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A square function involving the center of mass and rectifiability

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Abstract

For a Radon measure μ on $\mathbb{R}^d$, define $C_\mu^n(x, t) = \left(\frac{1}{t^n} \left| \int_{B(x,t)} \frac{x-y}{t} \, d\mu(y) \right| \right)$. This coefficient quantifies how symmetric the measure μ is by comparing the center of mass at a given scale and location to the actual center of the ball. We show that if μ is n-rectifiable, then

$$\int_0^\infty |C_\mu^n(x, t)|^2 \frac{dt}{t} < \infty \ \mu\text{-almost everywhere.}$$

Together with a previous result of Mayboroda and Volberg, where they showed that the converse holds true, this gives a new characterisation of n-rectifiability. To prove our main result, we also show that for an n-uniformly rectifiable measure, $|C_\mu^n(x, t)|^2 \frac{dt}{t} d\mu$ is a Carleson measure on $\mathrm{spt}(\mu) \times (0, \infty)$. We also show that, whenever a measure μ is 1-rectifiable in the plane, then the same Dini condition as above holds for more general kernels. We also give a characterisation of uniform 1-rectifiability in the plane in terms of a Carleson measure condition. This uses a classification of Ω-symmetric measures from Villa (Rev Mat Iberoam, 2019).

Keywords Rectifiability · Tangent points · Beta numbers · Hausdorff content

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1 Introduction

Rectifiable sets are a main object of study in geometric measure theory. These are sets that can be covered by (countably many) Lipschitz images of the Euclidean space, up to a set of Hausdorff measure zero. More generally, a *measure* μ in $\mathbb{R}^d$ is n-rectifiable if there exists an n-rectifiable set E and a Borel function $f : \mathbb{R}^d \to \mathbb{R}$ so that $\mu = f \mathcal{H}^n|_E$.

In the first part of this paper, however, we will use a quantitative notion of rectifiability: that of *uniformly rectifiable* (or UR) sets. First introduced by Guy David and Stephen Semmes in [6], uniform rectifiability is a stronger property than rectifiability: one has control on how much mass in any given ball centered on the set can be covered by *just one* Lipschitz image - in contrast to rectifiable sets, where in some places one may need very many images to cover just a small portion of the original set. This sub-area of GMT has very strong ties to Harmonic analysis. Indeed, not only '*is there an obvious analogy between rectifiability property of sets* (this is GMT) *and differentiability properties of functions* (see [7], Introduction), but also the problems themselves that originally motivated the development of the theory of UR sets are harmonic analytic in nature, although this problems concerns more singular integrals and analytic capacity. Here's an example: consider $f : \mathbb{R} \to \mathbb{R}$ in $L^2(\mathbb{R})$. Then (see [17]) f is locally absolute continuous and $f' \in L^2$ if and only if $\int_0^\infty \int_\mathbb{R} t^{-2}|f(x+t) + f(x-t) - 2f(x)|^2 \, dx \frac{dt}{t} < \infty$. The quantity $t^{-1}(f(x+t) + f(x-t) - 2f(x))$ can be seen as a measurement of how far f is from being affine. Something along these lines is the following result by Dorronsoro [9]. Let $f \in W^{1,2}(\mathbb{R}^d)$ and define $\Omega_f(x,t) := \left(\inf_A t^{-d} \int_{B(x,t)} t^{-2} \left(f(y) - A(y)\right)^2 \, dy\right)^{\frac{1}{2}}$, where the infimum is taken over all affine functions A. Then $\int_{\mathbb{R}^d} \int_0^\infty \Omega_f(x,t)^2 \frac{dt}{t} dx \sim \|\nabla f\|_2^2$. Since rectifiable sets are composed of Lipschitz images, it is natural to ask whether similar quantities can be designed for sets. Take a subset $E \subset \mathbb{R}^n$ and let

$$\beta_{E,p}^n(x,t) := \inf_L \left(\frac{1}{t^n} \int_{B \cap E} \left(\frac{\text{dist}(y, L)}{t} \right)^p d\mathcal{H}^n(y) \right)^{\frac{1}{p}}, \qquad (1.1)$$

A similar quantity was first introduced by Peter Jones in [12] as a tool to prove his Analyst's Traveling Salesman theorem (and so it is called the Jones-beta number), and further developed by David and Semmes [6,7], to show that if a set E is n-Ahlfors–David regular (ADR), that is, if there exists a constant $c \geq 1$ such that $c^{-1} r^n \mathcal{H}^n(B(x,r) \cap E) \leq c\, r^n$ for all $x \in E$, $r > 0$, then E is UR if and only if

$$\int_{B_R} \int_0^R \beta_{E,p}^n(x,t)^2 \frac{dt}{t} d\mathcal{H}^n|_E(x) \leq C\, R^n. \qquad (1.2)$$

Note the analogy with the result by Dorronsoro: there we studied regularity properties of functions, here those of sets—in both cases such properties are characterised quantitatively by measuring the linear approximation properties of our objects. David and Semmes also show that a set E is UR if and only if a wide class of Calderón–Zygmund kernels are $L^2(E)$ bounded; this is precisely the problem which originated the field: on what sort of sets are Calderón–Zygmund kernels L^2 bounded?

Let us go back for a moment to the world of functions: we may find further inspiration. Recently, Alabern, Mateu and Verdera showed in [1] that if $f \in W^{1,p}(\mathbb{R}^d)$, then $\|S(f)\|_p \sim \|\nabla f\|_p$, where

$$S(f)^2(x) := \int_0^\infty \left| t^{-d} \int_{B(x,t)} t^{-1}(f(x) - f(y))\, dy \right|^2 \frac{dt}{t}.$$

Appealing to analogy once more ('*you can prove anything using the Analogy*'[1]), one may define a corresponding quantity for sets, or, more generally for a Radon measure μ on $\mathbb{R}^d$: we let the *n-dimensional C-number* to be defined as

$$C_\mu^n(x,t)^2 = C_\mu(x,t)^2 := \left(\frac{1}{t^n} \left| \int \frac{x-y}{t} d\mu(y) \right| \right)^2.$$

While a β number gives us information on the linear approximation properties of a set or a measure, C_μ tells us about the geometry of measures by capturing how far the center of mass of our object is from the center of the ball where we are focusing our attention. This quantity appears for different reasons other than pure analogy: Mattila, in [14], while investigating for what kind of measures μ in $\mathbb{C}$ does the Cauchy transform exists μ-almost everywhere (in the sense of principal values), gives a complete characterisation of what he calls *symmetric measures*; that is, measures that satisfy $C_\mu(x,t) = 0$ for all $x \in \text{spt}(\mu)$ and for all $t > 0$. He shows that any symmetric locally finite Borel measure on $\mathbb{C}$ is either discrete or continuous. In the latter case, it is either the 2-dimensional Lebesgue measure (up to a multiplicative constant) or a countable sum of 1-dimensional Hausdorff measures restricted to equidistant affine lines. Mattila needed such characterisation to understand the geometry of tangent measures of a measure μ for which the Cauchy transform exists μ-almost everywhere (in the sense of principal values) - thus to understand the geometry of μ itself. Briefly after, Mattila and Preiss (see [15]) generalised this to the higher dimensional equivalent.

More recently, Mayboroda and Volberg in [16] proved the following.

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[1] See U.K. Le Guin, *The Dispossessed*. However, the reader should be warned: analogy can be misleading, see [6], Introduction.

Theorem 1.1 *Let μ be a finite measure with finite and positive n-dimensional upper density μ-almost everywhere. If*

$$\int_0^\infty |C_\mu^n(x,t)|^2 \frac{dt}{t} < \infty \ \text{ for } \mu - \text{almost all } x \in \mathbb{R}^d, \tag{1.3}$$

then μ is n-rectifiable.

Recall that the n-dimensional upper density of μ at x is given by

$$\theta^{*,n}(x,\mu) := \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^n}. \tag{1.4}$$

The same result appeared as a corollary of the work of Jaye, Nazarov and Tolsa in [11], Subsection 1.6.

In this paper we prove the converse.

Theorem 1.2 *Let μ be an n-rectifiable measure in $\mathbb{R}^d$. Then*

$$\int_0^\infty |C_\mu^n(x,t)|^2 \frac{dt}{t} < \infty \ \text{ for } \mu - \text{almost all } x \in \mathbb{R}^d, \tag{1.5}$$

Thus, together with Theorem 1.1, we have a characterisation of rectifiability.

Corollary 1.3 *Let μ be a measure on $\mathbb{R}^d$ so that $0 < \theta^{*,n}(x,\mu) < \infty$ for μ-almost all $x \in \mathbb{R}^d$. Then μ is n-rectifiable if and only if*

$$\int_0^\infty |C_\mu^n(x,t)|^2 \frac{dt}{t} < \infty \ \ \mu - \text{almost all } x \in \mathbb{R}^d, \tag{1.6}$$

Theorem 1.2 will follow from the other result of this paper.

Theorem 1.4 *Let μ be an n-ADR measure on $\mathbb{R}^d$. If μ is uniformly rectifiable then, for each ball B centered on $\text{spt}(\mu)$ and with radius r_B,*

$$\int_B \int_0^{r_B} |C_\mu(x,t)|^2 \frac{dt}{t} d\mu(x) \lesssim r_B^n. \tag{1.7}$$

The condition in (1.7) is analogous to (1.2); it says that $|C_\mu(x,t)|^2 d\mu(x)\frac{dt}{t}$ is a Carleson measure on $\text{spt}(\mu) \times (0,\infty)$.

It is natural to ask whether a similar characterisation can be proven for more general kernels. Note that $K(x) = |x|Id\left(\frac{x}{|x|}\right)$, where $Id : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ is the identity map of the sphere to itself. Now suppose we perturb the identity, in the following sense. Let $\Omega : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ be an odd, twice continuously differentiable map which is also bi-Lipshitz with constant $1 + \delta_\Omega$. Given a Radon measure μ, one can define the corresponding perturbed C_μ^n number, i.e.

$$C_{\Omega,\mu}^n(x,t)^2 := \left(\frac{1}{t^n}\left|\int \frac{|x-y|\Omega\left(\frac{x-y}{|x-y|}\right)}{t} d\mu(y)\right|\right)^2.$$

We prove the following characterisation of uniform rectifiability in the plane.

Theorem 1.5 *Let Ω as above and suppose that δ_Ω is sufficiently small[2]. Let μ be an Ahlfors 1-regular measure on $\mathbb{C}$. Then μ is uniformly 1-rectifiable if and only if $|C_{\Omega,\mu}^1(x,t)|^2 \frac{dt}{t}d\mu(x)$ is a Carleson measure on $\text{spt}(\mu) \times (0,\infty)$.*

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[2] We require δ_Ω to be smaller than a universal constant. For example, $\delta_\Omega < 1/10$ will work.

Remark 1.6 In the case $\Omega = Id$, that the Carleson measure condition implies uniform rectifiability follows from [11]. However, for general Ω, this is novel and is an application of Theorem 1.4 in [22].

A counterpart of Theorem 1.2 holds in this case, too.

Theorem 1.7 *Let μ be a 1-rectifiable measure on $\mathbb{C}$. Then*

$$\int_0^\infty |C_{\Omega,\mu}(x,t)|^2 \, \frac{dt}{t} < \infty \text{ for μ-almost all } x \in \mathbb{C}.$$

Remark 1.8 We will not prove Theorems 1.5 and 1.7 in great details, since the proofs are very similar to those for the case where $\Omega = Id$; we will highlight the places where a slight change is needed. However, we will briefly illustrate on how to apply Theorem 1.4 from [22] in proving one direction of Theorem 1.5.

Remark 1.9 Now, a remark added two year later, in 2022. *A posteriori*, this work finds the following further motivation. In an upcoming article with J. Azzam and M. Mourgoglou [2], we show that a form of Dorronsoro's theorem holds for functions defined on uniformly rectifiable sets. As explained there, the current work could (work in progress) be used to show a variation of [2], based on [1], where the coefficients involved *are purely metrical*. This might be of interest in the context of analysis in metric space.

1.1 Outline of the proof

We first show that if a set is uniformly rectifiable, then we have the Carleson estimate (1.7). To show this, we follow the strategy in [3], where the same is shown for a different square function, involving differences in densities. To prove Theorem 1.2, we then borrow the techniques and ideas from [21], where the same is shown, but again, for the square function mentioned before.

2 Preliminaries

We collect some notions and theorem from the literature and some lemmas which will be needed later.

2.1 Notation

We gather here some notation and some results which will be used later on. We write $a \lesssim b$ if there exists a constant C such that $a \leq Cb$. By $a \sim b$ we mean $a \lesssim b \lesssim a$. In general, we will use $d \in \mathbb{N}$ to denote the dimension of the ambient space $\mathbb{R}^d$, while we will use n, with $n \leq d - 1$, to denote the 'dimension' of a measure μ, in the sense of n-Ahlfors regularity.

For sets $A, B \subset \mathbb{R}^n$, we let

$$\text{dist}(A, B) := \inf_{a \in A, b \in B} |a - b|.$$

For a point $x \in \mathbb{R}^n$ and a subset $A \subset \mathbb{R}^n$,

$$\text{dist}(x, A) := \text{dist}(\{x\}, A) = \inf_{a \in A} \text{dist}(x, a).$$

We write

$$B(x, t) := \{y \in \mathbb{R}^n \mid |x - y| < t\},$$

and, for $\lambda > 0$,

$$\lambda B(x, t) := B(x, \lambda t).$$

At times, we may write $\mathbb{B}$ to denote $B(0, 1)$. When necessary we write $B^n(x, t)$ to distinguish a ball in $\mathbb{R}^n$ from one in $\mathbb{R}^d$, which we may denote by $B^d(x, t)$.

