Light Scattering
by
Solar System Dust:
Image Reconstruction
of the
Lunar Sunrise Sketches
Drawn by the Apollo 17 Crew
Abstract

In 1972, astronauts on board Apollo 17 saw a lunar sunrise while they were in orbit around the Moon. They made drawings of the distribution of light above the lunar horizon, which has been called "the lunar sunrise sketches". The sunrise sketches by the Apollo 17 crew have been interpreted to contain light scattered by interplanetary dust as well as light scattered by dust in the close vicinity of Moon. In this thesis the light scattering by interplanetary dust and by a micrometeoroid impact-generated lunar ejecta cloud at the time of Apollo 17 sunrise sketches is modeled and an image is reconstructed for the same observation geometry that can be compared to the sunrise sketches. To determine the contribution of light scattered by different dust populations, we need to know their dust number densities, the respective light scattering properties of the dust as well as the precise geometry of the observations.

A model for the interplanetary dust distribution is constructed using the results of COBE DIRBE team. Scattering properties of interplanetary dust are described with a combination of empirical scattering function for intermediate and high scattering angles and with a Fraunhofer diffraction for small scattering angles. For the lunar ejecta cloud we use a model derived from \textit{in situ} measurements of the lunar dust cloud by LADEE probe. Scattering properties of the dust grains in the lunar ejecta cloud are calculated using Lorenz-Mie theory. The location of the Apollo 17 spacecraft is determined based on the orbital information given in the flight report. Finally images of the lunar orbital sunrise are reconstructed using the known sight-line geometries and dust configurations together with their scattering properties. In the reconstructed images zodiacal light appears dominant and similar to that in the sunrise sketches. Light scattered by the lunar ejecta cloud is instead weak and not visible in front of the zodiacal light.
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1 Introduction

In 1972 three astronauts on board Apollo 17 draw sketches of a lunar sunrise while on an orbit around the Moon shortly before heading back to Earth. The most famous of these sketches is the one made by the commander of the flight, Eugene Cernan. It consists of a series of drawings representing a time sequence of a sunrise as he saw it while on orbit around the Moon (Figure 1).

The sketch contains five distinct drawings made 6 minutes, 3 minutes, 2 minutes, 1 minute, and 5 seconds prior the Sun became visible. In the drawings we can see the solar corona/zodiacal light (CZL) growing in size in the first four of the drawings, a glow near the lunar horizon in drawings 2, 3 and 4, and ray like features originating from the Sun in the last three of the drawings.

The sketch by Cernan is often used in articles about possible observations of a light scattered by the lunar exosphere, but it has not been rigorously reconstructed before. Recently, the lunar dust cloud caused by micrometeoroid impacts has been successfully measured for the first time [1], which makes it possible now for the first time to quantify its contribution to the brightness in the geometry of Apollo 17 sunrise observations.

In this thesis light scattered by dust at the time of Apollo 17 sunrise sketches is reconstructed. The image reconstruction is made by calculating how sunlight is scattered from interplanetary particles, causing the well known zodiacal light and how sunlight is scattered from particles in the impact-generated lunar exosphere. The location of the observer and the illumination geometry we get from the Apollo 17 trajectory information [2].

In order to model light scattered by dust, we need to know the spatial distribution of the dust as well as its light scattering properties. For the spatial distribution of interplanetary particles a model derived from COBE infrared observations [3] is used. For the scattering characteristics of interplanetary particles in visible light regime a Henyey-Greenstein scattering function + Fraunhofer diffraction is used as proposed by Hong [4]. For the lunar exosphere we use a simplified version of a model for the impact-generated lunar dust cloud derived by Szalay and Horanyi which is based on measurements of dust around the Moon by the LADEE orbiter [5]. For the scattering characteristics of dust in the lunar exosphere Lorenz-Mie theory is applied.
Figure 1: Lunar orbital sunrise sketch by Eugene Cernan. The sketch consists of five drawings representing brightness above the lunar horizon approximately 232 h 55 min after the planned lift-off of the Apollo 17. The times marked on the upper corner of each drawing stand for time before the Sun became visible. (Image: NASA)
Figure 2: The same sketch by Eugene Cernan as in Figure 1 with highlighted boundaries for different assumed light scattering regimes. Solar corona/zodiacal light boundary are highlighted with red, a glow near the lunar horizon with blue, and the ray-like features which originate from the approximate location of the Sun in the drawing are highlighted with green. (Image: NASA)
2 Zodiacal Light

2.1 Interplanetary dust

Interplanetary space is populated by numerous small, rocky and icy objects sizing from tens of nanometers to hundreds of microns known as interplanetary dust. The dust density is the highest within a few astronomical units of the Sun. The dust is mainly of cometary and asteroidal origin but small interstellar particles also contribute. Dust of asteroidal origin is produced in collisions of asteroids while cometary dust derives from comets passing through the inner Solar system [6].

The orbits of interplanetary dust grains are constantly affected by a range of gravitational and non-gravitational forces. The most important one of these forces, in addition to solar gravity, is the solar radiation pressure pushing dust grains outwards from the Sun. A common way in the literature to express the ratio between the force exerted by solar radiation pressure and solar gravity is through a dimensionless quantity

\[
\beta(r) = \frac{F_r}{F_g} = \frac{3LQ_{pr}}{16\pi GMc\rho s},
\]

where \(L\) is the solar luminosity, \(Q_{pr}\) the radiation pressure efficiency factor, \(G\) the gravitational constant, \(M\) the solar mass, \(c\) the speed of light and \(\rho\) and \(s\) are the density and the radius of the considered dust grain [7], respectively.

All grains initially on Keplerian orbits with \(\beta\) greater than 0.5 will leave the solar system on hyperbolic orbits; such grains are known as \(\beta\)-meteoroids [8]. For grains with \(\beta\) smaller than 0.5, the net effect of solar radiation on their orbits may be opposite to the repulsive effect for \(\beta\) larger than 0.5, and the grains might end up spiralling towards the Sun. This spiralling towards the Sun is due to the Poynting-Robertson effect [9], i.e. the component of the force exerted by the solar radiation tangential to the orbit of a particle, as seen in the reference frame of a particle, causes a decrease in the semi-major axis. For dust grains microns to tens of microns in size, at 1 AU, it typically takes about \(10^5\) years to spiral into the Sun [10].

2.2 Light scattered by the interplanetary dust cloud

Sunlight scattered by the interplanetary dust cloud can be seen as zodiacal light. On Earth, the zodiacal light can be seen from favorable sites just before sunrise or just after sunset as a faint cone shaped light rising above the horizon approximately along the ecliptic [11]. Figure 3 shows zodiacal light as observed from ESO La Silla Observatory in Chile. The brightness of
the zodiacal light diminishes with increasing Sun elongation angle because of the strong forward scattering property of zodiacal cloud particles. The alignment of zodiacal light roughly along the ecliptic follows from the number density of zodiacal cloud particles being highest near the ecliptic plane.

Figure 3: Zodiacal light over La Silla (Image: ESO/Y.Beletsky)

The radiance, also called the surface brightness, of zodiacal light seen by the observer in a direction given by angles \( \theta \) and \( \varphi \) (Figure 4) is an integral over scattering volume elements along the line of sight. Single scattering is assumed which is a very good approximation due to a low typical value of the zodiacal cloud optical thickness of \( 10^{-6} \) [11].

The surface brightness contribution \( dI \) of a scattering volume element \( dV \) on a line of sight with heliocentric coordinates \( r \) and \( \beta \) is

\[
dI_{ZD} = \frac{F}{r^2} \Phi(\phi) \sigma(r, \beta) \frac{dV}{r^2},
\]

where \( F \) is the solar radiant flux, \( \Phi(\phi) \) is the volume scattering function averaged over the particle sizes expressing how much of incoming radiation to the volume element is scattered at an angle \( \phi \). \( \sigma(r, \beta) \) is the cross-section
density of particles at \((r, \beta)\), \(dV\) is the volume of the volume element and \(l\) is the distance between the volume element and the observer.

![Figure 4: Zodiacal light geometry. The observer is located in the lower left corner of the plot near the x-axis. The Sun is located at the origin. The volume element of scattering is located on the line of sight at a distance \(l\) from the observer. \(r\) is the distance of the volume element from the Sun, \(R\) is the distance of the observer from the Sun, \(\beta\) is the angle between the line of \(r\) and the ecliptic and \(\phi\) is the phase angle of the volume element as seen by the observer. \(\theta\) is the angle between the line of sight and a plane parallel to the ecliptic through the observer location. In that plane \(\varphi\) is the angle between the projections of the solar direction and the direction to the volume element.](image)

The cross-section density of particles at \((r, \beta)\) can be expressed as

\[
\sigma(r, \beta) = n(r, \beta)\bar{\sigma},
\]

where \(n(r, \beta)\) is the number density of particles at \((r, \beta)\) and \(\bar{\sigma}\) is the average cross-section of a zodiacal cloud particle. Using observer-centered spherical
coordinates the volume element is

\[ dV = \Omega l^2 dl, \]  

(4)

where \( \Omega \) is the solid angle under which the volume element is seen by the observer. For \( F \) we have

\[ F = r_{AU}^2 S_0, \]  

(5)

where \( S_0 = 1361 \text{ W m}^{-2} \) is the Solar irradiance at a distance of one astronomical unit from the Sun. \[12\]

With equations (2) – (5) we get the zodiacal light surface brightness

\[ I_{ZD} = \int_0^\infty \frac{dI_{ZD}}{\Omega} = \sigma S_0 r_{AU}^2 \int_0^\infty \frac{\Phi(\phi) n(r, \beta)}{r^2} dl. \]  

(6)

Now, the brightness \( I_{ZD} \) is somewhat straightforward to calculate through numerical integration if the averaged volume scattering function \( \Phi(\phi) \) and the dust number density \( n(r, \beta) \) are known.

### 2.2.1 Dust number density function

Determining the averaged volume scattering function \( \Phi(\phi) \) or the dust number density function \( n(r, \beta) \) in (6) from zodiacal light observations alone is a difficult task. The difficulty arises from having the product of two unknown functions under an integral, and for an observer in a fixed heliocentric distance, e.g. on the Earth, a solution is only possible if one of the two unknown functions is assumed \[11\].

Zodiacal light observations from a fixed heliocentric distance, however, are not the only method to constrain the population of interplanetary dust. Since the beginning of the space age, observations not bound to the Earth orbit became possible as well as \textit{in situ} measurements of interplanetary dust grains by spacecraft dust detectors. These new methods together with zodiacal light imaging also in infrared and at ultraviolet wavelengths have revolutionized the interplanetary dust research.

