\alpha_\beta \in \Lambda'_1$ such that, whenever $\alpha \in \Lambda'_1$, $\alpha \succeq \alpha_\beta$

$$(\xi_1(\alpha), \xi_2(\beta)) \in \mathcal{U}_2$$

$$\|x_{\xi_1(\alpha)\xi_2(\beta)} - a_{\xi_1(\alpha)} b_{\xi_2(\beta)}\| < \varepsilon \quad \text{and} \tag{3.3}$$

$$\left| f(a_{\xi_1(\alpha)} b_{\xi_2(\beta)}) - \lim_{\alpha \in \Lambda'_1} f(a_{\xi_1(\alpha)} b_{\xi_2(\beta)}) \right| < \varepsilon. \tag{3.4}$$

We can find in the same way for every $\alpha \in \Lambda'_1$, $\beta_\alpha \in \Lambda'_2$ such that, whenever $\beta \in \Lambda'_2$, $\beta \succeq \beta_\alpha$:

$$(\xi_1(\alpha), \xi_2(\beta)) \in \mathcal{U}_1$$

$$\|x_{\xi_1(\alpha)\xi_2(\beta)} - a_{\xi_1(\alpha)} b_{\xi_2(\beta)}\| < \varepsilon \quad \text{and} \tag{3.5}$$

$$\left| f(a_{\xi_1(\alpha)} b_{\xi_2(\beta)}) - \lim_{\beta \in \Lambda'_2} f(a_{\xi_1(\alpha)} b_{\xi_2(\beta)}) \right| < \varepsilon. \tag{3.6}$$

Now, for each $\alpha \in \Lambda'_1$, put $L_\alpha = \lim_{\beta \in \Lambda'_2} f(a_{\xi_1(\alpha)} b_{\xi_2(\beta)})$ and, for each $\beta \in \Lambda'_2$, $M_\beta = \lim_{\alpha \in \Lambda'_1} f(a_{\xi_1(\alpha)} b_{\xi_2(\beta)})$. Since $\lim_{\alpha \in \Lambda'_1} L_\alpha = \lim_{\beta \in \Lambda'_2} M_\beta$, one can find $\alpha_0 \in \Lambda'_1$ and $\beta_0 \in \Lambda'_2$ such that $|L_\alpha - M_\beta| < 2\varepsilon$ whenever $\alpha \succeq \alpha_0$ and $\beta \succeq \beta_0$. If we combine this inequality with inequalities (3.4) and (3.6), we see that, for any $\alpha \succeq \alpha_0$ and $\beta \succeq \beta_0$,

$$|f(a_{\xi_1(\alpha_\beta)} b_{\xi_2(\beta)}) - f(a_{\xi_1(\alpha)} b_{\xi_2(\beta_\alpha)})| < 4\varepsilon$$

and this implies, using now inequalities (3.3) and (3.5), that, for any $\alpha \succeq \alpha_0$ and $\beta \succeq \beta_0$,

$$|f(x_{\xi_1(\alpha_\beta)\xi_2(\beta)}) - f(x_{\xi_1(\alpha)\xi_2(\beta_\alpha)})| \leq 2\|f\|\varepsilon + 4\varepsilon. \tag{3.7}$$

Then,

$$\begin{aligned} \left\| \mathcal{I}(\mathbf{c}) - f|_{\langle X \rangle} \right\|_{\langle X \rangle^*} &= \left\| \psi_{\mathbf{c}} - f|_{\langle X \rangle} \right\|_{\langle X \rangle^*} \\ &\geq \frac{1}{2M} \left(\left| \psi_{\mathbf{c}}(x_{\xi_1(\alpha_\beta)\xi_2(\beta)}) - f(x_{\xi_1(\alpha_\beta)\xi_2(\beta)}) \right| + \left| \psi_{\mathbf{c}}(x_{\xi_1(\alpha)\xi_2(\beta_\alpha)}) - f(x_{\xi_1(\alpha)\xi_2(\beta_\alpha)}) \right| \right) \\ &\geq \frac{1}{2M} \left| \psi_{\mathbf{c}}(x_{\xi_1(\alpha_\beta)\xi_2(\beta)}) - f(x_{\xi_1(\alpha_\beta)\xi_2(\beta)}) - \psi_{\mathbf{c}}(x_{\xi_1(\alpha)\xi_2(\beta_\alpha)}) + f(x_{\xi_1(\alpha)\xi_2(\beta_\alpha)}) \right| \\ &\geq \frac{1}{2M} \left(\left| \psi_{\mathbf{c}}(x_{\xi_1(\alpha_\beta)\xi_2(\beta)}) - \psi_{\mathbf{c}}(x_{\xi_1(\alpha)\xi_2(\beta_\alpha)}) \right| - \left| f(x_{\xi_1(\alpha_\beta)\xi_2(\beta)}) - f(x_{\xi_1(\alpha)\xi_2(\beta_\alpha)}) \right| \right) \\ &\geq \frac{1}{M} \left| \mathbf{c}(\lambda) \right| - \frac{\|f\|}{M} \varepsilon - \frac{2}{M} \varepsilon, \end{aligned}$$

where for the last inequality we have used (3.7) and that $\psi_{\mathbf{c}}(x_{\xi_1(\alpha_\beta)\xi_2(\beta)}) = -\mathbf{c}(\lambda)$ and $\psi_{\mathbf{c}}(x_{\xi_1(\alpha)\xi_2(\beta_\alpha)}) = \mathbf{c}(\lambda)$. We conclude therefore that

$$\left\| \mathcal{I}(\mathbf{c}) - f|_{\langle X \rangle} \right\|_{\langle X \rangle^*} \geq \frac{1}{M} \|\mathbf{c}\|_\infty, \quad \text{for every } \mathbf{c} \in \ell^\infty(\eta) \text{ and every } f \in \mathcal{WAP}(\mathcal{A}).$$

This and (3.1) show that \mathcal{I} is a linear isomorphism preserved by $\mathcal{WAP}(\mathcal{A})|_{\langle X \rangle}$. It thus induces, by Lemma 3.8, a linear isomorphism $\mathcal{J}: \ell^\infty(\eta) \rightarrow \frac{\langle X \rangle^*}{\mathcal{WAP}(\mathcal{A})|_{\langle X \rangle}}$. The proof is complete. \square

COROLLARY 3.10. *If the hypothesis of Theorem 3.9 are satisfied, then there is a bounded linear map of $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ onto $\ell^\infty(\eta)$. If, in addition, $d(\mathcal{A}) \leq \eta$, then \mathcal{A} is enAr.*

Proof. The isomorphism of Theorem 3.9 and the injectivity of $\ell^\infty(\eta)$ as a Banach space (see, for example, [1, Proposition 2.5.2]), give a linear surjective map $\mathcal{S}: \langle X \rangle^*/\mathcal{WAP}(\mathcal{A})|_{\langle X \rangle} \rightarrow \ell^\infty(\eta)$ and the restriction map $\mathcal{A}^* \rightarrow \langle X \rangle^*$ induces a surjective bounded linear map $\mathcal{R}: \frac{\mathcal{A}^*}{\mathcal{WAP}(\mathcal{A})} \rightarrow \frac{\langle X \rangle^*}{\mathcal{WAP}(\mathcal{A})|_{\langle X \rangle}}$.