We will also write

$$\beta_\mu^{p,d}(x, t) := \beta_\mu^{p,d}(B(x, t)).$$

Let $A \in \mathbb{R}^n$ and $0 < \delta \leq \infty$. Set

$$\mathcal{H}_\delta^d(A) := \inf \left\{ \sum (\mathrm{diam}(A_i))^d \mid A \subset \cup_i A_i \text{ and } \mathrm{diam}(A_i) \leq \delta \right\}.$$

The d-dimensional Hausdorff measure of A is then defined by

$$\mathcal{H}^d(A) := \lim_{\delta \to 0} \mathcal{H}_\delta^d(A).$$

2.2 Intrinsic cubes with small boundaries

The following construction, due to David [5], provides us with a dyadic decomposition of the support of an AD-regular measure. Such construction has been extended by Christ [4] to spaces of homogeneous type and further refined by Hytönen and Martikainen [10]. Here is the construction.

Theorem 2.1 *Let μ be an n-AD regular measure in $\mathbb{R}^d$. There exists a collection $\mathcal{D}_\mu$ of subsets $Q \subset \mathrm{spt}(\mu)$ with the following properties.*

(1) *We have*

$$\mathcal{D}_\mu = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_\mu^j,$$

where $\mathcal{D}_\mu^j$ can be thought as the collection of cubes of sidelength 2^{-j}.
(2) *For each $j \in \mathbb{Z}$,*

$$\mathrm{spt}(\mu) = \bigcup_{Q \in \mathcal{D}_\mu^j} Q.$$

(3) *If $j \leq i$, $Q \in \mathcal{D}_\mu^j$, $Q' \in \mathcal{D}_\mu^i$, then either $Q \subset Q'$ or else $Q \cap Q' = \emptyset$.*
(4) *If $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_\mu^j$, then there exists a constant $C_0 \geq 1$ so that*

$$C_0^{-1} 2^{-j} \leq \mathrm{diam}(Q) \leq C_0 2^{-j}, \text{ and}$$
$$C_0^{-1} 2^{-jn} \leq \mu(Q) \leq C_0 2^{-jn}.$$

(5) *If $j \in \mathbb{Z}$, $Q \in \mathcal{D}_\mu^j$ and $0 < \tau < 1$, then*

$$\mu(\{x \in Q \mid \mathrm{dist}(x, \mathrm{spt}(\mu) \backslash Q)\}) \leq C \tau^{\frac{1}{C}} 2^{-nj}.$$

For a proof of this, see Appendix 1 in [5].

Notation 2.2 *For $Q \in \mathcal{D}_{\mu}^{j}$, we set*

$$\ell(Q) := 2^{-j}. \tag{2.1}$$

We will denote the center of Q by z_Q. Furthermore, we set

$$B_Q := B(z_Q, 3 \operatorname{diam}(Q)). \tag{2.2}$$

2.3 The weak constant density condition

We follow the definition given in [3]. Let μ be an n-AD regular measure on $\mathbb{R}^d$. We denote by $A(c_0, \epsilon)$ the set of points $(x, t) \subset \operatorname{spt}(\mu) \times (0, \infty)$ such that there exists a Borel measure $\sigma = \sigma_{x,t}$ which satisfies the following three conditions.

(1) $\operatorname{spt}(\sigma) = \operatorname{spt}(\mu)$.
(2) The measure σ is n-AD regular with constant c_0.
(3) It holds

$$|\sigma(B(y, r)) - r^n| \leq \epsilon r^n$$
$$\text{for all } y \in \operatorname{spt}(\mu) \cap B(x, t) \text{ and for all } 0 < t < r. \tag{2.3}$$

Definition 2.3 A Borel measure μ on $\mathbb{R}^d$ is said to satisfy the weak constant density condition (WCD), if there exists a constant $c_0 > 0$ such that the complement in $\operatorname{spt}(\mu) \times (0, \infty)$ of the set $A(c_0, \epsilon)$ defined above is a Carleson set for every $\epsilon > 0$, that is, for every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ so that

$$\int_0^R \int_{B(x,R)} \mathbb{1}_{(\operatorname{spt}(\mu) \times (0 \infty)) \setminus A(c_0, \epsilon)}(x, t) \, d\mu(x) \, \frac{dt}{t} \leq C(\epsilon) \, R^n.$$

for all $x \in \operatorname{spt}(\mu)$ and $R > 0$.

The WCD condition was firstly introduced by David and Semmes in [6], Section 6. There it was proven that if a set E is n-uniformly rectifiable, then the n-dimensional Hausdorff measure restricted to E satisfies WCD. In that case, $\sigma_{x,t}$ was simply the push forward measure of $\mathcal{H}^d|_E$ onto the best approximating plane. Shortly after, in [7], they proved the converse for the dimensions $n = 1, 2, d - 1$. More recently, Tolsa [19], proved the converse for all dimension. We thus have the following theorem

Theorem 2.4 [6,7,19] *Let $n \in (0, d)$ be an integer. An AD-regular measure μ on $\mathbb{R}^d$ us n-uniformly rectifiable if and only if it satisfies WCD.*

2.4 The β and the α numbers

As mentioned in the introduction, β numbers were firstly used by Jones in [12] to understand the geometry a set or of a measure. Below we will need another quantity, introduced by Tolsa some ten years ago in [18]. They are the so called α numbers or coefficients, and they are defined as follows. For two Radon measures μ, ν on $\mathbb{R}^d$, and a ball B with radius r_B, set

$$\operatorname{dist}_B(\mu, \nu) := \sup \left\{ \left| \int f \, d\mu - \int f \, d\nu \right| \mid \operatorname{Lip}(f) \leq 1 \text{ and } \operatorname{spt}(f) \subset B \right\}. \tag{2.4}$$

One can see that dist_B defines a metric on the set of Radon measures supported on B. Fix now an n-ADR measure on $\mathbb{R}^d$. Let Q be either an instrinsic cube, i.e. $Q \in \mathcal{D}_{\mu}$ or let Q be

an actual dyadic cube of $\mathbb{R}^d$ which intersects $\mathrm{spt}(\mu)$. Recall that $B_Q = B(z_Q, 3\,\mathrm{diam}(Q))$ where z_Q is the center of the cube. Then define

$$\alpha_\mu^n(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0, L} \mathrm{dist}_{B_Q}(\mu, c\mathcal{H}^n|_L). \tag{2.5}$$

Tolsa showed the following.

Theorem 2.5 ([18], Theorem 1.2) *Let μ be an n-ADR measure. The following are equivalent:*

(1) *μ is UR.*
(2) *For any cube $R \in \mathcal{D}_\mu$, we have*

$$\sum_{Q \in \mathcal{D}_\mu, Q \subset R} \alpha(Q)^2 \mu(Q) \leq C\mu(Q),$$

with C independent of R.

Tolsa's motivation to introduce the α coefficient was, again, the study of the relationship between UR measures and $L^2(\mu)$ boundedness of Calderón–Zygmund operators; for applications, see for example Theorem 1.3 in [3,13,20].

The two coefficient β and α are related by the following inequality. See [18], Remark 3.3.

$$\frac{1}{\ell(Q)^n} \int_{B(z_Q, 2\ell(Q))} \frac{\mathrm{dist}(y, L_Q)}{\ell(Q)} \, d\mu(y) \lesssim \alpha_\mu(Q). \tag{2.6}$$

We will need the following auxiliary lemma (where we merge two lemmas from [18]).

Lemma 2.6 ([18], Lemma 5.2 and Lemma 5.4) . *Let μ be a n-uniformly rectifiable measure on $\mathbb{R}^d$. For every $R \in \mathcal{D}_\mu$, we have*

$$\sum_{\substack{Q \in \mathcal{D}_\mu \\ Q \subset R}} \int_Q \left(\frac{\mathrm{dist}(x, L_Q)}{\ell(Q)} \right)^2 \leq C(n, C_0)\ell(R)^n.$$

2.5 Wavelets

We consider a family of tensor products of Debauchies-type compactly supported wavelets with three vanishing moments (in particular, they have zero mean). They have the following properties.

(1) Each element of the family belongs to $C^1(\mathbb{R}^n)$ and it is supported on $5I$, where $I \in \mathcal{D}_n$. Recall that by $\mathcal{D}_n$ we denote the standard dyadic grid in $\mathbb{R}^n$. Hence we index the family as $\{\varphi_I\}_{I \in \mathcal{D}_n}$.
(2) For each $I \in \mathcal{D}_n$,

$$\|\varphi_I\|_{L^2(\mathbb{R}^n)} = 1. \tag{2.7}$$

(3) For each $I \in \mathcal{D}_n$ we have

$$\|\varphi_I\|_\infty \leq C \frac{1}{\ell(I)^{\frac{n}{2}}}; \tag{2.8}$$

$$\|\nabla\varphi_I\|_\infty \leq C \frac{1}{\ell(I)^{1+\frac{n}{2}}}.. \tag{2.9}$$

Then any function $f \in L^2(\mathbb{R}^n)$ can be expressed as

$$f = \sum_{I \in \mathcal{D}_n} \langle f, \varphi_I \rangle_{L^2} \, \varphi_I.$$

2.6 Preliminaries on C_μ and $C_{\mu,\phi}$

We introduce a 'smooth' version of the C_μ numbers. The reason for doing so is that such a quantity is, in general, easier to work with. Moreover, the smooth version is, in some sense, smaller than the original version (this is because smooth cut offs can be bounded above by convex combinations of cut offs).

For each $N \in \mathbb{N}$, let $\phi_N : \mathbb{R}^d \to \mathbb{R}$ be given by

$$\phi_N(x) = e^{-|x|^{2N}}. \tag{2.10}$$

We will omit the subscript N as it is unimportant for the discussion to follow. For $t > 0$, set

$$\phi_t(x) := \frac{1}{t^n} \phi\left(\frac{x}{t}\right).$$

Definition 2.7 (Smooth C number) Let μ be an AD-regular measure on $\mathbb{R}^d$. Set

$$\Phi_t(x) := \frac{x}{t} \phi_t(x).$$

Then we define the smooth version of C_μ as

$$C_{\mu,\phi}(x, t) := \Phi_t * \mu(x)$$
$$= \int \Phi_t(x - y) \, d\mu(y)$$

The following it's all well known.

Lemma 2.8 *Let μ be an n-AD-regular measure with constant c_0 on $\mathbb{R}^d$. Take ϕ_t as given above. Then*

$$t^n |C_{\mu,\phi}(x, t)| \leq C(n, c_0). \tag{2.11}$$

Proof It is easy to see that

$$t^n \int \phi_t(x - y) \, d\mu(y) \leq c_0 \Gamma\left(\frac{n+2}{2}\right) t^n.$$

Let us prove this for the sake of completeness. We use the so called layer-cake decomposition:

$$\int t^n \, \phi_t(x - y) \, d\mu(y) = \int_0^1 \int_{\mathbb{R}^d} \mathbb{1}_{\{x \in \mathbb{R}^d \mid \phi_t(x-y) > s\}}(y) \, d\mu(y) \, ds.$$

Notice that

$$\phi_t(x - y) > t \iff |x - y| < (-\ln(s))^{1/2} t.$$

Thus

$$\int_{\mathbb{R}^d} \mathbb{1}_{\{x \in \mathbb{R}^d \mid \phi_t(x-y) > s\}}(y) \, d\mu(y) = \int_{\mathbb{R}^d} \mathbb{1}_{B(x, (-\ln(s))^{1/2} t)}(y) \, d\mu(y)$$
$$\leq c_0 (-\ln(s))^{1/2} t)^n.$$

Now,

$$\int_0^1 (-\ln(s))^{\frac{n}{s}}\, ds = \Gamma\left(\frac{n+2}{2}\right).$$

If we now argue as in [8], we obtain the result. □

Remark 2.9 Clearly, the very same holds for the quantity $C^n_{\Omega,\mu,\phi}$.

3 Uniform rectifiability implies a Carleson condition

This section will be devoted to prove the following proposition.

Proposition 3.1 *Let μ be an AD n-regular measure on $\mathbb{R}^d$ with constant c_0. If μ is uniformly n-rectifiable, then for each $R \in \mathcal{D}_\mu$, we have that*

$$\sum_{\substack{Q \in \mathcal{D}_\mu \\ Q \subset R}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |C_\mu(x,t)|^2\, \frac{dt}{t} d\mu(x) \lesssim_{c_0} \mu(R). \tag{3.1}$$

The same techniques apply to prove one direction of Theorem 1.5.

Proposition 3.2 *Let μ be an AD 1-regular measure on $\mathbb{C}$ with constant c_0. If μ is uniformly 1-rectifiable, then for each $R \in \mathcal{D}_\mu$, we have that*

$$\sum_{\substack{Q \in \mathcal{D}_\mu \\ Q \subset R}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |C^1_{\Omega,\mu}(x,t)|^2\, \frac{dt}{t} d\mu(x) \lesssim_{c_0} \mu(R). \tag{3.2}$$

Let $\delta > 0$ be a small constant. To prove (3.1), we may sum only on those cubes for which

$$\alpha_\mu(1000Q) \leq \delta^2 \tag{3.3}$$

holds. Indeed, we see that

$$\sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) > \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |C_\mu(x,t)|^2\, \frac{dt}{t}, d\mu(x)$$

$$\lesssim_{c_0} \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) > \delta^2}} \int_R \int_{\ell(Q)}^{2\ell(Q)} \left(\frac{\alpha_\mu(1000Q)}{\delta^2}\right)^2 \frac{dt}{t} d\mu(x)$$

$$\lesssim_{c_0} \frac{1}{\delta^4} \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) > \delta^2}} \alpha_\mu(1000Q)^2 \mu(Q)$$

$$\leq \frac{1}{\delta^4} \sum_{Q \in \mathcal{D}_\mu(R)} \alpha_\mu(1000Q)^2 \mu(Q)$$

$$\lesssim_\delta \mu(R).$$

The last inequality follows from Tolsa's Theorem 1.2 in [18].

