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DIRECT METHODOLOGY FOR ESTIMATING THE RISK NEUTRAL PROBABILITY DENSITY FUNCTION

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Abstract

The target of the study is to find out if the direct methodology could provide same information about the parameters of the risk neutral probability density function (RND) than the reference RND methodologies. The direct methodology is based on for defining the parameters of the RND from underlying asset by using futures contracts and only few at-the-money (ATM) and/or close at-the-money (ATM) options on asset. Of course for enabling the analysis of the feasibility of the direct methodology the reference RNDs must be estimated from the option data. Finally the results of estimating the parameters by the direct methodology are compared to the results of estimating the parameters by the selected reference methodologies for understanding if the direct methodology can be used for understanding the key parameters of the RND.

The study is based on S&P 500 index option data from year 2008 for estimating the reference RNDs and for defining the parameters from the reference RNDs. The S&P 500 futures contract data is necessary for finding the expectation value estimation for the direct methodology. Only few ATM and/or close ATM options from the S&P 500 index option data are necessary for getting the standard deviation estimation for the direct methodology. Both parametric and non-parametric methods were implemented for defining reference RNDs. The reference RND estimation results are presented so that the reference RND estimation methodologies can be compared to each other. The moments of the reference RNDs were calculated from the RND estimation results so that the moments of the direct methodology can be compared to the moments of the reference methodologies.

The futures contracts are used in the direct methodology for getting the expectation value estimation of the RND. Only few ATM and/or close ATM options are used in the direct methodology for getting the standard deviation estimation of the RND. The implied volatility is calculated from option prices using ATM and/or close ATM options only. Based on implied volatility the standard deviation can be calculated directly using time scaling equations. Skewness and kurtosis can be calculated from the estimated expectation value and the estimated standard deviation by using the assumption of the lognormal distribution.

Based on the results the direct methodology is acceptable for getting the expectation value estimation using the futures contract value directly instead of the expectation value, which is calculated from the RND of full option data, if and only if the time to maturity is relative short. The standard deviation estimation can be calculated from few ATM and/or at close ATM options instead of calculating the RND from full option data only if the time to maturity is relative short. Skewness and kurtosis were calculated from the expectation value estimation and the standard deviation estimation by using the assumption of the lognormal distribution. Skewness and kurtosis could not be estimated by using the assumption of the lognormal distribution because the lognormal distribution is not correct generic assumption for the RND distributions.

Keywords
Risk neutral probability density function, RND, direct methodology, S&P index options
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1 INTRODUCTION

The derivative markets provide dynamic information about market situation. The derivative markets are typically sensitive for any static and dynamic changes in the markets. The derivative markets provides useful source of information for gauging the current state and the trends of the markets. The derivative is a financial product, which price is depending on or derived from one or more underlying assets. The derivative is basically contract agreement between two or more parties. The value of the derivative is determined finally by markets but the value of the underlying asset is determining the value of the derivative. The valuation of the derivatives is typically complicated process from theoretical point of view. Futures contracts, forward contracts, options and swaps are the most common types of derivatives. The derivatives are interesting financial instruments, which are typically used as an instrument for hedging financial risks. The derivatives can also be used for speculative purposes.

The option payoffs depend on the future development of the underlying asset. The prices of the different option contracts reflect the market view of the probability that the contracts yield positive payoffs. The prices of the options on a specific asset with different strike prices but with same time-to-maturity are indicating market assessment of the probability of the payoff across strike prices. The risk neutral probability density function (RND) can be recovered from the option pricing information. The RNDs can be referred to additional information about the asset value evolution, which is critical for many different type of financial analysis, because the estimated RND is predicting the distribution of the asset value in future. The RND can be used for risk analysis and for making forecast about the future trends of the asset value.

The target of the study is to find out if the direct methodology could provide same information about the parameters of the risk neutral probability density function (RND) than the reference RND methodologies. The direct methodology is based on for defining the parameters of the RND from underlying asset by using futures contracts and only few at-the-money and/or close at-the-money options on specific asset. Of course for enabling the analysis of the feasibility of the direct methodology the reference RNDs must be estimated from the option data. Finally the results of estimating the parameters by the direct methodology are compared to the results of
estimating the parameters by the selected reference methodologies for understanding if the direct methodology can be used for understanding the key parameters of the RND.

The study is targeting to find out if the direct methodology can be used for finding the moments of the risk neutral probability density function from underlying asset by using futures contracts and only few at-the-money options on that specific asset. Futures contracts are used in the direct methodology for getting the expectation value estimation of the RND. Only few at-the-money and/or close at-the-money options are used for getting the standard deviation estimation of the RND. The implied volatility is calculated from option prices using at-the-money and/or close at-the-money options only. Based on implied volatility the standard deviation can be calculated directly using time scaling equations. Skewness and kurtosis can be calculated from the estimated expectation value and the estimated standard deviation by using the assumption of the lognormal distribution.

Different numerical methodologies have been used for defining the RND from options on the specific asset. Many of these numerical techniques are relative difficult to implement for defining the RNDs and the moments of the RNDs. Many numerical techniques have limitations for defining the RND because of potential problems with numerical issues. Many articles are available for comparing different techniques but typically only a few techniques are compared in same articles. The literature on the extraction of the RNDs is unsettled so that there are not real consensus of the best RND estimation techniques. Reliable results are not always available because many typical techniques are based on some tuning of parameters for estimating the RNDs.

As summary the earlier studies for defining RNDs are focusing for the comparison of only limited set of RND techniques, which is problematic for getting reliable results for comparing different techniques to each other. Typically option data set is different in different studies. The implementation details are also missing related to the implementation of the different techniques. As conclusion it’s very difficult to get reliable overview of the performance of the different techniques.

The direct methodology is studied for finding out if the direct methodology could provide same information about the key parameters of the RND than the reference
RND methodologies. Most of studies related to defining RNDs are focused for comparing different techniques for finding RNDs. Of course for enabling the analysis of the direct methodology the reference RNDs must be calculated from option data.

Three different numerical techniques were selected as reference methods so that both parametric and non-parametric methods were implemented for defining reference RNDs. The reference RND estimation results are presented so that the reference RND estimation methodologies can be compared to each other. The RND results are telling more visually about the form of the distribution. The moments of the reference RNDs were calculated from the RND estimation results so that the moments of the direct methodology can be compared to the moments of the reference methodologies.

Finally the results of using the direct methodology were compared to the results of the selected reference methods for understanding if the direct methodology can be used for understanding the key parameters of the RND.

The study is based on S&P 500 index option data from year 2008 for estimating the reference RNDs and for defining the reference moments from the reference RNDs. The S&P 500 futures contract data is necessary for finding the expectation value estimation for the direct methodology. Only at-the-money and/or close at-the-money options from the S&P 500 index option data are necessary for getting the standard deviation estimation for the direct methodology. VIX index option data from 2008 is necessary for the direct methodology for finding standard deviation estimation for reference.

Based on the analysis of the results the direct methodology is acceptable for getting the expectation value estimation using the futures contract value directly instead of the expectation value, which is calculated from the RND of full option data, if and only if the time-to-maturity is relative short. The standard deviation estimation can be calculated from only few at-the-money and/or close at-the-money options instead of calculating the RND using all options in reference methodology. Based on the analysis of the results the direct methodology is acceptable for getting the standard deviation estimation, which is calculated using only few at-the-money and/or close at-the-money options instead of calculating the standard deviation of the RND of full option data, if the time-to-maturity is relative short. The results were also compared to the VIX index
for getting other standard deviation estimation. Skewness and kurtosis were calculated from the expectation value estimation and the standard deviation estimation by using the assumption of the lognormal distribution but the results were not acceptable because the results of calculation are providing only positive values for skewness and incorrect values for kurtosis. As conclusion values for skewness and kurtosis were not close to the values of the reference methods. Skewness and kurtosis could not be estimated by using the assumption of the lognormal distribution because lognormal distribution is not correct generic assumption for RNDs.

The option pricing theory is introduced in chapter 2. Mathematics of statistical distributions for understanding the results of the analysis in more details is presented in chapter 3. The base theory for defining the RNDs from option prices is presented in chapter 4 for understanding reference methodology. The overview of the different type of techniques for defining the RNDs with different methodologies is presented in chapter 5. The theory of the reference methodologies are presented in chapter 6 for understanding implemented techniques in more details for defining the RNDs. The direct methodology for finding estimation of the parameters is presented in chapter 7. More details about data and methodology are presented in chapter 8. The results of the analysis are presented in chapter 9 by comparing the results of the direct methodology to the results of the reference techniques. Conclusion is presented in chapter 10.
2 OPTION PRICING THEORY

A call option gives the holder of the option the right to buy an asset by a certain date for a certain price. A put option gives the holder of the option the right to sell an asset by a certain date for a certain price. The date specified in the contract is known as the expiration date or the maturity date. The price specified in the contract is known as the exercise price or the strike price. European options can be exercised only on the expiration date. American options can be exercised at any time up to the expiration date. (Hull 2011: 194.)

2.1 Option price – generic view

The prices of European call and put options at time $t$ can be defined as generic form as the discounted sums of all expected future payoffs (Cox and Ross 1976):

$$
c(X, t) = e^{-rt} \int_{X}^{\infty} q(S_T) (S_T - X) dS_T \tag{1}
$$

$$
p(X, t) = e^{-rt} \int_{0}^{X} q(S_T) (X - S_T) dS_T \tag{2}
$$

where $c$ is price of call option, $p$ is price is put option, $X$ is strike price, $S_T$ is asset price at time $T$, $r$ is risk-free interest rate and $t$ is time. The prices of options can be calculated by integrating equations directly if and only if the form of the density function $q(S_T)$ is available. In practice, the real density function is relative difficult to define directly.

2.2 Stock price - process view

Stochastic processes can be classified as discrete time or continuous time. Stochastic processes can also be classified as continuous variable or discrete variable. Continuous time and continuous variable stochastic process is critical for understanding the pricing of options and other more complicated derivatives. The price of stock option is a function of the underlying stock price and time. The price of any derivative is a
function of the stochastic variables underlying the derivative and time. (Hull 2012: 280-289.)

Stock price process modeling is based on Hull (2012: 286-294) presentation. Model for stock price behavior, which is based on geometric Brownian motion, can be determinate as continuous time version:

\[
dS = \mu S \, dt + \sigma S \, dz
\]  

(3)

Same model can be determinate as discrete time version:

\[
\Delta S = \mu S \Delta t + \sigma S \, \varepsilon \sqrt{\Delta t}
\]  

(4)

where variable \( \mu \) is the stock expected rate of return, the variable \( \sigma \) is the volatility of the stock price and the variable \( \Delta S \) is the change in the stock price \( S \) over time interval \( \Delta t \). The function \( \varepsilon \) is a standard normal distribution with mean of zero and standard deviation of one.

Based on Ito’s lemma, the price of derivative \( f \), which depends on the price of stock and time, comply with Ito’s process and can be defined as continuous time version:

\[
df = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz
\]  

(5)

and discrete time version:

\[
\Delta f = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z
\]  

(6)

Based on Ito’s lemma lognormal model for asset price behavior can be defined by equation:

\[
\ln S_T \approx \Phi \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right]
\]  

(7)
where $S_T$ is the asset price at future time $T$ and $S_0$ is the asset price at time 0. The asset price has lognormal distribution if the natural logarithm of the asset price has normal distribution. The stochastic process, which is typical assumption, is based on geometric Brownian motion (GBM). The Black-Scholes-Merton model is based on the geometric Brownian motion assumption. (Hull 2012: 293-294.)

### 2.3 Stock option price

The type of stochastic process is important in the valuation of options. The option pricing is typically based on the assumption of lognormal diffusion process. Cox and Ross (1976) studied the importance of the type of the stochastic process by presenting diffusion processes in more details and also different jump type processes for the valuation of options. The commonly used Black and Scholes model (1973) for option pricing assumes that the underlying asset price is tracking a lognormal diffusion process.

The payoff of the European option at the maturity can be defined if there are not transaction costs:

$$ c(S_T, T, X) = \max(S_T - X ; 0) \quad (8) $$

$$ p(S_T, T, X) = \max(X - S_T ; 0) \quad (9) $$

The Black and Scholes model (1973) for option pricing assumes that the underlying asset price has a lognormal distribution. Geometric Brownian motion (GBM) is stochastic process with a constant expected return and a constant volatility so that the parameters $\mu$ and $\sigma$ are assumed to be constant. Black and Scholes model assumes the constant volatility during the term of the option and the same volatility across the total range of the strike prices.