The composition of \mathcal{S} with \mathcal{R} shows that $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ has $\ell^\infty(\eta)$ as a quotient.

Corollary 2.5 proves then the last statement. \square

4. Algebras with bai's and algebras with TI's

Our first application of Theorem 3.9 involves bounded approximate identities. In particular, we obtain an improvement of the main result of Ülger's inspiring paper [40], see Theorem 4.4.

DEFINITION 4.1. Let \mathcal{A} be a Banach algebra. We say that a bounded net $\{e_\alpha: \alpha \in \Lambda\}$ is a *bounded approximate identity* (bai for short) if

$$\lim_{\alpha} \|ae_\alpha - a\| = \lim_{\alpha} \|e_\alpha a - a\| = 0 \quad \text{for each } a \in \mathcal{A}.$$

We now see that bai's induce sets that approximate triangles.

THEOREM 4.2. *Let \mathcal{A} be a Banach algebra, which contains a bai $\{a_\alpha : \alpha \in \Lambda\}$ of true cardinality η which is an $\ell^1(\eta)$ -base.*

- (i) *Then there is a linear bounded map of $\mathcal{A}^*/\mathcal{W}\mathcal{A}\mathcal{P}(\mathcal{A})$ onto $\ell^\infty(\eta)$.*
- (ii) *In particular, \mathcal{A} is non-Arens regular.*
- (iii) *If in addition $d(\mathcal{A}) \leq \eta$, then \mathcal{A} is enAr.*

Proof. By Lemma 3.5, we can assume that $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$ and with both subsets Λ_1 and Λ_2 cofinal in Λ of true cardinality η . Put $A = \{a_\alpha : \alpha \in \Lambda_1\}$ and $B = \{a_\alpha : \alpha \in \Lambda_2\}$. For each $\alpha \in \Lambda_1$ and $\beta \in \Lambda_2$ with $\alpha \preceq \beta$, define $x_{\alpha\beta} = a_\alpha$. For each $\alpha \in \Lambda_1$ and $\beta \in \Lambda_2$ with $\beta \preceq \alpha$, define $x_{\alpha\beta} = a_\beta$. Then the sets

$$X_1 = \{x_{\alpha\beta} : \alpha \in \Lambda_1, \beta \in \Lambda_2, \beta \succ \alpha\} \text{ and } X_2 = \{x_{\alpha\beta} : \alpha \in \Lambda_1, \beta \in \Lambda_2, \alpha \succ \beta\}$$

approximate segments in T_{AB}^u and T_{AB}^l , respectively. To see this we observe that, for a given $\alpha \in \Lambda_1$, we can find $\beta_\alpha \in \Lambda_2$ with $\beta_\alpha \succeq \alpha$, and so the approximate identity property yields

$$0 = \lim_{\beta} \|a_\alpha - a_\alpha a_\beta\| = \lim_{\substack{\beta \in \Lambda_2 \\ \beta \succeq \alpha}} \|a_\alpha - a_\alpha a_\beta\| = \lim_{\beta \in \Lambda_2} \|x_{\alpha\beta} - a_\alpha a_\beta\|.$$

A similar observation shows that, for each $\beta \in \Lambda_2$,

$$\lim_{\alpha \in \Lambda_1} \|x_{\alpha\beta} - a_\alpha a_\beta\| = 0.$$

The two first conditions of Theorem 3.9 are then satisfied.

Since $a_\alpha \neq a_{\alpha'}$ when $\alpha \neq \alpha'$, X_1 is horizontally injective and X_2 is vertically injective, and so Condition (iii) of Theorem 3.9 holds also. The theorem then follows from Corollary 3.10. \square

The proof of next lemma is modelled on [40, Lemma 3.2] and the proof of [40, Theorem 3.3]. Since we are assuming neither Arens regularity nor weak sequential completeness of the algebra, as done in [40], a full proof is provided.

LEMMA 4.3. *Let \mathcal{A} be an infinite-dimensional Banach algebra with a bai and a separable subspace S . Then \mathcal{A} contains a separable closed subalgebra containing S and a sequential bai. If \mathcal{A} is non-unital, then this subalgebra can be chosen also non-unital.*

Proof. The first part of the lemma is proved in [40, Lemma 3.2]. As done there, denote this subalgebra by \mathcal{B}_1 and the sequential bai in \mathcal{B}_1 by (a_n) . If \mathcal{B}_1 is non-unital, then the second part of the claim follows too. Otherwise, let e_1 be the unit in \mathcal{B}_1 and note that e_1 is the norm limit of (a_n) (since $\lim_n \|a_n - e_1\| = \lim_n \|a_n e_1 - e_1\| = 0$), and so $\|e_1\|$ is bounded by the bound of (a_n) . Since \mathcal{A} is non-unital, we may pick $x_1 \in \mathcal{A} \setminus \mathcal{B}_1$ such that $x_1 e_1 \neq x_1$ or $e_1 x_1 \neq x_1$, and consider the set $\{x_1\} \cup \mathcal{B}_1$. Then again by [40, Lemma 3.2], this set is contained in a separable closed subalgebra \mathcal{B}_2 which has a sequential bai. If \mathcal{B}_2 is non-unital, we are done. Otherwise, we let e_2 be the unit in \mathcal{B}_2 and note that $e_2 \neq e_1$. Then pick $x_2 \in \mathcal{A} \setminus \mathcal{B}_2$ such that $x_2 e_2 \neq x_2$ or $e_2 x_2 \neq x_2$, consider the set $\{x_2, e_2\} \cup \mathcal{B}_1$, and repeat the process. If the subalgebras keep being unital at every stage, we obtain a sequence of distinct idempotents (e_n) in \mathcal{A} bounded by the bound of the original bai and satisfying

$$e_n e_m = e_m e_n = e_{\min\{n,m\}} \text{ for every } n, m \in \mathbb{N}.$$

Note that this implies that the idempotents e_n , $n \in \mathbb{N}$, are linearly independent. Let now \mathcal{B} be the closed linear span of $\{e_n : n \in \mathbb{N}\} \cup \mathcal{B}_1$. Since $b e_n = e_n b = b$ for every $b \in \mathcal{B}_1$ and

$n \in \mathbb{N}, \mathcal{B}$ is a subalgebra of \mathcal{A} . Moreover, if x is in the linear span of $\{e_n : n \in \mathbb{N}\} \cup \mathcal{B}_1$, then $x = b + \sum_{i=1}^m c_i e_i$ for some $b \in \mathcal{B}_1$ and some $m \in \mathbb{N}$, and

$$x e_n = e_n x = x \quad \text{for every } n \geq m,$$

which implies that (e_n) is a bai for \mathcal{B} .

Suppose finally that the subalgebra \mathcal{B} has a unit e , say. Then, (e_n) being a bai would converge in norm to the unit e . But this is not possible, since $e - e_n$ being an idempotent makes $\|e - e_n\| \geq 1$ for every $n \in \mathbb{N}$. Therefore, the subalgebra \mathcal{B} cannot have a unit, as required for our claim. □

Statement (ii) in the following theorem was proved in [40, Theorem 3.3].