Remark 3.3 A recurring issue when working with the C_μ numbers is that there is no quasi-monotonicity formula (as there is for the β numbers) as we increase the radius of the ball over which we are integrating. In other words, we do not know how $C_\mu(x, t)$ compares with $C_\mu(x, 2t)$, for example. For this reason, when wanting to compare μ with, say, the Hausdorff measure on the best approximating plane, we cannot simply consider the push forward measure through an orthogonal projection. The C_μ numbers are unstable under such space deformations. Because of this, we will use instead a *circular* projection, which we learnt from [13] (see eq. 54) and from [3].

Notation 3.4 *For $x = (x_1, ..., x_n, x_{n+1}, ..., x_d) \in \mathbb{R}^d$, set*

$$x^H := (x_1, ..., x_n) \in \mathbb{R}^n.$$

We define the circular projection $\Pi^\circlearrowleft : \mathbb{R}^d \to \mathbb{R}^n$ by setting

$$\Pi^\circlearrowleft(x) := \frac{|x|}{x^H} x^H.$$

If we view $\mathbb{R}^n = V$ as a subspace of $\mathbb{R}^d$, we may write

$$\Pi^\circlearrowleft(x) = \frac{|x|}{|\Pi_V(x)|} \Pi_V(x),$$

where here Π_V is the standard orthogonal projection onto V.

For us, the fundamental property of $\Pi^\circlearrowleft$ (and the reason to use it, see the remark above) is that

$$|\Pi^\circlearrowleft(x)| = |x|.$$

This implies, in particular, that

$$\mathbb{1}_{\mathbb{B}_d} = \left(\mathbb{1}_{\mathbb{B}_n} \circ \Pi^\circlearrowleft \right). \tag{3.4}$$

Let $x \in \mathbb{R}^d$. Let L^x be an affine n-plane such that $x \in L$. We define $\Pi_{L^x}^\circlearrowleft$ as follows. Let R_{L^x} be the rotation so that $R_{L^x}(L^x)$ is parallel to $\mathbb{R}^n$. For $y \in \mathbb{R}^d$, we set

$$\Pi_{L^x}^\circlearrowleft(y) := R_{L^x}^{-1} \left(\Pi^\circlearrowleft \left(R_{L^x}(x - y) \right) + R_{L^x}(x) \right).$$

Again the fundamental property is satisfied.

$$\left| \Pi_{L^x}^\circlearrowleft(y) - x \right| = \left| R_{L^x}^{-1} \left(\Pi^\circlearrowleft \left(R_{L^x}(x - y) \right) \right) \right| = |\Pi^\circlearrowleft(R_{L^x}(x - y))|$$
$$= |R_{L^x}(x - y)| = |x - y|.$$

Fix a cube $Q \in \mathcal{D}_\mu(R)$ with $\alpha_\mu(1000Q) \leq \delta^2$ and $(x, t) \in Q \times [\ell(Q), 2\ell(Q)]$. Let L_Q be the minimising n-plane for $\alpha_\mu(Q)$ and c_Q the minimising constant. Denote by L_Q^x the n-plane parallel to L_Q so that

$$x \in L_Q^x.$$

Let $\Omega : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ be an odd, $\mathcal{C}^2$ map which is also bi-Lipschitz. For $x \in \mathbb{R}^d$, set

$$K_t(x) := \frac{|x|\Omega\left(\frac{x}{|x|}\right)}{t}.$$

Note that

$$|\nabla K_t(x)| \leq \frac{C(\Omega)}{t}. \tag{3.5}$$

Remark 3.5 We carry out some estimates at this level of generality[3]. Soon, however, we will have to restrict to the case where $\Omega = Id$ or $d = 2$. We will highlight when and why this happen below.

We write

$$t^{-n} \left| \int_{B(x,t)} K_t(x-y) \, d\mu(y) \right| \leq t^{-n} \left| \int_{B(x,t)} K_t(x-y) - K_t(x - \Pi^{\circlearrowleft}_{L^x_Q}(y)) \, d\mu(y) \right|$$

$$+ t^{-n} \left| \int_{B(x,t)} K_t(x - \Pi^{\circlearrowleft}_{L^x_Q}(y)) \, d\mu(y) \right|$$

$$=: I(x,t) + II(x,t). \tag{3.6}$$

3.1 Estimates for *I(x, t)*

Without loss of generality we may take $x = 0$ and assume that L^0_Q is parallel to $\mathbb{R}^n$. We further split our integral, so to compare the support of μ with the plane containing 0. In this way, we may be able to use the β numbers.

$$I(x,t) \leq t^{-n} \left| \int_{B(x,t)} K_t(x-y) - K_t(x - \Pi_{L^0_Q}(y)) \, d\mu(y) \right|$$

$$+ t^{-n} \left| \int_{B(x,t)} K(x - \Pi^{\circlearrowleft}_{L^0_Q}(y)) - K(x - \Pi_{L^0_Q}(y)) \right|$$

$$:= I_1(x,t) + I_2(x,t).$$

3.1.1 Estimates for *I₁*

We make one more splitting: recall that L_Q is the minimising plane of $\alpha(Q)$ and L^0_Q is its translate containing 0. Using (3.5), we write

$$I_1(x,t) \lesssim \frac{1}{t^n} \int_{B(0,t)} |y - \Pi_{L_Q}(y)| t^{-1} \, d\mu(y)$$

$$+ \frac{1}{t^n} \int_{B(0,t)} |\Pi_{L_Q}(y) - \Pi_{L^0_Q}(y)| t^{-1} \, d\mu(y) =: I_{1,1} + I_{1,2}.$$

Now, recalling the definitions in Sect. 2.4,

$$I_{1,1} = \beta^{1,n}_\mu(Q, L_Q) \lesssim \alpha_\mu(Q).$$

On the other hand, we see that

$$I_{1,2} \lesssim_{c_0} \frac{\text{dist}(0, L_Q)}{\ell(Q)}.$$

---

[3] Most of them actually hold more generally, for example it would suffice to assume that K is an odd kernel with $\|\nabla K\|_\infty \leq C$.

3.1.2 Estimates for I_2

If $y \in \mathbb{R}^d$, let θ_y be the angle between the plane $\mathbb{R}^n$ and the line segment $[0, y]$. Then,

$$|\Pi_{L_Q^0}(y) - \Pi_{L_Q^0}^{\circlearrowleft}(y)| = |\Pi_{L_Q^0}^{\circlearrowleft}(y)|(1 - \cos(\theta_y)) = |\Pi_{L_Q^0}^{\circlearrowleft}(y)| \left(\sin(\theta_y/2)^2\right)$$

$$\leq |\Pi_{L_Q^0}^{\circlearrowleft}(y)| \left(\sin(\theta_y/2)\right) = |\Pi_{L_Q^0}^{\circlearrowleft}(y)| \frac{\text{dist}(y, L_Q^0)}{|\Pi_{L_Q^0}^{\circlearrowleft}(y)|}$$

$$= \text{dist}(y, L_Q) + \text{dist}(0, L_Q). \tag{3.7}$$

Thus, as before, $I_2 \lesssim_{c_0} \alpha_\mu(Q) + \frac{\text{dist}(0, L_Q)}{\ell(Q)}$. All in all, we have $I \lesssim_{c_0} \alpha_\mu(Q) + \frac{\text{dist}(0, L_Q)}{\ell(Q)}$. Hence for general $x \in Q$, we have that

$$I \lesssim_{c_0} \alpha_\mu(Q) + \frac{\text{dist}(x, L_Q)}{\ell(Q)}.$$

Recall Lemma 2.6 and Theorem 2.5. Then we see that

$$\left(\sum_{Q \in \mathcal{D}_\mu(R)} \int_Q \int_{\ell(Q)}^{2\ell(Q)} I(x, t)^2 \, d\mu(x) \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\lesssim \left(\sum_{Q \in \mathcal{D}_\mu(R)} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left(\alpha_\mu(Q) + \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 \, d\mu(x) \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\lesssim \left(\sum_{Q \in \mathcal{D}_\mu(R)} \alpha_\mu(Q)^2 \mu(Q) \right)^{\frac{1}{2}}$$

$$+ \left(\sum_{Q \in \mathcal{D}_\mu(R)} \int_Q \left(\frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 \, d\mu(x) \right)^{\frac{1}{2}} \lesssim_{c_0} \mu(R)^{\frac{1}{2}}.$$

3.2 Estimates for $II(x, t)$

For this estimate, we let $\Omega = Id$. We will comment later on the estimate for the case $\Omega \neq Id$ and $d = 2, n = 1$. Recall that (with $\Omega = Id$),

$$II(x, t) = t^{-n} \left| \int_{B(x,t)} t^{-1} \left(\Pi_{L_Q^x}^{\circlearrowleft}(y) - x \right) d\mu(y) \right|.$$

Due to the property (3.4) of the circular projection, we notice that

$$\int_{B(x,t)} \left(\Pi_{L_Q^x}^{\circlearrowleft}(y) - x \right) d\mu(y) = \int_{\Pi_{L_Q^x}^{\circlearrowleft}(B_d(x,t))} (y - x) \, d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y)$$

$$= \int_{B_n(x,t)} (y - x) \, d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y). \tag{3.8}$$

For $y \in L_Q^x$, we may write

$$\int_{B_n(x,t)} (y - x) \, d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y) = \sum_{i=1}^{d} \left(R_{L_Q^x}^{-1} e_i \right) \int_{B_n(x,t)} (y - x) \cdot R_{L_Q^x}^{-1}(e_i) \, d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y),$$

where here $\{e_i\}_{i=1}^{d}$ is the standard orthonormal basis of $\mathbb{R}^d$. Notice that for $i = n + 1, ..., d$, and for $y \in L_Q^x$, $(x - y) \cdot R_{L_Q^x}^{-1}(e_i) = 0$. For each $i \in \{1, ..., d\}$ define

$$g_i^0 : \mathbb{R}^n \to \mathbb{R}$$

by

$$g_i^0(y) := y \cdot e_i \mathbb{1}_{B_n}(y).$$

Notice that

$$\|g_i^0\|_\infty \leq 1. \tag{3.9}$$

We then see that

$$\int_{B_n(x,t)} \left(\frac{y - x}{t} \right) d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y)$$

$$= \sum_{i=1}^{n} \left(R_{L_Q^x}^{-1} e_i \right) \int t^{-1}(x - y) \cdot R_{L_Q^x}^{-1}(e_i) \, d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y)$$

$$= \sum_{i=1}^{n} \left(R_{L_Q^x}^{-1} e_i \right) t^{-1} \int_{B_n(x,t)} (R_{L_Q^x}(y - x)) \cdot e_i \, d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y)$$

$$= \sum_{i=1}^{n} \left(R_{L_Q^x}^{-1} e_i \right) t^{-1} \int g_i^0(R_{L_Q^x}(y - x)) \, d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y)$$

Recall now Sect. 2.5. For each $i \in \{1, ..., n\}$, we may decompose g_i^0 through the wavelets basis $\{\varphi_I\}$. That is, we write g_i^0 as

$$g_i^0(y) = \sum_{I \in \mathcal{D}_n} a_I \varphi_I(y).$$

Thus,

$$\int g_i^0 \left(R_{L_Q^x} \left(\frac{y - x}{t} \right) \right) d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y)$$

$$= \sum_{I \in \mathcal{D}_n} a_I \int \varphi_I \left(R_{L_Q^x} \left(\frac{y - x}{t} \right) \right) d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y).$$

Moreover,

$$\int \varphi_I \left(R_{L_Q^x} \left(\frac{y - x}{t} \right) \right) d\Pi_{L_Q^x}^{\circlearrowleft}[\mu](y) = \int \varphi_I \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y - x) \right)}{t} \right) d\mu(y). \tag{3.10}$$

The following lemma gives us estimates on the coefficients a_I. Recall that

$$a_I = \langle g_i^0, \varphi_I \rangle_{L^2} = \int_{\mathbb{R}^n} g_i^0(y) \, \varphi_I(y) \, dy.$$

Lemma 3.6 *Let a_I, φ_I as above. Then,*

(1) *If $5I \cap \partial\mathbb{B} = \emptyset$, then $a_I = 0$.*
(2) *If $\ell(I) \lesssim 1$, $|a_I| \lesssim \ell(I)^{n/2}$.*
(3) *If $\ell(I) \gtrsim 1$, $|a_I| \lesssim \ell(I)^{-n/2-1}$.*

Proof

(1) If $5I \subset \left(\overline{\mathbb{B}}\right)^c$, then φ_I and g_i^0 have disjoint support and thus $a_I = 0$. Suppose that

$$5I \subset \mathbb{B} \text{ but that } 5I \cap \partial\mathbb{B} = \emptyset. \tag{3.11}$$

Then by Fubini's theorem,

$$
\begin{aligned}
\int_{\mathbb{R}^n} g_i^0(y)\varphi_I \, dy &= \int_{\mathbb{B}} y_i \, \varphi_I(y) \, dy \\
&\overset{(3.11)}{=} \int_{\mathbb{R}^n} y_i \, \varphi_I(y) \, dy \\
&= \int_{\mathbb{R}^n} y_i \prod_{j=1}^n \varphi_I^j(y_j) \, dy_1 \cdots dy_n \\
&= \left(\prod_{\substack{j=1 \\ j \neq i}}^n \int_{\mathbb{R}} \varphi_I^j(y_j) \, dy_j \right) \left(\int_{\mathbb{R}} y_i \varphi_I^i(y_i) \, dy_i \right) \\
&= 0
\end{aligned}
$$

since for each $i \in \{1, ..., n\}$, φ_I^i has zero mean.

