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PERFORMANCE OF THE BLACK-SCHOLES OPTION PRICING MODEL
– EMPIRICAL EVIDENCE ON S&P 500 CALL OPTIONS IN 2014

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This paper evaluates performance of the Black-Scholes option pricing model on European call options that are written on U.S. S&P 500 equity index in year 2014. Main purpose is to show empirical evidence about false assumptions contained in the model and complete it by relaxing unconditional restrictions. Analysis consists of investigating biasedness and heteroscedasticity properties by complementing the Black-Scholes model with GARCH(1,1) method based on maximum likelihood estimations. Varying volatility is studied also through implicit volatility surface.

Depending on their characteristics, call options are categorized into specific groups according to their moneyness and maturity for further analysis. Using common econometrics and statistical methods, the paper shows that assumption about constant volatility is false, that the Black-Scholes model exhibits a bias which leads to mispricing of certain type of options and that assumption about normally distributed error term is false. Volatility is estimated through historical and implicit methods, of which the latter one uses GARCH(1,1) method to capture especially time-series characteristics of varying volatility.

Findings regarding performance of the Black-Scholes option pricing model were expected and are in line with prior literature.
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1 INTRODUCTION

The Black-Scholes option pricing model was a groundbreaking step that revolutionized option pricing in finance in the beginning of 1970s. Prior that option pricing formulas were based on and derived by taking discounted expectations. In their model Black and Scholes (1973) assume that the price of underlying asset follows a geometric Brownian motion and in addition to that they also define constant volatility for pricing dynamics. Also the principle of no-arbitrage pricing in the Black-Scholes model is obvious.

The model enjoyed of wide acceptance for a decade before some criticism appeared. If the law-of-one-price holds in practice and options were redundant, it would be unlikely that they actually would be traded as separate assets (Cochrane, 2005: 327). Major audience woke up into insufficient performance of the model in 1987 when the market crash appeared after which the Black-Scholes model was incapable to explain why the observed option prices deviated from the predicted ones. By using constant volatility estimate conducted for example from the past returns of the underlying asset resulted in mispricing for certain options: far out-of-the-money and in-the-money options traded on the market were continuously “overpriced” compared to at-the-money options. From efficient market hypothesis point of view suggested by Fama (1965), this mispriced state should be corrected quickly by arbitrageurs harvesting their profits until mispricing disappear and, in this particular case, volatility for all type of options with the same strike price and maturity equal to each other.

Since the Black-Scholes model was unable to explain option price deviations and mispricing from the model’s perspective in practice, accuracy of the model has been questioned by some researchers even before the crash in 1987 and by many others after that. For example Rubinstein (1985) argue that the Black-Scholes model suffer from pricing biases related to maturity and the strike price of an option, and Hull & White (1987) suggest that it is the assumption about constant volatility in the Black-Scholes model causing failure to value options accurately. The latter one, among others, extend their option pricing model to incorporate the fact that volatility of an underlying asset vary over time. In his article Rubinstein (1994) conclude that before the stock market
crash in 1987, the Black-Scholes model produced very applicable results but inconsistency between predicted theoretical price and actual prices was growing wider when applied to data constructed after the crash.

![Figure 1. Typical implied volatility smile for S&P 500 index options before (left) and after the stock market crash in 1987. (Rubinstein, 1994).](image)

Basically due to its limitations and strict assumptions involved with the Black-Scholes model, several more sophisticated and generalized option pricing models have been introduced recently. Early simple extension to the original Black-Scholes model covered definition of dividend paying stock (Merton 1973), then became general acceptance of the implied volatility surface which still formulates estimated volatility in deterministic way. Today focus is on stochastic volatility consisting of mean reverting process and discontinued jump processes in the underlying asset price, and most recent option pricing models incorporate the volatility smile as well as the bias related to maturity of the option.

Recent empirical studies show that volatility smile in currency options and smirk in case of options written on equity or stock indices are widely recognized, which have resulted in more advanced option pricing models with different explanatory characteristics and calibration methods. Just to mention couple, Dumas et al. (1998) introduce so called Practitioner Black-Scholes model that fits in the implied volatility surface, Bakshi et al. (1997) use a six-parameter random volatility jump-diffusion model while Heston (1993) uses four-parameter continuous-time stochastic volatility model. In addition to their complexity compared to the Black-Scholes model,
successor models require several additional assumptions and extra relaxations that need to be made in order to get volatility estimates eventually converge.

Need for more precise volatility estimate pursues from the recognized biases related to option’s time-to-maturity and moneyness. For example Bates (2003) among others summarizes that the volatility smile can be observed across the before mentioned categories and that these biases are associated with the Black-Scholes option price model. He concludes that the implicit standard deviations derived from option prices are not identical across different strike prices and maturities. Furthermore, the practitioners report that the Black-Scholes model is performing better for certain type of options when performance is evaluated out-of-sample. Even non-parametric models have been reported to be more consistent with the data than the formulas formed by other widely used option pricing models (Lin et al., 2016). Typical method for evaluating consistency of the model would be to measure magnitude of pricing error, like a root mean square pricing error (RMSE). However, aligning a proper loss function is crucial when evaluating out-of-sample performance of a certain option pricing model. This is pointed out by Christoffersen and Jacobs (2004) as they conclude that performance can be improved significantly when in-sample calibration parameters are estimated using an appropriate loss function and out-of-sample evaluation is made accordingly.

Despite its lack in performance, the Black-Scholes model is seen as an easy and simple framework to start with for an investor trading options, since all other parameters except the volatility can be observed at the market. Thus to form as accurate price estimation as possible, it is always eventually about how accurate one can be in predicting future volatility. However, this paper does not aim to test or measure size of inaccuracy of any specific option pricing model but the focus will lay in proving potentially inadequate performance of the Black-Scholes model using real market data, and that there is common understanding about presence of the biases and failure mechanism.
It is interesting to see whether it is evident or not that the Black-Scholes model exhibit dynamics which may eventually result in mispricing the options. Thus, research problem set to this paper is as follow:

*Does the Black-Scholes model still provide inconsistent price information for the options, and can this be empirically proven with complementary models relying on maximum likelihood estimators pursuing from real market data?*

Furthermore we complement problem setting with the following research questions:

1. Does the Black-Scholes model underestimate tail probabilities of the return distribution? That is, does it have a bias which misprices options?
2. In analysis of the volatility-affecting properties in the Black-Scholes model, is it the assumption about homoscedasticity that causes failure in pricing?
3. Is GARCH(1,1) estimation model implemented via maximum likelihood method able to capture certain stochastic behavior in returns of the underlying and thus, capable of providing evidence about the volatility smile?