With the new improvements in observations, our picture of the spatial interplanetary dust distribution has sharpened. Traditionally, zodiacal cloud models are often based on the assumption of a radial power-law, \( n(r) \sim r^{-\nu} \), for the dust density variation with heliocentric distance. Further they assume axial symmetry of the zodiacal cloud \[13\]. A coincidence of the zodiacal cloud midplane with the ecliptic is also often assumed. The part of the zodiacal cloud which dust number density follows a radial power-law is called a smooth cloud.
Later zodiacal cloud research has revealed fine structure in the zodiacal cloud such as dust bands and a circumsolar ring along the orbit of the Earth. The dust bands consist of debris from asteroid collisions about 2 AU from the Sun, which is spiraling towards the Sun. The circumsolar ring consist of dust particles which the Earth has temporarily trapped into resonant orbits near 1 AU. [3] The center of the zodiacal cloud has been interpreted to be offset from the Sun [3] and a small, \(\sim 2^\circ\) inclination of zodiacal cloud midplane to the ecliptic [14] has been found.

For the zodiacal light observations considered in this work, the Sun elongation angles are relatively low, the maximum elongation angle being under 30° [15]. Since the elongation angles are clearly under 90° and the observer is located closer to the Sun than the dust bands and the circumsolar ring, the light observed from these fine structures has traveled a long journey and is back-scattered. Intensity of back-scattered zodiacal light is far weaker than forward-scattered (Figure 6) and the long distance light has to travel from the Sun to reach the fine structures further decreases their visibility. Since the dust densities in the fine structures are also orders of magnitude lower than that of the smooth cloud [3], we can safely neglect the contribution of the light scattered from the fine structures in our zodiacal cloud model.

Our zodiacal cloud model consists of the smooth cloud part of the interplanetary dust cloud model developed by the COBE (Cosmic Background Explorer) DIRBE (Diffuse Infrared Background Experiment) team [3], where we have further neglected for simplicity the offset of the zodiacal cloud center from the Sun and its small inclination to the ecliptic. Omission of the inclination is quantitatively not significant for the comparison of our results to images and the omission of the small, \(\sim 0.01\) AU, offset of the zodiacal cloud center from the Sun has no effect on our results.

The zodiacal cloud model from the DIRBE team is constructed from observation with a wide spectral and surface coverage. The smooth cloud part of the model is a relatively simple analytical function. Infrared emission of interplanetary dust is also significantly easier to simulate than that at visible wavelengths leading to easier convertibility of the observations to dust densities. Moreover, the DIRBE interplanetary dust model has been found to be in good agreement with other interplanetary dust distribution research. Specifically, \textit{in situ} interplanetary dust measurements by dust detectors on board Galileo and Ulysses spacecrafts and the interplanetary meteoroid flux model by Grün et al. [16] were found to be consistent with the DIRBE interplanetary dust model [6].

Our model for the interplanetary dust number density is

\[
n(X, Y, Z) = n_0 R^{-1.34} f(\zeta),
\]  

(7)
where $X$, $Y$, $Z$ are heliocentric Cartesian coordinates, $n_0 = 1.13 \times 10^{-7} \frac{1}{m^3}$ is the dust number density at a distance of 1 AU from the Sun in the ecliptic. Here $R$ is the radial distance from the Sun i.e. $R = \sqrt{X^2 + Y^2 + Z^2}$, $\zeta = |\frac{Z}{R}|$ and

$$f(\zeta) = e^{-\beta g^\gamma},$$

(8)

where

$$g = \begin{cases} \frac{\zeta^2}{2\mu} & \text{for } \zeta < \mu, \\ \zeta - \frac{\mu}{2} & \text{for } \zeta \geq \mu, \end{cases}$$

(9a)

(9b)

with parameter values $\beta = 4.14$, $\gamma = 0.942$ and $\mu = 0.189$.

Figure 5: Interplanetary dust number density contours following from the simplified DIRBE model. Horizontal axis is the solar distance along the ecliptic and the vertical axis is the distance perpendicular to the ecliptic. Blue contours denote dust number densities with respect to the dust number density at the orbit of the Earth, which is marked with the red contour. The dust number density value at the orbit of the Earth is $1.13 \times 10^{-7} \frac{1}{m^3}$.

A contour plot of the interplanetary dust number densities for this model is shown in Figure 5 where the dust number density exponentially increases as solar distance decreases. Also, the dust being concentrated near the ecliptic can be seen, as the dust number density decreases much faster with an increasing distance from the ecliptic than with an increasing solar distance along the ecliptic.
2.2.2 Averaged volume scattering function

The averaged volume scattering function in equations (2) and (6) can be built under the assumption that the scattering properties and size distribution of the interplanetary dust particles are independent of their location within the zodiacal cloud. When in addition spherical shape of the zodiacal cloud particles and scattering properties dependence on just one parameter \( m \) is assumed, the averaged volume scattering function is

\[
\Phi(\phi, \lambda, m) = \frac{\int_a^b \Phi(s, m, \lambda, \phi)n(s)ds}{\int_a^b n(s)ds}, \tag{10}
\]

where \( \Phi(s, m, \lambda, \phi) \) is the volume scattering function of particles with radius \( s \), refractive index \( m \) and wavelength of the incident light \( \lambda \). \( n(s) \) is the particle number density where \( a \) is the lower limit and \( b \) is the upper limit for the grain size [11, 17].

Now the average volume scattering function could be calculated by using Lorenz-Mie theory if composition and size distribution of interplanetary dust grains were known. An easier way, however, to achieve a reasonable agreement between calculated and measured brightnesses is to use an empirical averaged volume scattering function. Specifically, an empirical volume scattering function can be fitted to the measured zodiacal light surface brightnesses, assuming the dust spatial density function \( n(r, \beta) \) in (6). Therefore the reverse operation, i.e. calculation of zodiacal light surface brightness \( I \) using an empirical volume scattering function and dust number density, which was used to derive the empirical scattering function, by construction reproduces in the actual measured zodiacal light surface brightness.

In this work, we apply an empirical scattering function derived by Hong [4]. The averaged scattering function by Hong consists of a linear superposition of three Henyey-Greenstein functions

\[
\psi_{hg}(\phi; g) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g\cos(\phi))^2}, \tag{11}
\]

each with a different value of \( g \). The parameter \( g \) in (11) takes values between \(-1\) and \(1\) and it measures the asymmetry of the scattering function. For \( g = 1 \), all scattering is in the forward direction and for \( g = -1 \), all scattering is backward. The intermediate case \( g = 0 \) represents isotropic scattering [18]. Features such as a forward peak, isotropic middle, and backward enhancement in a scattering pattern can be reproduced in this way, using linear combinations with different \( g \).
For diffraction-dominated small scattering angles, a linear substitution of Henyey-Greenstein functions is, however, a somewhat inefficient way to build the scattering function, since a sum over many Henyey-Greenstein functions would be needed [4]. Therefore we follow the proposition by Hong and use Fraunhofer diffraction as done by Van de Hulst in his pioneering zodiacal light study [19] to describe the scattering properties of interplanetary dust particles in the forward scattering direction.

Lastly, following Hahn et al. in their work with the zodiacal light images taken by the Clementine probe [20], we write the averaged volume scattering function $\Phi(\phi)$ in the equations (2) and (6) as a product $a\psi(\phi)$, where $a$ is the average geometric albedo of interplanetary dust grains and $\psi(\phi)$ is a phase function normalized to unity when integrated over a full solid angle. This formulation is valid only within the limits of geometric optics, which is indeed a good approximation since the majority of dust particles responsible for the zodiacal light lie in the size range $10 \mu m - 100 \mu m$ [16].

It is worth noting that the mean volume scattering function in the paper by Hong does not coincide with our averaged volume scattering function $\Phi(\phi)$, but with our phase law $\psi(\phi)$. The explanation for this is that the mean total scattering cross-section by Hong equals $a\sigma$, i.e. the product of averaged geometric albedo and the average cross-section of a zodiacal cloud particle, in our terminology.

Now we are ready to build the phase law of interplanetary dust particles. First for the small scattering angles, following Van de Hulst, we have, for a particle of radius $s$ and light of a wavelength $\lambda$, a scattering intensity $I = I_r(\phi) + I_d(\phi, s, \lambda)$, which is the sum of a reflection and a Fraunhofer diffraction term. Diffraction is dependent on the particle size and wavelength of the incident light. Hence to construct the phase law of interplanetary particles averaged over the particle size distribution, we need to use (11) to calculate the phase law for small angles. We also average the diffraction term over the visible solar spectrum. This results in a reasonable results for all visible wavelengths since the color of the zodiacal light is close to that of the solar light, i.e. all visible wavelengths are scattered roughly in a similar manner [21]. Reflection, which is negligible for low scattering angles, is thought to be Lambertian and is independent of grain size and the wavelength.
The phase law for small angles is

\[
\psi_{V,dH}(\phi) = K_1 \left[ \frac{2\gamma}{3\pi^2} (\sin(\phi) - \phi \cos(\phi)) + \int_{\lambda_1}^{\lambda_2} \int_a^b \frac{J^2_1(2\pi s \phi)}{\pi \phi^2} n_s(s)n_\lambda(\lambda)d\lambda d\phi \right],
\]

(12)

where \( K_1 \) is a normalization constant, \( \gamma = 0.1 \) is the albedo [20], \( J_1 \) is a Bessel function of the first kind, \( n_s(s) = n_{a0}s^{-t} \) is the size distribution of interplanetary particles and

\[
n_\lambda(\lambda) = \frac{2hc}{\lambda^5} \frac{1}{e^{hc/(\lambda kT)} - 1}
\]

(13)

is the wavelength distribution of the solar visible spectrum following Plack’s law. The value of the exponent \( t \) is \(-2.2\) [11] and lower and upper limit of the size distribution are 10\( \mu \)m and 100\( \mu \)m [16]. In (13) \( h \) is the Planck constant, \( c \) is the speed of light, \( k \) is the Boltzmann constant and \( T \) is the effective temperature of the Sun, for which the value 5778\( K \) [22] is used.

The phase law for intermediate and large scattering angles is constructed with the equations (10), (11a), (13) and (14) by Hong. The phase law by Hong is

\[
\psi_{Hong}(\phi) = K_2 \left[ (1 - \nu) \cos(\phi) - \sin(\phi) \frac{d}{d\phi} \right] \sum_{k=1}^{3} w_k H(\phi; g_k),
\]

(14)

where \( K_2 \) is a constant, \( \nu \) is the power law exponent 1.34 from the dust spatial density function, the prefactors \( w_k \) are weights of different Henyey-Greenstein functions and

\[
H(\phi; g_k) = \frac{1}{\sin^2(\phi)} \frac{1 - g_k}{g_k} \frac{1 + g_k}{(1 + g_k^2 - 2g_k \cos(\phi))^2} - 1,
\]

(15)

where \( g_k \) are asymmetry factors. Values for factors \( w_k \) and \( g_k \) are gotten from Table 1 by Hong as \( g_k = 0.70, -0.20 \) and \(-0.81 \) and \( w_k = 0.665, 0.330 \) and 0.005.