(2) We see that, by (3.9) and recalling that $\|\varphi_I\|_{L^2} = 1$ for each I,

$$\left| \int_{\mathbb{R}^n} \varphi_I(y) \, g_i^0(y) \, dy \right| \leq \int |\varphi_I(y)| \, dy \leq \ell(I)^{\frac{n}{2}} \|\varphi_I\|_{L^2} = \ell(I)^{\frac{n}{2}}.$$

(3) We use the fact that

$$\int y \cdot e_i \mathbb{1}_{\mathbb{B}}(y) \, dy = 0.$$

Then we see that

$$
\begin{aligned}
\left| \int g_i^0(y)\varphi_I(y) \, dy \right| &= \left| \int g_i^0(y)(\varphi_I(y) - \varphi_I(0)) \, dy \right| \leq \int_{\mathbb{B}} |g_i^0(y)||\varphi_I(y) - \varphi_I(0)| \, dy \\
&\leq \int_{\mathbb{B}} \|\nabla\varphi_I\|_\infty |y| \, dy \\
&\lesssim \|\nabla\varphi_I\|_\infty \lesssim \ell(I)^{-\frac{n}{2}-1}
\end{aligned}
$$

$\square$

Now, notice that

$$\text{spt}\left(\varphi_I\left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowright}\left(R_{L_Q^x}(y-x)\right)}{t} \right) \right) \subset (R_{L_Q^x} \circ \Pi_{\mathbb{R}^n}^{\circlearrowright})^{-1}(t \cdot 5I) + x. \tag{3.12}$$

We see that if an n-cube satisfies either

$$5I \cap \partial \mathbb{B} = \emptyset, \tag{3.13}$$

or

$$\left[(R_{L_Q^x} \circ \Pi_{\mathbb{R}^n}^{\circlearrowleft})^{-1} (t \cdot 5I) + x \right] \cap \mathrm{spt}(\mu) = \emptyset, \tag{3.14}$$

then I is negligible for our calculations. In the first case, (3.13), the wavelet coefficients a_I will vanish identically, as from Lemma 3.6. In the second case, (3.14), the integrand on the right hand side of (3.10) and the measure μ will have disjoint support.

Notation 3.7 *We say that $I \in \mathcal{D}_n$ belongs to the subfamily of non negligible cubes $\mathcal{NG}(x, t)$ if it satisfies the two following conditions.*

$$5I \cap \partial \mathbb{B} \neq \emptyset \quad and \quad \left[(R_{L_Q^x} \circ \Pi_{\mathbb{R}^n}^{\circlearrowleft})^{-1} (t \cdot 5I) + x \right] \cap \mathrm{spt}(\mu) \neq \emptyset \tag{3.15}$$

Remark 3.8 From now on we will only consider n-cubes $I \in \mathcal{NG}(x, t)$. Moreover, following the general strategy laid out in [3], we will distinguish between n-cubes with small side length and n-cubes with large side length.

Notation 3.9 *Let $0 < \eta < 1$. We set*

$$\mathcal{BNG}(x, t) := \{I \in \mathcal{D}_n \mid I \in \mathcal{NG}(x, t) \ and \ \ell(I) \geq \eta\} \tag{3.16}$$

$$\mathcal{SNG}(x, t) := \{I \in \mathcal{D}_n \mid I \in \mathcal{NG}(x, t) \ and \ \ell(I) < \eta\} \tag{3.17}$$

3.3 $II(x, t)$: estimates when $I \in \mathcal{BNG}(x, t)$

Without loss of generality, we may let $x = 0$ and assume that L_Q^0 is parallel to $\mathbb{R}^n$.

Pick a $P = P(I) \in \mathcal{D}_\mu$ so that

$$\left(\Pi_{L_Q^0}^{\circlearrowleft} \right)^{-1} (t \cdot 5I) \subset 3P(I),$$

and so that

$$\ell(P(I)) \sim \ell(Q)\ell(I).$$

Consider a smooth cut off function χ_P so that $\mathbb{1}_{3P} \leq \chi_P \leq \mathbb{1}_{B_P}$ and such that

$$\|\nabla \chi_P\|_\infty \lesssim \frac{1}{\ell(P)}. \tag{3.18}$$

Then, for each $i \in \{1, ..., n\}$,

$$e_i \int g_i^0 \left(\frac{y}{t} \right) d\Pi_{L_Q^0}^{\circlearrowleft}[\mu](y) = e_i \sum_{I \in \mathcal{D}_n} a_I \int \varphi_I \left(\frac{\Pi_{L_Q^0}^{\circlearrowleft}(y)}{t} \right) d\mu(y)$$

$$= \sum_{I \in \mathcal{D}_n} e_i \, a_I \int \chi_P(y) \varphi_I \left(\frac{\Pi_{L_Q^0}^{\circlearrowleft}(y)}{t} \right) d\mu(y).$$

Let c_P be the constant which infimises $\alpha_\mu(P)$. As done previously, we want to compare the measure μ, its push forward through the circular projection, its push forward through the

orthogonal projection and the n-dimensional Hausdorff measure on the best approximating plane and its translate: we write

$$
\left| \int \chi_P(y)\, \varphi_I \left(\frac{\Pi_{L_Q^0}^{\circlearrowleft}(y)}{t} \right) d\mu(y) \right|
$$

$$
\leq \left| \int \chi_P(y) \left(\varphi_I \left(\frac{\Pi_{L_Q^0}^{\circlearrowleft}(y)}{t} \right) - \varphi_I \left(\frac{\Pi_{L_Q^0}(y)}{t} \right) \right) d\mu(y) \right|
$$

$$
+ \left| \int \chi_P(y)\, \varphi_I \left(\frac{\Pi_{L_Q^0}(y)}{t} \right) d\left(\mu - c_P \mathcal{H}_{L_Q^0}^n \right)(y) \right|
$$

$$
+ \left| \int \chi_P(y) \varphi_I \left(\frac{\Pi_{L_Q^0}(y)}{t} \right) c_P \, d\mathcal{H}_{L_Q^0}^n(y) \right|
$$

$$
=: II_1 + II_2 + II_3.
$$

We want to obtain the following estimate, which resembles Lemma 5.3 in [3].

Lemma 3.10 *Let* $I \in \mathcal{BNG}(x, t)$. *Take P as above. Then*

$$
II_1 + II_2 + II_3 \lesssim \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2} \left(\frac{\operatorname{dist}(0, L_Q)}{\ell(P)} + \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(2S) \right) \ell(P)^n. \qquad (3.19)
$$

Proof We arrived at a point where the quantities which we need to bound do not depend on the kernel which we started with, that is, the center of mass. We will solely use the properties of the wavelets decomposition. Thus the proof is, almost *verbatim*, the proof of Lemma 5.3 in [3].

We include it for the sake of completeness.

Estimates for II_1. We see that

$$
II_1 \leq \frac{\|\nabla \varphi_I\|_\infty}{t} \int_{B_P} \left| \Pi_{L_Q^0}^{\circlearrowleft}(y) - \Pi_{L_Q^0}(y) \right| d\mu(y).
$$

Now, as in (3.7), we have that

$$
\left| \Pi_{L_Q^0}^{\circlearrowleft}(y) - \Pi_{L_Q^0}(y) \right| \lesssim \operatorname{dist}(y, L_Q^0).
$$

Let $y \in B_P \cap \operatorname{spt}(\mu)$. Let L_P denote the n-plane which infimises $\alpha_\mu(P)$. Let $q_P \in L_P$ be so that

$$
\operatorname{dist}(y, L_P) = \operatorname{dist}(y, q_P).
$$

Notice that, because $\alpha_\mu(1000Q) \leq \delta^2$, $q_P \in B_P$ and $L_Q \cap B_P \neq \emptyset$. Then

$$
\inf_{q_Q \in L_Q} \operatorname{dist}(y, q_Q) \leq \operatorname{dist}(y, q_P) + \inf_{q_Q \in L_Q \cap B_P} \operatorname{dist}(q_P, q_Q)
$$

$$
\leq \operatorname{dist}(y, L_P) + \operatorname*{dist}_H \left(L_P \cap B_P, L_Q \cap B_P \right).
$$

Thus

$$\mathrm{dist}(y, L_Q^0) \le \mathrm{dist}(0, L_Q) + \mathrm{dist}(y, L_P) + \underset{H}{\mathrm{dist}}\left(L_P \cap B_P, L_Q \cap B_P\right).$$

By Lemma 5.2 in [18], and recalling that $P(I) \supset Q$, we see that

$$\underset{H}{\mathrm{dist}}\left(L_P \cap B_P, L_Q \cap B_P\right) \le \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(S)\ell(S). \tag{3.20}$$

Hence

$$\int_{B_P} \left| \Pi_{L_Q^0}^{\circlearrowright}(y) - \Pi_{L_Q^0}(y) \right| d\mu(y)$$

$$\lesssim \int_{B_P} \mathrm{dist}(0, L_Q) d\mu(y) + \int_{B_P} \mathrm{dist}(y, L_P)\, d\mu(y) + \int_{B_P} \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(S)\ell(S)\, d\mu(y)$$

$$\lesssim \mathrm{dist}(0, L_Q)\mu(P) + \int_{B_P} \mathrm{dist}(y, L_P)\, d\mu(y) + \ell(P)^{n+1} \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(S)$$

$$\lesssim \mathrm{dist}(0, L_Q)\mu(P) + \ell(P)^{n+1} \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(2S).$$

The last inequality is due to Remark 3.3 in [18].

All in all, we get that

$$II_1 \lesssim \|\nabla\varphi_I\|_\infty \frac{\ell(P)}{\ell(Q)} \left(\frac{\mathrm{dist}(0, L_Q)}{\ell(P)} + \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(2S) \right) \ell(P)^n.$$

Now, because $\ell(P) \sim \ell(Q)\ell(I)$, and by the bound $\|\nabla\varphi_I\|_\infty \lesssim \frac{1}{\ell(I)^{1+n/2}}$, we see that

$$\|\nabla\varphi_I\|_\infty \frac{\ell(P)}{\ell(Q)} \lesssim \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2}.$$

Thus

$$II_1 \lesssim \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2} \left(\frac{\mathrm{dist}(0, L_Q)}{\ell(P)} + \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(2S) \right) \ell(P)^n. \tag{3.21}$$

Estimates for II_2 Once again, we add and subtract the quantities which we are interested in comparing.

$$II_2 = \left| \int \chi_P(y)\, \varphi_I \left(\frac{\Pi_{L_Q^0}(y)}{t} \right) d\left(\mu - c_P \mathcal{H}_{L_Q^0}^n \right)(y) \right|$$

$$\leq \left| \int \chi_P(y)\, \varphi_I \left(\frac{\Pi_{L_Q^0}(y)}{t} \right) d\left(\mu - c_P \mathcal{H}_{L_P}^n \right)(y) \right|$$

$$+ \left| \int \chi_P(y)\, \varphi_I \left(\frac{\Pi_{L_Q^0}(y)}{t} \right) d\left(c_P \mathcal{H}_{L_P}^n - c_P \mathcal{H}_{L_Q^0}^n \right)(y) \right|$$

$$=: II_{2,1} + II_{2,2}.$$

Notice that, recalling the properties of the wavelets φ_I (see (2.8) and (2.9)),

$$\left| \nabla \left(\chi_P(y)\, \varphi_I \left(\frac{\Pi_{L_Q^0}(y)}{t} \right) \right) \right| \overset{(3.18)}{\lesssim} \frac{1}{\ell(P)} \|\varphi_I\|_\infty + \left| \nabla \varphi_I \left(\frac{\Pi_{L_Q^0}(y)}{t} \right) \right|$$

$$\lesssim \frac{1}{\ell(P)} \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2} + \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2+1} \frac{1}{\ell(Q)}$$

$$= \frac{1}{\ell(Q)} \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2+1}.$$

Hence

$$II_{2,1} \lesssim \left\| \nabla \left(\chi_P\, \varphi_I \left(\frac{\Pi_{L_Q^0}(\cdot)}{t} \right) \right) \right\|_\infty \alpha_\mu(P)\ell(P)^{n+1}$$

$$\lesssim \frac{1}{\ell(Q)} \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2+1} \alpha_\mu(P)\ell(P)^{n+1}$$

$$= \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2} \alpha_\mu(P)\ell(P)^n.$$

Similarly, and recalling (3.20) and the subsequent discussion,

$$II_{2,2} \lesssim \frac{1}{\ell(Q)} \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2+1} \operatorname*{dist}_H \left(B_P \cap L_P, B_P \cap L_Q^0 \right) \ell(P)^n$$

$$\left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2} \left(\frac{\operatorname{dist}(0, L_Q)}{\ell(P)} + \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(2S) \right) \ell(P)^n.$$

Thus

$$II_2 \lesssim \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2} \left(\frac{\operatorname{dist}(0, L_Q)}{\ell(P)} + \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(2S) \right) \ell(P)^n. \tag{3.22}$$

Estimates for II_3 Because φ_I has zero mean, we see that

$$II_3 = 0. \tag{3.23}$$

Putting together (3.21), (3.22) (3.23), we get that, when $I \in \mathcal{BNG}(x, t)$,

$$II_1 + II_2 + II_3 \lesssim \left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2} \left(\frac{\text{dist}(0, L_Q)}{\ell(P)} + \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P}} \alpha_\mu(2S)\right) \ell(P)^n. \tag{3.24}$$

$\square$

3.4 $II(x, t)$: estimates when $I \in \mathcal{SNG}(x, t)$

Remark 3.11 What we mentioned above holds here, too: the quantity we are estimating do not depend on the kernel, but rather on the properties of the wavelets decomposition. Thus this subsection will resemble very closely Subsection 5.3 in [3]: in the stopping time argument below, we stop only depending on angles, just as in [3]; the subsequent estimates will follow as they follow in [3]. As a matter of fact, the reader might end up under the impression of doing shopping at 'The Other Mathematicians' Tools Warehouse'.