THEOREM 4.4. *Let \mathcal{A} be a Banach algebra that contains a Banach algebra \mathcal{B} , which is non-unital, weakly sequentially complete and has a bai.*

- (i) *There is a linear bounded map from the quotient space $\mathcal{A}^*/\mathcal{W}\mathcal{A}\mathcal{P}(\mathcal{A})$ onto ℓ^∞ .*
- (ii) *In particular, \mathcal{A} is non-Arens regular.*
- (iii) *If \mathcal{A} is, in addition, separable, then \mathcal{A} is enAr.*

Proof. By Lemma 4.3, there is a separable non-unital subalgebra \mathcal{B}_1 of \mathcal{B} which admits a sequential bai $\{x_n : n \in \mathbb{N}\}$. This sequence cannot have any weak Cauchy subsequence because, otherwise, its weak limit $e \in \mathcal{B}_1^{**}$ would be the unit in \mathcal{B}_1 . This latter fact is checked by observing that, since \mathcal{B}_1 is weakly sequentially complete, $e \in \mathcal{B}_1$ and

$$\begin{aligned} \lim_n \langle x_n a, f \rangle &= \lim_n \langle x_n, a f \rangle = \langle e, a f \rangle = \langle e a, f \rangle \quad \text{and} \\ \lim_n \langle a x_n, f \rangle &= \lim_n \langle x_n, f a \rangle = \langle a e, f \rangle, \end{aligned}$$

for any $a \in \mathcal{B}_1$ and $f \in \mathcal{B}_1^*$. This implies that $\lim_n x_n a = e a$ and $\lim_n a x_n = a e$ weakly. Since, on the other hand, $\{x_n : n \in \mathbb{N}\}$ being a bai implies that $\lim_n x_n a = a$ and $\lim_n a x_n = a$ in norm, hence weakly, we conclude that $e a = a e = a$, for every $a \in \mathcal{B}_1$, which makes \mathcal{B}_1 unital.

Not having any weak Cauchy subsequences, the sequence $\{x_n : n \in \mathbb{N}\}$ must have a subsequence $\{x_{n(k)} : k \in \mathbb{N}\}$ that is an ℓ^1 -base in \mathcal{B} , by Rosenthal’s ℓ^1 -theorem (see [38, Theorem 1]). Since $\{x_{n(k)} : k \in \mathbb{N}\}$ is a bai in \mathcal{A} , the rest of the proof follows from Theorem 4.2. □

The second application of Theorem 3.9 involves (weak) TI-nets.

DEFINITION 4.5. In a Banach algebra \mathcal{A} , a bounded net $\{a_\alpha : \alpha \in \Lambda\}$ is a *weak TI-net* if

$$\lim_\alpha \|a_\alpha a_\beta - a_\alpha\| = \lim_\alpha \|a_\beta a_\alpha - a_\alpha\| = 0 \quad \text{for each } \beta \in \Lambda.$$

If \mathcal{A}^* is a von Neumann algebra and we require that $\lim_\alpha \|a_\alpha a - a_\alpha\| = 0$ for every normal state a of \mathcal{A}^* , and not only for members of the net itself, then we obtain the familiar concept of a *TI-net*. Here TI stands for topological invariance, the term was introduced by Chou [5] and is related to the notions of *famille moyennante* of Lust–Picard [35] and of *nets converging to invariance* of Day [10], as stated by Greenleaf [26].

THEOREM 4.6. *Let \mathcal{A} be a Banach algebra, η be an infinite cardinal number and suppose that \mathcal{A} contains a weak TI-net of true cardinality η , which is an $\ell^1(\eta)$ -base.*

- (i) *Then, there is a linear bounded map of $\mathcal{A}^*/\mathcal{W}\mathcal{A}\mathcal{P}(\mathcal{A})$ onto $\ell^\infty(\eta)$.*
- (ii) *In particular, \mathcal{A} is non-Arens regular.*
- (iii) *Ifn in addition, $d(\mathcal{A}) = \eta$, then \mathcal{A} is enAr.*

Proof. Let $\{a_\alpha : \alpha \in \Lambda\}$ be a weak TI-net in \mathcal{A} . As with the bai's in Theorem 4.2, weak TI-nets also approximate segments in the upper and lower triangles defined by them. If one takes a partition of Λ , $\Lambda = \Lambda_1 \cup \Lambda_2$, with both Λ_1 and Λ_2 cofinal and of true cardinality η , the only difference with the proof of Theorem 4.2 is that we now define $x_{\alpha\beta} = a_\beta$ when $\alpha \preceq \beta$ and $x_{\alpha\beta} = a_\alpha$ when $\beta \preceq \alpha$. As before, the sets

$$X_1 = \{x_{\alpha\beta} : \alpha \in \Lambda_1, \beta \in \Lambda_2, \alpha \prec \beta\} \text{ and } X_2 = \{x_{\alpha\beta} : \alpha \in \Lambda_1, \beta \in \Lambda_2, \beta \prec \alpha\}$$

can be used again to approximate segments in T_{AB}^u and T_{AB}^l , respectively. Since $a_\alpha \neq a_{\alpha'}$ when $\alpha \neq \alpha'$, X_1 is horizontally injective and X_2 is vertically injective. The proof of this theorem then runs exactly along the same lines as that of Theorem 4.2. \square

5. Algebras in harmonic analysis

As already observed in [40], most (if not all) of the known cases of non-Arens regular Banach algebra follow from Statement (ii) of Theorem 4.2. For an infinite locally compact group, the list includes the group algebra $L^1(G)$ [41], the weighted group algebra $L^1(G, w)$ (for any weight when G is either non-discrete or discrete and uncountable and for diagonally bounded weights when G is discrete and countable [8]) and the Fourier algebra $A(G)$ (when G is either non-discrete or is a discrete group containing an infinite amenable subgroup, [18, 32]). The list includes also the semigroup algebra $\ell^1(S)$ for cancellative semigroups and in general the weighted semigroup algebra $\ell^1(S, w)$ whenever the weight w is diagonally bounded on an infinite subset of S (see [3, 8]). Since these algebras have bounded approximate identities and are preduals of von Neumann algebras (they are therefore weakly sequentially complete) Theorem 4.4 can be applied to all of them. In fact, with preduals of von Neumann algebras an alternative approach is possible. We intend to address it in a forthcoming paper [14].

In the rest of the paper, we focus on three other important algebras defined in harmonic analysis, the 1-Segal Fourier algebra $S^1A(G)$, the Lebesgue-Fourier algebra $LA(G)$ and the Figà-Talamanca Herz algebra $A_p(G)$.

The 1-Segal Fourier algebra and the Lebesgue-Fourier algebra. Both algebras are built on the same Banach space, $L^1(G) \cap A(G)$ with norm $\|f\|_S = \|f\|_1 + \|f\|_{A(G)}$. With convolution product as multiplication, this Banach space is made into a Banach algebra, known as the *Lebesgue-Fourier algebra* and denoted by $LA(G)$. It is a Segal algebra with respect to $L^1(G)$, see [21, Proposition 2.2].