The objective of the paper is to show that using real option data in accordance to implicitly form volatility estimates with the Black-Scholes model might still exhibit recognized inconsistencies. Method to evaluate answers for the research questions is to test against the Black-Scholes model related assumptions, which are assumed lead to empirically seen characteristics of the volatility. Conclusion about the constraints related to the Black-Scholes model could be that these constraints might be necessary to ease in order to get proper option price, or at least existence of them must be taken into account when trading with the options.

On the way to empirically prove performance of the Black-Scholes model, the first step in this paper will be establishing an estimate for future volatility of an underlying asset. This can be done by using historical methods based on past returns of the underlying asset or a Constant Mean Model of returns with GARCH(1,1) conditional variance. Another way is to implicitly estimate volatility using observed option prices. Both historical methods and their estimates for volatility are presented as a starting
point and reference. For the rest part of the paper, a maximum likelihood estimator is introduced and used in forming more robust volatility estimate to be used in option pricing. Aim is to complement the original Black-Scholes option price formula with characteristics that have statistical significance and which are implicitly derived from the market data.

As the Black-Scholes model assumes normally distributed pricing error, it is the next thing to test and find out if hypothesis about normally distributed pricing error can be rejected. Furthermore, the empirical option pricing model can be extended by introducing the biasedness property with classifications like moneyness of the option or taking into account different maturities as explanatory variables. Again statistical significance of an intercept will be tested to see whether or not the Black-Scholes model is biased and thus misprices options.

Next property of heteroscedasticity is defined by allowing variance of disturbance term in the model vary over the sample. To evaluate heteroscedasticity, the Black-Scholes option pricing model is tested against with two different characteristics, moneyness and maturity. Finally, statistical significance is tested against to prove implied volatility’s potential dependency on option’s moneyness property in form of a volatility smirk.

The paper is organized as follow: Section 2 describes briefly fundamentals behind option pricing and volatility, Section 3 describes methods for estimating volatility, Section 4 describes hypothesis for potential biases of interest and statistical tests for them, Section 5 describes the data, Section 6 describes empirical results and findings, while Section 7 concludes.
2 FUNDAMENTALS BEHIND OPTION PRICING

This chapter first summarizes basic concepts and assumptions related to option pricing, after which the above-mentioned information is used to introduce theory of the Black-Scholes (BS) option pricing model. In addition, relevant time-series characteristics are also described to further support evaluation and testing volatility in coming chapters.

2.1 Concepts and definitions

Option is a financial instrument and its value is derived from the values of other underlying variables. Options can be written on different kind of assets like stocks, indices, currencies or interest rates, and in such case option’s price is exclusively dependent on the price of that particular underlying asset. Thus, they can be used to hedge against an unwanted exposure or to speculate about future price movements of the underlying. Or in addition, options can be used to leverage the investment. For example an at-the-money call option typically gives a beta of about 10 which in turn enables different strategies: an investor can use it according to his/her preferences for hedging or leveraging purposes. (Cochrane 2005: 314, Hull 2008: 179.)

There exist several type of options depending on how their value is ultimately defined, or whether the option can be exercised only at the end (European) or before the expiration (American). Moreover, an option may have varying payoff rules calculated from its price development over the entire life-time and in such a case the option might be referred as an exotic one. Basic classification for an individual option is its strike price, maturity and whether the option is a call or put. The former’s payoff is positive when the price of an underlying is above the strike price at exercise, and the latter has positive payoff when price development of an underlying has been negative and is below the strike price at exercise. So the value of an option is limited to zero and cannot be negative, which in turn reduces the downward risk in contrast to investing directly in the underlying asset.
Option price is affected by different factors of which before-mentioned price of the underlying asset, option’s time-to-maturity and strike price are the key ones. All in all six price-affecting factors can be distinguished (Hull 2008: 215):

1. value of the underlying asset
2. strike/exercise price of an option
3. time-to-maturity
4. interest rate
5. dividends payable for the underlying
6. future volatility of the underlying asset

When calculating option price the first five listed characteristics above can be observed directly from the market but the sixth one, volatility, is the only unknown parameter and must somehow be estimated. So option pricing is basically only about forecasting the volatility of the underlying asset.

During the past decades scientists have developed more and more detailed and sophisticated option pricing models but eventually it has always culminated in estimating the future volatility of an underlying asset and how accurately the model has succeeded. To generalize, the volatility is a measure of uncertainty that is used in checking riskiness of the return realized on an asset. It is necessary information in pricing and trading options or any asset in general.

*Historical volatility* is defined as the sample standard deviation for logarithmic asset prices over a fixed time window that precedes the option transaction. Typically it is calculated for preceding 12 months unless otherwise defined. *The implicit volatility* is referred to as forward looking measure which is often used to express future volatility of an underlying asset. It is a value for the annualized standard deviation that equates certain theoretical option pricing formula like the BS model with the observed option price (Hull 2008: 296). Thus when volatility is conducted from the market prices, it can be seen as of great interest for the option traders since it reflects market’s opinion or belief about expected volatility in the future.
Asset price process is another important assumption in option pricing. The following chapter describes one widely used assumption related to price process in finance and that is called geometric Brownian motion.

2.1.1 Geometric Brownian motion

When assuming markets to be efficient and that they are dominated by normal events meaning that we can exclude very rare events like market crashes of which probability to occur moves close to zero, a geometric Brownian motion can be used to describe asset price dynamics. It is a continuous-time stochastic process where extreme events occur only according to probabilities in the tails of the normal distribution (Hull 2008: 266).

In generalized Wiener process stock price has a constant expected drift rate and a constant variance rate. This is not, however, the case in real life because it fails to capture key aspects of stock prices and thus, in empirical science stock price process is generally defined to follow geometric Brownian motion. That is based on an assumption that investors’ expected rate of return from a stock is constant and independent of the price itself. (Hull 2008: 265.)