Notice that, if before we had that $Q \subset P(I)$, because $\ell(I) \leq \eta$, now we have the opposite containment (up to a constant):

$$P(I) \subset CQ.$$

Choosing $\eta > 0$ appropriately, we may pick $C = 1000$. For a fixed $Q \in \mathcal{D}_\mu$, we introduce a stopping time condition on the $P \in \mathcal{D}_\mu$: let $P \in \textbf{Good}$ if the following two conditions hold true.

(1) We have

$$P \subset 1000Q. \tag{3.25}$$

(2) We have that

$$\sum_{\substack{S \in \mathcal{D}_\mu \\ P \subset S \subset 1000Q}} \alpha_\mu(100S) \leq \delta. \tag{3.26}$$

Let now **Term** be the subfamily of cubes in $\mathcal{D}_\mu \backslash \textbf{Good}$ (these are 'bad' cubes!) which are maximal with respect to inclusion. It is a well known issue that adjacent cubes belonging to **Term** may have wildly different size. This can cause troubles; we resort to a well known smoothing procedure, which will output a family of maximal 'bad' cubes which have comparable size if they are close. We define

$$\text{for } y \in \mathbb{R}^d, \ d(y) := \inf_{P \in \textbf{Good}} \left(\ell(P) + \text{dist}(y, P)\right), \tag{3.27}$$

$$\text{for } z \in \mathbb{R}^n, \ D(z) := \inf_{y \in (\Pi_{\mathbb{R}^n}^{\circ})^{-1}\{z\}} d(y). \tag{3.28}$$

Lemma 3.12 ([3], Lemma 5.6) *The function d is 1-Lipschitz and the function D is 3-Lipschitz. Moreover, if the δ in (3.3) is chosen small enough, $d(y) < \infty$ ans $D(z) < \infty$.*

Proof See Lemma 5.6 in [3] and the comment immediately after the proof. $\square$

Now set

$$\mathcal{F} := \left\{ I \in \mathcal{D}_n \mid t\ell(I) \leq \frac{1}{5000} \inf_{z \in t \cdot I} D(z) \right\}.$$

Moreover, we let **Reg** to be the subfamily of $\mathcal{F}$ of maximal cubes. That is,

$$\mathbf{Reg} := \{ I \in \mathcal{F} \mid \text{if } J \in \mathcal{F} \text{ and } J \cap I \neq \emptyset, \text{ then } I \supset J \}.$$

Lemma 3.13 *The cubes in* **Reg** *are pairwise disjoint. Moreover, if* $I, J \in$ **Reg** *and*

$$20J \cap 20I \neq \emptyset,$$

then

$$\ell(I) \sim \ell(J).$$

Proof This follows as in [6], Lemma 8.7. □

We now define two subfamilies of cubes in $\mathcal{SNG}(x, t)$, which will need each one its own treatment.

Let $I \in \mathbf{Tree}(x, t) \subset \mathcal{D}_n$ if

$$I \in \mathcal{SNG} \text{ and there is no } J \in \mathbf{Reg}(x, t) \text{ so that } J \supset I.$$

Let $I \in \mathbf{Stop}(x, t) \subset \mathcal{D}_n$ if

$$I \in \mathcal{SNG}(x, t) \cap \mathbf{Reg}(x, t).$$

We write

$$\sum_{I \in \mathcal{SNG}(x,t)} \frac{a_I}{t^n} \int \varphi_I \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y - x) \right)}{t} \right) d\mu(y)$$

$$= \sum_{I \in \mathbf{Tree}(x,t)} \frac{a_I}{t^n} \int \varphi_I \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y - x) \right)}{t} \right) d\mu(y)$$

$$+ \sum_{I \in \mathbf{Stop}(x,t)} \sum_{\substack{J \in \mathcal{SNG}(x,t) \\ J \subset I}} \frac{a_J}{t^n} \int \varphi_J \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y - x) \right)}{t} \right) d\mu(y). \qquad (3.29)$$

3.4.1 $II(x, t)$: estimates when $I \in \mathbf{Tree}(x, t)$

Lemma 3.14 *Let* $I \in \mathbf{Tree}(x, t)$. *There exists a* $P := P(I) \in \mathcal{D}_\mu$ *with*

$$\ell(P) \sim \ell(I)t \text{ and } \mathrm{spt}(\mu) \cap \left(x + \left(\Pi_{L_Q^x}^{\circlearrowleft} \right)^{-1} (t \cdot 5I) \right) \subset 3P.$$

Proof See Lemma 5.7 in [3] and the proof of it. Notice that the hypothesis

$$5I \cap (\partial B_n(0, 1) \cup \partial B_n(0, 2)) \neq \emptyset$$

is substituted with

$$5I \cap \partial B_n(0, 1) \neq \emptyset. \qquad \qquad \square$$

Lemma 3.15 *Let $I \in \mathbf{Tree}(x, t)$ and let $P := P(I)$ be a μ-cube satisfying the conclusions of Lemma 3.14. Then*

$$\left| \int \varphi_I \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft}(R_{L_Q^x}(y - x))}{t} \right) d\mu(y) \right| \lesssim \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2} \left(\sum_{\substack{S \in \mathcal{D}_\mu \\ P \subset S \subset Q}} \alpha(S) + \frac{\mathrm{dist}(x, L_Q)}{\ell(Q)} \right) \ell(P)^n. \tag{3.30}$$

Proof The proof of Lemma 5.8 in [3] can easily be adapted to our current situation; we include it for the sake of completeness. The idea, as in previous lemmata, is to bound the wavelet with the sum of angles between n-planes, or α_μ numbers, which are under control because the measure μ is uniformly rectifiable.

Without loss of generality we may take $x = 0$ and assume that L_Q is parallel to $\mathbb{R}^n$. Denoting by L_P the plane which minimises $\alpha_\mu(P)$, let $q_P \in B_P \cap \mathrm{spt}(\mu)$ be so that

$$\mathrm{dist}(q_P, L_P) \lesssim \alpha(P)\ell(P). \tag{3.31}$$

Such a point exists: for some constant $c^* > 0$, set

$$A(c*) = A$$
$$:= \left\{ y \in B_P \cap \mathrm{spt}(\mu) \mid \mathrm{dist}(y, L_P)\ell(P)^{-1} < c^*\ell(P)^{-n} \int_{B(z_P, 2\ell(P))} \frac{\mathrm{dist}(y, L_P)}{\ell(P)} \, d\mu(y) \right\}.$$

Then, by Chebyshev inequality, we see that

$$\mu(B_P \setminus A) \leq \frac{1}{c^* \beta_{\mu,1}(P)} \int_{B_P} \frac{\mathrm{dist}(y, L_P)}{\ell(P)} \, d\mu(y)$$
$$\leq \frac{\ell(P)}{c^*}.$$

Choosing c^* large enough (not depending on P), we see that $A \neq \emptyset$. Finally, recalling (2.6), we verify the existence of a point $y_P \in \mathrm{spt}(\mu) \cap B_P$ so that (3.31) holds.

Recall that L_Q^0 is the plane parallel to L_Q but containing 0. Now, denote by $\tilde{L}_P$ the plane parallel to L_Q^0 (and thus to $\mathbb{R}^n$) which moreover contains q_P. As before, let χ_P be a smooth bump function with

$$\mathbb{1}_{B_P} \leq \chi_P \leq \mathbb{1}_{3B_P} \quad \text{and} \quad \|\nabla \chi_P\|_\infty \lesssim \frac{1}{\ell(P)}. \tag{3.32}$$

Since $\alpha(P)$ is assumed to be very small, we have that

$$\left(\Pi^{\circlearrowleft} \right)^{-1} (5I \cdot t) \cap \tilde{L}_P \subset B_P. \tag{3.33}$$

Recall that $\Pi^{\circlearrowleft} = \Pi_{\{}^{\circlearrowleft} \mathbb{R}^n \}$. Set

$$\sigma_P := c_P \mathcal{H}^n|_{L_P} \text{ and}$$
$$\tilde{\sigma}_P := c_P \mathcal{H}^n|_{\tilde{L}_P}.$$

We split the integral on the left hand side of (3.30) so to compare the measure μ to σ_P, and then σ_P to $\tilde{\sigma}_P$.

$$\int \varphi_I \left(\frac{\Pi^{\circlearrowleft}(y)}{t} \right) d\mu(y) = \int \chi_P(y) \varphi_I \left(\frac{\Pi^{\circlearrowleft}(y)}{t} \right) d\mu(y)$$

$$= \int \chi_P(y) \, \varphi_I \left(\frac{\Pi^{\circlearrowleft}(y)}{t} \right) d\,(\mu - \sigma_P)\,(y)$$

$$+ \int \chi_P(y) \, \varphi_I \left(\frac{\Pi^{\circlearrowleft}(y)}{t} \right) d\,(\sigma_P - \tilde{\sigma})\,(y)$$

$$+ \int \chi_P(y) \, \varphi_I \left(\frac{\Pi^{\circlearrowleft}(y)}{t} \right) d\tilde{\sigma}_P(y)$$

$$=: A_1 + A_2 + A_3.$$

(1) *Estimates for A_1.* First, notice that since I has small size and it intersects the boundary of the unit ball on the plane, and moreover, Q lies close to L_Q^0 (that is, $\alpha_\mu(1000Q) \leq \delta$), then B_P is far from $(\Pi^{\circlearrowleft})^{-1}(\{0\})$. Thus we have

$$\|\nabla \Pi^{\circlearrowleft}\|_\infty \lesssim 1 \text{ on } B_P. \tag{3.34}$$

Moreover, recall that $\ell(P) \sim \ell(I)\,t \sim \ell(I)\,\ell(Q)$. Hence this together with the properties of the Debauchies wavelets give us

$$|A_1| \lesssim \left\| \nabla \left(\chi_P(y)\,\varphi_I \left(\frac{\Pi^{\circlearrowleft}(y)}{t} \right) \right) \right\|_\infty \alpha(P)\ell(P)^{n+1}$$

$$\overset{(3.32),(2.8),(2.9)}{\lesssim} \left(\frac{1}{\ell(P)\,\ell(I)^{n/2}} + \frac{1}{\ell(I)^{n/2+1}} \frac{1}{t} \right) \alpha(P)\ell(P)^{n+1}$$

$$\sim \left(\frac{\ell(Q)}{\ell(P)} \right)^{\frac{n}{2}} \alpha(P)\ell(P)^n. \tag{3.35}$$

(2) *Estimated for A_2.* As above, we write

$$|A_2| \lesssim \left(\frac{1}{\ell(P)\,\ell(I)^{n/2}} + \frac{1}{\ell(I)^{n/2+1}} \frac{1}{t} \right) \underset{B_P}{\text{dist}}(\sigma_P, \tilde{\sigma}_P)$$

$$\lesssim \left(\frac{1}{\ell(P)\,\ell(I)^{n/2}} + \frac{1}{\ell(I)^{n/2+1}} \frac{1}{t} \right) \ell(P)^n \underset{H}{\text{dist}}(L_P \cap B_P, \tilde{L}_P \cap B_P). \tag{3.36}$$

Now, from [3], Lemma 5.2, and using here our choice of q_P, i.e. (3.31), we see that

$$\frac{1}{\ell(P)} \underset{H}{\text{dist}}(L_P \cap B_P, \tilde{L}_P \cap B_P) \lesssim \sum_{P \subset S \subset Q} \alpha(S).$$

Hence, together with (3.36) and as in (3.35), we obtain

$$|A_2| \lesssim \left(\frac{\ell(Q)}{\ell(P)} \right)^{\frac{n}{2}} \alpha(P)\ell(P)^n. \tag{3.37}$$

(3) *Estimates for A_3.* Let B be a ball centered on L_Q^0 containing the support of $\varphi_I\left(\frac{\cdot}{t} \right)$ and with $r(B) \lesssim \ell(P)$. Let $0 < \tilde{c}$ be a constant which will be chosen below. Recall that $\tilde{L}_P$ is

the plane parallel to L_Q but containing q_P. We want to compare $\tilde{\sigma}$ to the n-dimensional Hausdorff measure restricted to L_Q^0: we further split the integral A_3 in the following way.

$$|A_3| \overset{(3.33)}{=} \left| \int \varphi_I \left(\frac{\Pi^{\circlearrowleft}(y)}{t} \right) d\tilde{\sigma}_P(y) \right|$$

$$= \left| \int \varphi_I \left(\frac{y}{t} \right) d\Pi^{\circlearrowleft}[\tilde{\sigma}_P](y) \right|$$

$$\leq \left| \int \varphi_I \left(\frac{y}{t} \right) d\Pi^{\circlearrowleft}[\tilde{\sigma}_P](y) - \tilde{c}\, c_P \int \varphi_I \left(\frac{y}{t} \right) d\mathcal{H}^n|_{L_Q^0}(y) \right|$$

$$+ \tilde{c}\, c_P \left| \int \varphi_I \left(\frac{y}{t} \right) d\mathcal{H}^n|_{L_Q^0}(y) \right|.$$

Notice that since $0 \in L_Q^0$, the second term on the right hand side equals to 0. The first term can be bounded as in (3.36) (recalling that $c_P \lesssim 1$):

$$|A_3| \lesssim \frac{1}{\ell(I)^{n/2+1}\, t} \underset{B}{\operatorname{dist}}(\Pi^{\circlearrowleft}[\mathcal{H}^n|_{\tilde{L}_P}], \tilde{c}\, \mathcal{H}^n|_{L_Q^0}).$$

We need to bound $\operatorname{dist}_B(\Pi^{\circlearrowleft}[\mathcal{H}^n|_{\tilde{L}_P}], \tilde{c}\, \mathcal{H}^n|_{L_Q^0})$.