To combine the phase laws we first determine the normalization constant \( K_1 \) by requiring the small angle phase law to fulfill the normalization condition

\[
\int_{4\pi} \psi(\phi)d\Omega = 2\pi \int_0^\pi \psi(\phi) \sin(\phi)d\phi = 1.
\]

(16)

Secondly, we determine the constant \( K_2 \) and the limiting angle \( \phi_{lim} \) between the phase laws by requiring the phase laws to combine and that the first
derivative of the combined phase law is continuous. Lastly, we normalize the combined phase law with the normalization condition (16) and we get the phase law

$$\psi(\phi) = \begin{cases} K\psi_{VdH}(\phi) & \text{for } \phi < \phi_{lim} \\ K\psi_{Hong}(\phi) & \text{for } \phi \geq \phi_{lim}, \end{cases}$$

where $K$ is a normalization constant.

Figure 6: Phase law for our model of interplanetary dust. On the $y$-axis is the value of the scattering function $\psi$ and on the $x$-axis the scattering angle $\phi$ in degrees, see (Figure 3). At an approximate scattering angle of $40^\circ$ lies the boundary between the small scattering angle Fraunhofer diffraction law, see equation (12), and the empirical scattering law by Hong, see equation (14). The grain size and wavelength averaging integrals in the diffraction term in the first part of the phase law were calculated with a trapezoidal rule over 180 evenly spaced grain sizes and 320 evenly spaced wavelengths.

In Figure 6 the phase law (17) is plotted as a function of the scattering angle. The $y$-axis of the plot is in logarithmic scale because the forward Fraunhofer diffraction scattering is very strong compared to the scattering at higher angles. The boundary between the two phase laws lies approximately at a scattering angle value of $40^\circ$. Towards the back-scattering the value of the phase law rises again from the minimum at an approximate scattering angle of $80^\circ$. This increased back-scattering gives rise to the phenomenon of increased brightness centered near the antisolar point, known as the gegenschein [20].
3 Lunar Horizon Glow

3.1 Dust in the lunar exosphere

Each year the Moon is bombarded by about $10^6$ kg of interplanetary micrometeoroids, which excavate about 1000 times their own mass from the lunar soil. This collision ejecta forms a perpetual dust cloud around the Moon [23], which has lately been investigated by the Lunar Atmosphere and Dust Environment Explorer (LADEE) probe.

The science observation period of LADEE lasted from October 2013 to April 2014 at altitudes in the range of 20-260 km, following a near-equatorial orbit. In situ measurements of the lunar dust cloud carried out by the Lunar Dust Experiment (LDEX) dust detector on board LADEE have directly measured the lunar dust cloud. The impact-generated lunar dust cloud has been found to temporally vary with the annual meteor showers and to be asymmetric in shape, with a strong enhancement near the morning terminator between 5 and 7 local time, with a peak density cantled sunward. [1][24]

The measured asymmetric shape of the lunar dust cloud follows from the dominance of high-speed particles of cometary origin as a source of the lunar ejecta cloud. That is, lower velocity particles of asteroidal origin could only sustain a much weaker and more azimuthally symmetric ejecta cloud contrary to LDEX observations. [1]

In addition to bombardment by interplanetary dust, electrostatic lofting of charged dust grains from the lunar surface has also been suggested as a prominent factor behind the dust configuration around the Moon [25][26]. The most significant mechanisms behind the lunar dust charging are photoemission of electrons by solar ultraviolet radiation and collection of charged particles from the ambient solar wind plasma. These mechanisms can possibly charge dust grains enough to create an electrostatic force sufficient to overcome the lunar gravity. [27][28]

Laboratory experiments indicate that electrostatic lofting of dust particles from the lunar surface may indeed happen as a consequence of ultraviolet radiation and plasma exposure. The heights, however, which these electrostatically lofted dust grains were found to be able to jump to from the lunar surface were only about 11 cm [28]. Besides the experiments of electrostatic dust lofting, the lunar dust measurements by LDEX also do not support the existence of a significant electrostatically lofted high-altitude dust population around the Moon. Specifically, no evidence of electrostatically lofted dust grains were found by these in situ lunar dust measurements at an altitude range from 3 km to 250 km above the lunar surface [29].
3.1.1 Lunar dust cloud number density function

A model for the lunar dust cloud based on local time averaged in situ measurements by LDEX has been derived by Szalay and Horanyi. They found an exponential fit to be in good agreement with the measurements by LDEX. In their model, Szalay and Horanyi also derived the azimuthal dependence of the dust number density. In our model for the lunar exosphere we neglect, however, the azimuthal dependence of the dust number density and use a simplified spherically symmetric dust cloud model instead.

The number density of dust particles with a radius greater than \( r > 0.3 \) \( \mu m \) in the lunar exosphere as a function of altitude in a spherically symmetric version of the model by Szalay and Horanyi is

\[
n(h) = n_{Ld0} e^{h/\lambda_{Ld}}, \tag{18}
\]

where \( n_{Ld0} \) is the dust number density on the lunar surface, \( h \) is the altitude above the lunar surface in kilometers and \( \lambda_{Ld} = 200 \) km is the scale height.

3.1.2 Scattering properties of the lunar dust cloud grains

In order to model light scattering by the lunar dust cloud particles, we must fix the size distribution of the grains. To derive this size distribution, we first consider the respective cumulative mass distribution, which is given in the Extended Data Table by Horanyi et al. [1].

The number of dust particles with a radius greater than \( r > s \) is

\[
N_s(s) = K_3 m(s)^{-\alpha} = K_3 \left( \frac{4\pi s^3}{3} \right)^{-\alpha} = K_4 s^{-3\alpha} = \int_s^{s_2} n(s') ds'. \tag{19}
\]

Differentiating (19) with respect to \( s \) gives the differential number density of dust particles as a function of radius:

\[
n_s(s) = 3\alpha K_4 s^{-3\alpha-1}, \tag{20}
\]

where \( \alpha = 0.91 \) is the mass distribution slope and \( K_4 \) is a constant which can be calculated by comparing the equations (19) and (20).

By applying the size distribution (20) with lower and upper limits of the grain radii as 0.3 \( \mu m \) and 100 \( \mu m \) [1], we get an average cross-section of about 1.04 \( \mu m^2 \). The corresponding grain radius is about 0.58 \( \mu m \), which is at the wavelength of yellow light.

In order to describe light scattering by particles whose size is similar to the wavelength of incident light, a formal solution of Maxwell’s equations...
with appropriate boundary conditions is required. In the case of an isotropic and homogeneous sphere, a solution to this problem has been derived independently by Gustav Mie and Ludwig Lorenz among a few other physicists in the late nineteenth and early twentieth century [30]. Later, a solution was also found for more complex geometries and compositions.

Although the particles in the lunar dust cloud are expected to be complex in shape with heterogeneous compositions, and more sophisticated methods to describe their scattering properties have been applied before [31], we use the Lorenz-Mie theory to describe their light scattering features in order to retain some simplicity. In Figure 7, the intensity of light scattered by the lunar dust cloud particles is plotted as a function of scattering angle. The scattering curve is calculated with the Lorenz-Mie code spher.f by M. Mishchenko [32], using refractive index $1.6 + i0.003$ [31], wavelength 505 nm and a size distribution (21) with lower and upper limits 0.3 $\mu$m and 100 $\mu$m for the grain radii.

![Figure 7: Phase law of the lunar exosphere dust. On the logarithmic y-axis is the intensity of the scattered light and on the linear x-axis is the corresponding scattering angle. The scattering curve was calculated by the Lorenz-Mie scattering code spher.f by M. Mishchenko and normalized to unity when integrated over a full solid angle, see equation (16).](image)

The scattering curve in Figure 7 features a strong enhancement in the forward-direction and a milder one in the backward-direction. In contrast to the phase law for interplanetary dust particles (Figure 6), which showed the weakest scattering at an approximate scattering angle of 80°, the weakest
scattering for lunar dust cloud particles is at an approximate scattering angle of 120°.

3.2 Light scattered by dust in the lunar exosphere

A glow near the lunar horizon, which is not related to the solar corona-zodiacal light, was first seen in the images of lunar sunsets taken by the surveyor probes in the late 60's. In the lunar sunset images by Surveyor 7 (Figure 8), a wide line of light extends about 5.5° along the western lunar horizon following a lunar sunset. Rennilson & Criswell [25] have analyzed the Surveyor 7 images and suggested that the glow in the images follows from forward-scattered sunlight by dust grains at a low altitude above the lunar surface. As a dominant mechanism behind the levitation of the dust population, Rennilson & Criswell proposed electrostatic lofting [25].

Figure 8: A glow above the western lunar horizon following a local sunset as seen by the Surveyor 7 television system in 1968. Image: NASA

The next observations of a glow near the lunar horizon came from the Apollo flights. Each of the three last Apollo missions, i.e. A15-A17, included a series of photographic sequences which were designed to map the solar corona and the inner zodiacal light. Surprisingly, an anomalous excess brightness unrelated to the inner coronal and zodiacal light, but instead well correlated with the lunar horizon, occurred in several of these data sets [33]. As a part of the Solar Corona/Zodiacal Light Experiments of the Apollo program, the Apollo 17 crew also drew sketches of their visual observations of the solar corona-zodiacal light prior to orbital sunrise. The most famous one of these sketches is the one drawn by the commander of Apollo 17, Eugene Cernan (Figures 1 and 2). In addition to the zodiacal light, the sketch by
Cernan also includes a glow near the lunar limb, which first became visible about three minutes prior to sunrise.

As a part of the Clementine lunar mapping mission in 1994, the lunar horizon glow was also searched using its star tracker navigation cameras. The geometry in the possible lunar horizon glow observations by Clementine was similar to those of the Apollo flights. These images were studied by Glenar et al. [34], but no sign of a glow correlated with the lunar horizon was found in the Clementine images after subtraction of the solar corona-zodiacal light. [34]

If lunar horizon glow was sunlight scattered by the lunar dust cloud, its surface brightness would be an integral over scattering volume elements along the line of sight. The geometry of the observations is shown in Figure 9. For an observer in the shadow of the Moon, the first volume elements on the line of sight are not illuminated by sunlight. The distance which sunlight has traveled before it gets scattered by the dust is practically the same for all volume elements due to the significantly longer distance between the Sun and the Moon than between the observer and the scattering volume element, which is at the order of a lunar radius. Also the scattering angle $\phi$ remains essentially constant for volume elements along a given line of sight.

The surface brightness contribution $dI_{HG}$ of a scattering volume element $dV'$ on a line of sight with selenographic coordinates $r'$ and $\theta'$ is

$$dI_{HG} = \frac{F}{r_{1AU}^2} \Phi_{Ld}(\phi) \sigma_{Ld}(r', \theta') \frac{dV'}{l'^2},$$

(21)

where $F$ is the solar radiant flux and $\Phi_{Ld}(\phi)$ is the averaged volume scattering function expressing how much incoming radiation on the volume element in the lunar exosphere is scattered at an angle $\phi$. $\sigma_{Ld}(r', \theta')$ is the cross-sectional density of lunar dust particles at $(r', \theta')$, $dV'$ is the volume element and $l'$ is the distance between the volume element and the observer.