When the standard deviation of the change observed in very short time period is proportional to the stock price, the stock price behavior can be described with the following model:

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \]  

where \( S_t \) is stock price at time \( t \), \( \mu \) is the expected rate of return, \( \sigma \) is the volatility of the stock price, and \( W_t \) is the standard Wiener process (Hull 2008: 266). When dividing the equation (1) by \( S_t \) we get:

\[ \frac{dS_t}{S_t} = \mu \, dt + \sigma \, dW_t \]  

(2)
where the first term defines deterministic (expected value) return of the asset within a short time interval and the second term takes into account random changes (stochastic component) in asset price. This stochastic differential equation is known as geometric Brownian motion and it is, by definition,

“A stochastic process often assumed for asset prices where the logarithm of the underlying variable follows a generalized Wiener process.” (Hull 2008: 782.)

In other words, the Black-Scholes option pricing formula (introduced in chapter 2.2) among other financial instruments assume the price of an asset follows

$$\ln P_t - \ln P_{t-1} = \mu + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2_u)$$

(3)

where $P_t$ is price of an asset at time $t$ and $P_{t-1}$ is the previous observation, $\mu$ is mean of expected return and $\varepsilon_t$ is an error term. Under this process the return in a small time periods is normally distributed and returns measured during two non-overlapping periods are independent.

When estimating volatilities directly from observed option prices later in chapter 6.2 with maximum likelihood method, the use of cumulative normal distribution stems from the lognormality assumption (can be derived by using Itô’s lemma).

2.1.2 Risk-neutral valuation

In analysis of derivatives the definition of risk neutral valuation is one of the most important tools (Hull 2008: 289). When using risk-neutrality, the value of any derivative can be calculated by discounting its expected payoff at risk-free rate. This simplifies the analysis of derivatives considerably. In a risk-neutral world where risk-preferences are already included in the price of underlying asset, the no-arbitrage limitation for a call option that matures in time period $T$ with delivery price $K$ must hold:

$$p_c \geq S_0 - K e^{-rT}$$

(4)
In the equation above $p_c$ is the price of a call option, $K$ is the strike price of that particular option and $r$ is the risk-free interest rate. For options with long maturity $T$ and written on a dividend paying asset, the first term spot price $S_0$ must be discounted with continuously compounding dividend yield.

### 2.2 Theory of the Black-Scholes option pricing model

Starting point for the Black-Scholes option pricing model is that it defines a perfectly hedged and risk-neutral portfolio which eliminates systematic risk and by which it eventually calculates the price of the option (Black & Scholes, 1973). To make derivation possible, the model requires some assumptions related to asset price movements of the underlying asset. First important assumption is that the price of underlying moves according to geometric Brownian motion defined in equation (2) in chapter 2.2. Secondly, there must not exist any riskless arbitrage opportunities, by which one can assume that price movements in underlying are immediately reflected to market price of the derivative. Other necessary assumptions defined by Black and Scholes are: no transaction costs or taxes, trading is continuous and short-selling is permitted, risk-free rate of interest is constant and the same for all maturities, and that there are no dividends paid for underlying stock before expiration of the derivative.

A lot of literature exist\(^1\) from where it is possible to find exact derivation of the BS model and thus, it is not expressed here in details. However an important notation to say is that the differential BS equation is independent of risk preferences, which eventually makes it possible to use any possible set of risk preferences. In the world of derivatives where a decision about risk neutral investors can be made, valuation of an instrument providing a payoff at particular time can be conducted. This assumption about risk-neutral valuation is fundamental in the Black-Scholes option pricing model (Hull 2008: 289).

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\(^1\) Proof for the BS formula in Hull 2008: 307.
The original BS option pricing model introduced in 1973 is defined for non-dividend paying stocks. Since option data\(^2\) used in this thesis are written on US S&P 500 equity index and it covers one calendar year, it is justifiable to take into account dividends paid on that particular year. Assumption about deterministic dividend yield on options is acceptable when we take into account that options generally have short life-time and majority of the companies, in this case big S&P 500 companies, follow stable dividend policy within short time horizon. Thus, the following extended Black-Scholes option pricing equations for European call (5) and put (6) options at time \(t\) include continuously payable dividend rate, which were originally introduced by Merton (1973).

\[
\begin{align*}
  c_t &= S_t e^{-qtT} \phi(d_1) - Ke^{-rT} \phi(d_2) \\
  p_t &= Ke^{-rT} \phi(-d_2) - S_t e^{-qtT} \phi(-d_1)
\end{align*}
\]  

(5)  

(6)  

where

\[
\begin{align*}
  d_1 &= \frac{\ln \left( \frac{S_t}{K} \right) + (r_t - q_t + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \\
  d_2 &= \frac{\ln \left( \frac{S_t}{K} \right) + (r_t - q_t - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}
\end{align*}
\]

In equations above \(S_t\) is the price of underlying at time \(t\), \(\phi(d_n)\) is a cumulative probability distribution function for a standardized normal distribution with value \(d_n\), \(K\) is the strike price, \(r_t\) is the continuously compounded risk-free rate, \(\sigma\) is the annualized volatility of underlying’s price, \(q_t\) is continuously payable dividend rate, and \(T\) is option’s time-to-maturity (TTM) in years.

As can be seen from the formulas above, the BS model does not take into account any other type of volatility curve but constant, meaning that historical volatility applies for the future as well. In addition, assumption about log-normally distributed asset price

\(^2\) More detailed description about the data can be found in Chapter 5.
is also severe since financial data is known to suffer from excess kurtosis, or fat tails (Fama 1965). Researchers have shown that stringent assumptions related to the BS model are not realistic and therefore popular extension to the BS framework is to introduce different market imperfections for option pricing. Relaxing major assumptions of the Black-Scholes option pricing model in order to get more realistic volatility estimate is the main focus in this paper.

As empirical research has recognized variation and skewness in volatility since 1987, analytical conclusion might be that implied probability distribution for equity options must be different from log-normal distribution. The following Figure 2 illustrates expected difference compared to log-normal distribution.