By considering a portfolio comprising one unit of a derivative asset and a short position of $\Delta$ units of the underlying asset, it’s possible to apply the partial differential equation of this portfolio getting the Black and Scholes partial equation:
\[
\frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + \sigma S_t \frac{\partial f}{\partial S_t} r + \frac{\partial f}{\partial t} r f = 0
\]  
(10)

The value of the option depends on risk-free rate \( r \) and standard deviation \( \sigma \) and the boundary condition of the option contracts in equations for calls and puts. By solving the partial differential equation in equation (10) with the boundary conditions results the Black and Scholes pricing formulas for call and put options:

\[
c(S; t) = S N(d_1) - X e^{-rT} N(d_2)
\]  
(11)

\[
p(S; t) = X e^{-rT} N(-d_2) - S N(-d_1)
\]  
(12)

with \( d_1 \) and \( d_2 \) can be calculated using equations:

\[
d_1 = \frac{\ln(S/X) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}}
\]  
(13)

and

\[
d_2 = \frac{\ln(S/X) + (r - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}}
\]  
(14)

The parameter \( \mu \) is not in equation, which means that the expected return does not appear in the Black and Scholes equation. Consequently, the value of the option does not depend on the investors risk preferences.

The Black and Scholes model assumes that the price of the underlying asset follows a stochastic model with constant expected return and constant volatility. The parameter regarding of the instantaneous volatility \( \sigma \) in the underlying return of the asset is not available directly. However, instantaneous volatility can be estimated from inverting Black and Scholes equation in terms of implied volatility \( \sigma \). In practice the implied volatility calculated for each strike price is different so that the implied volatilities are different across maturities, which is not consistent with the Black and Scholes lognormal assumptions that define volatility as being constant across the total range of
strike prices and maturities. The implied volatilities observed in the market are a function of strike prices, which creates the phenomenon called volatility smile.

Black and Scholes and Merton seminal work (1973) was critical for understanding theoretically that the risk free rate should be used for discounting instead of the expected return on asset. Based on theory in complete markets investors can hedge investment position of an option by an offsetting position in the stock and the bond. The expected return should be only the risk free rate if any investor can cost efficiently eliminate the risk of the option position. The risk of the investment can be eliminated in this approach because any risk could be hedged. Black-Scholes realized that the expected return on the asset did not appear in the option pricing equation anymore. The risk free rate turned out to be convenient term for taking into account discounting in risk neutral situation.

Black-Scholes equation for standard call option can be presented as combined version for getting better overall view of components:

\[
c = S \left[ \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] - e^{-rT} \left[ \frac{\ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \]

Black-Scholes equation for standard put option:

\[
p = e^{-rT} \left[ \frac{\ln \left( \frac{S}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] - S \left[ \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right]
\]

Black-Scholes option pricing equation was breakthrough theoretically and practically for understanding the risk-neutral pricing because all necessary inputs to the Black-Scholes equation were observable except the implied volatility parameter, which could be estimated from historical asset returns.
3 DISTRIBUTION PARAMETERS

3.1 Moments of distributions

The uncentered moments of random variable $X$ can be defined:

$$m_k = E[X^k] = \int_{-\infty}^{\infty} x^k f(x) \, dx$$  \hspace{1cm} (17)

The centered moments of random variable $X$ can be defined:

$$\mu_k = E[(X - m)^k] = \int_{-\infty}^{\infty} (x - m)^k f(x) \, dx$$  \hspace{1cm} (18)

The properties of probability distributions can be described by the moments of the distributions. 1st moment (M1) is expectation value of the distribution. 2nd moment (M2) is variance of the distribution. 3rd moment (M3) is skewness of the distribution and 4th moment (M4) is kurtosis of the distribution. Any high-order moments are not included in analysis.

Skewness is 3rd moment of a distribution. Skewness describes the asymmetry of the statistical distribution, in which the distribution curve comes out distorted or skewed either to the left or to the right. Skewness can be quantified based on the distribution difference from the normal distribution. Negative skewness means that the distribution is skewed to left. If a distribution is skewed to the left, the tail on the curve's left-hand side is longer than the tail on the right-hand side, and the mean is less than the mode. Positive skewness means that the distribution is skewed to the right. If a distribution is skewed to the right, the tail on the curve's right-hand side is longer than the tail on the left-hand side, and the mean is greater than the mode. Overview of negative and positive skewness is presented in figure 1.

Kurtosis is 4th moment of a distribution. Kurtosis is typically measured with respect to the normal distribution. Kurtosis captures the tail thickness of the distributions. A distribution that is behaving in the same way as any normal distribution is mesokurtic distribution. Kurtosis value of the leptokurtic distribution is more than the kurtosis
value of the mesokurtic distribution (positive kurtosis). The statistics of the
distribution in leptokurtic case is more concentrated about mean than normal
distribution. The tails of the leptokurtic distributions, to both the right and the left, are
typically slim and light. Kurtosis value of the platykurtic distribution is less than the
kurtosis value of the mesokurtic distribution (negative kurtosis). The statistics of the
distribution in platykurtic case is less concentrated about mean than normal
distribution. The tails of the platykurtic distributions, to both the right and the left, are
typically thick and heavy. Overview of negative and positive kurtosis is presented in
figure 1.

![Figure 1. Overview of skewness and kurtosis of probability distribution.](image)

**3.2 Lognormal distribution**

The parameter $\mu$ in the lognormal distribution is the mean of the distribution. The
parameter $\sigma$ is the standard deviation of the distribution. On the logarithmic scale $\mu$ is
location parameter and $\sigma$ is scale parameter. The mean and standard deviation of the
non-logarithmic values are denoted $m$ and $s$.

A lognormal distribution location parameter $\mu$ can be calculated from mean $m$ and
standard deviation $s$ of the non-logarithmic distribution:

$$\mu = \ln \left( \frac{m^2}{\sqrt{s^2 + m^2}} \right)$$  \hspace{1cm} (19)
A lognormal distribution scale parameter $\sigma$ can be calculated from mean $m$ and standard deviation $s$ of the non-logarithmic distribution:

$$
\sigma = \sqrt{\ln \left( 1 + \frac{s^2}{m^2} \right)}
$$

(Moments of the distributions can be calculated from location parameter $\mu$ and scale parameter $\sigma$ by making assumption of lognormal distribution. Basic moments can be calculated using equations (21) – (24).

The expectation value on lognormal distribution parameters $\mu$ and $\sigma$:

$$
\mu_d = e^{\mu + \sigma^2 / 2}
$$

The standard deviation based on lognormal distribution parameters $\mu$ and $\sigma$:

$$
\sigma_d = \sqrt{e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)}
$$

Skewness based on lognormal distribution parameter $\sigma$:

$$
\gamma_1 = \sqrt{e^{\sigma^2} - 1} (2 + e^{\sigma^2})
$$

Kurtosis (excess) based on lognormal distribution parameter $\sigma$:

$$
\gamma_2 = e^{4\sigma^2} + 2 e^{\sigma^2} + 3 e^{2\sigma^2} - 6
$$

### 3.3 Moments of RNDs

The moments of the RND were calculated from the reference RND methods so that the moments of the reference RNDs can be compared to the moments of the direct methodology. Key moments must be calculated from the reference RNDs. Expectation value is calculated directly using equations (25) for discrete case and (26) for
continuous case. Variance is calculated using equations (27) for discrete case and (28) for continuous case. Standard deviation is used as measure of dispersion in the analysis of the results. Standard deviation, which is square root of the variance of the distribution, is calculated directly by taking the square root of the variance. Skewness S is calculated directly using equations (29) for discrete case and (30) for continuous case. Kurtosis K is calculated directly using equations (31) for discrete case and (32) for continuous case.

Expectation value E(X) for discrete random variable X with probability distribution f(x):

\[ E(X) = \sum_{i=1}^{n} x_i f(x) \]  \hspace{1cm} (25)

Expectation value E(X) for continuous random variable X with probability distribution f(x):

\[ E(X) = \int_{-\infty}^{\infty} x f(x) \, dx \]  \hspace{1cm} (26)

Variance \( D^2(X) \) for discrete random variable X with probability distribution f(x):

\[ D^2(X) = \sigma^2 = E(X - \mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 f(x) \]  \hspace{1cm} (27)

Variance \( D^2(X) \) for continuous random variable X with probability distribution f(x):

\[ D^2(X) = \sigma^2 = E(X - \mu)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \]  \hspace{1cm} (28)

Skewness \( S_d(X) \) for discrete random variable X with probability distribution f(x):

\[ S_d(X) = E \left[ \left(\frac{X-\mu}{\sigma}\right)^3 \right] = \frac{\mu_3}{\sigma^3} = \sum_{i=1}^{n} (x_i - \mu)^3 f(x) \]  \hspace{1cm} (29)

Skewness \( S \) for continuous random variable X with probability distribution f(x):
\[ S_c(X) = E \left[ \left( \frac{X-\mu}{\sigma} \right)^3 \right] = \frac{\mu_3}{\sigma^3} = \int_{-\infty}^{\infty} (x - \mu)^3 f(x) \, dx \]  \hspace{1cm} (30)

Kurtosis \( K \) for discrete random variable \( X \) with probability distribution \( f(x) \):

\[ K_d(X) = E \left[ \left( \frac{X-\mu}{\sigma} \right)^4 \right] = \frac{\mu_4}{\sigma^4} = \sum_{i=1}^{n} (x_i - \mu)^4 f(x) \]  \hspace{1cm} (31)

Kurtosis \( K \) for continuous random variable \( X \) with probability distribution \( f(x) \):

\[ S_c(X) = E \left[ \left( \frac{X-\mu}{\sigma} \right)^4 \right] = \frac{\mu_4}{\sigma^4} = \int_{-\infty}^{\infty} (x - \mu)^4 f(x) \, dx \]  \hspace{1cm} (32)

In practice, the calculations are based on using discrete versions of equations for determining the moments of the reference RNDs. But theoretical point of view continuous versions are more suitable for analysis.
4 RND ESTIMATION – BASE THEORY

4.1 Discounted risk-neutral density

The options on the same underlying asset, with the same time-to-maturity, but with the different exercise prices, can be combined to modeling state-contingent claims. The prices of other state-contingent securities, which can be modeled by options, represent investor assessments of the probabilities of the specific states occurring in the future. An elementary claim, an Arrow-Debreu security, is a derivative security that is paying one unit at future time T if the underlying asset or the portfolio of assets takes a specific value or state $S_T$ at time T and is paying nothing otherwise. The prices of Arrow-Debreu securities, state prices, at each possible state are directly proportional to the risk-neutral probabilities of each of the states occurring. (Bahra 1997)

The relation between probabilities and the price of a state-contingent claim was initially proposed in Arrow (1964) who applied a contingent claim model to the securities market. The prices of the elementary claim, Arrow-Debreu security, are proportional to the risk-neutral probabilities attached to each of the states. Arrow-Debreu security has an important theoretical value. The price of the elementary claim can be modeled with a combination of call options, called butterfly spread, which consists of a long position in the two call options with strikes $(X - \Delta M)$ and $(X + \Delta M)$ and a short position in the two calls with strike $(X)$ where $\Delta M > 0$.

Ross (1976) first demonstrated the relationship between call option prices and state prices. Ross’s finding was enabling the definition of the risk-neutral densities. Breeden and Litzenberger (1978) and Banz and Miller (1978) showed that if the underlying price at time T has a continuous probability distribution then the state price at state $S_T$ is determined by the second partial derivative of the European call option pricing function for the underlying asset with respect to exercise price $\partial^2 C/\partial^2 X$ evaluated an exercise price of $X = S_T$. The $\partial^2 C/\partial^2 X$ is directly proportional to the risk-neutral probability density function of $S_T$. Many of the techniques for estimating the RNDs from the option prices can be related to this result. The Breeden and Lietzenberger and Banz and Miller approach, which were developed within a time-state preference...
framework, provide the most general approach to pricing state-contingent claims. Pricing procedure is model free so that the risk neutral distribution doesn’t depend on any specific pricing model.

Breeden and Lietzenberger (1978) applied the developments by Arrow and Debreu by using a state contingent claim in the form of a butterfly spread to demonstrate that the second partial derivate of an option pricing function with respect to the strike prices yields the discounted RND. In fact, a butterfly spread centered on X implies a payoff of $\Delta M$ if the price of the underlying asset at maturity $T$ is equal to $X$.