The averaged volume scattering function $\Phi_{HG}(\phi)$ can be written as a product

$$\Phi_{HG}(\phi) = a_{Ld} \psi_{Ld}(\phi),$$

(22)

where $a_{Ld}$ is the albedo of the lunar dust and $\psi_{Ld}(\phi)$ is the phase law.

The cross-sectional density of particles at $(r', \theta')$ can be expressed as

$$\sigma_{Ld}(r', \theta') = n_{Ld}(r', \theta') \bar{\sigma}_{Ld},$$

(23)

where $n_{Ld}(r', \theta')$ is the number density of particles at $(r', \theta')$ and $\bar{\sigma}_{Ld}$ is the average cross-section of the lunar exosphere particles. We get the number density $n_{Ld}(r', \theta')$ from (18) where we use parameter values that best fit
Figure 9: Lunar horizon glow observation geometry. On the left side is shown the projection to the lunar equator plane and on the right side the view from the (135°E, 0°N) direction. The location of the observer in spherical selenographic coordinates (φ', θ', r') is (135°W, 10°S, 1737 + 500 km) and it is marked with a magenta dot. Between the observer and the center of the Moon is a magenta line. The line of sight from the observer to a direction specified by (φ, θ) (Figure 4) is marked with a green line. Sunlight is marked with a red line. φ is the scattering angle towards the observer from the volume element at a distance l' from the observer, and its projections to the left and right projection planes are φ' and φ''. The volume elements which are closer to the observer than l'' are in the shadow of the Moon. The yellow line on the Moon is the morning terminator, the blue circle on the surface of the Moon is the prime meridian and the blue line in the +x -z-direction is pointing towards the Earth. The z-axis is towards the north pole of the Moon and the x-axis is defined by the condition that the Earth lies in positive x direction of the xz-plane.

the LADEE lunar dust cloud measurements [5]. Using observer-centered spherical coordinates the volume element is

\[ dV' = \Omega' l'^2 dl', \tag{24} \]

where \( \Omega' \) is the solid angle under which the volume element is seen by the observer.

For the solar radiant flux \( F \) we have

\[ F = r_{\text{AU}}^2 S_0, \tag{25} \]
where $S_0 = 1361 \text{W} m^{-2}$ is the solar irradiance at a distance of one astronomical unit from the Sun [12].

With equations (21) – (25) we get the surface brightness of the lunar horizon glow

$$I_{HG} = \int_{l'_{\text{min}}}^{l'_{\text{max}}} dI_{HG}/\Omega'$$

$$= \sigma_{LD} S_0 a_{LD} n_{LD}\psi_{LD}(\phi) \int_{l'_{\text{min}}}^{l'_{\text{max}}} e^{r(i')-r_M}/\lambda_{LD} dl',$$

where $l'_{\text{min}}$ is the distance from the observer to the first volume element on a line of sight which is not in the shadow. $l'_{\text{max}}$ is the upper limit of integration for which a relevant value is a few times the radius of the Moon.

4 Reconstruction of the Lunar Sunrise Images

4.1 Apollo 17 trajectory around the Moon

The last Apollo mission, Apollo 17, was launched on 7.12.1972 at 05:33:00 G.m.t from Kennedy Space Center, Florida. After approximately three hours in Earth orbit the translunar injection was initiated and the spacecraft inserted into a retrograde lunar orbit at a GET 86:14:23. The notion GET stands for Ground Elapsed Time, which means hours:minutes:seconds after the launch of the mission, i.e. lift-off from the Kennedy Space Center. In the Apollo 17 flight report, times are given in GET. [2]

The lunar module separated from the command module at GET 107:47:56 and landed on the Moon. After about 75 hours on the lunar surface the lunar module ascended and at GET 187:37:15 the modules redocked. Soon after the docking, the lunar module was jettisoned and led to hit the Moon. The command module stayed in a lunar orbit for about two days before the transearth injection took place at GET 234:02:09. On 19.12.1972, about three days after the transearth injection, the Apollo 17 command module landed in the Pacific Ocean, finishing the Apollo program. [2]

In order for us to determine the spacecraft’s location when the sunrise sketches (Figure 1) were drawn, we first need to know the time. In the upper right corner of the sunrise sketch by Cernan (Figure 1) reads GET 232 h 55min. We have, however, reason to believe that this is not the time since the actual lift-off from the Kennedy Space Center. Instead, the time Cernan wrote is the time since the planned lift-off. Specifically, the lift-off of Apollo 17 was delayed by about 2 h 40 minutes due to a hardware failure. [2] The
mission timeline on the page 10-64 of the flight report (Figure 11) also shows that at GET 232 h 55 min, the command module was clearly on the bright side of the Moon [2]. If one instead subtracts two hours and forty minutes of the time written by Cernan, one obtains a time when orbital sunrise could be observed from the command module (see Figure 10). Hence, we believe that the sunrise sketches were drawn at an approximate Ground Elapsed Time of 230 h 15 min.

To determine the location of the command module at GET 230:15 we need to know the spacecraft’s location and velocity at a reasonably close time. The engines of the spacecraft should also not have been used between 230:15 and the time we choose for the location calculation. In that case the calculation of the trajectory from the as a two-body problem gives a reasonable result. On pages 3-4 of the flight report (Figure 12 and 13), locations and velocities of the lunar and command modules are given at different parts of the mission. The closest time to 230:15 for which trajectory parameters are given is 234:02:09, when the transearth injection ignition took place. Between the transearth injection and the sunrise sketching the engines were indeed not used. [2]

As can be seen from the flight report (Figure 12 and 13), at GET 234:02:09, the location of the command module was (19.65°S, 170.02°W) 62.1 nautical miles (115 km) further from the center of the Moon than the landing site of the Apollo 17 lunar module. The command module was heading towards south-west, 257.32° east from north, with a flight-path angle of −0.18°. Flight-path angle is the angle between the velocity vector and a body-centered local horizontal plane and its positive side is above the plane. The velocity of the spacecraft was 5337.1 ft s (1626.7 m s).

In order to determine the spacecraft’s position at GET 230:15 we solve the related spacecraft-Moon two-body problem. Within the time span of about four hours between the transearth-injection and the drawing of the sunrise sketches (Figure 1), the determination of the trajectory from the two-body problem does not deviate significantly from a more realistic trajectory modeling. Better precision in the trajectory calculation would be achieved by also taking into account the gravity of the Earth and the Sun as well as the oblateness of the Moon.

Following the two-body problem treatment in the book by Murray and Dermott [35], we evaluate the spacecraft’s trajectory on the lunar orbit. The first step, to determine the spacecraft position at GET 230:15 is to find the corresponding eccentric anomaly $\epsilon$. It can be found from Kepler’s equation

$$\epsilon - \epsilon \sin(\epsilon) = n(t - \tau), \quad (28)$$
where \( e \) is the eccentricity and

\[
\tau = \frac{e \sin(\epsilon_0) - \epsilon_0}{n}
\]  

(29)

is the moment of pericenter passage. The mean motion, \( n \), is equal to the Kepler frequency if we neglect oblateness of the Moon

\[
n = \sqrt{\frac{GM}{a^3}}.
\]  

(30)

Here \( G \) is the gravity constant, \( M \) the mass of the Moon and \( a \) is the semi-major axis of the orbit.

The eccentricity, \( e \), we obtain by first solving the semi-major axis of the spacecraft’s orbit, \( a \), from a relation for the total energy, \( E \), of the spacecraft

\[
E = \frac{v^2}{2} - \frac{GM}{r} = -\frac{GM}{2a},
\]  

(31)

where \( v \) is the velocity of the spacecraft and \( r \) its distance from the center of the Moon. Values for \( v \) and \( r \) we get the from the Apollo 17 flight report. With the semi-major axis solved from (31), we obtain the eccentricity from

\[
a = \frac{h^2}{GM(1 - e^2)},
\]  

(32)

where \( h \) is the angular momentum of the spacecraft.

Our method to solve \( e \) from the transcendental Kepler equation is to first set \( \epsilon_1 = n(t - \tau) \) and then solve

\[
e_2 = e \sin(\epsilon_1) + n(t - \tau).
\]  

(33)

This gives us \( \epsilon_2 \), which we can further substitute into (33) and get an even better estimation of the actual \( e \). This iteration converges finally to \( e \).

From \( e \) solved, we obtain the true anomaly \( f \) from

\[
\cos(f) = \frac{\cos(e) - e}{1 - e \cos(e)}
\]  

(34)

and then further the spacecraft’s distance from the center of the Moon,

\[
r' = \frac{a(1 - e^2)}{1 + e \cos(f)}.
\]  

(35)
Figure 10: Part of the page 10 – 64 of the Apollo 17 mission report [2]. The numbers on the left side show the Ground Elapsed Time and the bar on the right shows whether the command module was in the lunar shadow or seeing the Sun.
Figure 11: Same as Figure 10 but for the following two hours.
Figure 12: Apollo 17 trajectory parameters for the lunar orbit phase of the mission [2].

<table>
<thead>
<tr>
<th>Event</th>
<th>Reference body</th>
<th>Time, hr:min:sec</th>
<th>Latitude, deg:min</th>
<th>Longitude, deg:min</th>
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</thead>
<tbody>
<tr>
<td>Lunar orbit insertion</td>
<td>Moon</td>
<td>86:14:23</td>
<td>11.33 S</td>
<td>177.35 E</td>
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<tr>
<td>Ignition</td>
<td>Moon</td>
<td>86:20:56</td>
<td>6.81 S</td>
<td>151.54 E</td>
</tr>
<tr>
<td>Cutoff</td>
<td>Moon</td>
<td>90:31:37</td>
<td>11.40 S</td>
<td>164.16 E</td>
</tr>
<tr>
<td>First descent orbit insertion</td>
<td>Moon</td>
<td>90:31:59</td>
<td>11.06 S</td>
<td>163.04 E</td>
</tr>
<tr>
<td>Ignition</td>
<td>Moon</td>
<td>107:47:56</td>
<td>5.02 S</td>
<td>135.91 E</td>
</tr>
<tr>
<td>Cutoff</td>
<td>Moon</td>
<td>109:17:29</td>
<td>20.03 S</td>
<td>149.17 W</td>
</tr>
<tr>
<td>Command and service module/lunar module separation</td>
<td>Moon</td>
<td>109:17:33</td>
<td>20.02 S</td>
<td>149.30 W</td>
</tr>
<tr>
<td>Command and service module circularization</td>
<td>Moon</td>
<td>109:22:42</td>
<td>19.22 S</td>
<td>165.78 W</td>
</tr>
<tr>
<td>Ignition</td>
<td>Moon</td>
<td>109:23:04</td>
<td>19.12 S</td>
<td>166.77 W</td>
</tr>
<tr>
<td>Second descent orbit insertion</td>
<td>Moon</td>
<td>178:54:05</td>
<td>12.37 S</td>
<td>124.36 E</td>
</tr>
<tr>
<td>Ignition</td>
<td>Moon</td>
<td>179:54:14</td>
<td>12.01 N</td>
<td>57.38 W</td>
</tr>
<tr>
<td>Cutoff</td>
<td>Moon</td>
<td>185:28:56</td>
<td>21.91 N</td>
<td>19.60 E</td>
</tr>
<tr>
<td>Ascent (insertion)</td>
<td>Moon</td>
<td>185:32:12</td>
<td>22.90 N</td>
<td>8.22 E</td>
</tr>
<tr>
<td>Vernier adjustment maneuver</td>
<td>Moon</td>
<td>186:15:58</td>
<td>15.12 S</td>
<td>129.53 W</td>
</tr>
<tr>
<td>Docking</td>
<td>Moon</td>
<td>191:18:31</td>
<td>19.47 N</td>
<td>31.23 E</td>
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<tr>
<td>Lunar module jettison</td>
<td>Moon</td>
<td>192:58:14</td>
<td>0.26 S</td>
<td>86.97 E</td>
</tr>
<tr>
<td>Lunar module ascent stage deorbit</td>
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<td>193:00:10</td>
<td>2.02 N</td>
<td>81.68 E</td>
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<tr>
<td>Ignition</td>
<td>Moon</td>
<td>234:02:09</td>
<td>19.65 S</td>
<td>170.02 W</td>
</tr>
<tr>
<td>Cutoff</td>
<td>Moon</td>
<td>234:04:33</td>
<td>21.52 S</td>
<td>179.69 W</td>
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</table>
Figure 13: Apollo 17 trajectory parameters for the lunar orbit phase of the mission [2].