Assumption of constant volatility in the BS model results in log-normal probability distribution. But the expected volatility smirk for equity options in real life could correspond to the implied probability distribution as shown in figure above (Hull 2008: 394–395). For example Jackwerth and Rubinstein (1996) examine distribution changes around 1987 market crash. They report a distinct change in shape of the probability distributions between the pre- and post-crash: while pre-crash distribution exhibits lognormal form, the post-crash distribution is leptokurtosis and exhibits negative skewness. This may be interpreted in so that one should seriously consider
option pricing models where assumptions related to constant volatility and log-normal price distribution are relaxed.

Popular stochastic volatility model introduced by Heston (1993) relaxes assumptions about fixed volatility and price process of underlying itself, while Bakshi et al. (1997) includes stochastic jumps in their model. Bakshi et al. continue to show strong support for including jumps in the model as those are able to capture the leptokurtic nature of the returns.

2.3 Time-series characteristics of returns

Assumption about lognormality in the BS model is strict and it is known to fail. Returns tend to exhibit several characteristics that can be seen as time-dependent and which eventually causes deviation from the normal distribution. One additional observation related to asset return behavior is that they might reveal information on the variance of returns and exhibit positive autocorrelation when analyzing squared returns. This means that if asset price makes a large move today, it is more likely to make a big move tomorrow as well. More specifically, for the volatility this means that there is tendency to move in the same direction as previously observed change occurred. This feature makes variance of returns and volatility potentially predictable.

Another recognized feature of variance is that if it spikes, it will gradually drop back to approximately the same level as it was before the sudden incident. It means that variance has a drift (negative or positive) that pulls it back to a long-run average level (Hull 2008: 482). This kind of memory effect, or mean reversion, is important to take into account when estimating tomorrow’s volatility forecast. Mean reversion is recognized for example by the GARCH(1,1) model.

The third and fourth moments of returns are called skewness and kurtosis, respectively. If log-normal distribution is assumed for asset price, then returns are assumed to have normal distribution. It is an important assumption and if normality under \( i.i.d \) hypothesis is not present, then another moments need to be considered as well. Negative skewness indicates that the left tail of distribution is longer and fatter than
the right one. This kind of behavior is typical for equity markets and results in a volatility smirk. Returns typically exhibit excess kurtosis compared to a normal distribution, meaning it has fatter tails and kurtosis of higher than 3. Typically one can achieve better estimate for volatility by using a model like GARCH(1,1) which attempts to capture autocorrelation, mean reversion and the excess kurtosis all at the same time. This is why the GARCH(1,1) model based on maximum likelihood estimator is used in this paper to form estimates for volatility and it will be discussed more in details in chapter 3.1.2.

Another empirical observation in equity markets is that negative price movements have different impact on future volatility than positive ones. Engle and Ng (1993) study impact of news on volatility with numerous asymmetric volatility forecast models and find that majority of the models were able to predict negative shocks to introduce more volatility than positive shocks. They fitted the models to daily Japanese stock return data starting from 1980 and ending in 1988. This kind of asymmetric reaction in volatility shows that it is not only the magnitude of change but also the sign, negative or positive, which in addition affects future volatility estimate. To capture asymmetry, there exist models like EGARCH developed by Nelson (1991) or GJR-GARCH introduced by Glosten et al. (1993) and other variations as well. However, Engle and Ng (1993) conclude that although asymmetric models yielded in better estimates with GJR-GARCH model performing best, the modeled asymmetry is not adequate and difference between models culminate when extreme shocks occurred.

2.4 Potential explanations for existence of the volatility smirk

As empirical studies showed volatility varies depending on the option’s strike price and time-to-maturity, the smileys and smirks have been persistent for decades and can thus reflect something that might be related to investor’s behavior and assumptions about future price development. Several articles study the equity volatility smirk phenomenon since 1987 and try to find scientific explanations and reasons behind of it. For example Rubinstein (1994) proposes that reason for the volatility smirk in equity options may be called “crash-o-phobia” indicating that the traders are pricing options in accordance to hedge against another crash. García-Machado and Rybczyński (2017)
summarize it as volatility skew in general contains at least three levels of information: the likelihood of a negative price jump, the expected magnitude of it, and the premium to compensate investors for both the risk of a jump to occur and that it could be large.

Another explanation is related to company’s financial strength and considers the leverage in general. Toft and Prucyk (1997) find that the higher the firm’s financial leverage, the steeper the implied volatility smirk. When the company’s own equity declines it means that the underlying company’s leverage must increase and the equity becomes more risky. In finance this reflects an increase in volatility and thus, the leverage phenomenon can be illustrated as the volatility skew in Figure 2 is moving horizontally depending whether leverage is increasing or decreasing.

One argument in close relation to leverage effect has been investors’ appetite for in-the-money calls as instant replacement for stock purchases. That can offer increased ROI (return-on-investment), and greater demand results in increased implied volatility at lower strike prices. Eventually anything from investors’ sentiment (Han 2008) to the market liquidity (Chou et al. 2009) have been proposed as root causes for existence of the volatility smile and smirk, so results are somewhat mixed regarding this.

Despite incoherent and alternating explanations for existence of the volatility smirk, it is good to notice that the pattern of implied volatility compared to moneyness may differ across different markets over the world. For instance Garcia-Machado and Rybczyński (2017) point out that it is not consistent when a volatility smile can be observed instead of a smirk. They add as an example that while the US stock options usually exhibit a smile, for US equity index it is typical to exhibit a volatility smirk instead.
3 ESTIMATING VOLATILITIES

In this chapter we describe two methods for estimating volatility from historical data and an implicit method that is based on maximum likelihood estimation. Eventually complementary empirical models are presented in order to evaluate imperfections included in the original Black-Scholes equation.

3.1 Historical methods

When volatility is estimated empirically from historical data, the first step would be in defining observation time interval which can be anything from a minute to for example one month. Typically volatility is expressed per annum and for that the monitoring time interval can be one day or month. Hull (2008: 283) points out that more data usually lead to more accuracy, but in case of volatility as it tends to change over time, too old data might become useless in predicting future volatility. He argues that it is typically a compromise regarding sample frequency and presents several rule of thumbs to choose proper time interval depending on length of forward looking period for volatility estimate.