$\square$

Sublemma 3.16 *With the notation as above, we have*

$$\underset{B}{\operatorname{dist}}(\Pi^{\circlearrowleft}[\mathcal{H}^n|_{\tilde{L}_P}], \tilde{c}\, \mathcal{H}^n|_{L_Q^0}) \lesssim \left(\sum_{P \subset S \subset Q} \alpha(S) + \frac{\operatorname{dist}(0, L_Q)}{\ell(Q)} \right) \ell(P)^{n+1}.$$

Proof of sublemma This is done in [3], see the proof of Lemma 5.8, the paragraph below it and Lemma 5.9. $\square$

This and the previous estimates for A_1 and A_2 give the desired result. $\square$
Set now

$$\mathbf{upTree}(x, t) := \left\{ P \in \mathcal{D}_\mu \mid P = P(I) \text{ for } I \in \mathbf{Tree}(x, t) \right\}.$$

Lemma 3.17 *Keep the notation as above. Then*

$$\left| \sum_{I \in \mathbf{Tree}(x,t)} \frac{a_I}{t^n} \int \varphi_I \left(\frac{\Pi^{\circlearrowleft}_{\mathbb{R}^n} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|$$

$$\lesssim \sum_{P \in \mathbf{upTree}(x,t)} \left(\alpha(aP) + \frac{\operatorname{dist}(x, L_Q)}{\ell(Q)} \right) \frac{\mu(P)}{\mu(Q)},$$

for some absolute constant $a \geq 1$.

Proof Again, one can easily adapt the proof of Lemmas 5.10 and 5.11 in [3] to the current situation. $\square$

Lemma 3.18 *With notation as above, we have*

$$
\sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \le \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left(\left| \sum_{I \in \mathbf{Tree}(x,t)} \frac{a_I}{t^n} \int \varphi_I \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \right) \frac{dt}{t} d\mu(x)
$$
$$
\lesssim \mu(R).
$$

Proof This follows as Lemma 5.12 in [3].   □

This concludes the estimates for $I \in \mathbf{Tree}(x,t)$.

3.4.2 B: estimates for $I \in \mathbf{Stop}(x,t)$

Lemma 3.19 *Keep the notation as above. Then*

$$
\sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \le \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathbf{Stop}(x,t)} \sum_{\substack{J \in \mathcal{SNG}(x,t) \\ J \subset I}} \frac{a_J}{t^n} \int \varphi_J \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)
$$
$$
\lesssim \mu(R).
$$

This follows as Lemmas 5.13 and 5.14 in [3].

3.5 Final estimates

We have that

$$
\sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \le \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} II(x,t)^2 \frac{dt}{t} d\mu(x)
$$

$$
= \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \le \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{i=1}^{d} \sum_{I \in \mathcal{NG}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)
$$

$$
\le \sum_{i=1}^{d} \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \le \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathcal{NG}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)
$$

For each $i \in \{1, ..., d\}$, we write

$$\sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \leq \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathcal{NG}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)$$

$$\overset{(3.16),(3.17)}{\lesssim} \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \leq \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathcal{BNG}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)$$

$$+ \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \leq \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathcal{SNG}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)$$

$$\overset{(3.29)}{\lesssim} \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \leq \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathcal{BNG}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)$$

$$+ \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \leq \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathbf{Tree}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)$$

$$+ \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \leq \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathbf{Stop}(x,t)} \sum_{\substack{J \in \mathcal{SNG}(x,t) \\ J \subset I}} \frac{a_J}{t^n} \int \varphi_J^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)$$

$$\lesssim \sum_{\substack{Q \in \mathcal{D}_\mu(R) \\ \alpha_\mu(1000Q) \leq \delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \left| \sum_{I \in \mathcal{BNG}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)$$

$$+ \mu(R),$$

by Lemmas 3.18 and 3.19.

Moreover, by (3.24), we have that

$$\left| \sum_{I \in \mathcal{BNG}(x,t)} \frac{a_I}{t^n} \int \varphi_I^i \left(\frac{\Pi_{\mathbb{R}^n}^{\circlearrowleft} \left(R_{L_Q^x}(y-x) \right)}{t} \right) d\mu(y) \right|^2$$

$$\lesssim \sum_{I \in \mathcal{BNG}(x,t)} \frac{a_I}{t^n} \left(\frac{\ell(Q)}{\ell(P(I))} \right)^{n/2} \left(\frac{\mathrm{dist}(0, L_Q)}{\ell(P(I))} + \sum_{\substack{S \in \mathcal{D}_\mu \\ Q \subset S \subset P(I)}} \alpha_\mu(2S) \right) \ell(P(I))^n$$

Arguing as in Lemma 5.4 and Lemma 5.5 in [3] one obtains the desired Carleson estimate. This concludes the proof of Proposition 3.1.

3.6 Estimates for $II(x, t)$ when $\Omega \neq Id$

Let us consider the estimate of $II(x, t)$ (as given in (3.6)). Recall that from Sect. 3.2 onward we assumed that $\Omega = Id$. Assume now that $d = 2$, $n = 1$ and that Ω and μ are as in the statement of Theorem 1.5.

Recall that $II(x, t) = t^{-1} \left| \int \right|_B (x, t) K_t(\Pi^{\circlearrowleft}_{L^0_Q}(y) - x) \, d\mu(y)$. We proceed as in (3.8) to arrive at

$$t^{-1} \int_{B_1(x,t)} K_t(y - x) \, d\Pi^{\circlearrowleft}_{L^0_Q}[\mu](y).$$

Note that because $K(x) = |x|\Omega(x/|x|)$, $K(L^x_Q - x)$ is a line through the origin. This is not the case if L^x_Q is an n-plane in $\mathbb{R}^d$ for $n \neq 1$ or $d \neq 2$. However, with the assumptions of Theorem 1.5, this holds, and thus we can proceed as in Subsection 3.2 to define the appropriate function g_1 as in (3.9), where this time $\{e_1, e_2\}$ will be taken to be the standard basis of $K(L^x_Q - x) \times K(L^x_Q - x)^{\perp}$. The rest of the argument goes through unchanged. This gives Proposition 3.2, and thus one direction of Theorem 1.5.

4 Rectifiability implies finiteness of square function

Define the operator C_μ by

$$C_\mu(f)(x) := \left(\int_0^\infty \left| C_{f\mu}(x, t) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Here $x \in \mathbb{R}^d$.

Here we will prove the following.

Proposition 4.1 *Let μ be a finite n-rectifiable measure on $\mathbb{R}^d$. Then*

$$C_\mu(x) < \infty \quad \text{for } \mu\text{-almost every } x \in \mathrm{spt}(\mu).$$

Remark 4.2 The same proof goes through almost verbatim for the case where $K(x) = |x|\Omega(x/|x|)$, and Ω is as in the statement of Theorem 1.7. Actually, the proof below would work in any dimensions n, d. However, it rests upon Proposition 3.1. Since we only have Proposition 3.2 for $n = 1$ and $d = 1$, we can prove Theorem 1.7 only in this case.

To prove Proposition 4.1, we will show that if μ is a finite n-uniformly rectifiable measure on $\mathbb{R}^d$, and if ν is a Borel measure, then

$$\mu\left(\left\{x \in \mathbb{R}^d \mid C_\nu(x) > \lambda\right\}\right) \leq \frac{\|\nu\|}{\lambda} \tag{4.1}$$

Now we let $\mathrm{spt}(\mu)$ be decomposed into a countable compact subsets, say $\{E_n\}$, and we let $\nu = \mu|_{E_n}$. Then, assuming (4.1),

$$\mathcal{H}^n|_{E_n}\left(\left\{x \in \mathbb{R}^d \mid C_\mu(x) > \lambda\right\}\right) \leq \frac{\|\mu\|}{\lambda}.$$

4.1 $L^2(\mu)$ boundedness of C_μ

This subsection will be devoted to proving the following proposition.

Proposition 4.3 *Let μ be an n-uniformly rectifiable measure on $\mathbb{R}^d$ such that $\mu(\mathbb{R}^d) < \infty$. The operator C_μ is bounded from $L^2(\mu)$ to $L^2(\mu)$.*

Set

$$C_{\mu,k}(f)(x) := \left(\int_{2^{-k-2}}^{2^{-k-1}} |C_{f\mu}(x,t)|^2 \, \frac{dt}{t} \right)^{1/2}.$$

We then may write

$$C_\mu(f)(x)^2 = \int_0^\infty |C_{f\mu}(x,t)|^2 \, \frac{dt}{t} = \sum_{k\in\mathbb{Z}} C_{\mu,k}(f)(x)^2.$$

We have that

$$\int |C_\mu(f)(x)|^2 \, d\mu(x) = \int \sum_{k\in\mathbb{Z}} |C_{\mu,k}(f)(x)|^2 \, d\mu(x) \tag{4.2}$$

$$= \sum_{k\in\mathbb{Z}} \int |C_{\mu,k}(f)(x)|^2 \, d\mu(x) \tag{4.3}$$

$$= \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_\mu^k} \int_Q |C_{\mu,k}(f)(x)|^2 \, d\mu(x) \tag{4.4}$$

$$= \sum_{Q\in\mathcal{D}_\mu} \int_{\mathbb{R}^d} \left| \mathbb{1}_Q(x) C_{\mu,J(Q)}(f)(x) \right|^2 \, d\mu(x). \tag{4.5}$$

Notation 4.4 *For $Q \in \mathcal{D}_\mu$,*

$$C_Q(f)(x) := \mathbb{1}_Q(x) C_{\mu,J(Q)}(f)(x).$$

Writing this out:

$$C_Q(f)(x) = \mathbb{1}_Q(x) C_{\mu,J(Q)}(f)(x)$$

$$= \mathbb{1}_Q(x) \int_{\ell(Q)/4}^{\ell(Q)/2} \left| \frac{1}{t^n} \int_{B(x,t)} \left(\frac{y-x}{t} \right) f(y) \, d\mu(y) \right|^2 \, \frac{dt}{t}.$$

Thus, for fixed $x \in Q$, $|x - y| \leq \ell(Q)/2$ and thus

$$y \in N_{\ell(Q)/2}(Q),$$

the $\frac{\ell(Q)}{2}$-neighbourhood of Q. If we set

$$\text{Neig}(Q) := \left\{ P \in \mathcal{D}_\mu \mid \ell(P) = \ell(Q) \text{ and } \text{dist}(P, Q) \leq \ell(Q) \right\},$$

we see therefore that

$$y \in \bigcup_{P\in\text{Neig}(Q)} P. \tag{4.6}$$

Notation 4.5 *For $Q \in \mathcal{D}_\mu$, set*

$$N(Q) := \bigcup_{P\in\text{Neig}(Q)} P.$$

With (4.6), we see that

$$C_{\mu, J(Q)}(f)(x) = C_{\mu, J(Q)}(\mathbb{1}_{N(Q)} f)(x).$$

Now, we may decompose $\mathbb{1}_{N(Q)} f$ through a martingale decomposition. That is, we may write

$$\mathbb{1}_{N(Q)}(x) f(x) = \sum_{R \in \text{Neig}(Q)} \left(\mathbb{1}_R(x) f_R + \sum_{P \in \mathcal{D}_\mu(R)} \Delta_P f(x) \right).$$

Recall that

$$\Delta_P f(x) = \sum_{S \in \mathcal{D}_\mu^1(P)} (f_S - f_P) \mathbb{1}_S(x).$$

We write

$$\sum_{R \in \text{Neig}(Q)} \left(\mathbb{1}_R(x) f_R + \sum_{P \in \mathcal{D}_\mu(R)} \Delta_P f(x) \right)$$

$$= \sum_{R \in \text{Neig}(Q)} \mathbb{1}_R(x) f_R + \sum_{R \in \text{Neig}(Q)} \sum_{P \in \mathcal{D}_\mu(R)} \Delta_P f(x)$$

$$= \sum_{R \in \text{Neig}(Q)} \mathbb{1}_R(x) f_R + \sum_{R \in \text{Neig}(Q)} \mathbb{1}_R(x) f_Q - \sum_{R \in \text{Neig}(Q)} \mathbb{1}_R(x) f_Q + \sum_{R \in \text{Neig}(Q)} \sum_{P \in \mathcal{D}_\mu(R)} \Delta_P f(x)$$

$$= \mathbb{1}_{N(Q)}(x) f_Q + \sum_{R \in \text{Neig}(Q)} \mathbb{1}_R(x)(f_R - f_Q) + \sum_{R \in N(Q)} \sum_{P \in \mathcal{D}_\mu(R)} \Delta_P f(x).$$

Hence we split $C_{\mu, J(Q)}$ as follows.

$$C_{\mu, J(Q)}(f)(x)$$

$$= \mathbb{1}_Q(x) \left(\int_{\ell(Q)/4}^{\ell(Q)/2} \left| t^{-n} \int_{B(x,t)} \left(\frac{y - x}{t} \right) (\mathbb{1}_{N(Q)} f)(y) d\mu(y) \right|^2 \frac{dt}{t} \right)^{1/2}$$

$$\leq \mathbb{1}_Q(x) \left(\int_{\ell(Q)/4}^{\ell(Q)/2} \left| t^{-n} \int_{B(x,t)} \left(\frac{y - x}{t} \right) (\mathbb{1}_{N(Q)} f_Q)(y) d\mu(y) \right|^2 \frac{dt}{t} \right)^{1/2}$$

$$+ \mathbb{1}_Q(x) \left(\int_{\ell(Q)/4}^{\ell(Q)/2} \left| t^{-n} \int_{B(x,t)} \left(\frac{y - x}{t} \right) \left(\sum_{R \in \text{Neig}(Q)} \mathbb{1}_R(y)(f_R - f_Q) \right) d\mu(y) \right|^2 \frac{dt}{t} \right)^{1/2}$$

$$+ \mathbb{1}_Q(x) \left(\int_{\ell(Q)/4}^{\ell(Q)/2} \left| t^{-n} \int_{B(x,t)} \left(\frac{y - x}{t} \right) \left(\sum_{R \in \text{Neig}(Q)} \sum_{P \in \mathcal{D}_\mu(R)} \Delta_P f(y) \right) d\mu(y) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