If pointwise product is used, another Banach algebra is obtained. Borrowing the terminology from [20], we shall refer to this algebra as the *1-Segal Fourier algebra* and denote it by $S^1A(G)$. As shown in [21, Proposition 2.5], $S^1A(G)$ is a Segal algebra with respect to $A(G)$. Note that in this latter reference both the 1-Segal Fourier algebra and the Lebesgue-Fourier algebra are referred to as the Lebesgue-Fourier algebra and denoted by $LA(G)$.

The other algebra we will be concerned with will be the Figà-Talamanca Herz algebra $A_p(G)$, where $1 < p < \infty$, is the algebra of all functions $u \in C_0(G)$ which have a series expansion

$$u = \sum_{i=1}^{\infty} g_i * \check{f}_i, \quad f_i \in L^p(G), g_i \in L^q(G), (1/p + 1/q = 1)$$

(where $\check{f}(x) = f(x^{-1})$) with the property that $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$. The norm in $A_p(G)$ is then given by

$$\|u\|_{A_p(G)} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q : u = \sum_{i=1}^{\infty} g_i * \check{f}_i, \quad f_i \in L^p(G), g_i \in L^q(G) \right\}.$$

Regarding the group algebra $L^1(G)$ as an algebra of convolution operators on $L^p(G)$, its closure with respect to the weak operator topology in $\mathcal{B}(L^p(G))$ (the bounded operators on $L^p(G)$) is $PM_p(G)$, the so-called space of p -pseudo-measures on G . $PM_p(G)$ may be identified with the Banach dual of $A_p(G)$. When $p = 2$, $A_2(G)$ is the Fourier algebra $A(G)$ of G and $PM_2(G)$ is the group von Neumann algebra $VN(G)$ of G .

Although it is not known whether the algebra $A_p(G)$ is weakly sequentially complete in general, Forrest managed to prove that $A_p(G)$ is non-Arens regular when G is non-discrete, see [18, Theorem 3.2]. The key in the proof is, however, still weak sequential completeness, in this case of the subalgebra $A_p^E(G)$ of $A_p(G)$ for compact subsets $E \subseteq G$ (see [24, Lemma 18] or the proof of Theorem 5.4 for the definition). Based on this remarkable result of Granirer, we shall improve Forrest’s theorem in Theorem 5.5. We rely on weak TI-sequences to see that the quotient space $PM_p(G)/\mathcal{W}\mathcal{A}\mathcal{P}(A_p(G))$ is non-separable, and in particular, $A_p(G)$ is enAr when G is second countable and non-discrete.

Non-discreteness of G is essential for the existence of weak TI-sequences and for the definition of the weakly sequentially complete subalgebra $A_p^E(G)$ of $A_p(G)$. If G is discrete (and infinite), the first difficulty can be overcome if G is an amenable group, for weak TI-sequences can be replaced by bounded approximate identities, always available in this case [27]. For the second difficulty, that only arises when $p \neq 2$, we will have to content ourselves to argue as in [18, Proposition 3.5] assuming that $A_p(G)$ is weakly sequentially complete (or $A_p(H)$ for a subgroup H of G). It remains an open problem whether, for $p \neq 2$, $A_p(G)$ contains a non-trivial weakly sequentially complete subalgebra when G is discrete.

5.1. A ‘canonical’ TI-net for convolutions

We construct here a weak TI-net that works both in $S^1A(G)$ and in $A_p(G)$, for any non-discrete locally compact group

We start with an elementary fact which makes clear that a neighbourhood base with the properties required in the subsequent Lemma 5.2 is always available.

LEMMA 5.1. *Let G be a locally compact group and U a relatively compact neighbourhood of the identity. Then, there is another neighbourhood of the identity V with $\overline{V} \subseteq U$ and $\lambda_G(U) \leq 2\lambda_G(V)$, where λ_G denotes the left Haar measure on G .*

Proof. By regularity we can find $K \subseteq U$, compact, with $2\lambda_G(K) \geq \lambda_G(U)$. Apply [28, Theorem 4.10] to find a neighbourhood W_1 of the identity such that $KW_1 \subseteq U$. If W_2 is another neighbourhood of the identity with $\overline{W_2} \subseteq W_1$, then $V = KW_2$ is the required neighbourhood. \square

LEMMA 5.2. *Let $\{U_\alpha : \alpha \in \Lambda\}$ be a base at the identity e made of relatively compact symmetric open sets, such that for some $M > 0$, $\lambda_G(U_\alpha) \leq M$ for every $\alpha \in \Lambda$. Let Λ be directed by set inclusion $\beta \succeq \alpha$ if and only if $U_\beta \subseteq U_\alpha$. Choose for each $\alpha \in \Lambda$, a neighbourhood V_α of the identity with $\overline{V_\alpha} \subseteq U_\alpha$ and $\lambda_G(U_\alpha) \leq 2\lambda_G(V_\alpha)$. Define, for each $\alpha \in \Lambda$,*

$$\varphi_\alpha = \frac{1}{\lambda_G(V_\alpha)} \mathbb{1}_{U_\alpha} * \mathbb{1}_{V_\alpha}.$$

Then $\{\varphi_\alpha : \alpha \in \Lambda\}$ is a weak TI-net in both algebras $A_p(G)$ and $S^1A(G)$.

Proof. It is clear that the net (φ_α) belongs to both algebras $A_p(G)$ and $S^1A(G)$. To see that this net is bounded in both norms, we check that on the one hand

$$\|\varphi_\alpha\|_{A_p(G)} \leq \frac{1}{\lambda_G(V_\alpha)} \|\mathbb{1}_{U_\alpha}\|_p \|\mathbb{1}_{V_\alpha}\|_q$$

$$\begin{aligned} &\leq \frac{1}{\lambda_G(V_\alpha)} \lambda_G(U_\alpha)^{1/p} \lambda_G(V_\alpha)^{1/q} \\ &\leq \frac{1}{\lambda_G(V_\alpha)} (2\lambda_G(V_\alpha))^{1/p} \lambda_G(V_\alpha)^{1/q} = 2^{1/p}. \end{aligned}$$

And then we observe that

$$\|\varphi_\alpha\|_1 = \frac{1}{\lambda_G(V_\alpha)} \|\mathbb{1}_{U_\alpha} * \mathbb{1}_{V_\alpha}\|_1 = \frac{1}{\lambda_G(V_\alpha)} \lambda_G(V_\alpha) \lambda_G(U_\alpha) = \lambda_G(U_\alpha) \leq M.$$

In particular, for $p = 2$, this yields $\|\varphi_\alpha\|_S \leq M + 2$ for every $\alpha \in \Lambda$.

As to the weak TI-net property, for each $\alpha \in \Lambda$, we choose $\beta(\alpha) \in \Lambda$ such that

$$U_{\beta(\alpha)}^2 V_\alpha \subseteq U_\alpha$$

(one may apply [28, Theorem 4.10] for this).