3.1.1 Volatility estimate based on returns of the underlying asset

Returns of underlying are first standardized to correspond one time period. In finance when monitoring something with trending property, closing prices at the end of each interval are first changed to logarithmic ones. This results in return $u_i$ as follow:

$$u_i = \log\left(\frac{S_i}{S_{i-1}}\right) = \log(S_i) - \log(S_{i-1})$$

where $S_i$ is the price of the underlying asset. From here the standard deviation $s_r$ of the return can be defined as

$$s_r = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2}$$
where $\bar{u}$ is the mean of $u_i$ and $n$ is the number of most recent observations. When the standard deviation of the $u_i$ is $\sigma \sqrt{n}$ and the variable $s$ is therefore an estimate, it follows from the definition of the variance $\hat{\sigma}_u^2$ that the annualized volatility can be estimated as

$$\hat{\sigma} = \frac{s_i}{\sqrt{T}} = \sqrt{n \hat{\sigma}_u^2}$$  \hspace{1cm} (9)

where $T$ is length of a sample period expressed in years. (Hull 2008: 282 – 283.)

It is worth noting that replacing $n - 1$ by $n$ in Equation 8 is acceptable when $n$ is large. In that case the estimate is changing to a maximum likelihood estimate (MLE). Using MLE method as a volatility estimate is discussed more in details in chapter 3.2.1.

3.1.2 Constant mean model with GARCH(1,1) conditional variance

GARCH stands for Generalized Autoregressive Conditional Heteroscedasticity. The GARCH models are discrete-time models and thus they can be used to estimate volatility for a variety types of financial time series data. Establishing a financial model by using GARCH method it is possible to generate such a simulation that recognizes and attempts to track changes in volatility and correlation over time.

The GARCH(1,1) model was originally introduced by Bollerslev (1986). In it the variance estimate is calculated from a long-run average variance and in addition to that, it takes into account the most recent observations of variance and the price of market variable as well. All of these three variables are assigned with different weight ending in the following variance estimate:

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$ \hspace{1cm} (10)

where $\gamma$ is the weight for long-term variance $V_L$, $\alpha$ is the weight for previously observed percent change of asset and $\beta$ is the weight for latest variance forecast. Normally the first term is set to $\omega = \gamma V_L$ and altogether, the sum of weights must equal to 1. Weights
on these three coefficients for variance forecast eventually determine how fast the variance estimate changes to follow new information provided by the market and how fast it gets back to its long-run average value. (Hull 2008: 481 – 482.)

Furthermore, the constant mean model of returns using GARCH(1,1) conditional variance can be estimated with the following model:

\[ y_n = \gamma + \varepsilon_n \]
\[ \sigma_{y,n}^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{y,n-1}^2 \]
\[ \varepsilon_n \sim iid N(0, \sigma_{y,n}^2) \]

where \( y_n \) is the change in logarithmic price, and \( \gamma \) is the mean return. Bollerslev (1986) shows that the long-term variance can be calculated using estimates given by GARCH(1,1) model and it is as follow:

\[ \sigma_y^2 = \frac{\omega}{1 - \alpha - \beta} \quad (11) \]

Annualized volatility is given when applying previous variance estimate to equation (9) with corresponding amount of observations \( n \). When estimating unknown weighting parameters \( \alpha, \beta \) and \( \omega \) from the data, a maximum likelihood method is relevant approach and in that case correct expression of a volatility estimate is with an additional hat accent mark.

3.2 Implicit method

Since options are publicly traded at the exchanges, order books and trade recordings are generally available for an investor. Several commercial actors like Option Metrics and Thomson DataStream can even provide massive amounts of historical option price information upon request. From these observed market prices one can form an implied estimate for volatility \( \sigma \). It is value of volatility that must be used to constitute the option price formed by investors trading at the market. Since it is not possible directly to invert the Black-Scholes option pricing equations (5) and (6) in order to find
corresponding volatility, option traders typically approximate it for a certain option by using a volatility table with interpolated estimates. Or alternatively they can use some numerical and iterative approach like *Newton-Raphson* method to solve nonlinear equations in their volatility models. Another method is to use maximum likelihood estimator which is described next.

3.2.1 Maximum likelihood estimator

With maximum likelihood method one can estimate the best-matching values for parameters in the statistical model so that the probability of a set of historical data consisting of individual observations to occur is maximized. That is, having a realized dataset the maximum likelihood estimator (MLE) searches potential parameter values which most likely generate the observed data. Adversely, other way is to form an objective function or a loss function for parameter of interest, like measurement error, which can then be minimized.

Using maximum likelihood estimator requires some knowledge about the distribution of dependent variable. In option pricing when assuming log-normal asset price and furthermore normally distributed returns, the MLE method can be used to estimate the variance of normally distributed pricing error. Assuming log-returns are normally distributed with mean $\mu$ and variance $\sigma^2$, then probability density function of the constant mean model is defined as:

$$f(r_t; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_t - \mu)^2}{2\sigma^2}}$$  \hspace{1cm} (12)

This is the density function from which it is possible further to derive log-likelihood function used in MLE. If solved analytically, the log-likelihood function is then maximized by taking the first derivative (Gradient) with respect to unknown parameter $\theta$. Result is checked with the second derivative (Hessian) to ensure that the estimate maximizes log-likelihood function. Other way to solve the MLE is numerical with some starting values if the first order condition does not have analytical solution.
3.2.2 Implied volatility with MLE based on observed option prices

Next an empirical model for a call option is defined for further analysis. To estimate the unknown parameters in vector $\theta = \{\mu, \sigma_i^2\}$ of the empirical option pricing model

$$c_i = c_i^{BS}(\sigma) + \varepsilon_i$$  \hspace{1cm} (13)

by maximum likelihood method, the average log-likelihood function is given by:

$$
\begin{align*}
\log L(\theta) &= \frac{1}{N} \sum_{i=1}^{N} \log f(c_i; \theta) \\
&= \frac{1}{N} \sum_{i=1}^{N} \log \left( \frac{1}{\sqrt{2\pi \sigma_i^2}} e^{-\frac{(c_i - c_i^{BS}(\sigma))^2}{2\sigma_i^2}} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \log \frac{1}{\sqrt{2\pi \sigma_i^2}} - \frac{1}{N} \sum_{i=1}^{N} \frac{(c_i - c_i^{BS}(\sigma))^2}{2\sigma_i^2} \\
&= -\frac{1}{2} \log(2\pi \sigma_i^2) - \frac{1}{2N} \sum_{i=1}^{N} \left( \frac{c_i - c_i^{BS}(\sigma)}{\sigma_i} \right)^2 \\
& \hspace{1cm} \text{ (14)}
\end{align*}
$$

In equation (13), which equals to the base model without any relaxations to the Black-Scholes model as $E[\varepsilon_i] = 0$, $c_i$ represents the observed price of the $i^{th}$ call option and $c_i^{BS}(\sigma)$ is the Black-Scholes price as described in equation 5. $N$ in equation (14) is the number of observations in the sample.