<table>
<thead>
<tr>
<th>Altitude, mile</th>
<th>Space-fixed conditions</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>Velocity ft/sec</td>
</tr>
<tr>
<td>75.8</td>
<td>6110.2</td>
</tr>
<tr>
<td>51.2</td>
<td>5912.0</td>
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<tr>
<td>51.1</td>
<td>5512.7</td>
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<td>50.9</td>
<td>5322.1</td>
</tr>
<tr>
<td>47.2</td>
<td>5342.8</td>
</tr>
<tr>
<td>53.8</td>
<td>5279.9</td>
</tr>
<tr>
<td>53.8</td>
<td>5294.9</td>
</tr>
<tr>
<td>59.6</td>
<td>5274.5</td>
</tr>
<tr>
<td>59.6</td>
<td>5267.0</td>
</tr>
<tr>
<td>8.7</td>
<td>5550.3</td>
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<tr>
<td>64.9</td>
<td>5315.1</td>
</tr>
<tr>
<td>60.5</td>
<td>5341.1</td>
</tr>
<tr>
<td>8.0</td>
<td>5542.3</td>
</tr>
<tr>
<td>9.4</td>
<td>5534.7</td>
</tr>
<tr>
<td>46.6</td>
<td>5333.3</td>
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<tr>
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<td>5341.7</td>
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<td>5337.1</td>
</tr>
<tr>
<td>63.1</td>
<td>5374.3</td>
</tr>
</tbody>
</table>
Now the location of the spacecraft in the orbital plane at the time \( t \) is given by the vector \( \omega_2 = r' \cos(f + \varpi) \hat{i}' + r' \sin(f + \varpi) \hat{j}' \), where \( \varpi \) is the argument of pericenter. Both vectors \( \hat{i}' \) and \( \hat{j}' \) have their origin at the center of the Moon. The vector \( \hat{i}' \) is pointing to the location of the spacecraft at GET 234:02:09, see Figures 12 and 13. The vector \( \hat{j}' \) is pointing to a direction given by the projection of the velocity vector of the spacecraft at GET 234:02:09 to a local horizontal plane, i.e. the direction of the velocity vector of the spacecraft if the flight-path angle is set to zero. We get the vector \( \omega_1 \) specifying the location of the spacecraft in selenographic Cartesian coordinates with the coordinate transformation

\[
\omega_1 = R^{-1} \omega_2,
\]

where the rotation matrix is given by

\[
R = \begin{bmatrix}
1 & 0 & 0 \\
0 & \sin \beta_0 & \cos \beta_0 \\
0 & -\cos \beta_0 & \sin \beta_0
\end{bmatrix}
\begin{bmatrix}
\cos \theta_0 & 0 & \sin \theta_0 \\
0 & 1 & 0 \\
-\sin \theta_0 & 0 & \cos \theta_0
\end{bmatrix}
\begin{bmatrix}
\cos \varphi_0 & \sin \varphi_0 & 0 \\
-\sin \varphi_0 & \cos \varphi_0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

(37)

Here \( \beta_0 \) is the initial heading angle and \( \theta_0 \) and \( \varphi_0 \) are the initial latitude and longitude of the spacecraft. The components of the matrix \( R \) are set by rotations around the coordinate axes. The angles \( \varphi_0, \theta \) and \( \beta_0 \) are Tait-Bryan angles defined by intrinsic rotations around axes in \( z-y'-x'' \) order where the initial coordinate system is Cartesian selenographic. The orientation of the orbital plane in space in the Apollo 17 flight report is given through these angles instead by longitude of ascending node and inclination as in the standard treatment.

In Figure (14), the trajectory of the command module is plotted from the time of the sunrise sketches to the time of the translunar injection. At the time of the sunrise sketches, the command module was under the equator, heading approximately westwards.
Figure 14: The trajectory of the Apollo 17 command module around the Moon between Ground Elapsed Times 230:15-234:02:09 projected to the lunar equator plane. The orbit is retrograde, i.e. clockwise in this projection. The red line shows the part of the orbit when the spacecraft was above the equatorial plane and the green line shows the part when it was below the equatorial plane. The magenta dot denotes the spacecraft’s position at GET 230:15. The black circles denote selenographic latitudes 0°, 30° and 60°. The black lines denote selenographic longitudes 45°, 90°, 135°, 225°, 270° and 315°. The blue line is the prime-meridian and the green square shows the point where selenographic latitude and longitude are zero. The z-axis is upwards from the plot and the green line shows the direction of the Earth which lies in positive x-direction. The yellow ellipse on the Moon shows the morning terminator and the yellow line coming from the upper right is the direction of the Sun. With increasing time the morning terminator develops clockwise.
4.2 Image reconstruction

The brightness of each pixel in the reconstructed image is a sum of brightness contributions of the solar corona/zodiatic light and light scattered by the lunar dust cloud, see equations (6) and (27),

\[ I = I_{ZD} + I_{HG}. \]  

In order to construct from the brightnesses \( I \) measured from different directions a two-dimensional image we need a projection of the three-dimensional observations to a plane.

When a three-dimensional view is projected into a two-dimensional plane, there is always something wrong with angles or distances in the resultant image. In the view we are reconstructing (Figure 1) the most important parts are close to the axes: Zodiatic light is close to the vertical axis and the horizon glow is near the horizontal axis. Therefore in the projection we choose the areas near the axes as the most important to be represented with as small errors as possible.

Figure 15 shows the observation geometry and Figure 16 its projection as a two-dimensional image. The vertical extent of the reconstructed image is \( n \) pixels and its width on the x-axis is \( 2m - 1 \) pixels, where \( n \) and \( m \) in this simple example (Figures 15 and 16) are both 10. Every point of the view is located on an observer centered spherical surface.

The dots on every horizontal line of the reconstructed image lie on an observer centered isosphere separated by equal angular distances. The number of pixels on the horizontal lines decreases as the angle \( A \) of the line increases because the sides of the view are parts of observer centered isospheres which cross the point directly above the observer, i.e. the point where \( A = 90^\circ, \beta = 0^\circ \).

In the reconstructed image in Figure 16 pixels on the same horizontal line appear equally high. This is despite the fact that the angle between the x-axis of the view and the dot considered decreases as the distance from the y-axis of the view increases. This can be seen in Figure 15, where the angle \( \alpha \) is smaller than the angle \( A \). This causes the areas near the upper corners of the view to seem too large and not properly aligned, but the areas near the axes are not strongly affected.
Figure 15: Observation geometry. The Sun is at the origin and the observer’s heliocentric Cartesian coordinates are (0.8 AU, 1.5 AU, −0.4 AU). $\gamma$ is the azimuth angle of the observer in heliocentric spherical coordinates. We take the central pixel in the lowest image line as a reference point. Then $\theta$ is the angle between the line of sight to the reference point and a plane parallel to the ecliptic through the observer location. In that plane $\varphi$ is the angle between the projections of the solar direction and the direction to the lower central point of the view. The blue, red and green dots are all equally far from the observer and they denote pixels in the view. The points highlighted with green denote the positive $x$-axis of the view and the red points the positive $y$-axis. The black point is an example point, whose angular coordinates in the view are $\alpha$ and $\beta$. $\alpha$ is an angle measured upwards from the $x$-axis of the view and $\beta$ is an angle measured to the right from the $y$-axis of the view. $\epsilon$ tells how much the view is tilted, i.e. the angle between the planes where the angle $A$ is measured and the plane where the angle $\theta$ is measured.
4.3 Reconstructed lunar sunrise images

In order to calculate the brightness of each pixel in the reconstructed images we need to know the constants in the integrals (6) and (27). In the zodiacal light brightness integral, (6), the average cross-section of a zodiacal cloud particle, $\bar{\sigma}$, we get from the size distribution $n_s(s) = n_0 s^{-2.2}$ [11] with lower and upper limits for the grain radius, $s$, $a = 10$ $\mu$m and $b = 100$ $\mu$m. The average cross-section of a zodiacal cloud dust grain is thus

$$\bar{\sigma} = \frac{\int_a^b s^{-2.2} \sigma s^2 \, ds}{\int_a^b s^{-2.2} \, ds} = 2670 \mu m^2.$$  (39)

The value of the dust number density at Earth orbit, $n_0$ from the zodiacal cloud model by the COBE DIRBE team is $1.13 \times 10^{-7}$ [3]. For the albedo of
the zodiacal cloud particles, \(a\), see section 2.2.2, we use the value 0.06 which is derived by Ishiguro et al. from high resolution gegenschein images [36]. The value of the solar constant \(S_0\) in (6) was already given in chapter 2.2. But instead of using \(S_0\), we express the evaluated brightnesses in terms of solar surface brightnesses

\[
B_\odot = S_0 \Omega,
\]

where \(\Omega = 6.8 \times 10^{-5}\) is the solid angle under which the Sun is seen from the Earth [20].

In the lunar horizon glow brightness integral, (27), the albedo \(a_{Ld} = 0.93\) which we get from the Mie-scattering code spher.f by M.Mishchenko [32] which we used to calculate the phase law of the dust grains in the lunar exosphere in section 3.1.2. For the dust number density on the lunar-surface, \(n_{Ld}\), we use a value \(8.5 \times 10^{-3} \frac{1}{m^3}\) which has been derived by Szalay and Horanyi for a lunar dust cloud model based on the lunar exosphere measurements by LADEE [5]. For the lunar radius \(r_M\) we use a value 1737 km [37] and the value of the scale height constant, \(\lambda_{Ld}\), is 200 km as given in section 3.1.1. The average cross-section of a dust grain in the lunar dust cloud is 1.04 \(\mu m^2\) as calculated in section 3.1.2. As with the zodiacal light, we also express the brightness of the lunar horizon glow in terms of the surface brightnesses of the Sun.