Thus, also recalling (4.2), we see that

$$
\int \left| C_\mu(f)(x) \right|^2 d\mu(x)
$$

$$
\leq \sum_{Q \in \mathcal{D}_\mu} \int_Q |f_Q|^2 |C_\mu(\mathbb{1}_{N(Q)})(x)|^2 d\mu(x)
$$

$$
+ \sum_{Q \in \mathcal{D}_\mu} \int_Q \left| \left(\sum_{R \in \mathrm{Neig}(Q)} f_R - f_Q \right) C_\mu(\mathbb{1}_R)(x) \right|^2 d\mu(x)
$$

$$
+ \sum_{Q \in \mathcal{D}_\mu} \int_Q \left| \sum_{R \in \mathrm{Neig}(Q)} C_\mu \left(\sum_{P \in \mathcal{D}_\mu(R)} \Delta_P f \right)(x) \right|^2 d\mu(y)
$$

$$
=: A + B + C.
$$

4.1.1 Estimates for A and B

The terms A and B may be estimated as the first and second term in equation 4.1, page 6 of [21], to obtain

$$
A + B \lesssim \|f\|_{L^2(\mu)}^2.
$$

4.1.2 Estimate for C

We write

$$
\sum_{Q \in \mathcal{D}_\mu} \int_Q \int_{\ell(Q)/4}^{\ell(Q)/2} \left| \sum_{R \in \mathrm{Neig}(Q)} t^{-n} \int_{B(x,t)} \left(\frac{y-x}{t} \right) \left(\sum_{P \in \mathcal{D}_\mu(R)} \Delta_P f \right)(y) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x)
$$

$$
\leq \sum_{Q \in \mathcal{D}_\mu} \sum_{P \in \mathrm{Neig}(Q)} \int_Q \int_{\ell(Q)/4}^{\ell(Q)/2} \left| \sum_{P \in \mathcal{D}_\mu(R)} t^{-n} \int_{B(x,t)} \left(\frac{y-x}{t} \right) \Delta_P(y) d\mu(y) \right|^2 \frac{dt}{t} \mu(x)
$$

$$
\lesssim \sum_{Q \in \mathcal{D}_\mu} \sum_{R \in \mathrm{Neig}(Q)} \int_Q \int_{\ell(Q)/4}^{\ell(Q)/2} \left| \sum_{\substack{P \in \mathcal{D}_\mu(R) \\ P \cap \partial B(x,t) = \emptyset}} t^{-n} \int_{B(x,t)} \left(\frac{y-x}{t} \right) \Delta_P(y) d\mu(y) \right|^2 \frac{dt}{t} \mu(x)
$$

$$
+ \sum_{Q \in \mathcal{D}_\mu} \sum_{R \in \mathrm{Neig}(Q)} \int_Q \int_{\ell(Q)/4}^{\ell(Q)/2} \left| \sum_{\substack{P \in \mathcal{D}_\mu(R) \\ P \cap \partial B(x,t) \neq \emptyset}} t^{-n} \int_{B(x,t)} \left(\frac{y-x}{t} \right) \Delta_P(y) d\mu(y) \right|^2 \frac{dt}{t} \mu(x)
$$

$$
=: C_1 + C_2.
$$

Notice that if $P \subset B(x,t)^c$, then it is negligible for our computation. The term C_2 may be estimated as the third term in in equation 4.1, page 6 of [21] (and recalling Proposition 3.1).

To estimate C_1, we notice that, because $P \subset B(x,t)$, then

$$
\int_{B(x,t)} x \Delta_P f(y) d\mu(y) = 0.
$$

Thus, letting c_P to be the center of P, recalling that spt $(\Delta_P f) \subset P$, and by Cauchy - Scwhartz, we have

$$
\left| \sum_{\substack{P \in \mathcal{D}_\mu(R) \\ P \cap \partial B(x,t) \neq \emptyset}} t^{-n} \int_{B(x,t)} \left(\frac{y-x}{t} \right) \Delta_P(y) \, d\mu(y) \right|
$$

$$
= \left| \sum_{\substack{P \in \mathcal{D}_\mu(R) \\ P \cap \partial B(x,t) \neq \emptyset}} t^{-n} \int_{B(x,t)} \left(\frac{c_P - y}{t} \right) \Delta_P(y) \, d\mu(y) \right|
$$

$$
\lesssim \sum_{\substack{P \in \mathcal{D}_\mu(R) \\ P \cap \partial B(x,t) = \emptyset}} \frac{1}{\mu(Q)} \frac{\ell(P)}{\ell(Q)} \| \Delta_P f \|_{L^1(\mu)}
$$

$$
\lesssim \sum_{\substack{P \in \mathcal{D}_\mu(R) \\ P \cap \partial B(x,t) = \emptyset}} \frac{1}{\mu(Q)} \frac{\ell(P)}{\ell(Q)} \ell(P)^{\frac{n}{2}} \| \Delta_P f \|_{L^2(\mu)}
$$

Thus

$$
C_1 \lesssim \sum_{Q \in \mathcal{D}_\mu} \sum_{R \in \mathrm{Neig}(Q)} \int_Q \int_{\ell(Q)/4}^{\ell(Q)/2} \left(\sum_{\substack{P \in \mathcal{D}_\mu(R) \\ P \cap \partial B(x,t) = \emptyset}} \frac{1}{\mu(Q)} \frac{\ell(P)}{\ell(Q)} \ell(P)^{\frac{n}{2}} \| \Delta_P f \|_{L^2(\mu)} \right)^2
$$

$$
\lesssim \sum_{Q \in \mathcal{D}_\mu} \sum_{R \in \mathrm{Neig}(Q)} \sum_{P \in \mathcal{D}_\mu(R)} \left(\frac{\ell(P)}{\ell(Q)} \ell(P)^{\frac{n}{2}} \| \Delta_P f \|_{L^2(\mu)} \right)^2 \frac{1}{\mu(Q)}
$$

$$
\le \sum_{Q \in \mathcal{D}_\mu} \sum_{R \in \mathrm{Neig}(Q)} \left(\sum_{P \in \mathcal{D}_\mu(R)} \frac{\ell(P)}{\ell(Q)} \| \Delta_P f \|_{L^2(\mu)}^2 \right) \left(\sum_{P \in \mathcal{D}_\mu(R)} \frac{\ell(P)^{n+1}}{\ell(Q)} \right) \frac{1}{\mu(Q)}
$$

We see that

$$
\sum_{P \in \mathcal{D}_\mu(R)} \frac{\ell(P)^{n+1}}{\ell(Q)} \lesssim \mu(Q).
$$

Finally, by Fubini,

$$
\sum_{Q \in \mathcal{D}_\mu} \sum_{R \in \mathrm{Neig}(Q)} \left(\sum_{P \in \mathcal{D}_\mu(R)} \frac{\ell(P)}{\ell(Q)} \| \Delta_P f \|_{L^2(\mu)}^2 \right)
$$

$$
= \sum_{P \in \mathcal{D}_\mu} \| \Delta_P f \|_{L^2(\mu)}^2 \sum_{\substack{Q \in \mathcal{D}_\mu \\ Q \supset P}} \sum_{R \in \mathrm{Neig}(Q)} \frac{\ell(P)}{\ell(Q)}.
$$

Since the cardinality of $N(Q)$ is bounded above by a universal constant, we see that

$$
\sum_{\substack{Q \in \mathcal{D}_\mu \\ Q \supset P}} \sum_{R \in \mathrm{Neig}(Q)} \frac{\ell(P)}{\ell(Q)} \lesssim \sum_{\substack{Q \in \mathcal{D}_\mu \\ Q \supset P}} \frac{\ell(P)}{\ell(Q)} \lesssim 1.
$$

Thus we obtain

$$C_1 \lesssim \sum_{P \in \mathcal{D}_\mu} \|\Delta_P f\|^2_{L^2(\mu)}.$$

This together with the estimates on A, B and C_2 proves Proposition 4.3

4.2 $L^{1,\infty}(\mu)$ boundedness of C_μ

Let $M(\mathbb{R}^d)$ be the set of finite Borel measures on $\mathbb{R}^d$ and $\nu \in M(\mathbb{R}^d)$. Let the operator C on $M(\mathbb{R}^d)$ be given by

$$C_\nu(x) := \left(\int_0^\infty |C_\nu(x,t)|^2 \, \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Proposition 4.6 *Let μ be an n-uniformly rectifiable measure on $\mathbb{R}^d$. Then,*

$$\mu\left(\left\{ x \in \mathbb{R}^d \mid C_\nu(x) > \lambda \right\} \right) \leq \frac{\|\nu\|}{\lambda}$$

for each $\nu \in M(\mathbb{R}^d)$ with compact support and $\lambda > 0$.

We will use the Calderón–Zygmund decomposition for $\nu \in M(\mathbb{R}^d)$. See Theorem in [?].

Theorem 4.7 *Let μ be an n-AD-regular measure on $\mathbb{R}^d$. For every $\nu \in M(\mathbb{R}^d)$ with compact support and every $\lambda > 2^{d+1} \|\nu\| / \|\mu\|$ we have:*

(1) *There exists a finite or countable collection of dyadic cubes $\{D_j\}_{j \in J} \subset \mathcal{D}_d$ with*

$$\sum_{j \in J} \mathbb{1}_{D_j} \lesssim 1, \tag{4.7}$$

and a function $f \in L^1(\mu)$ such that, for each $j \in J$,

$$\mu(2D_j) < \frac{2^{d+1}}{\lambda} |\nu|(D_j). \tag{4.8}$$

$$\frac{2^{d+1}}{\lambda} \nu(\eta D_j) \leq \mu(2\eta D_j) \text{ for every } \eta > 2. \tag{4.9}$$

$$\nu = f\mu \quad \text{in } \mathbb{R}^d \setminus \left(\bigcup_{j \in J} D_j \right) \quad \text{with } |f| \leq \lambda \ \mu - a.e. \tag{4.10}$$

(2) *For each $j \in J$, set*

$$R_j := 6D_j \tag{4.11}$$

$$w_j := \mathbb{1}_{D_j} \left(\sum_{i \in J} \mathbb{1}_{D_i} \right)^{-1}. \tag{4.12}$$

There exsits a family of functions $\{b_j\}_{j \in J}$ with

$$\text{spt}(b_j) \subset R_j, \tag{4.13}$$

each one with constant sign, such that

$$\int b_j d\mu = \int w_j d\nu \tag{4.14}$$

$$\|b_j\|_{L^\infty(\mu)} \mu(R_j) \le c|\nu|(D_j) \tag{4.15}$$

$$\sum_{j\in J} |b_j| \le c\lambda. \tag{4.16}$$

Proof of Proposition 4.6 We keep the notation as in Theorem 4.7. Let $\nu \in M(\mathbb{R}^d)$ and fix $\lambda > 2^{d+1} \frac{\|\nu\|}{\|\mu\|}$. We write

$$d\nu = \mathbb{1}_{\mathbb{R}^d \setminus \cup_{j\in J} D_j} d\nu + \mathbb{1}_{\cup_{j\in J} D_j} d\nu = \mathbb{1}_{\mathbb{R}^d \setminus \cup_{j\in J} D_j} f\, d\mu + \sum_{j\in J} w_j\, d\nu$$

$$= \mathbb{1}_{\mathbb{R}^d \setminus \cup_{j\in J} D_j} f\, d\mu + \sum_{j\in J} b_j\, d\mu + \sum_{j\in J} (w_j d\nu - b_j d\mu).$$

Thus if we set

$$F := \bigcup_{j\in J} D_j, \quad g := f\mathbb{1}_{F^c} + \sum_{j\in J} b_j, \quad d\beta := \sum_{j\in J} (w_j d\nu - b_j d\mu),$$

we may let

$$d\nu = g\, d\mu + d\beta. \tag{4.17}$$

Let $2F = \cup_{j\in J} 2D_j$. Notice first that, because (4.8),

$$\mu\left(\{x \in 2F \mid C\nu(x) > \lambda\}\right) \le \mu(2F) \le \sum_{j\in J} \mu(2D_j) \le \frac{2^{d+1}}{\lambda} |\nu|(D_j) \lesssim_d \frac{\|\nu\|}{\lambda}.$$

Hence we may work on $(2F)^c$ only. We split C_ν as suggested by (4.17):

$$C_\nu(x) = \left(\int_0^\infty \left| t^{-n} \int_{B(x,t)} \left(\frac{x-y}{t}\right) (g(y)d\mu(y) + d\beta(y)) \right|^2\right)^{1/2}$$

$$\le \left(\int_0^\infty \left| t^{-n} \int_{B(x,t)} \left(\frac{x-y}{t}\right) g(y)\, d\mu(y) \right|^2\right)^{1/2}$$

$$+ \left(\int_0^\infty \left| t^{-n} \int_{B(x,t)} \left(\frac{x-y}{t}\right) d\beta(y) \right|^2\right)^{1/2}$$

$$= C_\mu(g)(x) + C_\beta(x).$$

Then

$$\mu\left(\{x \in (2F)^c \mid C_\nu(x) > \lambda\}\right) \le \mu\left(\left\{x \in (2F)^c \mid C_\mu(g)(x) > \frac{\lambda}{2}\right\}\right)$$

$$+ \mu\left(\left\{x \in (2F)^c \mid C_\beta(x) > \frac{\lambda}{2}\right\}\right)$$

$$=: A + B. \qquad \square$$

4.2.1 Estimates for *A*

This is easily done by noticing that $g \in L^2(\mu)$, since $g \in L^\infty(\mu)$ and $\mu(\mathbb{R}^d) < \infty$. Thus, in particular,

$$\mu\left(\left\{x \in (2F)^c \mid C_\mu(g)(x) > \lambda/2\right\}\right) \leq \frac{C}{\lambda^2} \int |g|^2 \, d\mu \lesssim \frac{1}{\lambda} \int |g| \, d\mu,$$

since, by (4.10) and (4.16),

$$\|g\|_{L^\infty(\mu,)} \leq \|f \mathbb{1}_{F^c}\|_{L^\infty(\mu)} + \left\|\sum_{j \in J} b_j\right\|_{L^\infty(\mu)} \leq c\lambda.$$