Let now $\gamma \succeq \beta(\alpha)$. Then $U_\gamma V_\gamma \subseteq U_{\beta(\alpha)}^2$, so that when $s \in U_\gamma V_\gamma$, we have

$$sV_\alpha \subseteq U_{\beta(\alpha)}^2 V_\alpha \subseteq U_\alpha,$$

and so $\mathbb{1}_{U_\alpha} * \mathbb{1}_{V_\alpha}(s) = \lambda_G(U_\alpha \cap sV_\alpha) = \lambda_G(V_\alpha)$. Hence

$$\varphi_\gamma(s) \varphi_\alpha(s) = \varphi_\gamma(s) \text{ if } s \in U_\gamma V_\gamma.$$

Since, obviously, $\varphi_\gamma(s) \varphi_\alpha(s) = \varphi_\gamma(s) = 0$, if $s \notin U_\gamma V_\gamma$, we conclude that $\varphi_\gamma \varphi_\alpha = \varphi_\gamma$ whenever $\gamma \succeq \beta(\alpha)$. The net is therefore a weak TI-net in both norms. \square

5.2. The 1-Segal Fourier algebras $S^1A(G)$

We first need a technical lemma.

LEMMA 5.3. *Every non-discrete locally compact group possesses a sequence $\{U_n : n \in \mathbb{N}\}$ of neighbourhoods of the identity such that $U = \bigcap_{n < \omega} U_n$ has an empty interior.*

Proof. We provide two different proofs for the lemma. Since G is not discrete, $\lambda_G(\{e\}) = 0$. By regularity of λ_G , there is for each $n \in \mathbb{N}$, a neighbourhood of the identity U_n such that $\lambda_G(U_n) \leq 1/n$. If $U = \bigcap_{n < \omega} U_n$, it follows that $\lambda_G(U) = 0$ and, therefore, U has empty interior.

We can also give a lower level proof of this Lemma without invoking Haar measure.

Let N be a countably infinite, relatively compact subset of G , such that e is a limit point of N , that is, such that $e \in \overline{N \setminus \{e\}}$. To find such N , it is enough to recall that compact sets are limit point compact, if we start with any countable relatively compact set M and a limit point p of M , $N = p^{-1}(M \setminus \{p\})$ will have the identity as a limit point.

Enumerate $N \setminus \{e\} = \{x_n : n \in \mathbb{N}\}$ and define, for each $n \in \mathbb{N}$, $W_n = G \setminus \{x_1, \dots, x_n\}$. Noting that W_n is a neighbourhood of the identity, one can pick another symmetric neighbourhood of the identity, U_n , such that $U_n^2 \subseteq W_n$. Suppose we can find $x \in \text{int}(\bigcap_n U_n)$. Then

$$e \in x^{-1} \text{int} \left(\bigcap_n U_n \right) \subseteq \bigcap_n U_n^2 \subseteq \bigcap_n W_n.$$

But this goes against our choice of N , for, in that case, e would be an interior point of $\bigcap_n W_n = G \setminus (N \setminus \{e\})$. \square

Statement (ii) of the following theorem was proved in [21, Theorem 5.2].

THEOREM 5.4. *Let G be a non-discrete, locally compact group.*

- (i) *Then the quotient space $S^1A(G)^*/\mathcal{WAP}(S^1A(G))$ has ℓ^∞ as a quotient.*
- (ii) *In particular, $S^1A(G)$ is not Arens regular.*
- (iii) *$S^1A(G)$ is enAr when G is second countable.*

Proof. Let $\{U_n : n < w\}$ be one of the collections of relatively compact symmetric neighbourhoods of the identity e in G such that $\bigcap_{n < w} U_n$ has an empty interior that Lemma 5.3 provides.

Define then a decreasing sequence of relatively compact neighbourhoods of the identity V_n such that $V_n^2 \subseteq U_n$. Apply Lemma 5.1 to find a neighbourhood W_n of the identity with $\overline{W_n} \subseteq V_n$ and $\lambda_G(V_n) \leq 2\lambda_G(W_n)$. If $\varphi_n = \frac{1}{\lambda_G(W_n)} \mathbb{1}_{V_n} * \mathbb{1}_{W_n}$, we know from Lemma 5.2 that $\{\varphi_n : n \in \mathbb{N}\}$ is a weak TI-net in $S^1A(G)$. Suppose that $\{\varphi_{n(k)} : k < w\}$ is a weak $\sigma(S^1A(G), S^1A(G)^*)$ -Cauchy subsequence of $\{\varphi_n : n < w\}$. This subsequence would also be $\sigma(A(G), A(G)^*)$ -Cauchy and weak sequential completeness of $A(G)$ would imply that there is $\psi \in A(G)$ with $\psi = \lim_k \varphi_{n(k)}$ in the $\sigma(A(G), A(G)^*)$ -topology. Observing that $\varphi_n(x) = 0$ if $x \notin V_n^2$, one easily deduces that $\psi(x) = 0$, if $x \notin \bigcap_{n < w} V_n^2$, which goes against ψ being continuous, $\text{int}(\bigcap_{n < w} U_n)$ being empty and $\psi(e) = \lim_k \varphi_{n(k)}(e) = 1$.

We deduce that $\{\varphi_n : n \in \mathbb{N}\}$ does not admit any $\sigma(S^1A(G), S^1A(G)^*)$ -Cauchy subsequence. By Rosenthal’s ℓ^1 -theorem (see [38, Theorem 1]), $\{\varphi_n : n \in \mathbb{N}\}$ must then contain a subsequence which is an ℓ^1 -base. With this subsequence being also a weak TI-net, it only remains to apply Theorem 4.6 to deduce Statement (i). The second statement is now straightforward. For Statement (iii), use the fact that $d(S^1A(G)) = \omega$ when G is second countable. \square

5.3. *The Lebesgue–Fourier algebra $LA(G)$*

The algebra $LA(G)$ is Arens-regular if and only if G is compact. This was proved in [22, Theorem 5.1] for unimodular groups. Later on, $LA(G)$ was proved to be non-Arens regular whenever G is not unimodular see [20, Corollary 3.6].

The theorem in this section proves that the Lebesgue–Fourier algebra is not only non-Arens regular, but extremely so, for any non-compact locally compact group.

As usual, $\kappa(G)$ will be the minimal number of compact sets required to cover G and $\chi(G)$ the minimal cardinality of an open base at the identity e of G .

THEOREM 5.5. *Let G be a non-compact, locally compact group and assume that $LA(G)$ is equipped with convolution product.*

- (i) *Then the quotient space $LA(G)^*/\mathcal{WAP}(LA(G))$ has $\ell^\infty(\kappa(G))$ as a quotient.*
- (ii) *In particular, $LA(G)$ is not Arens regular.*
- (iii) *If $\kappa(G) \geq \chi(G)$, then $LA(G)$ is enAr.*

Proof. Without loss of generality, we can assume that G contains two normal subgroups $H \leq N$ such that N/H is unimodular and $\kappa(N/H) = \kappa(G)$. The choices of N and H depend on the size of $\ker \Delta_G$, the kernel of the modular function. If $\kappa(\ker \Delta_G) = \kappa(G)$, one takes $N = \ker \Delta_G$ and $H = \{e\}$; if $\kappa(G/\ker \Delta_G) = \kappa(G)$ and $\ker \Delta_G$ is not compact, one takes $N = G$ and $H = \ker \Delta_G$ and if $\kappa(G/\ker \Delta_G) = \kappa(G)$ and $\ker \Delta_G$ is compact, one takes $N = G$ and $H = \{e\}$. The latter choice is possible because, when $\ker \Delta_G$ is compact, G is an SIN group (as the extension of a compact group by an Abelian one) and hence unimodular.