In addition to the constants in the brightness integrals we also need the location of the Moon and Earth at the time of the sunrise sketches. The location of the Moon is needed to calculate the coordinates of points along the lines of sight for each pixel, see Figure 15. The points on the lines of sight are then needed for the evaluation of the integral (6). The coordinates of the Earth are needed to fix the lunar coordinate system. Specifically, the x-axis in selenographic coordinates, i.e. in coordinates where the origin is at the center of the Moon, is defined so that the Earth lies on the positive-x-side of the xz-plane where the z-axis is towards the north pole of the Moon, (Figure 9).

The location of the Moon and the Earth in heliocentric Cartesian coordinates, where the x-axis is pointing towards vernal equinox and the z-axis is towards the ecliptic pole in right-handed convention, we get with the calculator by Don Cross [38]. The location of the Moon on 16.12.1972 was (0.0883668,0.9816074,0.0002128) and the location of the Earth on the same day was (0.0863694,0.9802330,0). The unit of the coordinates is AU.

The values of the phase laws for the dust in the zodiacal cloud were tabulated in 0.5° steps, (see equation (17) and Figure 6), and for the dust in the lunar exosphere, see Figure 7. For each volume element on given line of sight we then calculate the corresponding scattering angle, \(\phi\) in (6) and (27),
and approximate it by the closest value for which the phase law is tabulated. For the length of the line of sight a value of 4 AU is used for the zodiacal light calculations and 4 lunar radii, 1737 km [37], are used for the lunar horizon calculations. The evaluation of the brightness integrals (6) and (27) is done using a trapezoidal rule with 40 integration points for the zodiacal light calculations and 400 for the lunar horizon calculations.

We set the vertical extent of the images on the y-axis, i.e. the total span covered by the red dots in Figures 15 and 16, to be 30°. This corresponds to the value 30° for the angle \( \phi \) when measured to the uppermost central point of the view, i.e. to the point \((m = 0, n = n_{\text{max}})\) in Figure 16. The horizontal extent of the line from the lower left to the lower right corner of the view, i.e. from the point \((m = -m_{\text{max}}, n = 0)\) to the point \((m = m_{\text{max}}, n = 0)\), we also set to 30°. This corresponds to a value of 15° for \( \beta \) in Figure 15 when measured to the lower right corner of the view. For the numbers specifying the resolution of the images \( m \) and \( n \), (see chapter 4.2), we use values 389 and 195 respectively.

For the view tilt angle \( \epsilon \), see Figure 15, we use a value of 90° minus selenographic latitude of the command module at the time of observation which results in the lunar horizon in the reconstructed images being approximately horizontally aligned. In each of the reconstructed images the Sun is located in the central lower point of the image, i.e. the \((0,0)\) point Figure 16. This implies zero values for the angles \( \varphi \) and \( \theta \), (see Figure 15), which specify the viewing direction.

The sketches drawn by the astronauts, (Figure 1), were drawn 6, 3, 2, 1 minutes and 5 seconds before the Sun first became visible [15]. In order to calculate the spacecraft’s position at the times the sketches were drawn we first need to determine the time when the Sun became visible. That time we find by reconstructing sunrise images with different values for the time \( t \) in (28) until the lower central point of the reconstructed image is no longer black as are the pixels for which the corresponding line of sight intersects the Moon. In this manner we find that the Sun first became visible 13,820 seconds before the trans-earth injection ignition took place at GET 234:02:09, (see Figures 12 and 13). The GET at which the Sun became visible is thus 230:11:49.

In Figures 17 - 21 the reconstructed images are given. The extent of the zodiacal light is growing in size as the spacecraft gets closer to the morning terminator. As expected, the zodiacal light is aligned along the ecliptic and therefore it appears to be tilted rightwards in the reconstructed images since the astronauts were observing it from southern lunar latitudes, (Figure 14). The zodiacal light 5 seconds before the sunrise, (Figure 21), might actually not have been seen by the astronauts anymore due to a much brighter solar
corona.

The light scattered by the lunar dust cloud has a very small contribution compared to the zodiacal light in any of the reconstructed images. In Figure 22, the light scattered by the impact-generated lunar dust cloud multiplied by a factor of one hundred is shown 2 minutes before the Sun became visible. It reaches a maximum of about one tenth of the zodiacal light brightness, (Figure 19), on the left and right side of the view just above the lunar horizon.

Figure 17: View 6min before orbital sunrise. A faint zodiacal light can be seen rising above the lunar horizon. Each pixel represents the brightness \( I \) from equation (36). The unit of the brightness contours is the solar surface brightness,( see equation (38)).
In contrast to the interpreted lunar horizon glow alignment with the lunar horizon in the sunrise sketches by Cernan, see Figure 2, the brightness contours of the lunar horizon glow we reconstructed, see Figure 22, are rather aligned along Sun centered spheres.

Figure 18: View 3min before orbital sunrise. Zodiacal light is seen above the horizon.
Figure 19: View 2min before orbital sunrise. Pixels that are brighter than $10^{-10}$ times the surface brightness of the Sun are shown equally bright.
Figure 20: View 1min before orbital sunrise.
Figure 21: View 5 sec before orbital sunrise.
Figure 22: Light scattered by the impact-generated lunar dust cloud with brightness of each pixel multiplied by a factor of one hundred 2min before the sunrise. The wiggles in the brightness contours come from numerical errors in the evaluation of the lower limits of integration in the brightness integral (27) for each line of sight.
5 Conclusions and Discussion

5.1 Summary of work and main hypothesis

The objective of this study was to reconstruct sketches of a lunar orbital sunrise drawn by the crew of Apollo 17 (Figure 1). We assumed the sketches to consist of light scattered by the zodiacal cloud and of light scattered by the micrometeoroid impact-generated lunar ejecta cloud. To reconstruct the images, the dust number densities in the zodiacal cloud and in the lunar ejecta cloud were needed. Further, the light scattering properties of the dust grains in these dust configurations were needed.

For the zodiacal cloud we used a model derived from COBE infrared observations [3]. For the lunar ejecta cloud we used a simplified version of an impact-generated lunar exosphere model derived from LADEE in situ measurements by Szalay and Horanyi [5]. To describe the scattering characteristics of the interplanetary dust grains we applied an empirical scattering function derived from zodiacal light measurements by Hong [4] for large and intermediate scattering angles and, as proposed by Hong, Fraunhofer diffraction [19] for small scattering angles. For the scattering properties of the dust in the lunar ejecta cloud Lorenz-Mie theory was applied.

Finally, we applied the information of the spatial densities and the scattering properties of the interplanetary and of the lunar ejecta dust cloud to the observation geometries behind the sketches to reconstruct the orbital sunrise view. The location of the spacecraft at the time of the sketches we calculated from orbital parameters given in the Apollo 17 flight report [2].

5.2 Summary of main results

Based on our work there is little doubt that the Apollo astronauts actually saw the zodiacal light. Zodiacal light is the dominant feature in each of the reconstructed images and its distribution over the sky is consistent with the zodiacal light in the sketches (Figure 1).

Light scattered by an impact-generated lunar dust cloud [1] in turn was most likely not seen by the astronauts of Apollo 17. Sunlight is scattered by the ejecta cloud but its intensity is very low compared to the light scattered by the zodiacal cloud. This low intensity of the scattered light follows directly from the low dust number density in the lunar exosphere.

Further, the shape of the glow caused by light scattered by the ejecta cloud (Figure 22) is not well aligned with the lunar horizon as is the glow in the sketches (Figure 2). Taking the plausible assumption that light scattering properties of the dust grains in the lunar exosphere are the same at all
altitudes, a glow aligned well with the horizon would require a dust population which number density drops very fast with the altitude from the lunar surface. This is not the case for the ejecta cloud.

The so-called streamers which appear in the sketch by Cernan (Figures 1 and 2) we find hard to believe were related to light scattering by the lunar ejecta cloud or any other dust population. Specifically, if the streamers were caused by mountains on the Moon casting shadows on the lunar exosphere, the light scattered by the lunar dust cloud should itself be bright enough to be seen. Only then could shadows produce a visible contrast between brighter parts. Because we found the light scattered by the lunar dust cloud not to be visible at all in front of the zodiacal light, we did not model shadows in the lunar exosphere.

If, on the other hand, the dust density at high altitudes above the Moon were significantly higher and light scattered by this dust were visible over the zodiacal light, then light scattered by the lunar exosphere should also be seen on a wide horizontal and vertical angular range. This wide angular range of glow caused by light scattered by the lunar ejecta cloud can be seen from the brightness contours of the reconstructed lunar horizon in Figure 22. Such a wide glow is not visible in the sunrise sequence sketch by Cernan (Figures 1 and 2) where the glow that was interpreted as light scattered by a lunar dust cloud is only visible at low altitudes.

5.3 Uncertainties

For the constants used in the zodiacal light brightness integral (6) we took values from different studies which each concentrated on different feature of the interplanetary dust. This is likely to result in a small quantitative deviation of our zodiacal light model from the measured configuration. But is not decisive for the qualitative interpretation of our results.

The dust number density for the lunar ejecta cloud which we used is an overestimation. Specifically, in the spherically symmetric model for the dust number density (18) we assumed the number density to be that of the morning terminator side of the Moon where the observed dust density peaks. The observations behind the sunrise sketches, see Figure 1, were, however, made near the evening terminator on the opposite side of the Moon (Figure 14). On the evening terminator side the number density should be at its minimum [5] although at this location LADEE dust detector could not make measurements. We intentionally used the maximal measured dust density to obtain a conservative upper limit for the brightness contribution by the lunar dust exosphere to our reconstructed image.

The dust density model used is, on the other hand, derived from the
measurements by LADEE from January 2014 to April 2014 when the meteoroid shower activity was at minimum [5]. The dust number density around the Moon naturally increases from the modeled one during an annual meteor shower. One such meteor shower is the Geminids which occur in mid-December [1] which is close to but not precisely the date December sixteenth when the sketches were drawn (see Figure 1).

To describe the scattering characteristics of the dust grains in the lunar ejecta cloud, we used Lorenz-Mie theory (see chapter 3.1.2). Scattering properties of the dust particles in the lunar exosphere have been examined in more detail by Richard et al. [31] by numerically simulating more complex geometries of the dust particles and using the Discrete Dipole Approximation (DDA) to compute their scattering behavior. The study by Richard et al. [31] shows that the forward scattered intensities by more realistic lunar dust grain shapes can be up to several times greater than those of volume-equivalent spherical grains. This stronger forward scattering arises mainly as a consequence of a larger scattering cross-section of an irregular grain compared to a spherical grain which has the largest possible ratio of volume to scattering cross-section. Nevertheless, even if the brightness from the Lunar cloud would be larger by a factor of several, the main conclusion remains unchanged, that the astronauts can not have seen the cloud directly.