As the BS price is a non-linear function of volatility $\sigma$ and the BS model cannot be inverted with respect to volatility, an iterative algorithm is needed to estimate the parameters $\theta$ through maximum likelihood method. In this paper most of the numerical estimations ran in R software are done with Broyden–Fletcher–Goldfarb–Shanno
(BFGS) algorithm\textsuperscript{3}, which among others is an iterative method for solving unconstrained non-linear optimization problems. Exceptionally Newton-Raphson root finding and maximization method\textsuperscript{4} is used when approximating volatility smile surface in chapter 6.5.

Shared feature for all numerical solving estimations and maximization methods is that an initial value for all parameters must be defined in order to run the algorithm successfully. Potential problem with a random initial value choice could be that one might find a local maximum instead of global maximum. Once the iteration is converging and parameters of interested are estimated, common way in addition is to test few different initial values for algorithm to verify that obtained result defines a global maxima.

\textsuperscript{3} More detailed information about BFGS algorithm can be found from the book “Practical methods of optimization” by Fletcher, Roger (1987).

4 TESTING PERFORMANCE OF THE BLACK-SCHOLES MODEL

In this chapter the tests and hypothesis needed to evaluate the BS model are described. Evaluated tests follow procedures described in Hurn, Martin, Phillips and Yu (forthcoming) in chapter 16. Theory here is presented as preliminary information for the coming chapter 6 in which results of the empirical analysis are presented.

4.1 Normally distributed pricing error and normality test

The baseline for assessing the BS option pricing performance is the equation (13). It can be interpreted so that we take the option’s market price as given, and then we compare it to a Black-Scholes price which is incremented with a complementary pricing error term.

The error, or a disturbance, term $\epsilon_i$ represents the pricing error and it is assumed to be normally distributed with variance $\omega^2$:

$$\epsilon_i \sim iid \ N (0, \omega^2)$$

One essential assumptions of the BS model is that the pricing errors $\epsilon_i$ are assumed to be normally distributed. Normality feature is the first to evaluate.

Null and alternative hypotheses to be tested relating normality are as follow:

$$H_0 : \epsilon_i \text{ is normally distributed}$$

$$H_1 : \epsilon_i \text{ is not normally distributed}$$

The null hypothesis is a joint hypothesis of that both the skewness and the excess kurtosis are being zero. Empirically the distribution is tested with the Jarque-Bera normality test for rejection of the null hypothesis. On the other words, Jarque-Bera test evaluates whether the sample data have distribution-related characteristics like skewness and kurtosis equal to a normal distribution.
4.2 Bias in the BS model and test for unbiasedness

The empirical option price model is extended to allow biasedness by taking again the BS price as a baseline and adding an unknown intercept $\beta_0$ to the formula. The model that allows for biasedness is defined as follows:

$$c_i = \beta_0 + c_i^{BS}(\sigma) + \epsilon_i$$  \hspace{1cm} (15)

Hypotheses to be tested regarding unbiasedness are

$H_0 : \beta_0 = 0$

$H_1 : \beta_0 \neq 0$

The empirical test for rejection of the null hypothesis can be $t$-statistic, a Wald test or a likelihood ratio test.

The biasedness property can be extended to take into account different contract types like dependence on option’s moneyness\(^5\). In such case the empirical model becomes:

$$c_i = \beta_0 + \beta_1 DUM_i^{ATM} + \beta_2 DUM_i^{ITM} + c_i^{BS}(\sigma) + \epsilon_i$$  \hspace{1cm} (16)

In the equation above dummy variables correspond to options with particular moneyness category while $\beta_0$ corresponds to the out-of-the-money option contracts.

Now a test for biasedness property is based on hypotheses:

$H_0 : \beta_0 = \beta_1 = \beta_2 = 0.$

$H_1 : \text{at least one of the restrictions above fails.}$

\(^5\) Moneyness categories are described and defined in chapter 5.2.2.
This hypothesis can be tested jointly using a Wald test.

In addition, biasedness can be evaluated by forming an empirical model via different strike prices or dividing option data into maturity-based categories. However, in this paper biasedness property is tested only with the previously presented models in equations (15) and (16).

### 4.3 Maturity-based heteroscedasticity

As the BS model assumes constant variance \( (\omega^2) \) for disturbance term across all type of contracts, next reasonable step would be to relax the empirical model further and allow it for heteroscedasticity. When evaluating whether the variance is depended on option’s maturity, the model becomes as follow:

\[
c_i = \beta_0 + c_i^{BS}(\sigma) + \epsilon_i
\]

\[
\omega_i^2 = \exp(\alpha_0 + \alpha_1 DUM_i^{SHORT} + \alpha_2 DUM_i^{MEDIUM})
\]

\[
\epsilon_i \sim iid \ N(0, \omega^2)
\]

where the unknown parameters for the estimation are \( \theta = \{\sigma^2, \beta_0, \alpha_0, \alpha_1, \alpha_2\} \) and dummy variables are respectively depended on the maturity\(^6\) of an option. Exponential function is just for ensuring variance to remain positive.

A test for heteroscedasticity is based on restriction that both \( \alpha_1 = \alpha_2 = 0 \) and it can be evaluated by using a Wald test or a likelihood ratio test.

Another way instead of maturity to examine heteroscedasticity is to use option’s moneyness-category, which is described in the next chapter.