Let κ be the initial ordinal associated to the cardinal $\kappa(G) = \kappa(G/H) = \kappa(N/H)$ with its natural order.

Fix an open relatively compact symmetric neighbourhood U of the identity in G/H such that $\lambda_{G/H}(U) \leq 1$. Since the unimodular function $\Delta_{G/H}$ is a continuous homomorphism into the multiplicative group \mathbb{R}^+ and \bar{U} is compact, there is a real number $M \geq 1$ such that

$$\frac{1}{M} \leq \Delta_{G/H}(t) \leq M, \quad \text{for all } t \in U. \tag{5.1}$$

Choose next a compact symmetric neighbourhood K of the identity of G/H such that

$$\lambda_{G/H}(K)\lambda_{G/H}(U) \geq M. \tag{5.2}$$

Take now any compact covering of G/H of cardinality κ . Since every member of this covering is covered by a finite union of open sets, we may extract a covering $(K_\alpha)_{\alpha < \kappa}$ of G/H made of relatively compact symmetric open subsets. Form then an increasing covering of G/H made of symmetric open sets by writing inductively

$$V_0 = KU^2K \quad \text{and} \quad V_\alpha = \bigcup_{\gamma < \alpha} K_\gamma.$$

Since N/H cannot be covered by fewer than κ compact subsets, it is possible to form inductively a set $\{x_\alpha : \alpha < \kappa\}$ in N/H such that

$$(V_\alpha x_\alpha V_\alpha) \cap (V_\beta x_\beta V_\beta) = \emptyset \quad \text{for every } \alpha < \beta < \kappa.$$

Consider, for each $\alpha < \kappa$, the following functions $\varphi_\alpha, \psi_\alpha \in LA(G/H)$

$$\varphi_\alpha = \mathbb{1}_{x_\alpha K} * \mathbb{1}_U \quad \text{and} \quad \psi_\alpha = \mathbb{1}_U * \mathbb{1}_{Kx_\alpha}.$$

*Claim 1: For $\alpha, \beta < \kappa$, the $LA(G)$ -norm of the functions $\varphi_\alpha * \psi_\beta$ satisfies the inequalities:*

$$1 \leq \|\varphi_\alpha * \psi_\beta\|_{LA(G)} \leq (1 + M)\|\varphi_\alpha * \psi_\beta\|_1 = (1 + M)\lambda_{G/H}(K)\lambda_{G/H}(U).$$

We first note that

$$\begin{aligned} \|\varphi_\alpha\|_1 &= \int_{G/H} \int_{G/H} \mathbb{1}_{x_\alpha K}(t)\mathbb{1}_U(t^{-1}s) \, d\lambda_{G/H}(t) \, d\lambda_{G/H}(s) \\ &= \lambda_{G/H}(K)\lambda_{G/H}(U) \geq M \geq 1. \end{aligned} \tag{5.3}$$

Taking into account that N/H is a unimodular, normal subgroup of G/H , and so that $\Delta_{G/H}(x_\alpha) = 1$ for every $\alpha < \kappa$,

$$\begin{aligned} \|\psi_\alpha\|_1 &= \int_{G/H} \int_{G/H} \mathbb{1}_U(t)\mathbb{1}_{Kx_\alpha}(t^{-1}s) \, d\lambda_{G/H}(t) \, d\lambda_{G/H}(s) = \lambda_{G/H}(U)\lambda_{G/H}(Kx_\alpha) \\ &= \lambda_{G/H}(U)\lambda_{G/H}(K) \geq M \geq 1. \end{aligned} \tag{5.4}$$

To obtain a similar estimate for the functions $\check{\psi}_\alpha$, we note that, for any $\alpha \in \kappa$,

$$\begin{aligned} \|\check{\psi}_\alpha\|_1 &= \int_{G/H} \int_{G/H} \mathbb{1}_U(t)\mathbb{1}_{Kx_\alpha}(t^{-1}s^{-1}) \, d\lambda_{G/H}(s) \, d\lambda_{G/H}(t) \\ &= \int_{G/H} \mathbb{1}_U(t)\lambda_{G/H}(x_\alpha^{-1}Kt^{-1}) \, d\lambda_{G/H}(t) \\ &= \int_{G/H} \mathbb{1}_U(t)\Delta_{G/H}(t^{-1})\lambda_{G/H}(K) \, d\lambda_{G/H}(t). \end{aligned}$$

Therefore, using (5.1) and (5.2),

$$1 \leq \frac{1}{M} \lambda_{G/H}(K)\lambda_{G/H}(U) \leq \|\check{\psi}_\alpha\|_1 \leq M\lambda_{G/H}(K)\lambda_{G/H}(U) = M\|\psi_\alpha\|_1.$$

So,

$$1 \leq \|\check{\psi}_\alpha\|_1 \leq M\|\psi_\alpha\|_1 \quad \text{for every } \alpha \prec \kappa. \tag{5.5}$$

Since, as easily checked, $0 \leq \varphi_\alpha, \psi_\alpha \leq \lambda_{G/H}(U) \leq 1$, we can deduce from (5.3) and (5.5) that

$$\|\varphi_\alpha\|_2 \leq \|\varphi_\alpha\|_1, \quad \|\check{\psi}_\alpha\|_2 \leq \|\check{\psi}_\alpha\|_1. \tag{5.6}$$

Relations (5.5)–(5.6) and the following equality (due again to a simple application of Fubini’s theorem),

$$\|\varphi_\alpha * \psi_\beta\|_1 = \|\varphi_\alpha\|_1\|\psi_\beta\|_1 \quad \text{for every } \alpha, \beta \prec \kappa, \tag{5.7}$$

yield the non-obvious parts of the following inequalities

$$\begin{aligned} 1 \leq \|\varphi_\alpha * \psi_\beta\|_{LA(G/H)} &= \|\varphi_\alpha * \psi_\beta\|_1 + \|\varphi_\alpha * \psi_\beta\|_{A(G/H)} \\ &\leq \|\varphi_\alpha * \psi_\beta\|_1 + \|\varphi_\alpha\|_2\|\check{\psi}_\beta\|_2 \\ &\leq \|\varphi_\alpha * \psi_\beta\|_1 + \|\varphi_\alpha\|_1\|\check{\psi}_\beta\|_1 \\ &\leq \|\varphi_\alpha * \psi_\beta\|_1 + M\|\varphi_\alpha\|_1\|\psi_\beta\|_1 \\ &= (1 + M)\|\varphi_\alpha * \psi_\beta\|_1. \end{aligned}$$

Claim 1 is proved.