The price of the elementary claim security in the discrete case:

$$P(M, T; \Delta M) = \frac{[c(M-\Delta T)-c(M,T)] - [c(M,T)-c(M+\Delta T)]}{\Delta M} \quad (33)$$

For continuous $M$ the price of the butterfly at state $M = X$ is the second partial derivative of the portfolio if call options with respect to $X$:

$$\lim_{\Delta M \to 0} \frac{P(M, T; \Delta M)}{\Delta M} = \frac{d^2 c(X; T)}{dx^2} \quad \| X = M \quad (34)$$

The price of an Arrow-Debreu security is equal to expected payoff, which is calculated by multiplying the present value by the risk-neutral probability related to specific state and finally by discounting using risk free rate. By applying this relation to range of continuous possible future values for the underlying asset, leads to the estimation of the discounted risk-neutral density:

$$\frac{d^2 c(X; T)}{dx^2} = e^{-rt} f(S_T) \quad (35)$$

where $r$ is the risk-free rate of interest over the time period $T$. The function $f(S_T)$ is the risk neutral probability density function of the asset price at future time $T$. Equation (35) is fundamental result for defining the RND from call option values. Same methodology is valid for defining the RND distribution from put option values. But methodology cannot be implemented directly if option functions $c(X, T)$ and $p(X, T)$
are not behaving correctly. \( C(X,T) \) must be monotonic decreasing and convex function. \( P(X,T) \) must be monotonic increasing and convex function so that there is no arbitrage opportunities and the RND could not be negative.

### 4.2 RND estimation from call and put options

Figlewski (2008) presented clearly issues related to extracting the RND from the option prices in theory. The methodology is based on curve fitting technique but the article is describing systemically the RND estimation from the call and put options. Only basic procedure is presented for both call and put options by using key equations but more details about problems in the RND estimation is available from same article.

The value of a call option is the expected value of option payoff on the expiration date \( T \) discounted back to the present. Under risk neutrality the expectation value is taken with respect to the risk neural probabilities. Typically discounting is at the risk free interest rate:

\[
c = e^{-rT} \int_K^\infty f(S_T) (S_T - K) \, dS_T
\]  

(36)

By taking the partial derivative of (36) with respect to \( X \):

\[
\frac{\partial c}{\partial X} = e^{rT} \int_X^\infty f(S_T) \, dS_T = -e^{-rT} [1 - F(X)]
\]  

(37)

By solving the risk neutral distribution \( F(X) \) from equation (37):

\[
F(X) = e^{rT} \frac{\partial c}{\partial X} + 1
\]  

(38)

In practice, an approximation solution to equation (38) can be obtained using finite differences of option prices observed at discrete exercise prices in the market. Let there be option prices available for maturity \( T \) at \( N \) different exercise prices from \( X_1 \) to \( X_N \). By using three options with sequential prices \( X_{n-1} \), \( X_n \) and \( X_{n+1} \) for getting an approximation to \( F(X) \) centered on \( X_n \):
\[ F(X) \approx e^{rT} \left[ \frac{c_{n+1} - c_{n-1}}{x_{n+1} - x_{n-1}} \right] + 1 \]  

(40)

By taking the derivative to \( X \) in second time in equation (40) yields to the risk neutral density function at \( X \):

\[ f(X) = e^{rT} \frac{\partial^2 C}{\partial X^2} \]  

(41)

The density \( f(X_n) \) can be approximated as

\[ f(X_n) \approx e^{rT} \frac{c_{n+1} - 2c_n - c_{n-1}}{(\Delta X)^2} \]  

(42)

The part of RND between \( X_2 \) and \( X_{N-1} \) can be extracted from a set of call option prices using previous equations. \( X_1 \) and \( X_N \) must be taken care for defining RND but for understanding RND the processing of the limit values are not critical.

Similar procedure can be derived for getting equations for put options:

\[ F(X) = e^{rT} \frac{\partial P}{\partial X} + 1 \]  

(43)

\[ F(X) \approx e^{rT} \left[ \frac{p_{n+1} - p_{n-1}}{x_{n+1} - x_{n-1}} \right] + 1 \]  

(44)

\[ f(X) = e^{rT} \frac{\partial^2 P}{\partial X^2} \]  

(45)

\[ f(X_n) \approx e^{rT} \frac{p_{n+1} - 2p_n - p_{n-1}}{(\Delta X)^2} \]  

(46)

The part of RND between \( X_2 \) and \( X_{N-1} \) can be extracted from a set of put option prices using previous equations. \( X_1 \) and \( X_N \) must be taken care for defining RND but for understanding RND the processing of the limit values are not critical.

The RND can be defined from call or put options only but by combining information from both call and put options it’s possible to get more accurate RND estimation. Of course by using both call and put options for defining the RDN the weighting of the different options near ATM is very critical decision for avoiding stability issues. The RND is typically more symmetrical if both call and put options are taking into account
for defining RND. Anyway, more information is better for getting more accurate RND estimation. Typically most the methodologies are using both call and put options for defining the RND.

4.3 Other critical issues

The generic target for defining optimal RND is based on finding minimum distance between the observed call and put prices, $c_i$ and $p_i$, $i = 1\ldots N$ and the estimated call and put prices from the estimated PDF, $\hat{c}$ and $\hat{p}$, $i = 1\ldots N$. The RND estimation process is based on equation (47).

The optimizing function can be defined:

$$
\min \left( \sum_{i=1}^{N_c} \frac{(c_i - \hat{c}_i(\phi))^2}{\eta_i^2} + \sum_{i=N_c+1}^{N_c+N_p} \frac{(p_i - \hat{p}_i(\phi))^2}{\eta_i^2} \right) \tag{47}
$$

where $\phi$ is the list of the estimated parameters. The equation (47) is based on the maximum likelihood framework, where the pricing errors are normally distributed with zero mean and variances $\eta_i^2$. The stability of the estimated RND is critical for getting acceptable results. The stability of the estimated RND has two components: the theoretical stability at the solution and the stability of the convergence to a solution. (Bliss and Panigirtzoglou 2002.)

Any errors in option pricing will have influence for estimated RND. First source of error in option pricing is related to non-synchronicity errors. The reason for non-synchronicity errors are arising from the use many simultaneous prices as input parameters for modeling. The non-synchronicity errors are critical if option prices and underlying asset prices are not behaving correctly. Second source of error in option pricing is related to liquidity premium errors. The liquidity premium errors are coming from the potential impacts of differential liquidity on prices, which might have critical impact for modeling accuracy. Third source of error in option pricing is related to data errors. The data errors are arising from any type of issues in the recording and reporting of prices for input parameters to modeling. Fourth source of error in option pricing is related to discreteness errors. The discreteness errors are coming from
quotation, trading and reporting of prices in discrete increments. Typically it’s possible to obtain evidence of pricing errors even if it is not always possible to determinate whether there is really a pricing error or which type of pricing error is under consideration. (Bliss and Panigirtzoglou 2002.)

A generic problem in derivative markets is the low liquidity for options, which are deep out of the money or deep in the money. The low liquidity of these options makes the option prices less reliable meaning reducing the accuracy of the RND estimation. Typical way to avoid reliability problem is using only liquid options for estimating the RND, which are options close at-the-money. But by using less out-of-the-money (OTM) and in-the-money (ITM) options for estimating the RND distribution is limiting the spectrum of the RND estimation meaning that the tails of distribution are not so accurate anymore. The tails of the distribution is more depending on the estimation technique than on the option data. But in practice most interesting part of the RND estimation is not located in the tails of the RND.

Either real time prices or settlement prices can be used for the RND estimations. The bid-ask spread might be problem for the accuracy of the estimated RND if the real time quotations are used for the RND estimation. The bid-ask spread problem can be avoided by using the settlement prices but non-synchronicity problems can be limiting the accuracy of the RND estimation. Anyway, it’s important to take account different type of errors which might have influence to the RND estimations even if the analysis of the errors are not included in study because all necessary information for the analysis of the errors are not typically available.
5 RND ESTIMATIONS TECHNIQUES

5.1 Background of RND estimations techniques

The RND estimation techniques can be classified in few different ways. Parametric models are targeting to find a direct modeling for the RND distribution without referring to any specific dynamics of the RND. Non-parametric models are trying to define a accurate modeling of the RND distribution directly without trying to define any specific form for the RND. Cont (1997), Bahra (1997), Jackwerth (1999) and Perignon, Villa (2002) and Santos and Guerra (2011) are good references for getting more overview of details of the different methodologies. Many studies have tried to compare the performance of the different methods relative to each other. More information is available from Campa et al. (1998), Coutant et al. (2000), MacManus (1999) and Sherrick at al. (1996). Cooper (1999) and Jondeau and Rockinger (2000) have compared many methods but no real conclusions. The literature on the extraction of RNDs is unsettled so that there are not real consensus of the best RND estimation techniques.

Jackwerth (2004) is good introduction for getting generic overview of earlier studies related to defining RNDs. Jackwerth summary is good reference for understanding basic concept and methodologies for finding RNDs. Jondeau et al. (2007: 383-416) is good reference for understanding more about different algorithms related to the RND definition.

5.2 Parametric methods

In the parametric case, the parameters of the risk-neutral probability distributions are selected by different methodologies for minimizing the pricing error between the observed and the estimated option prices. The parametric methods have potential drawbacks if the parametric distribution is not adaptive for matching the observations. Parametric methods have potential benefits by producing stable distributions if the parametric distribution is matching the observations. Inside the parametric methods, it’s possible to identify three different categories of methods – extension methods, generalized distribution methods and mixture methods. (Jackwerth 2004.)
The extension methods are based on the classical probability distribution as normal or lognormal distributions but the extension methods are adding different type of corrections terms to base distributions in order to make distributions more adaptive for meeting accuracy requirements. The correction terms do not guarantee the integrity of the probability density function so that checking the resulting distribution is always necessary for meeting generic probability density function requirements.

The generalized distribution methods are producing more adaptive distributions with additional parameters compared to the base parameters of the normal and lognormal distributions. The generalized distribution methods are using distribution functions with more than the typical two parameters for the mean and the volatility. Typically, skewness and kurtosis, parameters are added for making modeling more accurate. The generalized distributions describe many families of the adaptive distributions, which simplify to standard distributions for specific parameter constellations. Typically checking for meeting probability density function requirements is necessary.

The mixture methods model is based on combination of different type of base distributions with optimal parameters for getting more accurate modeling for distribution. The better adaptability is coming from the increase of the number of the parameters, which in practice means more demanding modeling and more demanding optimization algorithms. Typically checking for meeting probability density function requirements is necessary. Moreover, the mixture methods have risk to under fitting or over fitting if the number of mixing distributions is not correct for getting accurate modeling. The resulting risk-neutral densities might not be accurate if under fitting has occurred in modeling. The resulting risk-neutral densities might have disruptions in distribution if over fitting has occurred in modeling.

5.3 Non-Parametric methods

In the nonparametric case, instead of selecting parameters of parametric risk-neutral distribution, target is to define optimal risk-neutral probability density function either by build-up from linear segments or even from nonlinear segments. The number of parameters is much higher than in the parametric case. The direct processing of fitting the risk-neutral distribution is not often undertaken because of potential problems to
constrain the probability distribution for meeting the probability density function requirements. The resulting distribution should meet generic probability density function requirements for integrity. The probability density function must integrate correctly. Inside the non-parametric methods, it’s possible to identify three different categories of methods – maximum entropy methods, kernel methods and curve fitting methods. (Jackwerth 2004.)

The improvement over a techniques of fitting the risk-neutral distribution straightaway is based on fitting the function of option prices across strike prices and proceed by taking two derivatives of the option price function in respect to strike prices for obtaining the risk-neutral distribution. The critical problem with this approach is stability because direct derivation process might have numerical issues. Furthermore, it must be ensured that the fitted function does not violate the arbitrage bounds, which is requirement that often leads to numerical difficulties. Maybe better methodology is based on fitting the function of implied volatilities across the strike prices. Based on fitting process the calculation of the option prices from the modified implied volatilities is straightforward step before proceeding by taking two derivatives for getting final risk neutral distribution. The previous methods yield arbitrage-free risk-neutral probability distributions as long as the fitted volatility smiles do not have disruptions. Most of the non-parametric methods are using the procedure called curve fitting.