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\(^6\) Classification of maturity groups is presented in chapter 5.2.1.
4.4 Moneyness-based heteroscedasticity

To check whether heteroscedasticity is caused by moneyness of an option, we define empirical model as follow:

\[ c_i = \beta_0 + c_i^{BS}(\sigma) + \varepsilon_i \]  
\[ \omega_i^2 = \exp(\alpha_0 + \alpha_1 DUM_{iATM} + \alpha_2 DUM_{iTM}) \]  
\[ \varepsilon_i \sim iid \ N (0, \omega^2) \]  

where unknown parameters are the same as in maturity-based heteroscedasticity test and dummy variables correspond to option’s moneyness property. Test is as previously: to evaluate restriction \( \alpha_1 = \alpha_2 = 0 \), a Wald test or a likelihood test can be used.

4.5 Volatility smile

Compared to previous empirical models with just a single estimate of the volatility parameter, a model that allows volatility vary over the strike price or across other characteristics can be evaluated by grouping the data for example according to strike price and run a similar estimation as earlier. Since amount of different strike prices may be enormous (185 in the data used in this paper) in sample that covers whole calendar year, the estimation presented here focuses on running analysis by dividing the data first into three subcategories depending on option’s moneyness.

Thus to check whether volatility is caused by different moneyness of an option, we define empirical model as follow:

\[ c_i = \beta_0 + c_i^{BS}(\sigma) + \varepsilon_i \]  
\[ \sigma_i^2 = \exp(\gamma_0 + \gamma_1 DUM_{iATM} + \gamma_2 DUM_{iTM}) \]  
\[ \varepsilon_i \sim iid \ N (0, \omega^2) \]  

where the unknown parameters are now \( \theta = \{\omega^2, \beta_0, \gamma_0, \gamma_1, \gamma_2\} \) and dummy variables are respectively depended on the predefined option’s moneyness level. If the volatility
estimates are different across groups, it is an evidence against the BS model which restricts volatility to be constant. To re-estimate the model and evaluate restriction $\gamma_1 = \gamma_2 = 0$ a Wald test or a likelihood test can be used.
5 DATA

This chapter describes the option data and all filters including terms and restrictions that are applied. Finally, summary statistics of data are presented.

5.1 Sample description

The original raw data consist of options written on US S&P 500 equity index. Before filtering, it consists of 634029 put and 634398 call options with necessary daily closing price and trading information for entire calendar year 2014.

For both the index price and the option price, daily closing prices are used in analysis.

5.2 Defining option contract characteristics of interest

Basic factors affecting option prices are those that are present in the equations, like value of the underlying asset and exercise price of the option. It is evident that other factors like remaining life-time before expiration, risk-free interest rate and volatility of the underlying do have influence on the option price as well. In addition, other economic explanations and determinants for existence of the volatility skew in real life have been presence of transaction costs and that institutional investors are commonly seen hungry for puts as portfolio insurance (Bollen & Whaley 2004). Rough and Vainberg (2007) among others recognize that the shape of the implied volatility smile is affected most by two factors, moneyness and the maturity of the option.

Due to fact that researchers recognize some of the characteristics having more statistical significance in determining option price than others, main explanatory variables of interest for analyzing the BS option price model are introduced next.

5.2.1 Time-to-maturity

There are two ways to define option’s maturity. Depending on data source, available information regarding option’s maturity may be expressed in trading days or in
calendar days. In case number of trading days are used, time-to-maturity (TTM) measured in years is calculated by dividing number of remaining days before expiration by 252 which is generally accepted amount of trading days in the US. There was exactly 252 trading days in New York Stock Exchange (NYSE) in 2014.

However, the data source used in this thesis lists remaining calendar days before expiration for every option and since the BS model requires option’s remaining lifetime expressed in years, we convert it to years by dividing number with 365. Once TTM in years is available, next step is to categorize options in three sub-categories as follow depending on option’s TTM:

1) a call option is defined as short if its time-to-maturity (TTM) is < 31 days, or < 0.08493 in years
2) a call option is defined as medium if 31 ≤ TTM ≤ 120, or expressed in years 0.08493 ≤ TTM ≤ 0.328767
3) a call option is defined as long if 120 < TTM ≤ 180 days, or expressed in years 0.328767 < TTM ≤ 0.49315

Method to divide options in three above-mentioned categories is mainly adapted from Lin et. al (2016), except we make a slight change and do not fully follow day ranges specified in their paper where authors used 7–39 (short), 42–130 (medium) and 133–221 days (long). This is not expected to have any significant difference from the research question’s perspective as we only aim to prove that volatility vary over maturity. We could actually define even more sub-categories but three is enough to find dependency in question.

5.2.2 Moneyness

Moneyness of an option is more commonly defined as the ratio of the strike price to the current price of an underlying asset on which that particular option is written.\(^7\)

---

\(^7\) Several other ways to measure moneyness exist, for instance Bollen and Whaley (2004) use Black-Scholes delta and Ni (2008) uses total volatility-adjusted strike-to-stock price ratio.
Therefore an at-the-money (ATM) option would have moneyness level of one. Moneyness could be written also as

$$M = \frac{S_0}{K},$$  \hspace{1cm} (20)$$

where $S_0$ represents the price of an underlying index and $K$ the exercise price of an option. The moneyness can also be expressed at different levels in percentage, and to make more accurate classification one can define several more detailed categories like deep-in-the-money or very-deep-in-the-money for further analysis depending on option’s moneyness level.

In option pricing another commonly seen expression used by researchers is near-at-the-money. That near-ATM term refers to the moneyness range where most option series are actively traded and most likely with higher open interest (Tanha & Dempsey 2015). Since this paper aim to have simple approach with only three moneyness categories, we widen definition of an ATM option to cover also near-ATM options with a proper range of 3 % moneyness. It is good to mention that depending on scope of the survey some scientific studies focusing on volatility smile phenomenon might classify options as ATMs with narrower or even wider spread, or controversially exclude all options with moneyness higher than 6 % if investigating exact root causes behind option price formation\(^8\).