Claim 2: For each $\alpha, \beta \in \kappa$, there exist $\beta_\alpha \in \kappa$ and $\alpha_\beta \in \kappa$ such that

$$\text{supp}(\varphi_\alpha * \psi_\beta) \subseteq x_\alpha KU^2 Kx_\beta \subseteq V_\beta x_\beta \quad \text{for every } \beta \succ \beta_\alpha, \text{ and}$$

$$\text{supp}(\varphi_\alpha * \psi_\beta) \subseteq x_\alpha KU^2 Kx_\beta \subseteq x_\alpha V_\alpha \quad \text{for every } \alpha \succ \alpha_\beta.$$

It is obvious that $\text{supp}(\varphi_\alpha * \psi_\beta) \subseteq x_\alpha KU^2 Kx_\beta$. To prove the other inclusions in the claim, one just has to note that $(V_\gamma)_{\gamma \prec \kappa}$ is an increasing open cover of G and that $x_\alpha KU^2 K$ and $KU^2 Kx_\beta$ are compact subsets of G .

Now that Claim 2 is proved, we partition κ into two disjoint cofinal copies κ_1 and κ_2 , each of true cardinality $\kappa(G)$, (for this we use Lemma 3.5 and Remark 3.6). Note also that in this case the cardinality of κ_1 and κ_2 is κ . Define then

$$X_1 = \{\varphi_\alpha * \psi_\beta : \beta \in \kappa_2, \beta \succeq \beta_\alpha\} \quad \text{and}$$

$$X_2 = \{\varphi_\alpha * \psi_\beta : \alpha \in \kappa_1, \alpha \succeq \alpha_\beta\}.$$

Claim 3: $X_1 \cup X_2$ is an $\ell^1(\kappa)$ -base in both $L^1(G/H)$ and $LA(G/H)$.

Since by Claim 1, $X_1 \cup X_2$ is bounded in $LA(G/H)$, and $\|\cdot\|_{LA(G/H)} \geq \|\cdot\|_1$, the set $X_1 \cup X_2$ will be an $\ell^1(\kappa)$ -base in the algebra $LA(G/H)$ as soon as it is an $\ell^1(\kappa)$ -base in $L^1(G/H)$.

Now, as seen in (5.3) and (5.4), the functions φ_α and ψ_α are bounded away from 0. The identity (5.7) shows that the same is true for the elements of $X_1 \cup X_2$. So, for the functions in $X_1 \cup X_2$ to form an $\ell^1(\kappa)$ -base in $L^1(G/H)$, it will be enough that their supports are pairwise disjoint. For two different pairs (α, β) and (α', β') , the following possibilities arise:

Case 1: $\varphi_\alpha * \psi_\beta \in X_1$ and $\varphi_{\alpha'} * \psi_{\beta'} \in X_2$. In this case, $\beta \in \kappa_2$ and $\beta \succeq \beta_\alpha$, while $\alpha' \in \kappa_1$ and $\alpha' \succeq \alpha'_\beta$. By Claim 2, the supports of $\varphi_{\alpha'} * \psi_{\beta'}$ and $\varphi_\alpha * \psi_\beta$, are contained (respectively) in $x_{\alpha'} V_{\alpha'}$ and $V_\beta x_\beta$. Since, by construction, these sets are disjoint, so will be the supports of $\varphi_{\alpha'} * \psi_{\beta'}$ and $\varphi_\alpha * \psi_\beta$.

Case 2: Both $\varphi_\alpha * \psi_\beta$ and $\varphi_{\alpha'} * \psi_{\beta'}$ are in X_1 and $\beta \neq \beta'$. In this case, we have by Claim 2 the supports of $\varphi_\alpha * \psi_\beta$ and $\varphi_{\alpha'} * \psi_{\beta'}$ are contained, respectively, in $V_\beta x_\beta$ and $V_{\beta'} x_{\beta'}$, and these sets are disjoint by construction.

Case 3: Both $\varphi_\alpha * \psi_\beta$ and $\varphi_{\alpha'} * \psi_{\beta'}$ are in X_1 and $\beta = \beta'$. Now, the supports of $\varphi_\alpha * \psi_\beta$ and $\varphi_{\alpha'} * \psi_{\beta'}$ are contained, respectively, in $x_\alpha KU^2 K x_\beta$ and $x_{\alpha'} KU^2 K x_\beta$. These sets are disjoint because $x_\alpha KU^2 K \subseteq x_\alpha V_\alpha$ and $x_{\alpha'} KU^2 K \subseteq x_{\alpha'} V_{\alpha'}$ and $x_\alpha V_\alpha$ and $x_{\alpha'} V_{\alpha'}$ are disjoint, by construction.

Cases 4 and 5: Both $\varphi_\alpha * \psi_\beta$ and $\varphi_{\alpha'} * \psi_{\beta'}$ are in X_2 , with either $\alpha \neq \alpha'$ or $\alpha = \alpha'$. Repeating the arguments of Cases 2 and 3, one readily sees that the supports of $\varphi_\alpha * \psi_\beta$ and $\varphi_{\alpha'} * \psi_{\beta'}$ are also disjoint.

We have therefore checked that $X_1 \cup X_2$ is an $\ell^1(\kappa)$ -base in $LA(G/H)$. Claim 3 is proved.

We proceed now to lift this $\ell^1(\kappa)$ -base to $LA(G)$. Recall from the beginning of the proof that H is either $\{e\}$ or non-compact. Since in case $H = \{e\}$, our $\ell^1(\kappa)$ -base is already in $LA(G)$, we may assume that H is not compact.

To lift the functions in $X_1 \cup X_2$ to $LA(G)$, we consider the averaging operator

$$T_H: L^1(G) \rightarrow L^1(G/H), \quad T_H f(x) = \int_H f(xh) d\lambda_H(h).$$

The averaging operator is a quotient operator and a Banach algebra homomorphism, see [37, Section 3.4]. By [20, Theorem 3.4], when H is not compact the restriction of T_H to $LA(G)$ is still a quotient operator $T_H: LA(G) \rightarrow L^1(G/H)$. So, for any $\phi \in L^1(G/H)$ there are $k > 0$ and $(\phi)_H \in LA(G)$ with $T_H((\phi)_H) = \phi$ such that

$$\frac{1}{k} \|\phi\|_{L^1(G/H)} \leq \|(\phi)_H\|_{LA(G)} \leq k \|\phi\|_{L^1(G/H)}, \tag{5.8}$$

where we have used the notation $\|\cdot\|_{LA(G)}$ instead of $\|\cdot\|_S$ to be able to stress the difference between G and G/H . With the help of this lifting property, we define the sets

$$(X_1)_H = \{(\varphi_\alpha)_H * (\psi_\beta)_H : \alpha \in \kappa_1, \beta \in \kappa_2, \beta \succeq \beta_\alpha\} \quad \text{and}$$

$$(X_2)_H = \{(\varphi_\alpha)_H * (\psi_\beta)_H : \alpha \in \kappa_1, \beta \in \kappa_2, \alpha \succeq \alpha_\beta\}.$$