The maximum entropy methods are based on for finding the risk-neutral probability density function so that a prior probability distribution and a posterior probability distribution have correlation with maximum cross entropy. The maximum entropy methods are trying to minimize the amount of missing information, which is achieved by maximizing the cross-entropy, for getting optimal RND. The main problem with entropy methods is related to numerical issue. The maximum entropy methods require the use of the nonlinear optimization processing, which is in many times demanding to implement. More details about the maximum entropy methods are available from Buchen and Kelly (1996) and Rockinger and Jondeau (2002).

The kernel methods are based on statistical regressions for generating the relationship between the option price and the strike price. The kernel estimator is basically a
smoothing estimator for distribution by constructing an assumed probability function at each data point. The function is assumed to pass most likely right through the data point and a kernel measures the likelihood that the function passes by the data point at a distance. The overall density function is the weighted sum of the individual density functions. The kernel regressions tend to be data intensive and do not work correctly for data with missing data values. The kernel methods are typically difficult to implement because of data intensive processing and issues with missing data points. The generic description about kernel estimation process is available from Pritzker (1998) and more detail about implementation of kernel methods is available from Ait-Sahalia and Lo (1998).

The curve fitting methods are used primary to fit the implied volatility function with some adaptive smoothing function. The most typical criteria for the goodness of the fit are the sums of the squared differences in modeled and observed volatilities or the squared difference in modeled and observed option prices. Typical functions for curve fitting are polynomials of different degrees. Typically the trade-off is related to the order of the polynomials for generating optimal risk neutral probability density function. The low order polynomial is typically more stable but modeling accuracy in more limited. The high order polynomial is less stable but modeling accuracy is less limited. Splines are curves, which are typically required to be continuous and smooth, for generating stable risk neutral distribution. Splines are collecting together piecewise polynomial segments at knots by matching levels and derivatives at the knots. The selection of the location of the knots is difficult because too few knots points prevent the observed volatilities from being matched correctly and too many knots cause over fitting of the observed volatilities. Polynomials can have typically lower order than the splines but might have more issues with modeling. Splines are typically behaving more steadily from theoretical point of view. Splines typically should be about 1-2 orders higher for the probability distribution to turn out to be behaving correctly. There are many studies available, which have tested different type of IV processing, with different conclusion about acceptable IV processing techniques related to defining risk neutral distribution.
6 THEORY OF REFERENCE METHODS

The reference methods were selected for meeting requirements that there should not be any critical stability issues and the accuracy of the modeling should be acceptable. The tradeoff between stability and accuracy is critical selection criteria for many RND estimation methods.

The generic process for calculating the RNDs is based on option data that must be processed using different algorithms for getting the RNDs out. The process for defining the reference RND is presented in figure 2.

![Figure 2. Process view for the definition of the reference RNDs from the option data.](image)

The generic target for defining optimal RND is based on using different type of algorithms for finding the parameters for different methodology for minimum pricing error between the observed option prices and the estimated option prices from the estimated RND.

The generalized beta (GB) and the mixture of the lognormals (LN) were selected from parametric methods. The parametric methods should not have many numerical issues. Shimko’s method was selected from the non-parametric methods. Shimko’s method as curve fitting method might have stability issues but Shimko’s method is acceptable trade-off between stability and accuracy. Only key equations are presented for understanding the algorithms of the reference methods. Jondeau et al. (2007: 383-416) is good reference for understanding more numerical details about different algorithms related to the RND estimation. Clews et al. (2002) are presenting generic overview of most common methodologies for defining RNDs.
6.1 Generalized beta

Bookstaber and McDonald (1987) proposed the generalized beta distribution to model asset returns. Liu et al. (2007) use the generalized beta of second kind (GB2) distribution for defining the RNDs from option prices. The RNDs can be defined by relative simple modification of the parameters of the distribution. The GB2 is interesting case with only a few positive parameters but the GB2 distribution is anyway capable for the modeling of the different type of distributions.

The distribution function for the GB2:

$$g_{BG2}(x; a, b, p, q) = \frac{ax^{ap-1}}{b^{ap}B(p,q)(1+\left(\frac{x}{b}\right)^a)^{p+q}}$$

(48)

The distribution function GB2 is risk neutral if and only if equation:

$$F = \frac{b^{B\left(p+\frac{1}{a}q-\frac{1}{a}\right)}}{B(p,q)}$$

(49)

The moments of GB2 can be defined by equation:

$$E_{GB2}(S^p_T) = \frac{b^{hB\left(p+\frac{h}{a}q-h\right)}}{B(p,q)}$$

(50)

The generalized beta distribution is a multimodal type of distribution, which is making this type of distribution convenient for the modeling of the RNDs. The generalized beta distribution has a density function that makes the GB2 distribution suitable for many applications because the density function could be defined explicitly. The GB2 is a function of four parameters a, b, p and q. These parameters work interactively in defining the shape of the distribution. The power parameter (a) determinates the behavior of the tails of the density function. The scale parameter (b) determinates the value of the kurtosis. The parameters (p) and (q) define together the skewness of the distribution. The GB2 distribution, unlike the lognormal, has the necessary flexibility to modeling either positive or negative skewness. (Bookstaber and McDonald 1987.)
The GB2 includes the generalized gamma (GG) as a limiting case:

\[
GG(x; a, \beta, b) = \lim_{q \to \infty} GB2(x; a, \beta q, p, q)
\]  

(51)

The further limits applied to the GG leads to the lognormal density as a limiting case:

\[
LN(x; \mu, \sigma) = \lim_{q \to 0} GG(x; a, \beta = (\sigma^2 a^2)^\frac{1}{\alpha}, p = (a\mu + 1)/\beta^a)
\]  

(52)

GB2 is containing many different distributions so that wide range of different type of distributions can be expressed as limiting and special cases of the GB2.

Liu et al. (2007) defined closed form equation for call option price for estimating the RND. The theoretical pricing formula for European call option:

\[
C(X | r, T) = e^{-rt} \int_{x}^{\infty} (x - X) g_{GB2}(x | a, b, p, q) \, dx
\]

\[
= Fe^{-rt} \left[ 1 - G_{\beta} \left( z(X, a, b) | p + \frac{1}{a}, q - \frac{1}{a} \right) \right] - X e^{-rt} \left[ 1 - G_{\beta} (z(X, a, b) | p, q) \right]
\]  

(53)

Liu et al. were using put-call parity for taking into account the put options for defining RND distribution. The put-call parity equation could be used for calculation:

\[
C + Ke^{-rt} = P + S_0
\]  

(54)

Of course same type of equation for put option price could be defined but put-call parity was used in the RND distribution definition. The RND distribution based on the GB2 methodology can be calculated by using equations (48)-(50) and (53)-(54).
6.2 Mixture of lognormals

Instead of specifying the underlying asset price dynamics to make conclusion of the RND function, it is possible to make assumptions about the functional form of the RND function for finding the parameters of the RND estimation. The description of the probability density distribution as the combination of the other density distributions has been common way to improve fitting in many statistical problems. Bahra (1996), Melick and Thomas (1997) and Söderling and Svensson (1997) have used lognormals for describing the RND distribution. Chen (2010) has studied using of multilognormal technique for getting better accuracy for modeling RNDs. Only two lognormals density distributions were selected in study for analysis even if from theoretical point of view using more than two lognormals would give better accuracy for modeling RND.

The mixture of lognormals approach is based on the assumption that the distribution of the underlying asset is a weighted sum of many independent lognormal distributions. The double lognormal distribution is described by five parameters: two parameters for each lognormal distribution ($\alpha_i, \beta_i$) and a weighting parameter ($\theta_i$) for relative weight for each distribution. The parameters are selected in order to satisfy as well as possible constrains on the observed call and put options and the observed forward rate. Typically loss function, which is sum of the squared deviations from constrains, must be defined for finding correct solution for parameters. Additional benefit of the mixture of lognormals is the smooth behavior of the tail so that the tails declines monotonically and always decays relative quickly to prevent unreasonable kurtosis. On the other hand, this type of approach may determinate too unyielding structure on the RND estimation.

The prices of European call and put options at time t can be written as the discounted sums of all expected future payoffs:

\[
c(X, t) = e^{-rt} \int_{X}^{\infty} q(S_T) (S_T - X) dS_T
\]

\[
p(X, t) = e^{-rt} \int_{0}^{X} q(S_T) (X - S_T) dS_T
\]
In theory any functional form for the density function \( q(S_T) \) can be used in equations (55) and (56) for finding the parameters recovered by numerical optimization. The problem with other models than the Gaussian model is that the underlying price distribution could be changing as the holding time is changing. In the Gaussian case any arbitrary length holding time price distribution must be lognormal if daily prices are lognormal distributed. No other finite variance distribution is same way stable under addition of lognormals. (Bahra 1997.)

The framework suggested by Ritchey (1990) assumes that \( q(S_T) \) is the weighted sum of k-component lognormal density functions:

\[
q(S_T) = \sum_{i=1}^{k} [\theta_i L(\alpha_i, \beta_i; S_T)]
\] (57)

where \( L(\alpha_i, \beta_i; S_T) \) is the \( i^{th} \) lognormal density function in the k-component mixture with parameters:

\[
\alpha_i = \ln S + \left( \mu_i - \frac{\sigma_i^2}{2} \right) \tau
\] (58)

\[
\beta_i = \sigma_i \sqrt{\tau}
\] (59)

for each \( i \). The values of call and put options, given by equations (54) and (55), can be calculated by equations:

\[
c(X, \tau) = e^{-r\tau} \int_{X}^{\infty} [\theta_1 L(\alpha_1, \beta_1; S_T) + (1 - \theta) L(\alpha_2, \beta_2; S_T)](S_T - X) \, dS_T
\] (60)

\[
p(X, \tau) = e^{-r\tau} \int_{0}^{\tau} [\theta_1 L(\alpha_1, \beta_1; S_T) + (1 - \theta) L(\alpha_2, \beta_2; S_T)](X - S_T) \, dS_T
\] (61)

In the absence of the arbitrage opportunities, the mean of the RND should be equal the future price of the underlying asset. The target of the optimization is finding numerically minimum of equation:
\[
\sum_{i=1}^{n} [c(X_i, \tau) - c_i]^2 + \sum_{i=1}^{n} [p(X_i, \tau) - p_i]^2
\]
\[
+ \left[ \theta e^{\alpha_1 + 1/2\beta_1^2} + (1 - \theta) e^{\alpha_2 + 1/2\beta_2^2} - e^{\tau S} \right]^2
\]

(62)

subject to \( \beta_1 > 0 \) and \( 0 < \theta < 1 \) over the observed strike range \( X_1, X_2, \ldots X_n \).

Evaluating equation (55) and (56) numerically might have numerical problems due to upper limit of infinity. The values of call and put options, given by equations (55) and (56), can be calculated by using the closed-form solutions to equations (60) and (61):

\[
c(X, \tau) = e^{-\tau \theta} \left[ e^{\alpha_1 + \frac{1}{2}\beta_1^2} N(d_1) - X N(d_2) \right]
\]
\[
+ (1 - \theta) \left[ e^{\alpha_2 + \frac{1}{2}\beta_2^2} N(d_3) - X N(d_4) \right]
\]

(63)

\[
p(X, \tau) = e^{-\tau \theta} \left[ -e^{\alpha_1 + \frac{1}{2}\beta_1^2} N(-d_1) + X N(-d_2) \right]
\]
\[
+ (1 - \theta) \left[ -e^{\alpha_2 + \frac{1}{2}\beta_2^2} N(-d_3) + X N(-d_4) \right]
\]

(64)

where

\[
d_1 = \frac{-lnX + \alpha_1 + \beta_1^2}{\beta_1}
\]

(65)

\[
d_2 = d_1 - \beta_1
\]

(66)

\[
d_3 = \frac{-lnX + \alpha_2 + \beta_2^2}{\beta_2}
\]

(67)

\[
d_4 = d_3 - \beta_2
\]

(68)

This two-lognormal model is the weighted sum of two Black-Scholes solutions, where \( \theta \) is the weight parameter and \( \alpha_1, \beta_1, \alpha_2 \) and \( \beta_2 \) are the parameters of each of the lognormal components of the RND functions. The RND estimation based on the
double lognormals methodology can be estimated in closed form by using equations (63) - (68). The results of the double lognormals methodology were calculated using of call and put options.

6.3 Implied volatility methods – Shimko

In the implied volatility methods the RNDs are estimated by differentiating twice the modified option prices before discounting by riskless interest rate. The RND estimation process is presented in chapter 4.2 in more details. Shimko’s technique is special case of many implied volatility methodologies.