5.4 Comparison to other work

Our results of the analysis of the light scattering behind the Apollo 17 sunrise sketches (Figure 1) are consistent with the analysis of zodiacal images taken in similar geometry by the Clementine probe in 1994. Glenar et al. [34] have analyzed these zodiacal light images by Clementine and found no sign of a lunar horizon glow. An excess brightness correlated with a solar corona streamers was, in turn, seen in some of the Clementine images. Clementine took the zodiacal light images between February and April which corresponds to the time when LADEE measured the lunar dust cloud model [5]. The brightness contours of the zodiacal light we reconstructed are approximately the same as the ones measured by Clementine from lunar orbit [20][34].

Coronal photographic sequence images of Apollo 15 have been analyzed by Glenar et al. [33] where they found an excess brightness above the lunar horizon. While a dust cloud consisting of micrometeoroid impact ejecta does not seem to result in a dust population that generates a glow well aligned with the lunar horizon, Glenar et al. [33] proposed that a cloud of secondary impact ejecta could cause such a glow. Namely, when the primary ejecta from micrometeoroid impacts fall back onto the Moon, they may excavate secondary impact ejecta. The dust grains of the second generation would
then have lower speeds than the dust grains of the first generation. These lower velocities of the second generation dust grains would then naturally result in a dust density which is more concentrated near the lunar surface, thus being consistent with the requirement for the scattering media to cause a glow well aligned with the horizon. This kind of cascade-like process could then in principle be responsible for the dust population near the lunar surface, possibly scattering light as sketched by the Apollo 17 crew, (Figure 1 and 2). However, the problem with low number densities deriving from the low number density of primary ejecta would still persist, so that one does not expect the secondary ejecta to cause a brightness that stands in front of the zodiacal light.

In their study of the streamers in the Apollo 17 sunrise sketches, McCoy and Criswell [15] proposed that the streamers were sunlight scattered by dust in the vicinity of the Moon in which the mountains on the Moon casted shadows. McCoy and Criswell deduced the required solar plasma streamer length to produce the observed streamers to be over $75 \times 10^6$ km and that a disturbance propagating in the streamers suddenly brightening them could not be fast enough to cause the observed streamer brightening just before the sunrise. In contrast to the conclusions by McCoy and Criswell about the streamers in the sunrise sketches by the Apollo 17 crew, we find the streamers being in shadows casted on the lunar dust cloud hard to believe. Instead, we propose that they were in deed part of the solar corona. The solar corona streamers can easily be confused as being of local origin as stated by Glenar et al. [34] in their study of zodiacal light images by Clementine, where they were searching for signs of an excess brightness behind the zodiacal light.

But as pointed out by McCoy and Criswell [15], the observed brightening of the streamers with an angular extent of about $30^\circ$ just before the sunrise could not be caused by streamers of the solar corona. Therefore we are led to believe that the streamers were not in reality brightened in their full extent just before the sunrise, but simply the brighter parts of them extending a few degrees from the Sun became gradually visible. Streamers of the solar corona do show this behavior of brightening rapidly within a few degrees from the Sun [34]. This explanation for the streamers was also preferred by Harrison Schmitt, one of the astronauts on board Apollo 17, who right away considered the glow streamers to be a direct extension of the inner solar coronal structure. Further Schmitt suggested that the flash effect at sunrise was not real, but a reaction to the rapid increase in brightness at the horizon as the inner corona rose [15].
5.5 Improvements and future work

A better model for the zodiacal light could be achieved by also taking into account the small inclination of the zodiacal cloud to the ecliptic and the zodiacal cloud center offset from the Sun. The product of the constants that make the prefactor in the brightness integral (6) could then be determined directly from a comparison of the modeled, reconstructed zodiacal light to measurements.

The scattering characteristics of the dust grains in the lunar ejecta cloud could be described better with a more sophisticated scattering model such as the Discrete Dipole Approximation (DDA) as done by Richard et al. [31].

Based on our modeling of the lunar horizon glow and the zodiacal light, we consider that the best chances for a human observation of light scattered by the lunar dust cloud would possibly be on higher selenographic latitudes, maybe even near the lunar poles. Specifically, the zodiacal light brightness decreases rapidly when the observations are not along the ecliptic which roughly corresponds to the lunar equatorial plane, while the brightness of the lunar horizon glow should not be so latitude dependent. Particularly if the dust population mainly responsible for the lunar horizon glow would be caused by a cascade-like process of the impact ejecta dust and this dust would be somewhat evenly spread around the Moon, the lunar horizon glow would not be strongly latitude dependent.
6 Appendix

6.1 Lorenz-Mie scattering theory

In this chapter we first take a look at general scattering theory and then solve the problem of an electromagnetic wave scattered by a homogeneous sphere. The solution for the scattered wave is obtained from Maxwell’s equations with appropriate boundary conditions. The problem of determining an electromagnetic wave scattered by a homogeneous sphere was independently solved by Lorenz in 1890, by Mie in 1908 and by a few other physicists in the early twentieth century ([30] page 114). Here we refer to this solution as Lorenz-Mie theory, following Mishchenko ([30] page 114). We do not treat Lorenz-Mie theory in detail but rather sketch the solution following roughly Van de Hulst [39]. Van de Hulst used c.g.s units in his treatment of Lorenz-Mie theory but we use SI units following the notation by Stratton [40].

6.1.1 General scattering theory

We consider the electric field of an electromagnetic wave propagating in a vacuum in positive $z$-direction, $E_0 e^{ikz-i\omega t}$. When the wave encounters a particle on its way it may be absorbed or scattered. For the relation between the incident and the scattered electromagnetic wave we have at a distance $r$ from the scattering center ([30] page 43)

$$
\begin{bmatrix}
E_L \\
E_R
\end{bmatrix} = \frac{S}{R} e^{ikR} \begin{bmatrix}
E_{L0} \\
E_{R0}
\end{bmatrix} e^{ikz-i\omega t},
$$

(41)

where the matrix $\overline{S}$ is called the scattering amplitude matrix,

$$
\overline{S} = \begin{bmatrix}
S_2(\theta, \phi) & S_3(\theta, \phi) \\
S_4(\theta, \phi) & S_1(\theta, \phi)
\end{bmatrix}.
$$

(42)

The two components of the electric field are the component parallel to the plane of scattering ($E_L, E_{L0}$) and the component perpendicular to the plane of scattering ($E_R, E_{R0}$). $\phi$ and $\theta$ are the azimuthal and the polar angle in a scattering particle centered spherical coordinate system respectively.

In order to know the scattering properties of a particle, the components of the scattering amplitude matrix (42) need to be solved. In case of a spherical and homogeneous scatterer, it follows from the spherical symmetry properties of $\overline{S}$ that its components $S_3$ and $S_4$ equal zero ([39] pages 48-49).

The intensity $I$ of the electromagnetic radiation scattered by a sphere is ([30] p.140)

$$
I(\theta, \phi) = \psi(\theta, \phi)I_0 = \frac{4\pi}{C_{sca}} [S_1(\theta, \phi)^2 + S_2(\theta, \phi)^2]I_0,
$$

(43)
where $I_0$ is the intensity of the incident radiation, $\psi$ is the phase law fulfilling the normalization condition (16), and $C_{\text{Sc}}$ is the scattering cross-section of the scattering particle. Mishchenko has a factor $\frac{2\pi}{C_{\text{Sc}}}$ where we have $\frac{4\pi}{C_{\text{Sc}}}$ in (43). This difference follows from a different normalization of the phase law $\psi$. We have normalized the integral of the phase law over a full space angle to unity (see (16)), while Mishchenko has normalized the phase law to $1/2$ ([30] p.101). Other features of electromagnetic radiation are polarization and phase, but in this work we concentrate only on intensity.

6.1.2 Boundary conditions for an electromagnetic wave

When electromagnetic radiation propagates from one medium to another, the normal component of the magnetic field and the tangential component of the electric field are always continuous on the boundary between the media. Following Stratton ([40] pages 34 - 37), we can write these boundary conditions as

\begin{align*}
\pi \cdot (B_2 - B_1) &= 0 \quad (44) \\
\pi \times (E_2 - E_1) &= 0, \quad (45)
\end{align*}

where $B_1$ and $B_2$ are the magnetic fields in the media and $E_1$ and $E_2$ are the respective electric fields. $\pi$ is a unit vector normal to the boundary between the media.

When in addition there are neither currents nor charges on the boundary, also the tangential component of the magnetic field $H$ and the normal component of the electric displacement field $D$ are continuous,

\begin{align*}
\pi \times (H_2 - H_1) &= 0 \quad (46) \\
\pi \cdot (D_2 - D_1) &= 0. \quad (47)
\end{align*}

The assumption of the absence of surface currents is valid in case of media with low conductivity ([40] p. 37). In case of a conductor (i.e. metals) the surface currents will, however, play a significant role. Therefore (46) is not valid for metallic scatterers ([39] page 117).

6.1.3 Vector wave equation

Within a homogeneous and isotropic medium, free from electric charges, each of the fields $E$, $B$, $D$ and $H$ of an electromagnetic wave satisfies the vector wave equation

\[ \nabla^2 C - \mu \varepsilon \frac{\partial^2 C}{\partial t^2} - \mu \sigma \frac{\partial C}{\partial t} = 0, \quad (48) \]
where $\mu$ is the magnetic permeability, $\epsilon$ is the permittivity and $\sigma$ is the electric conductivity of the media. $C$ can be any of the fields $E$, $B$, $D$ and $H$ ([40] page 392).

In order to find general forms for the vector fields of an electromagnetic wave satisfying the wave equation (48), we define, following Stratton ([40] page 393), vectors fields $M$ and $N$ as

$$M = \nabla \times (R\psi) = \frac{1}{k} \nabla \times N,$$

$$kN = \nabla \times M,$$  \hspace{1cm} (49)

where $R$ is a radial distance from the origin of the coordinate system, $k$ is the wave number of the electromagnetic wave considered and $\psi$ is a scalar potential which fulfills the scalar wave equation equation

$$\nabla^2 \psi + k^2 \psi = 0.$$  \hspace{1cm} (51)

The components of $M$ and $N$ are then

$$M_1 = 0,$$  \hspace{1cm} (52)

$$M_2 = \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \phi},$$  \hspace{1cm} (53)

$$M_3 = -\frac{\partial \psi}{\partial \theta}$$  \hspace{1cm} (54)

and

$$kN_1 = \frac{\partial^2 (R\psi)}{\partial R^2} + k^2 R \psi,$$  \hspace{1cm} (55)

$$N_2 = \frac{1}{kR} \frac{\partial^2 (R\psi)}{\partial R \partial \theta},$$  \hspace{1cm} (56)

$$N_3 = \frac{1}{kR \sin \theta} \frac{\partial^2 (R\psi)}{\partial R \partial \phi}.$$  \hspace{1cm} (57)

The electric and magnetic fields of an electromagnetic wave can now be given in terms of $N$- and $M$-fields as

$$E = -\sum_n (a_n M_n + b_n N_n),$$  \hspace{1cm} (58)

$$H = -\sum_n \frac{k}{i \omega \mu} (a_n N_n + b_n M_n),$$  \hspace{1cm} (59)

where $a_n$ and $b_n$ are coefficients to be determined from the current distribution ([40] page 394).
In case of a spherical wave, such as an electromagnetic wave scattered by a sphere, the scalar potential $\psi$ fulfilling the equation (51) is

$$\psi = \psi_0 \cos(m\phi)P^m_n(\cos \theta)z_n(kR)e^{-i\omega t},$$

(60)

where $f_0$ is the characteristic solution of the scalar wave equation (51) ([40] page 414). $\omega$ denotes the choice between $\cos$ or $\sin$ in $\cos \theta$ and $P^m_n$ is an associated Legendre polynomial. $z_n$ is a spherical Bessel function derived from a Bessel function of the first kind $Z_{n+1/2}$, $R$ is the distance from the coordinate origin and $k$ is the wave number of the wave considered.