We know from Claim 3 that $X_1 \cup X_2$ is an ℓ^1 -base in both $L^1(G/H)$ and $LA(G/H)$. Using that T_H is a bounded homomorphism and the inequalities in (5.8), the ℓ^1 -property of $X_1 \cup X_2$ is acquired by $(X_1)_H \cup (X_2)_H$. More precisely, for $z_1, \dots, z_p \in \mathbb{C}$, we have that, for some $M > 0$,

$$\begin{aligned} \frac{1}{M} \sum_{n=1}^p |z_n| &\leq \left\| \sum_{n=1}^p z_n (\varphi_n * \psi_n) \right\|_{L^1(G/H)} \\ &= \left\| \sum_{n=1}^p z_n T_H((\varphi_n)_H) * T_H((\psi_n)_H) \right\|_{L^1(G/H)} \\ &= \left\| T_H \left(\sum_{n=1}^p z_n (\varphi_n)_H * (\psi_n)_H \right) \right\|_{L^1(G/H)} \\ &\leq k \left\| \sum_{n=1}^p z_n (\varphi_n)_H * (\psi_n)_H \right\|_{LA(G)}, \end{aligned}$$

where the first equality follows from the fact that the summand functions are part of an $\ell^1(\kappa)$ -base (Claim 3), the third equality from the fact that T_H is an algebra homomorphism and the

last inequality from the fact that T_H is bounded by k as seen in (5.8). The second inequality in (5.8) proves that $(X_1)_H \cup (X_2)_H$ is bounded in $LA(G)$.

Therefore, the set $(X_1)_H \cup (X_2)_H$ is also an $\ell^1(\kappa)$ -base in $LA(G)$ when H is not compact. We finally define the sets

$$A = \{(\varphi_\alpha)_H : \alpha \in \kappa_1\} \quad \text{and} \quad B = \{(\psi_\alpha)_H : \alpha \in \kappa_2\},$$

where the functions φ_α and ψ_α are the ones we got previously in $LA(G/H)$.

Property (5.8) and Claim 1 imply that A and B are bounded sets in $LA(G)$, both when H is not compact and when $H = \{e\}$. They are indexed by subsets of κ of true cardinality (and so of cardinality) $\kappa(G)$ and the sets $(X_1)_H$ and $(X_2)_H$ approximate segments in, respectively, T_{AB}^u and T_{AB}^l . Horizontal and vertical injectivity are guaranteed as we have seen that even the supports of $\varphi_\alpha * \psi_\beta$ and $\varphi_{\alpha'} * \psi_{\beta'}$ are disjoint when $(\alpha, \beta) \neq (\alpha', \beta')$.

We have therefore met all the conditions required to apply Corollary 3.10. The proof is then complete. □

5.4. The Figà-Talamanca Herz algebras $A_p(G)$

Statement (ii) of the following theorem was proved in [18, Theorem 3.2].

THEOREM 5.6. *Let G be an infinite, non-discrete, locally compact group. Then:*

- (i) *the quotient space $PM_p(G)/\mathcal{W} \mathcal{A} \mathcal{P}(A_p(G))$ has ℓ^∞ as quotient;*
- (ii) *in particular, $A_p(G)$ is not Arens regular;*
- (iii) *$A_p(G)$ is enAr when G is in addition second countable.*

Proof. Let $\{\varphi_n : n < w\}$ be a weak TI-sequence constructed as in Theorem 5.4 with each φ_n supported in a neighbourhood U_n of the identity in such a way that the sequence $\{U_n : n < w\}$ is decreasing and $\bigcap_{n < w} U_n$ has empty interior. By [24, Lemma 18],

$$A_p^{U_1}(G) = \{\varphi \in A_p(G) : \text{supp } \varphi \subseteq U_1\}$$

is weakly sequentially complete. Replacing weak sequential completeness of $A(G)$ by weak sequential completeness of $A_p^{U_1}(G)$, the argument of Theorem 5.4 now proves that a subsequence of $\{\varphi_n : n < w\}$ is both an ℓ^1 -base, and a weak TI-net. All the statements of the present theorem then follow from Theorem 4.6 as in Theorem 5.4. □

The analog of Statement (ii) of the following theorem was proved in [18, Proposition 3.5] for $H = G$ assuming that G is amenable and $A_p(G)$ is weakly sequentially complete, and in [19, Theorem 2] only assuming that H is Abelian. Our method works with any amenable subgroup H with $A_p(H)$ weakly sequentially complete.

THEOREM 5.7. *Let G be a locally compact group with an infinite amenable open subgroup H such that $A_p(H)$ is weakly sequentially complete.*

- (i) *Then each of the quotient spaces $PM_p(H)/\mathcal{W} \mathcal{A} \mathcal{P}(A_p(H))$ and $PM_p(G)/\mathcal{W} \mathcal{A} \mathcal{P}(A_p(G))$ has ℓ^∞ as a quotient.*
- (ii) *In particular, $A_p(H)$ and $A_p(G)$ are non-Arens regular.*
- (iii) *If G is, in addition, second countable, then $A_p(H)$ and $A_p(G)$ are enAr.*

Proof. Note first that the trivial extension $u \mapsto \overset{\circ}{u}$ where $\overset{\circ}{u}$ extends u by making it 0 off H , establishes a Banach algebra embedding $\mathcal{I} : A_p(H) \rightarrow A_p(G)$, [27, Proposition 5]. Hence $\mathcal{I}^* : PM_p(G) \rightarrow PM_p(H)$ is a surjective bounded linear mapping. Since \mathcal{I} is multiplicative,

we have that $\mathcal{I}^*(\mathcal{WAP}(A_p(G)) \subseteq \mathcal{WAP}(A_p(H))$ and, therefore, \mathcal{I}^* induces a surjective bounded linear mapping

$$\tilde{\mathcal{I}}: \frac{PM_p(G)}{\mathcal{WAP}(A_p(G))} \rightarrow \frac{PM_p(H)}{\mathcal{WAP}(A_p(H))}.$$

Now, since H is amenable, $A_p(H)$ has a bai (see [27, Theorem 6]). And since it is assumed to be weakly sequentially complete, we obtain from Theorem 4.4 a linear bounded map \mathcal{T}_H from $PM_p(H)/\mathcal{WAP}(A_p(H))$ onto ℓ^∞ . If \mathcal{T}_H is composed with $\tilde{\mathcal{I}}$, a linear bounded map from $PM_p(G)/\mathcal{WAP}(A_p(G))$ onto ℓ^∞ is obtained.

The second statement is now obvious, and the last statement follows directly from Lemma 2.4 since $d(A_p(G)) = w$. \square

REMARK 5.8. In 2008, Losert announced in his lectures [34] that $A(G)$ is not strongly Arens irregular (sAir) when G is either the compact group $SU(3)$ or the locally compact group $SL(2, \mathbb{R})$. By Theorem 5.5 (this also follows from [14, 25, 29]), $A(G)$ is enAr in each of these cases. So $A(G)$ is a Banach algebra which is enAr but not sAir for the compact group $SU(3)$ and the non-compact non-discrete group $SL(2, \mathbb{R})$.

Recently, in [33], Losert also proved that $A_2(G) = A(G)$ is not sAir when G is a discrete group containing the free group \mathbb{F}_r with r generators, where $r \geq 2$. So with Theorem 5.7, we have another example, this time a discrete group, for which $A(G)$ is enAr but not sAir.

COROLLARY 5.9. *The Fourier algebra $A(\mathbb{F}_r)$ is enAr but not sAir for every $r \geq 2$.*

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