Shimko (1993) introduced technique of the fitting the implied volatility with a quadratic function for getting continuous of call option prices as function of strike prices. Malz (1997) modified Shimko’s technique by interpolating the implied volatilities across option delta space instead of across strike prices. Malz modification by using implied volatility over delta has numerical advantage but translation process from delta space back to strike prices is numerically demanding. Campa et al. (1997) use a modification of Shimko’s approach providing better formability in the form of the volatility smile and hence the PDF. The method is based on replacing the quadratic with cubic splines – polynomial functions of order three or lower. The polynomials between any two points are selected so that the polynomials meet at a single data point – the first derivatives if the two functions are equal and differentiable. The cubic spline approach is attractive for its generality as the third-order polynomial used for fitting the volatility smile is enabling to change form over each interval. Bliss and Panigirtzoglou were using Malz (1997) theory by interpolating in the implied volatility over the delta space and Campa et al. (1998) theory by using the smoothing splines for fitting the function. Methodology is theoretical point of view adaptive for providing reliable results but implementation is relative demanding.

Many other curve-fitting versions were studied for getting understanding of different type of implied volatility processing but Shimko’s method was selected as curve fitting reference method. Shimko’s method as curve fitting method might have some stability issues but Shimko’s method is acceptable trade-off between stability and accuracy.
Shimko’s (1993) approach is based on getting information contained in the volatility smile via a polynomial \( \sigma(K) \). The polynomial \( \sigma(K) \) is a function of the strike price \( K \). RND is based on evaluation of \( \sigma(K) \). Price of the call option:

\[
C(S_t, K, \tau, r, \sigma) = C(S_t, K, \tau, r, \sigma(K)) \tag{69}
\]

where

\[
\sigma_i = \alpha_0 + \alpha_1 K_i + \alpha_2 K_i^2 \tag{70}
\]

for \( N=1…N \). \( N \) represents the number of observed prices. The parameters of this polynomial can be estimated using a nonlinear least square regression.

The price of the call option \( i \) depending of the strike price \( K_i \), \( i=1…N \):

\[
C^{SH}(t, S, K, T) = S \Phi(d_1(\sigma(K))) - e^{-r\tau} K \Phi(d_2(\sigma(K))) \tag{71}
\]

RND can be calculated using equation:

\[
q^{SH}(K) = e^{-r\tau} \frac{\partial^2 C(t, S, \sigma(K), T)}{\partial K^2} = e^{r\tau} S (d''_1(\Phi(d_1(K))) \) - \( (d'_1)^2 d_1 \Phi(d_1(K)) - d'_2 \Phi(d_2(K)) - K(d''_2(\Phi(d_2(K)) - (d'_2)^2 d_2 \Phi(d_2(K))) \tag{72}
\]

For quadratic case necessary equations are:

\[
d_1(K) = \frac{1}{\sigma(K)\sqrt{\tau}} \log \left( \frac{S}{Ke^{r\tau}} \right) + \frac{1}{2} \sigma(K) \sqrt{\tau} \tag{73}
\]

\[
d_2(K) = d_1(K) - \sigma(K) \sqrt{\tau} \tag{74}
\]

The first and second derivatives of \( d_1 \) and \( d_2 \) are given by:
$$d'_1(K) =$$
$$= \frac{\sigma'(K)\sqrt{\tau}}{\sigma^2(K)\sqrt{t}} \log \left( \frac{S}{K e^{r t}} \right) - \frac{1}{K \sigma(K)\sqrt{t}} + \frac{1}{2} \sigma'(K)\sqrt{\tau}$$

$$d'_2(K) = d'_1(K) - \sigma'(K)\sqrt{\tau}$$

$$d''_1(K) =$$
$$= \frac{\sigma''(K)\sigma'(K)\sqrt{\tau} - 2\sigma'(K)^2\tau}{(\sigma(K)\sqrt{\tau})^3} \log \left( \frac{S}{K e^{r t}} \right) + \frac{1}{K} \frac{\sigma'(K)\sqrt{\tau}}{\sigma^2(K)\tau}$$
$$+ \frac{\sigma(K)\sqrt{\tau} + K\sigma'(K)\sqrt{\tau}}{K^2 \sigma^2(K)\tau} + \frac{1}{2} \frac{\sigma''(K)\sqrt{\tau}}{\sigma^2(K)\tau}$$

$$d''_2(K) = d''_1(K) - \sigma''(K)\sqrt{\tau}$$

$$\sigma(K) = (a_0 + a_1 K + a_2 K^2)$$

$$\sigma'(K) = (a_1 + 2a_2 K)$$

$$\sigma''(K) = 2a_2$$

The RND estimation based on Shimko’s methodology can be calculated in closed form by using equations (72) - (81). Shimko’s results were calculated using of call options only. Put option case can be defined by same methodology.

Shimko’s method is relative straightforward curve fitting techniques. Shimko’s method has been used as reference techniques often. The generic problem with curve fitting techniques is that there are many versions available but not any generically accepted reference technique for implied volatility processing. Even if there are many version of different curve fitting methods, which are using different type of volatility smile processing, Shimko’s method was selected as classical curve fitting methodology.
6.4 Summary of selected methods

The generalized beta as parametric method was selected because the method is based on generic functionality for modeling different type of distributions using beta functions. Theoretical point of view modeling should provide reliable estimations for the RND without any critical issues with modeling process. The accuracy of the RND estimations should be acceptable level. The stability of the RND estimations should not have any problems. The mixture of the lognormals as other parametric method was selected because the method is commonly used reference method, which is typically providing reliable estimation for RND without any critical issues with modeling process. The accuracy of the RND estimations should be acceptable level. The stability of the RND estimation should not have any critical problems.

The selection of the curve fitting method was based on ad hoc type testing by implementing different type of implied volatility processing using polynomial functions, spline functions and finally smoothed spline functions. The implied volatility processing was implemented with different degree of polynomials from the degree of two to the degree of five. Based on ad hoc testing it was possible to find optimal processing methodology for specific data set but same processing was not necessary working optimally with other data set. The accuracy of the RND estimations should be acceptable level but the stability of the RND estimations might have potential problems. Shimko’s method was selected as baseline methodology by accepting that there might be issues with both stability and accuracy. Shimko’s method was selected from curve fitting methods because many other methods are based on Shimko’s theory. Shimko’s method is not necessary optimal from theoretical point of view.

The implementation of the RND estimations is more demanding in the parametric case than in the non-parametric case if the target of the RND estimations process is providing accurate RNDs. Potential stability issues are more problematic from implementation point of view in the non-parametric case than in the parametric case if the target of the RND estimations process is providing accurate RNDs.
Typically different methods generate quite same type of the RND results, even if there are typically differences in the RDN results if the requirement for accuracy is critical. There are more problems in defining accurate RND estimation if only limited amount of options are available for defining the RND estimation. Many studies have compared different methods for defining the RNDs. The literature on the extraction of RNDs is indicating that there is not real consensus of stable and accurate techniques. Typically there are tradeoffs between stability and accuracy, which must be taken care in selection and/or implementation of techniques. Reliable results are not always available because many typical techniques are based on some tuning of parameters for estimating RNDs. The selection of reference methods for defining the RNDs is not based on any systematic comparison but more like finding generic methods, which are stable and providing reliable and relative accurate RND estimations.
7 DIRECT METHODOLOGY

7.1 Overview of direct methodology

The direct methodology is targeting to find the key moments of the RND more directly than calculation the key moments of the RND from full option pricing data. The direct methodology for finding the moments of the RND is presented in figure 3. Futures contract is used as direct methodology for getting the expectation value of the RND. Only few at-the-money options are used for getting the standard deviation of the RND. The implied volatility is calculated from option prices using ATM and/or close to ATM options only. Based on the implied volatility standard deviation can be calculated directly using option pricing equations. Skewness and kurtosis can be calculated from location parameter $\mu$ and scale parameter $\sigma$ by using the assumption of the lognormal distribution. The VIX index is used only as reference for standard deviation estimation.

![Figure 3. Direct methodology for finding the moments of the reference RND.](image)

7.2 Futures contracts for getting expectation value

Futures contract is an agreement between two parties to buy or sell an asset at a certain time in the future for certain price. The futures contracts are based on the theory of the forward contracts. The futures contracts are normally traded on an exchange. The exchange specifies certain standardized features of the contract to make trading possible. The exchange also provides a mechanism for guaranteeing that the contract will be respected. (Hull 2012: 7.)

The traders of futures contracts must fulfill the requirement of the contract on the delivery date. The futures contracts are more binding agreement than options
contracts. Cash settlement of profit and loss instead of the delivery of the asset is more convenient between the futures traders in many situations. The traders of the futures contracts could close out contract obligations by taking the opposite position on another futures contract on the same asset and settlement date for exit the commitment prior to the settlement date. The difference in futures prices is a profit or loss, which must be settled for closing position.

The future price for an investment asset can be calculated by equation:

\[ F_0 = S_0 e^{rT} \]  \hspace{1cm} (82)

where \( F_0 \) is future price at time \( T_0 \) and \( S_0 \) is asset price at time \( T_0 \). If \( F_0 > S_0 e^{rT} \) riskless profit can be obtained by shorting the futures contract and buying the underlying asset. If \( F_0 < S_0 e^{rT} \) riskless profit can be obtained by shorting the underlying asset and buying the futures contract.

The European futures options can be valued by extending the general Black-Scholes model by assuming the same lognormal distribution. Black (1976) presented that the call option \( c \) and put option \( p \) of the European futures options can be valuated:

\[ c(S; t) = e^{-rt}[FN(d_1) - XN(d_2)] \]  \hspace{1cm} (83)

\[ p(S; t) = e^{-rt}[XN(-d_2) - FN(-d_1)] \]  \hspace{1cm} (84)

with

\[ d_1 = \frac{\ln\left(\frac{F}{X}\right) + \frac{1}{2}\sigma^2t}{\sigma \sqrt{t}} \]  \hspace{1cm} (85)

and

\[ d_2 = \frac{\ln\left(\frac{F}{X}\right) - \frac{1}{2}\sigma^2t}{\sigma \sqrt{t}} \]  \hspace{1cm} (86)
The parameter $\sigma$ is the volatility of the futures price and parameter $F$ is futures price of the contract. Black’s model does not require that the option contract and the futures contract mature at the same time. Black model for futures option pricing is not necessary for getting the expectation value estimation but model is useful for understanding the price process of the futures contracts.

The futures contract is used as direct methodology for getting the expectation value of the RND. The futures contract, as derivative, is sensitive instrument for getting market information. In the absence of arbitrage opportunities, the discounted futures contract price of the underlying asset must be equal to the expectation value of the RND. The future contract value is directly compared to the expectation values of the RND of the reference methods.

### 7.3 ATM options for getting standard deviation

Only few ATM and/or close ATM options are used for getting the standard deviation of RND. The implied volatility is calculated from option prices using ATM and/or close to ATM options only. Based on the implied volatility standard deviation can be calculated directly using the option pricing equations.

The implied volatility $\sigma$ can be calculated from call and put option pricing equations:

$$c(S; t) = SN(d_1) - Xe^{-t(R-t)}N(d_2) \quad (87)$$

$$p(S; t) = Xe^{-t(R-t)}N(-d_2) - SN(-d_1) \quad (88)$$

with

$$d_1 = \frac{\ln(\frac{S}{X}) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} \quad (89)$$

and
Relative accurate estimation of the implied volatility is critical for getting acceptable estimation of the standard deviation. Only few ATM and/or close to ATM options with different strike values K are used for getting the estimation of the implied volatility. Standard deviation estimation should be based on reliable estimation of the implied volatility. Reliable implied volatility value can be calculated using iterative process for finding correct value for implied volatility for each call and put options and calculating average value for final implied volatility value including both call and put options.

Standard deviation can be calculated from reliable implied volatility estimation by using equation:

\[
STD_{IV} = IV_{ATM} S_0 \sqrt{\tau - \tau_m}
\]  

(91)

where \( IV_{ATM} \) is implied volatility estimation from ATM only options.

Standard deviation based on the few ATM and/or close to ATM options is compared to standard deviation of the RND of the reference methods.

7.4 VIX index as reference

The VIX index in used as reference for the standard deviation of the RND. Typical index is based on rules that define the selection of the securities and necessary equations for calculating index values. The VIX index can be used as a reliable reference for understanding the standard deviation of the market. The VIX index is calculated by using expected volatility based on averaging S&P call and put options over wide range of strike prices. The VIX is a volatility index consisting of options rather than stocks with the price of each option reflection the expectation of the future volatility of the market. (CBOE 2009.)