With (52) - (57) we get the vectors defined in (49) and (50) corresponding to the scalar potential (60) as $M = m_0 \cos \theta \sin \phi \hat{i}_2 - j_n(k_2 R) \frac{\partial P^1_n(\cos \theta)}{\partial \theta} \cos \phi \hat{i}_3$ (61) and $N = n_0 \sin \theta \sin \phi \hat{i}_1$ (62)

$$= \frac{n(n+1)}{k_2 R} j_n(k_2 R) P^1_n(\cos \theta) \sin \phi \hat{i}_1 + j_n(k_2 R) \frac{\partial P^1_n(\cos \theta)}{\partial \theta} \cos \phi \hat{i}_2 + \frac{1}{k_2 R \sin \theta} [k_2 R j_n(k_2 R)]' P^1_n(\cos \theta) \sin \phi \hat{i}_3.$$  

(63)

The prime in (61)-(64) denotes differentiation with respect to the argument $k_2 R$ and $\omega$ denotes the choice between $\cos$ or $\sin$ in $\cos \theta$ and between $+$ and $-$ in $\pm$ respectively. $P^1_n$ is an associated Legendre polynomial, $j_n$ is a spherical Bessel function derived from a Bessel function of the first kind $J_{n+1/2}$, $R$ is the distance from the coordinate origin and the vectors $\hat{i}_1$, $\hat{i}_2$ and $\hat{i}_3$ are the unit vectors of spherical coordinates.

6.1.4 Solution of coefficients from boundary conditions

Following Stratton ([40] page 564-565), we consider a plane electric wave which is linearly polarized in the x-direction and is propagating in the direction of the positive z-axis. This wave is given in terms of vector spherical coordinates as

$$E_{x} = E_0 \sin \theta \cos \phi \hat{i}_x + E_0 \cos \theta \sin \phi \hat{i}_y - E_0 \cos \theta \sin \phi \hat{i}_z.$$  

(64)
wave functions as

\[ E_i = a_1 E_0 e^{ik_2 z - i \omega t} = E_0 e^{i \omega t} \sum_{n=1}^{\infty} i^n \frac{2n + 1}{n(n + 1)} (m_{01n}^{(1)} - i n_{e1n}^{(1)}), \]  

(65)

\[ H_i = a_2 \frac{k_2}{\mu_2} E_0 e^{ik_2 z - i \omega t} \]

(66)

\[ = - \frac{k_2 E_0}{\mu_2} \sum_{n=1}^{\infty} i^n \frac{2n + 1}{n(n + 1)} (m_{e1n}^{(1)} + i n_{e1n}^{(1)}), \]

(67)

where \( E_0 \) is the amplitude of the electric field, \( \omega \) is the angular frequency of the wave, \( k_2 \) is the propagation constant and \( \mu_2 \) is the magnetic permeability of the medium.

As the incident wave encounters a scattering grain, it induces an electric and a magnetic field inside it. Inside the scatterer, i.e. \( R < a \), we then have for the electric and the magnetic field

\[ E_t = E_0 e^{i \omega t} \sum_{n=1}^{\infty} i^n \frac{2n + 1}{n(n + 1)} (a_n^r m_{01n}^{(1)} - i b_n^r n_{e1n}^{(1)}), \]

(68)

\[ H_t = - \frac{k_1}{\omega \mu_1} E_0 e^{i \omega t} \sum_{n=1}^{\infty} i^n \frac{2n + 1}{n(n + 1)} (b_n^r m_{e1n}^{(1)} + i a_n^r n_{e1n}^{(1)}), \]

(69)

where the wave number \( k_2 \) in \( m \) and \( n \), see (61) - (64) has been replaced by a wave number of the interior of the sphere \( k_1 \).

For the scattered electric and magnetic field, i.e. \( R > a \), we have

\[ E_r = E_0 e^{i \omega t} \sum_{n=1}^{\infty} i^n \frac{2n + 1}{n(n + 1)} (a_n^r m_{01n}^{(3)} - i b_n^r n_{e1n}^{(3)}), \]

(70)

\[ H_r = - \frac{k_2}{\omega \mu_2} E_0 e^{i \omega t} \sum_{n=1}^{\infty} i^n \frac{2n + 1}{n(n + 1)} (b_n^r m_{e1n}^{(3)} + i a_n^r n_{e1n}^{(3)}), \]

(71)

where the spherical Bessel functions \( j_n \) in (61) - (64) have been replaced by \( h_n^{(1)} \), which is a spherical Bessel function derived from a Bessel function of a second kind \( H_n^{(1)} \).

Now we apply the boundary conditions (45) and (46) for the incident, transmitted and reflected fields as

\[ i_1 \times (E_i + E_r) = i_1 \times E_t \]

(72)

\[ i_1 \times (H_i + H_r) = i_1 \times H_t. \]

(73)

Transmitted and reflected fields stand for the fields inside and for the scattered fields in the notation by Stratton. By applying the definitions of the
electric and magnetic fields inside and outside the scattering sphere, see (65)-(71), we get at the boundary, i.e. at \( R = a \), two pairs of equations for the expansion constants of the scattered wave \( a^r_n \) and \( b^r_n \) as well as for the expansion constants of the wave inside the sphere \( a^i_n \) and \( b^i_n \):

\[
\begin{align*}
    a^r_n j_n(N\rho) - a^r_n h^{(1)}_n(\rho) &= j_n(\rho), \\
    \mu_2 a^r_n [N\rho j_n(N\rho)]' - \mu_1 a^r_n [\rho h^{(1)}_n(\rho)]' &= \mu_1 [\rho j_n(\rho)]', \\
    \mu_2 N b^r_n j_n(N\rho) - \mu_1 b^r_n h^{(1)}_n(\rho) &= \mu_1 j_n(\rho), \\
    b^r_n [N\rho j_n(N\rho)]' - NB^r_n [\rho h^{(1)}_n(\rho)]' &= N[\rho j_n(\rho)]'.
\end{align*}
\]

From (74) and (75) we can solve

\[
a^n_r = -\frac{\mu_1 j_n(N\rho)[\rho j_n(\rho)]' - \mu_2 j_n(\rho)[N\rho j_n(N\rho)]'}{\mu_1 j_n(N\rho)[\rho h^{(1)}_n(\rho)]' - \mu_2 h^{(1)}_n(\rho)[N\rho j_n(N\rho)]'}
\]

and from (76) and (77) we get

\[
b^n_r = -\frac{\mu_1 j_n(\rho)[N\rho j_n(N\rho)]' - \mu_2 N^2 j_n(\rho)[\rho j_n(\rho)]'}{\mu_1 h_n(\rho)[N\rho j_n(N\rho)]' - \mu_2 N^2 j_n(\rho)[\rho h^{(1)}_n(\rho)]'},
\]

where \( N = k_1/k_2 \) is called the relative reflective index and \( \rho = k_2a \) is the ration between the circumference of the sphere and the wavelength of the incident light. In the notation by Van de Hulst ([39] page 123) and Mishchenko ([30] p.154) \( a_n \) equals our \(-b^n_r \) and \( b_n \) equals our \(-a^n_r \).

With the expansion coefficients \( a^n_r \) and \( b^n_r \) of the scattered wave solved, we can now represent the \( i_2 \)-component of the scattered electric field, see (70), as

\[
E_{r2} = E_0 e^{-i\omega t} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} a^n_r \frac{1}{\sin \theta} h^{(1)}_n(k_2R) P^1_n(\cos \theta) \cos \phi
\]

\[
-ib^n_r \frac{1}{k_2R} [k_2 R h^{(1)}_n(k_2R)]' \frac{P^1_n}{\partial \theta} \cos \phi
\]

\[
= -E_0 \frac{i}{k_2 R} \cos \phi e^{ik_2R-i\omega t} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a^n_r \pi_n(\cos \theta) + b^n_r \tau(\cos \theta)]
\]

\[
= -E_0 \frac{i}{k_2 R} \cos \phi e^{ik_2R-i\omega t} S_2(\theta),
\]

where we have taken the asymptotic limit of

\[
h^{(1)}_n(k_2R) \sim \frac{(-i)^{n+1}}{k_2R} e^{ik_2R}.
\]
defined functions
\[ \pi_n(\cos \theta) = \frac{1}{\sin \theta} P_n^1(\cos \theta) \] (86)
and
\[ \tau_n(\cos \theta) = \frac{\partial P_n^1(\cos \theta)}{\partial \theta} \] (87)
as well as identified the \( S_2 \) component of the scattering amplitude matrix as ([39] page 125)
\[ S_2(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n^r \pi_n(\cos \theta) + b_n^r \tau(\cos \theta)]. \] (88)
Similarly, for \( i_3 \)-component of the scattered electric field (70) we have
\[ E_{r3} = E_0 e^{-i\omega t} \sum_{n=1}^{\infty} r^n \frac{2n+1}{n(n+1)} (-a_n^r h_n^{(1)}(k_2 R) \frac{\partial P_n^1(\cos \theta)}{\partial \theta} \sin \phi) \]
\[ + ib_n^r \frac{1}{k_2 R} [k_2 R h_n^{(1)}(k_2 R)'] P_n^1(\cos \theta) \sin \phi \] (90)
\[ = E_0 \frac{i}{k_2 R} e^{ik_2 R - i\omega t} \sin \phi \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n^r \tau(\cos \theta) + b_n^r \pi(\cos \theta)] \] (91)
\[ = E_0 \frac{i}{k_2 R} e^{ik_2 R - i\omega t} \sin \phi S_1(\theta), \] (92)
where we can distinguish the component \( S_1 \) of the scattering amplitude matrix as ([39] page 125)
\[ S_1(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n^r \tau(\cos \theta) + b_n^r \pi(\cos \theta)]. \] (93)
The scattering cross-section of the scattering particle is given by ([30] p.140)
\[ C_{scu} = \frac{2\pi}{k_1^2} \sum_{n=1}^{\infty} (2n+1) \text{Re} \{ |a_n|^2 + |b_n|^2 \}. \] (94)
Now we have everything we need to determine the phase law \( \psi \) in (43) characterizing the intensity of electromagnetic radiation as a function of scattering angle when scattered by a spherical and homogeneous grain.
References