Moreover, using some of the properties listed in Theorem 4.7,

$$\int |g| \, d\mu \leq \int |f \mathbb{1}_{F^c}| \, d\mu + \int \left|\sum_{j \in J} b_j \, d\mu\right| = |\nu|(F^c) + \sum_{j \in J} \int |b_j| \, d\mu \leq \|\nu\|$$

$$+ \sum_{j \in J} \|b_j\|_\infty \mu(R_j) \leq \|\nu\| + \sum_{j \in J} c|\nu|(D_j) \lesssim \|\nu\|.$$

Thus

$$\mu\left(\left\{x \in (2F)^c \mid C_\mu(g)(x) > \lambda/2\right\}\right) \lesssim \frac{\|\nu\|}{\lambda}.$$

4.2.2 Estimates for *B*

Set $E := (2F)^c$. By Chebyshev's inequality,

$$\mu\left(\left\{x \in E \mid C_\beta(x) > \lambda/2\right\}\right) \leq \frac{2}{\lambda} \int_E C_\beta(x) \, d\mu(x)$$

$$\leq \frac{2}{\lambda} \sum_{j \in J} \int_{\mathbb{R}^d \setminus 2R_j} C_{\beta_j}(x) \, d\mu(x)$$

$$+ \frac{2}{\lambda} \sum_{j \in J} \int_{2R_j \setminus 2D_j} C_{\beta_j}(x) \, d\mu(x) =: B_1 + B_2.$$

Estimates for B_1. We write

$$\int_{\mathbb{R}^d \setminus R_j} C_{\beta_j}(x) = \int_{\mathbb{R}^d \setminus R_j} \left(\int_0^\infty \left|t^{-n-1} \int_{B(x,t)} (x-y) \, d\beta_j(y)\right|^2 \frac{dt}{t}\right)^{1/2} d\mu(x)$$

$$= \int_{\mathbb{R}^d \setminus R_j} \left(\int_{\{t:R_j \subset B(x,t)^c\}} \left|t^{-n-1} \int_{B(x,t)} (x-y) \, d\beta_j(y)\right|^2 \frac{dt}{t}\right)^{1/2} d\mu(x)$$

$$+ \int_{\mathbb{R}^d \setminus R_j} \left(\int_{\{t:R_j \subset B(x,t)\}} \left|t^{-n-1} \int_{B(x,t)} (x-y) \, d\beta_j(y)\right|^2 \frac{dt}{t}\right)^{1/2} d\mu(x)$$

$$+ \int_{\mathbb{R}^d \setminus R_j} \left(\int_{\{t:R_j \cap B(x,t)\}} \left|t^{-n-1} \int_{B(x,t)} (x-y) \, d\beta_j(y)\right|^2 \frac{dt}{t}\right)^{1/2} d\mu(x)$$

$$:= B_{1,1} + B_{1,2} + B_{1,3}$$

Clearly $B_{1,1} = 0$; moreover, since $x \in \mathbb{R}^d \backslash 2R_j$, if $t \leq \text{dist}(x, R_j)$, then $t \in \{t : R_j \subset B(x, t)^c\}$.

We estimate $B_{1,2}$. First, notice that $\beta_j(R_j) = 0$. Thus, if $t \in \{t : R_j \cap B(x, t)\}$ and letting c_j denote the center of R_j, we get

$$\left| \int_{B(x,t)} t^{-n-1}(x - y)\, d\beta_j(y) \right| = \left| \int_{B(x,t)} t^{-n-1}(c_j - y)\, d\beta_j(y) \right|$$

$$\lesssim \left(\int_{B(x,t)} t^{-n} \frac{\ell(R_j)}{|x - c_j|}\, d|\beta_j|(y) \right) = t^{-n} \frac{\ell(R_j)|\beta_j|(R_j)}{|x - c_j|}.$$

Since, if $t \in \{t : R_j \cap B(x, t)\}$, then $t > \text{dist}(x, R_j)$, then

$$C_{\beta_j}(x)^2 \lesssim \int_{\text{dist}(x,R_j)}^{\infty} \left(t^{-n} \frac{\ell(R_j)|\beta_j|(R_j)}{|x - c_j|} \right)^2 \frac{dt}{t} = \frac{\ell(R_j)^2|\beta_j|(R_j)^2}{|x - c_j|^2} \int_{\text{dist}(x,R_j)}^{\infty} t^{-n-1}\, dt$$

$$\lesssim \frac{\ell(R_j)^2|\beta_j|(R_j)^2}{|x - c_j|^2} \frac{1}{|x - c_j|^{2n}}.$$

Recalling that μ is n-uniformly rectifiable and thus n-AD-regular,

$$\int_{\mathbb{R}^d \backslash 2R_j} \frac{1}{|x - c_j|^{n+1}}\, d\mu(x) = \sum_{k=0}^{\infty} \int_{2^k(2\ell(R_j)) \leq |c_j - x| \leq 2^{k+1}(2\ell(R_j))} \frac{1}{|x - c_j|^{n+1}}\, d\mu(x)$$

$$\leq \sum_{k=0}^{\infty} \frac{\mu(B(c_j, 2^{k+1}2\ell(R_j)))}{2^{(n+1)k}(2\ell(R_j))^{n+1}}$$

$$\lesssim_{c_0} \sum_{k=0}^{\infty} \frac{2^{(k+1)n}(2\ell(R_j))^n}{2^{(n+1)k}(2\ell(R_j))^{n+1}}$$

$$\lesssim \frac{1}{\ell(R_j)}.$$

All in all, we then see that

$$\int_{\mathbb{R}^d \backslash 2R_j} C_{\beta_j}(x)\, d\mu(x) \lesssim \ell(R_j)|\beta_j|(R_j) \int_{\mathbb{R}^d \backslash 2R_j} \frac{1}{|x - c_j|^{n+1}}\, d\mu(x) \lesssim |\beta_j|(R_j).$$

One can easily estimate $B_{1,3}$ by arguing as Tolsa and Toro [21]; see the treatment of the first term on the right hand side of equation 5.10 at page 10 there.

Estimates for B_2. Applying Cauchy–Schwartz, We write

$$\int_{2R_j \backslash 2R_j} C_{\beta_j}(x)\, d\mu(x) \leq \mu(R_j)^{n/2} \left(\int_{2R_j \backslash 2R_j} |C_{\beta_j}(x)|^2\, d\mu(x) \right)^{1/2}$$

$$\leq \mu(R_j)^{n/2} \left(\int_{2R_j \backslash 2R_j} |C_\mu(b_j)(x)|^2\, d\mu(x) \right)^{1/2}$$

$$+ \mu(R_j)^{n/2} \left(\int_{2R_j \backslash 2R_j} |C_\nu(w_j)(x)|^2\, d\mu(x) \right)^{1/2}.$$

Since $b_j \in L^\infty(\mu)$ with compact support, then $b_j \in L^2(\mu)$. Thus, by Proposition 4.3, we see that

$$\mu(R_j)^{n/2} \left(\int_{2R_j \setminus 2R_j} |C_\mu(b_j)(x)|^2 \, d\mu(x) \right)^{1/2} \lesssim \mu(R_j)^{n/2} \|b_j\|_{L^2(\mu)}$$

$$\leq \mu(R_j)^{n/2} \mu(R_j)^{n/2} \|b_j\|_\infty$$

$$\leq |\nu|(D_j).$$

On the other hand, recalling that $\mathrm{spt}(w_j) \subset D_j$ and $\|w_j\|_\infty \leq 1$,

$$C_\nu(w_j)(x)^2 = \int_0^\infty \left| t^{-n} \int_{B(x,t)} (x-y) \, w_j(y) \, d\nu(y) \right|^2 \frac{dt}{t}$$

$$\leq \int_{\ell(R_j)}^\infty \left| t^{-n} \int_{B(x,t)} (x-y) \, w_j(y) \, d\nu(y) \right|^2 \frac{dt}{t}$$

$$\leq \int_{\ell(R_j)}^\infty \frac{\ell(R_j)^2 |\nu|(D_j)^2}{t^{2n+2}} \, d\nu(y) \frac{dt}{t} \leq \frac{|\nu|(D_j)^2}{\ell(R_j)^{2n}}.$$

Hence, we see that

$$\mu(R_j)^{n/2} \left(\int_{2R_j \setminus 2R_j} |C_\nu(w_j)(x)|^2 \, d\mu(x) \right)^{1/2} \lesssim \mu(R_j)^{n/2} \mu(R_j)^{n/2} \frac{|\nu|(D_j)}{\ell(R_j)^n} \lesssim |\nu|(D_j).$$

All in all, recalling (4.7), we see that

$$B_1 + B_2 \lesssim \frac{1}{\lambda} \sum_{j \in J} |\nu|(D_j) \lesssim \frac{\|\nu\|}{\lambda}.$$

$\square$

As in the proof of Theorem 5.1 in [21], we have the following corollary.

Corollary 4.8 *Proposition 4.6 holds for any $\nu \in M(\mathbb{R}^d)$.*

Proof Fix N_1 be an arbitrary positive number. Pick $N_2 \geq 2N_1$ and such that $|N_1 - N_2| > \frac{2\|\nu\|}{\lambda}$. We see that, if $x \in B(0, N_1)$,

$$C_\nu(\mathbb{1}_{\mathbb{R}^d \setminus B(0,N_2)})(x)^2 = \int_0^\infty \left| t^{-n} \int_{B(x,t)} \left(\frac{x-y}{t} \right) \mathbb{1}_{\mathbb{R}^d \setminus B(0,N_2)} \, d\nu(y) \right| \frac{dt}{t}$$

$$\leq \int_{|N_1-N_2|}^\infty \left| t^{-n} \int_{B(x,t)} \left(\frac{x-y}{t} \right) \mathbb{1}_{\mathbb{R}^d \setminus B(0,N_2)} \, d\nu(y) \right| \frac{dt}{t}$$

$$\lesssim |\nu|(\mathbb{R}^d \setminus B(0, N_2)) \int_{|N_1-N_2|}^\infty \frac{1}{t^{2n+1}} \, dt$$

$$= \frac{|\nu|(\mathbb{R}^d \setminus B(0, N_2))}{|N_1 - N_2|^{2n}}.$$

Thus, by our choice of N_2, we see that

$$C_\nu(\mathbb{1}_{\mathbb{R}^d \setminus B(0,N_2)})(x) \leq C \frac{\lambda}{2}.$$

Notice that, here, the constant C does *not* depend on N_1.

Set $v_2 := v|_{B(0,N_2)}$. We see that

$$\mu\left(\{x \in B(0, N_1) \,|\, C_v(x) > \lambda\}\right) \leq \mu\left(\left\{x \in B(0, N_1) \,|\, C_{v_2}(x) > \lambda/2\right\}\right)$$
$$\lesssim \frac{\|v_2\|}{\lambda} \leq \frac{\|v\|}{\lambda}.$$

Since these estimates do not depend on N_1, we may let $N_1 \to \infty$. □

5 Proof of Theorem 1.5

What is left to do is to prove the remaining direction in Theorem 1.5. That is, we need to prove the following.

Proposition 5.1 *Let μ be an Ahlfors 1-regular measure in $\mathbb{C}$. Let Ω be as in the statement of Theorem 1.5. If $|C^1_{\Omega,\mu,\phi}(x,t)|^2 \frac{dt}{t} d\mu(x)$ is a Carleson measure on $\mathrm{spt}(\mu) \times (0,\infty)$, then μ is uniformly 1-rectifiable.*

Recall that

$$C^1_{\Omega,\mu,\phi}(x,t) := t^{-1} \int \frac{|x-y| \Omega((x-y)/|x-y|)}{t} e^{-\left|\frac{x-y}{t}\right|^{2N}} d\mu(y).$$

The proof of Proposition 5.1 is a standard compactness argument. For example, one can follow [3] almost verbatim. There is one important difference, however. In the limit, one ends up with a measure which satisfies $C^1_{\Omega,v,\phi}(x,t) = 0$ for all $t > 0$ and $x \in \mathrm{spt}(v)$. It then follows from an argument similar to the proof of Lemma 3.6 in [3], that v is an Ω-symmetric measure, that is, a measure satisfying $C_{\Omega,v}(x,t) = 0$ for all $t > 0$ and $x \in \mathrm{spt}(v)$. Then one needs to appeal to Theorem 1.4 in [22] to conclude that such a measure is flat. With this, one can close the compactness argument and continue as in [3].

The remaining direction of Theorem 1.5 follows from Proposition 5.1 since

$$\int_B \int_0^{r(B)} |C^1_{\Omega,\mu,\phi}(x,t)|^2 \frac{dt}{t} d\mu(x) \lesssim \int_B \int_0^{r(B)} |C^1_{\Omega,\mu}(x,t)|^2 \frac{dt}{t} d\mu(x).$$

This is a standard fact. A proof can be found for example in [3], Corollary 3.12.

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