The generalized formula used in the VIX calculation:
\[ \sigma^2 = \frac{1}{z} \sum_i \frac{\Delta K}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T} \left[ \frac{F}{K_0} - 1 \right]^2 \]  

(92)

where \( \sigma \) is VIX index divided by 100, \( T \) is time to expiration, \( F \) is forward index level derived from index option prices, \( K_0 \) is first strike below the forward index level \( F \). \( K_i \) is strike price of \( i \)th out-of-the-money option - call option strike price if \( K_i > K_0 \) and put option if \( K_i < K_0 \). Both call and put if \( K_i = K_0 \). \( \Delta K_i \) interval between strike prices:

\[ \Delta K = \frac{K_{i+1} - K_{i-1}}{2} \]  

(93)

\( R \) is risk-free interest rate to expiration and \( Q(K_i) \) is the mid-point of the bid-ask spread for each option with strike \( K_i \). (CBOE 2009.)

The VIX index measures 30-day expected volatility of the S&P 500 index. The components of the VIX are near and near-term call and put options, usually in the first and second SPX contract months.

Standard deviation can be calculated from the VIX index using equation:

\[ STD_{VIX} = VIX_i \cdot S_0 \cdot t^{1/2} - t_m \]  

(94)

where \( VIX_i \) is implied volatility estimation from the VIX index.

The VIX index is compared to standard deviation of the RND of the reference methods.

### 7.5 Mathematics for skewness and kurtosis

The location parameter \( \mu \) and scale parameter \( \sigma \) of the lognormal distribution can be defined from mean \( m \) and standard deviation \( s \) of the non-logarithmic distribution. The location parameter \( \mu \) and scale parameter \( \sigma \) of the lognormal distribution can be calculated by using equations (19) and (20). Skewness and kurtosis can be calculated from the location parameter \( \sigma \) and the scale parameter \( \mu \) by using the assumption of
the lognormal distribution. Skewness and kurtosis are calculated by using equations (23) and (24).

Skewness based on lognormal distribution parameter $\sigma$:

$$\gamma_1 = \sqrt{e^{\sigma^2} - 1} (2 + e^{\sigma^2})$$

(95)

Excess kurtosis based on lognormal distribution parameter $\sigma$:

$$\gamma_2 = e^{4\sigma^2} + 2 e^{3\sigma^2} + 3 e^{2\sigma^2} - 6$$

(96)

Skewness is compared to skewness of the RND of the reference methods. Kurtosis is compared to kurtosis of the RND of the reference methods.
8 DATA AND METHODOLOGY

S&P 500 index options are active index options with wide range of strike prices, which is useful for getting wide and accurate RND estimation. The S&P 500 index option trading is active so that trading volatility is high for ATM and near to ATM options. The options with active trading are necessary for the comparison of the direct methodology with the different reference methods. The S&P 500 index options were selected as reliable reference data. S&P 500 futures contracts data was selected for getting expectation value estimation for the RND. The VIX index was selected for getting reference for standard deviation estimation for the RND.

Year 2008 was selected for analysis because year 2008 was interesting from financial point of view because of financial crisis. The option data was also available for analysis. Financial point of view 2008 data were representing both optimistic and pessimistic view about the status of the market. Anyway, the analysis of the financial situation during year 2008 was not included in the results even if the financial situation might and most probably would have been helping for understanding the results more in details. The comparison of methodology is not directly depending on the status of the market.

The direct methodology analysis was based on the S&P 500 futures contracts and the S&P 500 index options data on every Wednesday using end-of-day data. The reference RND estimation analysis was based on the S&P 500 index options week level data on every Wednesday using end-of-day data. Simple average value of the bid-ask spread was selected for getting relative reliable estimation of S&P 500 index option prices. The risk-free interest rate data was based on the average value of the 3-month treasure bill rate.

R open source statistical computing software was used for data processing for RND estimations including input data pre-processing, implementation of the mandatory reference algorithms, post-processing of results and output data processing. Microsoft Excel was used for comparison of the direct methodology and the reference methods, presentation of final results and for crosschecking the direct methodology and the reference RND distribution results.
Three different numerical methods were selected as reference methods for defining the reference RNDs. The moments of the reference RNDs were calculated from the estimated reference RNDs so that the moments of the RNDs can be compared to direct methodology. The moments of the direct methodology were calculated using the direct methodology. The final analysis of study results was based on the comparison of the moments for understanding if the direct methodology could be used for estimating the parameters of the reference RNDs. Process view for comparison the direct methodology to the reference methodologies is presented in figure 4.

![Figure 4. Process for comparison the direct methodology to the reference methodologies.](image)

The S&P 500 index options data was used for defining the reference RND estimation. The reference RND estimations were based on options with prices less than ±20% from the underlying index price. In practice the most active options were included for defining the RND. More options are not necessary for getting reliable results. The selection of ±20% limit was based on testing with including options from ±10% to ±50% from underlying index price. The ±20% limit for selection of options was acceptable tradeoff for getting reliable estimation of the RND.

The S&P 500 futures contracts data was used as the direct methodology for getting the expectation value estimation of the RND. The S&P 500 ATM and/or close to ATM options only were used for getting the standard deviation estimation of the RND. The standard deviation estimations using the implied volatility were based on options with prices less than ±0.5% from the underlying index price. In practice only few options were included for standard deviation calculation. The implied volatility was calculated
from the option prices using ATM and/or close to ATM options only. The VIX index value was used directly for getting the standard deviation for the RND as reference case. Skewness and kurtosis can be calculated from the location parameter $\mu$ and the scale parameter $\sigma$ by using the assumption of the lognormal distribution.

The reference RND results in graphical form were presented so that methodologies could be compared to each other. The RND results by using graphics are telling more visually about form of the RND estimation. The results of the reference RND estimation analysis were presented by using graphics for providing overview of the behavior of the estimated RNDs. The numerical data in table format was presented for comparing the accuracy of the different reference RNDs.

The target of the study was find out if the direct methodology could provide same information about key parameters of the RND than the reference RND methodologies. Finally the results of using the direct methodology were compared to the results of the reference methods for understanding if the direct methodology can be used for understanding the key parameters of the RND distribution. The results of the analysis were presented by using graphics so that the comparison of the moments over time could be understood without focusing too many details. Based on the results the direct methodology can be compared directly to the reference methodology.
9 RESULTS OF ANALYSIS

9.1 RND results of reference – overview

Only two RND estimation results are presented for each quartile expiring 11 and 6 weeks before time-to-maturity. The RND estimation results were calculated with all reference methods for every weeks for getting better understanding of the RND results but presented results are providing overview of the RND results. The reference RND results are presented in figures 5. GB abbreviation means the RND estimation results based on the generalized beta method. LN abbreviation means the RND estimation results based on the mixture of lognormals method. SH abbreviation means the RND estimation results based on Shimko’s method.

Typically the RNDs with long time-to-maturity have more flat type distributions (more negative kurtosis) and the RNDs with short time-to-maturity have more peak type distributions (more positive kurtosis). It’s not possible to make real conclusion about performance of methodologies based on visual view of the RND results but some generic findings are visible from RND results.

The generalized beta method (GB) is producing quite flat type distributions without any disruptions, which is typical for generic modeling methodology. The mixture of the lognormals method (LN) is producing more peak type distributions with capability for modeling more dynamic distributions. Shimko’s method (SH) is providing stable distributions even if Shimko’s method should be capable for modeling different type of distributions.

The moments of the RNDs were calculated for different reference methods for comparison the results in more details. Simple mean and delta values are calculated for each moment using same RND distribution data for getting overview of the accuracy of reference methods. The comparison of the accuracy of the reference methods is presented in table 1 in more details.
Figure 5. Quartile reference RND results. 1st quartile expiration date is 21.03.2008. 2nd quartile expiration date is 21.06.2008. 3rd quartile expiration date is 20.09.2008. 4th quartile expiration date is 20.12.2008.
Table 1. Moments of the reference RNDs for the comparison of accuracy including arithmetic mean value and delta between minimum and maximum.

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The moments of the RNDs based on analysis are providing quite accurate results in typical market situation where time-to-maturity is at moderate range. The moments of the RND estimation results are more close to each other if time-to-maturity is long. The moments of the RND estimation results are less close to each other if time-to-maturity is short. The expectation values and the standard deviations calculated from the RNDs are very close to each other with all the RND estimation methodologies. Skewness and kurtosis based on analysis are close to each other too. Of course maybe with different market situation the results could not be so consistent.

9.2 RND results of reference – day-to-day analysis

The day level RND estimation results are presented for getting overview of day level variation of the RND estimation results during five sequential days. The day-to-day RND estimation results are presented in figure 6. The RND estimation results are calculated with all reference methods for 4 different five sequential days in every quartile for getting better understanding of the RND estimation results but presented results are providing overview of the RND estimation results only for five days in one quartile from 04.02.2008 to 08.02.2008.

The day level RND estimation results are not changing remarkable during presented time period, which was quite expected result. But of course, it’s not possible to make any generic conclusion based on this analysis. The results are more telling that the day-to-day level variations of the RNDs are quite moderate in normal market situation. Of course with different market situation the results could not be so consistent. The generalized beta method (GB) and the mixture of the lognormals (LN) method as parametric reference methods for defining the RND estimation results were working correctly based on presented day-to-day results. Shimko’s method (SH) as non-parametric reference method was providing same type of distributions than the GB and the LN methods based on presented day-to-day results.
A: date 04.02.2008.

B: date 05.02.2008.

C: date 06.02.2008.

D: date 07.02.2008.

E: date 08.02.2008.

Figure 6. Reference RND results for five sequential days. Reference RND from 04.02.2008 (A) to 08.02.2008 (E). Expiration date is 21.03.2008.

The generalized beta (GB) method is producing distributions without any disruptions. The mixture of the lognormals (LN) method is producing more peak type distributions but tails are clearly more asymmetric. Shimko’s method (SH) is producing relative stable distributions without any stability issues.
9.3 RND results of reference – quartile analysis

Quartile RND estimation results are presented only for one quartile for getting the overview of the variation of the RND estimation results as the function of the time-to-maturity. The RND distribution results are calculated with all reference methods for all quartiles but results are presented for 4Q quartile only for getting overview of the RND results. The 4\textsuperscript{th} quartile RND estimation results are presented for every second week but the last RND results are presented for every week. The quartile RND distribution results are presented in figure 7 for providing overview of the RND distribution as the function of the time-to-maturity.

Typically the RNDs with the long time-to-maturity have more flat type distributions (more negative kurtosis) and the RNDs with the short time-to-maturity have more peak type distributions (more positive kurtosis), which is clearly visible in the RND estimation results. The GB and the LN methods are providing very identical RND estimation results. Typically GB has more flat type distribution and LN has more peak type distribution. But the moments of the distributions are close to each other. Shimko’s method has negative skewness and lower kurtosis. But more critical issue with Shimko’s method has stability problems in short time-to-maturity because the RND distributions are focused close to ATM value and coming clearly unsymmetrical.

Only the 4\textsuperscript{th} quartile results were presented but the results of the other quartiles were behaving same way in case of time-to-maturity. The results are confirming common sense that the calculated RND estimation results are telling that uncertainly about underlying asset price is decreasing as the time-to-maturity is decreasing. Maybe more interesting result is that if the time-to-maturity is not close to the expiration date the distribution is changing relative slowly but if the time-to-maturity is coming close to the expiration date the distribution is changing relative quickly. Of course with different market situation the results could not be so consistent.
Figure 7. Reference RND results for quartile view. Reference RND results at week level for 4Q from 01.10.2008 (A) to 17.12.2008 (H). Expiration date is 20.12.2008.
9.4 Distribution parameters analysis

The expectation value analysis is based on comparing the direct methodology by using the expectation value of the futures contracts with the expectation value of the RND of the reference methods. The results of the expectation value analysis are presented in figures 8-11. GB mean abbreviation is the expectation value based on the generalized beta method. LN mean abbreviation is the expectation value based on the mixture of the lognormals method. SH mean abbreviation is the expectation value based on the Shimko’s method. FC mean abbreviation is the expectation value based on S&P 500 futures contracts.

Standard deviation analysis is based on comparing the direct methodology by using the standard deviation estimation of the direct methodology with the standard deviation of the RND of the reference methods. The results of the standard deviation analysis are presented in figures 8-11. GB std abbreviation is the standard deviation based on the generalized beta method. LN std abbreviation is the standard deviation based on the mixture of the lognormals method. SH std abbreviation is the standard deviation based on the Shimko’s method. IV std abbreviation means standard deviation calculated from implied volatility of only few ATM options using the direct methodology. VIX std abbreviation means standard deviation calculated directly from the VIX index value.

Skewness analysis is based on comparing skewness of direct methodology with skewness of the RND of the reference methods. The results of the skewness analysis are presented in figures 8-11. GB skewness is based on the generalized beta reference method. LN skewness is based on the mixture of the lognormals reference method. Skewness of the direct methodology is based on calculation of skewness from the expectation value and the standard deviation estimations. IV skewness is based on calculation of skewness from the expectation value and the standard deviation estimation from implied volatility of ATM options. VIX skewness means skewness calculated directly from the VIX index value.

Kurtosis analysis is based on comparing kurtosis of direct methodology with kurtosis of the RND of the reference methods. The results of the kurtosis analysis are presented
in figures 8-11. GB kurtosis is based on the generalized beta reference method. LN kurtosis is based on the mixture of the lognormals reference method. Kurtosis of the direct methodology is based on calculation of kurtosis from the expectation value and the standard deviation estimations. IV kurtosis is based on calculation of kurtosis from the expectation value and the standard deviation estimation from implied volatility of ATM options. VIX kurtosis means skewness calculated from directly from the VIX index value.

The results of GB and LN analysis were providing reliable results. There was no any numerical issues related stability of the RND estimation results. The results of the GB and LN analysis have good correlation based on the RND estimation.

The results of Shimko’s method were excluded because of numerical issues related to skewness and kurtosis results. The problem with Shimko’s method was related to the stability of the algorithm for calculation RND results. Typically results were correct but with specific data there were stability issues. Stability issues are quite typical for non-parametric methods. Stability type issues can be canceled by tuning parameters for numerical processing. But tuning parameters will have influence for results so that tuning must be taken care carefully for keeping results acceptable. Sensitivity analysis for parameters was carried out for understanding that numerical issues were the root cause for stability issues. Based on sensitivity type analysis Shimko’s method is not stable if the price of the underlying asset is not in-line with the price of options.

The results of using the direct methodology are compared to the results of the reference methods for understanding if the direct methodology can be used for understanding the key parameters of the RNDs. The results of comparison of the direct methodology with the reference methods are presented by using specific type of graphical approach only. The comparison is based on analysis by quartile. Numerical data is not included in results. Numerical data was excluded because of problems using numerical presentation is demanding because of comparing of many parameters as function of the time-to-maturity.
9.5 Distribution parameter analysis for 1st quartile

The direct methodology is compared to the reference methods by quartile. The first quartile results are presented in this chapter. The expectation value comparison results for the first quartile are presented in figure 8A. The standard deviation comparison results for the first quartile are presented in figure 8B.

A: The expectation value comparison between the futures contract and the reference RNDs for the 1st quartile. Expiration date is 21.03.08.

B: The standard deviation comparison of the reference RNDs for the 1st quartile including standard deviation of the IV and the VIX index. Expiration date is 21.03.08.

C: Skewness comparison of the reference RNDs for the 1st quartile including skewness based on the IV and the VIX index. Expiration date is 21.03.08.

D: Kurtosis comparison of the reference RNDs for the 1st quartile including kurtosis based on the IV and the VIX index. Expiration date is 21.03.08.

Figure 8. The comparison of the distribution parameters between the direct methodology and the reference RNDs for the 1st quartile. A is presenting the analysis of the expectation value. B is presenting the analysis of the standard deviation. C is presenting the analysis of the skewness and D is presenting the analysis of the kurtosis.
The expectation values based on the expectation values of the RNDs of the reference methods are indicating higher values than the expectation values based on the futures contract value at the same time. The difference in the expectation values is high at the long time-to-maturity. The difference in the expectation values is low at the short time-to-maturity. The expectation values of the reference methods are close to each other, which are indicating that the estimation methods are reliable for the expectation value analysis. The delta between the expectation values of the reference methods is changing during the observation time at different time-to-maturity. Based on data the delta is not behaving systematically. Typically the delta is more in the long time-to-maturity and less in the short time-to-maturity.

The standard deviation calculated from the IV of the only few ATM or close to ATM options and the VIX index are indicating higher standard deviation than the standard deviations from the reference RNDs. The difference between the direct methodology and the standard deviation of the reference RNDs is decreasing as the time-to-maturity is decreasing. The standard deviations of the reference methods are very close to each other, which are indicating that the estimation methods are reliable for the standard deviation analysis. The standard deviation calculated from the VIX index is typically higher than the standard deviation calculated from the IV of the only few ATM or close to ATM options.

Skewness comparison results for first quartile are presented in figure 8C. Kurtosis comparison results for first quartile are presented in figure 8D.

Skewness values from the GB reference method are systematically higher than skewness values from the LN reference method. Skewness results of the reference methods are clearly negative. Only at close to the expiration date the results are coming less negative. Skewness results of the direct methodology, which are calculated from the expectation values and the standard deviation values, are more positive at the long time-to-maturity and less positive at the short time-to-maturity. Skewness is depending on only the value of the distribution parameter based on equation for skewness by using the assumption of the lognormal distribution.
Kurtosis values from the LN reference method are systematically higher than kurtosis values from the GB reference method. Kurtosis values from the GB and the LN methods are both negative at the long time-to-maturity and positive at the short time-to-maturity. Based on the results kurtosis values are increasing as time-to-maturity is decreasing. Kurtosis results of the direct methodology, which are calculated from the expectation values and the standard deviation values, are more positive at the long time-to-maturity and less positive at the short time-to-maturity. Kurtosis is depending on only the value of the distribution parameter based on equation for kurtosis by using the assumption of the lognormal distribution.

9.6 Distribution parameter analysis for 2nd quartile

The direct methodology is compared to the reference methods by quartile. The second quartile results are presented in this chapter. The expectation value comparison results for the second quartile are presented in figure 9A. The standard deviation comparison results for the second quartile are presented in figure 9B.

The expectation values based on the expectation values of the RNDs of the reference methods are indicating typically higher values than the expectation values based on the futures contract value at the same time. The expectation value results of the reference methods are close to each other even if there is some difference visible. The delta between the expectation values of the reference methods is changing during the observation time at different time-to-maturity. Based on data the delta is not behaving systematically. Typically delta is more in the long time-to-maturity and less in the short time-to-maturity. Overall behavior of the expectation values is quite same than in earlier quartile.

The standard deviation calculated from the IV of the only few ATM or close to ATM options and the VIX index are indicating higher standard deviation than the standard deviations from the reference RNDs. The difference between the direct methodology and the standard deviation of the reference RNDs is decreasing as the time-to-maturity is decreasing. The standard deviations of the reference methods are close to each other. The standard deviation calculated from the VIX index is typically higher than the standard deviation calculated from the IV of the only few ATM or close to ATM
options. Overall behavior of the standard deviation is quite same than in earlier quartile.

Figure 9. The comparison of the distribution parameters between the direct methodology and the reference RNDs for the 2nd quartile. A is presenting the analysis of the expectation value. B is presenting the analysis of the standard deviation. C is presenting the analysis of the skewness and D is presenting the analysis of the kurtosis.

Skewness comparison results for second quartile are presented in figure 9C. Kurtosis comparison results for second quartile are presented in figure 9D.

Skewness values from the GB reference method are systematically higher than skewness values from the LN reference method. Skewness results of the reference
methods are clearly negative. Skewness values of the reference methods are not close to each other but there is anyway same type of overall behavior in skewness of the reference methods. Skewness results of the reference methods are not behaving systematically at the every time-to-maturity in relation to each other. Skewness results of the direct methodology, which are calculated from the expectation values and the standard deviation values, are more positive at the long time-to-maturity and less positive at the short time-to-maturity.

Kurtosis values from the LN reference method are systematically higher than kurtosis values from the GB reference method as earlier quartile. Kurtosis values from the GB and the LN methods are both negative at the long time-to-maturity and positive at the short time-to-maturity. Based on results kurtosis values are increasing as the time-to-maturity is decreasing. Kurtosis results of the direct methodology, which are calculated from the expectation values and the standard deviation values, are more positive at the long time-to-maturity and less positive at the short time-to-maturity.

9.7 Distribution parameter analysis for 3rd quartile

The direct methodology is compared to the reference methods by quartile. The third quartile results are presented in this chapter. The expectation value comparison results for the third quartile are presented in figure 10A. The standard deviation comparison results for the third quartile are presented in figure 10B.

The expectation values based on the expectation values of the RNDs of the reference methods are indicating typically higher values than the expectation values based on the futures contract value at the same time. The expectation values of the reference methods are close to each other even if there is some difference visible between different reference methods. The delta between the expectation values of the reference methods is changing during the observation time at different time-to-maturity. Based on data the delta is not behaving systematically. Typically delta is more in the long time-to-maturity and less in the short time-to-maturity. Overall behavior of the expectation values is quite same than in earlier quartiles.
A: The expectation value comparison between the futures contract and the reference RNDs for the 3rd quartile. Expiration date is 20.09.08.

B: The standard deviation comparison of the reference RNDs for the 3rd quartile including standard deviation of the IV and the VIX index. Expiration date is 20.09.08.

C: Skewness comparison of the reference RNDs for the 3rd quartile including skewness based on the IV and the VIX index. Expiration date is 20.09.08.

D: Kurtosis comparison of the reference RNDs for the 3rd quartile including kurtosis based on the IV and the VIX index. Expiration date is 20.09.08.

Figure 10. The comparison of the distribution parameters between the direct methodology and the reference RNDs for the 3rd quartile. A is presenting the analysis of the expectation value. B is presenting the analysis of the standard deviation. C is presenting the analysis of the skewness and D is presenting the analysis of the kurtosis.

The standard deviation calculated from the IV of the only few ATM or close to ATM options and VIX index are indicating higher standard deviation than the standard deviations from the reference RNDs. The difference between direct methodology and the standard deviation of the reference RNDs is decreasing as the time-to-maturity is decreasing. The standard deviations of the reference methods are close to each other. The standard deviation calculated from the VIX index is typically higher than the standard deviation calculated from the IV of the only few ATM or close to ATM
options. Overall behavior of the standard deviation is quite same than in earlier quartiles.

Skewness comparison results for third quartile are presented in figure 10C. Kurtosis comparison results for third quartile are presented in figure 10D.

Skewness values from the GB reference method are typically higher than skewness values from the LN reference method. There is different type of overall behavior in skewness of the reference methods. Skewness values of the reference methods are not typically close to each other. Skewness values of the reference methods are not close to each other at the long time-to-maturity but are close to each other at the short time-to-maturity. Anyway, at the end of the observation time the results of the GB and LN methods are about same level. Skewness results of the reference methods are clearly negative. Skewness results of the reference methods are not behaving systematically at the every time-to-maturity in relation to each other. Skewness results of the direct methodology, which are calculated from the expectation values and the standard deviation values, are more positive at the long time-to-maturity and less positive at the short time-to-maturity.

Kurtosis values from the LN reference method are systematically higher than kurtosis values from the GB reference method as earlier quartile. Kurtosis values from the GB and LN methods are both negative at the long time-to-maturity and positive at the short time-to-maturity. Based on results kurtosis values are increasing as the time-to-maturity is decreasing. Kurtosis results of the direct methodology, which are calculated from the expectation values and the standard deviation values, are more positive at the long time-to-maturity and less positive at the short time-to-maturity.

9.8 Distribution parameter analysis for 4th quartile

The direct methodology is compared to the reference methods by quartile. The fourth quartile results are presented in this chapter. The expectation value comparison results for the fourth quartile are presented in figure 11A. The standard deviation comparison results for the fourth quartile are presented in figure 11B.
A: The expectation value comparison between the futures contract and the reference RNDs for the 4th quartile. Expiration date is 20.12.08.

B: The standard deviation comparison of the reference RNDs for the 4th quartile including standard deviation of the IV and the VIX index. Expiration date is 20.12.08.

C: Skewness comparison of the reference RNDs for the 4th quartile including skewness based on the IV and the VIX index. Expiration date is 20.12.08.

D: Kurtosis comparison of the reference RNDs for the 4th quartile including kurtosis based on the IV and the VIX index. Expiration date is 20.12.08.

Figure 11. The comparison of the distribution parameters between the direct methodology and the reference RNDs for the 4th quartile. A is presenting the analysis of the expectation value. B is presenting the analysis of the standard deviation. C is presenting the analysis of the skewness and D is presenting the analysis of the kurtosis.

The expectation values based on the expectation values of the RNDs of the reference methods are indicating typically higher values than the expectation values based on the futures contract value at the same time. The expectation values of the reference methods are close to each other even if there is some difference visible. The delta between the expectation values of the reference methods is changing during the observation time at different time-to-maturity. Based on data the delta is not behaving
systematically. Typically delta is more in long time-to-maturity and less in short time-to-maturity. Overall behavior of the expectation values is quite same than in earlier quartiles.

The standard deviation calculated from the IV of the only few ATM or close to ATM options and VIX index are indicating remarkable higher standard deviations than the standard deviations from the reference RNDs during the fourth quartile. The difference between the direct methodology and the standard deviation of the reference RNDs is decreasing as the time-to-maturity is decreasing but the difference is not decreasing as fast as in earlier quartiles. The standard deviations of the reference methods are close to each other as in earlier quartiles. The standard deviation calculated from the VIX index is remarkable higher than the standard deviation calculated from the IV of the only few ATM or close to ATM options. Overall behavior of the standard deviation is quite same than in earlier quartiles even if the direct methodology is providing different type of results in the fourth quartile.

Skewness comparison results for fourth quartile are presented in figure 11C. Kurtosis comparison results for fourth quartile are presented in figure 11D.

Skewness values from the GB reference method are typically higher than skewness values from the LN reference method. The results of the GB and the LN reference methods are about same level at the end of the observation time. Skewness results of the reference methods are clearly negative. Skewness results of the reference methods are quite close to each other. Skewness results of the direct methodology, which are calculated from the expectation values and the standard deviation values, are more positive than in earlier quartiles. Skewness results of the VIX based direct methodology are remarkable more positive than skewness results of the IV based direct methodology during this quartile. The VIX and IV based skewness results of the direct methodologies in earlier quartile are behaving same way. Skewness is depending on only the value of the distribution parameter based on equation for skewness by using the assumption of the lognormal distribution.

Kurtosis values from the LN reference method are systematically higher than kurtosis values from the GB reference method as earlier quartiles. Kurtosis values from the GB
and the LN reference methods are both negative during the observation time. Kurtosis results of the reference methods are quite close to each other. Kurtosis results of the VIX based direct methodology are remarkable more positive than kurtosis results of the IV based direct methodology during this quartile. The VIX and IV based kurtosis results of the direct methodologies in earlier quartile are behaving same way. Kurtosis is depending on only the value of the distribution parameter based on equation for kurtosis by using the assumption of the lognormal distribution. Kurtosis results of the direct methodology, which are calculated from the expectation values and the standard deviation values, are very positive at the long time-to-maturity and less positive at the short time-to-maturity.

9.9 Summary of distribution parameter analysis

The expectation values based on the RND estimation results are indicating higher values than the expectation values based on the futures contract value at the same time. The difference is high at the long time-to-maturity. The difference is low at the short time-to-maturity. The difference is not important anymore as the time-to-maturity decrease. The expectation values, which are calculated from the reference RND estimation results, are typically close to each other. The expectation values of the reference methods are providing results, which are very close to each other, even if there is some minor difference visible in the results. Maybe basic reason for difference between the expectation values based on the futures contracts and the expectation value of the RND estimation results is coming from different type of the risks related to financial instruments. The futures contacts are more binding type contracts than option type contracts. Risk is typically higher at longer time-to-maturity than at shorter time-to-maturity.

The standard deviation calculated from the IV of the ATM and/or close ATM only options and the VIX index are indicating higher standard deviations than the standard deviations from the reference RND estimation results. The difference is not important anymore as the time-to-maturity decrease. The standard deviations, which are calculated from the reference RND estimation results, are typically close to each other. The standard deviations of the reference methods are providing results, which are very close to each other. The standard deviation calculated from the IV of the ATM and/or
close ATM only options have higher standard deviations than the standard deviations from the reference RND estimation results because IV is typically relative high with ATM and/or close ATM options. The IV of the ATM and/or close ATM options is typically higher at longer time-to-maturity than at shorter time-to-maturity. The standard deviation calculated from the IV of the ATM and/or close ATM only options is indicating high standard deviation because only few options are used for defining the standard deviation. The standard deviations of the RND estimation results are more accurate because many options are used for estimating the standard deviation.

Skewness values from the GB reference methods are systematically higher than skewness values from the LN reference methods. Skewness results of the reference methods are clearly negative during the observation time. Typically at close to expiration date results are coming less negative. Skewness results calculated from the expectation values and the standard deviation values are always positive because of the assumption of the lognormal distribution for calculating results. Skewness results calculated from the expectation values and the standard deviation values are more positive at the long time-to-maturity but less positive at the short time-to-maturity.

Kurtosis values from the LN reference methods are systematically higher than kurtosis values from the GB reference methods. Kurtosis values from the GB and the LN methods are typically both negative at the long time-to-maturity and positive at the short time-to-maturity. Based on the results kurtosis values are increasing as time-to-maturity is decreasing. Kurtosis results calculated from the expectation values and the standard deviation values are always positive because of the assumption of the lognormal distribution for calculating results. Kurtosis results calculated from the expectation values and the standard deviation values are more positive at the long time-to-maturity but less positive at the short time-to-maturity.
10 CONCLUSION

The risk neutral density functions (RND) can be referred as additional information about the asset value evolution, which is critical for the different type of financial analysis. The RND estimation results can be used for risk analysis and for making forecast about the future trends of the asset value.

The target of the study is to find out if the direct methodology could provide same information about the parameters of the risk neutral probability density function than the reference RND methodologies. The direct methodology is based on for defining the parameters of the RND from underlying asset by using futures contracts and only few at-the-money and/or close at-the-money options on specific asset. Of course for enabling the analysis of the feasibility of the direct methodology the reference RNDs must be estimated from the option data.

The futures contracts are used in the direct methodology for getting the expectation value estimations of the RND. Only few at-the-money and/or close at-the-money options are used for getting the standard deviation estimations of the RNDs. The implied volatility is calculated from the option prices using at-the-money and/or close at-the-money options only. Based on implied volatility the standard deviation can be calculated directly using time scaling equations. Skewness and kurtosis can be calculated from the expectation value and the standard deviation by using the assumption of the lognormal distribution.

Both parametric and non-parametric methods were implemented for defining reference RNDs. The reference RND estimation results are presented so that the reference RND estimation methodologies can be compared to each other. The moments of the reference RNDs were calculated from the RND estimation results so that the moments of the direct methodology can be compared to the moments of the reference methodologies.

Finally the results of estimating the parameters by the direct methodology are compared to the results of estimating the parameters by the reference methodologies.
for understanding if the direct methodology can be used for understanding the key parameters of the RNDs.

S&P 500 index options are active index options with wide range of strike prices, which is useful for getting wide and accurate RND estimation. The S&P 500 index options were selected as reliable reference data for calculating the reference RNDs. The S&P 500 index option trading is active so that trading volatility is high for ATM and close ATM options. The options with active trading are necessary for the comparison of the direct methodology with the reference methodology. The S&P 500 futures contract data and the S&P index option data were necessary for the direct methodology. S&P 500 futures contracts data was selected for getting expectation value estimation for the RND. The S&P 500 ATM and/or close to ATM options only were used for getting the standard deviation estimation for the RND. The VIX index was selected for getting reference for standard deviation estimation for the RND. Year 2008 was selected for analysis because year 2008 was interesting from financial point of view because of financial crisis.

The generalized beta method (GB) and the mixture of the lognormals (LN) method as reference methodology for defining the RND estimation were working correctly. The GB and the LN methods were providing equivalent RND estimation results so that the correlation between the RND estimation results of the GB and the LN methods was acceptable for making both the GB and the LN reliable reference methodologies. The parametric methods based on study are stable and relative accurate reference methodologies for defining the RND estimations.

Shimko’s method was providing reliable results in typical cases, which were in-line with other reference methods, but algorithm was too sensitive for the values of the parameters in some cases. Based on the sensitivity analysis Shimko’s method is not stable for good reference method if the price of the underlying asset is not in-line with the price of options. Basic issue with Shimko’s methods is related to processing of the implied volatility values for the RND determination. Stability issues are quite typical for non-parametric methods.
The futures contract can be used as good reference for getting the expectation value estimation for the RND at the short time-to-maturity but the futures contract should not be used as reference for getting the expectation value estimation for the RND at the long time-to-maturity. The standard deviation from the IV of the ATM and/or close ATM options can be used as reference for getting the standard deviation value estimation for the RND at the short time-to-maturity. The standard deviation from the IV of the ATM and/or close ATM options can’t be used as reference for getting the standard deviation value estimation for the RND at the long time-to-maturity. The standard deviation calculated from the VIX index is indicating higher standard deviation than the standard deviations from the reference RNDs. The difference is not so important anymore as the time-to-maturity decrease. Skewness and kurtosis results calculated directly from the expectation values and the standard deviation values are always positive because of assumption of lognormal distribution for calculating results. Skewness and kurtosis results calculated from the expectation values and the standard deviation values are more positive at long time-to-maturity but less positive at short time-to-maturity.

Based on the analysis of the results the expectation value of the RND can be estimated using information of the futures contract directly in the case of the relative short time-to-maturity. The standard deviation from the IV of the only few ATM and/or close to ATM options can be used as reference for getting the standard deviation estimation for the RND distribution at relative short time-to-maturity.

But based on the analysis the RND distributions have typically negative skewness and negative kurtosis so that the direct methodology for calculation skewness and kurtosis from the estimated expectation value and the estimated standard deviation could not yield correct results because of the assumption of the lognormal distribution. Skewness of the lognormal distribution is always positive and kurtosis of the lognormal distribution is always positive.

The findings of study were providing interesting information but based on the findings there are many open questions. Of course the analysis is based on only few reference RND methods. Based on study selected parametric methods were providing reliable RND estimation results but selected non-parametric IV based Shimko’s method was
not providing reliable RND estimation results in every situation because Shimko’s method is sensitive for parameters related to the processing of the implied volatility values for the RND determination. More advanced IV methods should be considered for future studies. Stability issues are quite typical for non-parametric methods.

The direct methodology could be studied more for understanding more about difference between the expectation value based on the futures contract and the expectation value based on the RND in the case of the relative long time-to-maturity. The direct methodology could be studied more for understanding that the standard deviation calculated from the IV of the ATM and/or ATM options is indicating higher standard deviation than the standard deviations from the reference RND in the case of the relative long time-to-maturity.

The direct methodology could be based on using the futures contract for defining the expectation values and only few ATM and/or close to ATM options for defining the standard deviation in the case of the relative short time-to-maturity. Anyway, for getting reliable skewness and kurtosis estimations only few ATM and/or close to ATM options is not really correct way to get reliable results.

Skewness and kurtosis could not be estimated by using the assumption of the lognormal distribution because the lognormal distribution is not correct generic assumption for RNDs. Maybe using some other definition for skewness and kurtosis, which are based on other parameters than only the expectation value and the standard deviation, could enable meaningful direct methodology for getting information about skewness and kurtosis of the RND without the full RND estimation by using reference methodology.

The outcome of the study was providing more understanding of implementation of the parametric and non-parametric reference methods for estimating the RNDs from option data. The results of the study were providing information for understanding the relation between the expectation value based on the futures contract and the expectation value based on the reference RNDs. The results of the study were providing information for understanding the relation between the standard deviation based on the VIX index and the standard deviation based on the reference RND.
Based on the analysis of the results the direct methodology is acceptable for getting the expectation value estimation using the futures contract value directly instead of the expectation value, which is calculated from the RND of full option data, if and only if the time-to-maturity is relative short. The standard deviation estimation can be calculated from only few at-the-money and/or close at-the-money options instead of calculating the RND of full option data. Based on the analysis of the results the direct methodology is acceptable for getting the standard deviation estimation, which is calculated using only few at-the-money and/or close at-the-money options instead of calculating the standard deviation of the RND of full option data, if the time-to-maturity is relative short. Skewness and kurtosis were calculated from the expectation value estimation and the standard deviation estimation by using the assumption of the lognormal distribution but the results were not acceptable because the results of calculation are providing only positive values for skewness and incorrect values for kurtosis. Skewness and kurtosis could not be estimated by using the assumption of the lognormal distribution because lognormal distribution is not correct generic assumption for RNDs. Based on the analysis the RNDs might have negative skewness and negative kurtosis so that the direct methodology of calculation skewness and kurtosis from the estimated expectation value and the estimated standard deviation could not yield correct results because of the assumption of the lognormal distribution.
REFERENCES