In this paper, however, extreme moneyness are excluded from the final data. So the analysis eventually includes call options with moneyness ranging of $0.7 \leq M \leq 1.3$. Categorizing moneyness between three levels a call option is classified as:

1) out-of-the-money (OTM) if the following holds

$$0.7 \leq \frac{S_0}{K} < 0.97$$

2) at-the-money (ATM) if the following holds

\(^8\) For instance Tanha and Dempsey (2015) classify options as ATM if the strike price is within 2 % of the index, and ITM or OTM if the strike price is within 32 % (but more than 2 %).
0.97 \leq \frac{S_0}{K} \leq 1.03

3) in-the-money (ITM) if the following holds

1.03 < \frac{S_0}{K} \leq 1.3

Moneyness is one major component in option pricing research and it has an explicit definition. However every researcher have their liberty to further define sub-categories regarding option’s moneyness including degree and range of it. For example Tanha and Dempsey (2015) are interest in moneyness of range \([0.8, 1.2]\), which we extend here by 0.1 in both directions to cover a slightly wider range. Similarly to maturity, here we are satisfied with three sub-classes although there exist no true limitation for that.

5.3 Filtering the data

In the absence of arbitrage opportunities and by assuming that the market price reflects the Black-Scholes option price, it follows from the call-put parity that the implied volatility for a European put and call options, with equal time-to-maturity and strike price, are the same\(^9\). Thus, to make final data processing slightly easier this thesis will concentrate on call options only and put options are deleted from the raw data at first.

It is worth noting though that in real life this kind of assumption about the same implied volatility for equal put and call options might not hold, of which Evnine and Rudd (1985) presents their early evidence based on intraday prices. They conclude that options frequently violate put-call parity and in addition are substantially mispriced. Since thesis is not aiming to measure accuracy of the BS model but just empirically to find evidence about inadequate performance of the model, this kind of potential violation regarding put-call parity theorem and focusing on call options only is not seen as fundamental limitation for the research objective itself.

The original raw data includes huge amount of options with wide range of maturity starting from 0 and ending to 1080 days. The longest maturity of 1080 days equals almost to three years and is not relevant from thesis’ perspective, so we will limit maturity to 180 calendar days. In addition to that, several other limitations are applied as well and all the filters applied in particular order are as follow:

1. Options with time-to-maturity longer than 180 days are excluded. This filter left 140 912 option prices outside the analysis.
2. The implied volatility (IV) for options with short TTM might be exceptional high. Thus, IV included in the data must be positive and below 100 %. After filter applied 182 856 option prices were removed.
3. Only options that have Best bid and Best offer values different from 0 are kept. In total 17 112 option prices were deleted after 3rd filter applied.
4. Call options very deep in-the-money (>1.3) or out-of-the-money (<0.7) are removed. That is, only options with an absolute moneyness less than or equal to 30 percent are kept. After the 4th filter 38 931 option prices were removed from the data.
5. The non-arbitrary condition defined by Merton (1973) must hold. This means that if the following lower boundary condition for a call option price is not satisfied, the option is excluded from the analysis:

\[ c_{t,\text{mid}} \geq S_t e^{-q_t(T-t)} - K e^{-r_t(T-t)} \]  \hspace{1cm} (18) 

where \( c_{t,\text{mid}} \) is average price calculated from the best bid and offer prices included in raw data, and \( q_t \) is continuously compounded dividend yield. According to Dumas et al. (1998) it is recommended to use average price instead of last trade price to reduce noise in the estimation of the volatility. Purpose of this filter is to ensure non-arbitrage conditions and after executing none of the option prices were removed, which was expected outcome.
6. Options with calculated average price of lower than $0.50 are excluded to avoid price inconsistency and causing potentially erroneous trade information. Bakshi et al. (1997) and Li (1999) both points out that it is important somehow to control for potential noise in option prices which in turn could be amplified
in any option pricing model, especially in implied binomial tree approach. Way to mitigate the noise is to remove options with very small price. This particular limitation removed 19 547 option prices.

It is worth to notice that after applying the final (6th) filter we could actually lose relevant information when it comes to out-of-the-money options especially in the analysis based on moneyness-category. Thus, every part of the comprehensive analysis will be run also with options less than $0.5 in value and result regarding informativeness of cheap options will be concluded in chapter 6.6.

5.4 Summary of the filtered data

In addition to option contract specific characteristics like moneyness or maturity, we need information related to market conditions and expectations. In the Black-Scholes model this information is covered by risk-free interest rate and dividend yield payable on underlying. Necessary daily information of annual risk-free rate and continuously compounded dividend yield on underlying equity index are fetched from the same data source as raw data itself.

After filtering and applying before mentioned restrictions in chapter 5.3, total of 235 040 option prices remain with relevant trading information. Below in Table 1 is presented summary statistics of all contract-dependent and market variables.

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option price [$]</td>
<td>0.50</td>
<td>483.35</td>
<td>135.05</td>
<td>94.35</td>
</tr>
<tr>
<td>Strike price [index value]</td>
<td>1375.0</td>
<td>2500.0</td>
<td>1852.7</td>
<td>1855.0</td>
</tr>
<tr>
<td>Time-To-Maturity [days]</td>
<td>2.0</td>
<td>180.0</td>
<td>54.6</td>
<td>49.0</td>
</tr>
<tr>
<td>Implied volatility* [%]</td>
<td>6.19</td>
<td>99.96</td>
<td>19.50</td>
<td>16.41</td>
</tr>
<tr>
<td>Risk-free interest rate [%]</td>
<td>0.118</td>
<td>0.292</td>
<td>0.187</td>
<td>0.193</td>
</tr>
<tr>
<td>Dividend yield [%]</td>
<td>1.56</td>
<td>1.99</td>
<td>1.88</td>
<td>1.91</td>
</tr>
</tbody>
</table>

*Information about implied volatility is delivered by the original data source, but the numerical value as such is used only in estimating volatility surface.
Applying certain filters might have affected range of listed variables. For example excluding options with really high implied volatility probably did limit TTM to start from 2 days although there were no specific filter applied to exclude short TTM. On the other words that is because deep-in-the-money options with short maturity most likely have IV of higher than 100 %.

The following Table 2 summarizes the S&P 500 call options that were categorized accordingly after filtering and used in the final analysis.

<table>
<thead>
<tr>
<th></th>
<th>Short</th>
<th>Medium</th>
<th>Long</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>In-the-money</td>
<td>37 331</td>
<td>86 983</td>
<td>7 618</td>
<td>131 932</td>
</tr>
<tr>
<td>At-the-money</td>
<td>23 825</td>
<td>38 894</td>
<td>2 744</td>
<td>65 463</td>
</tr>
<tr>
<td>Out-of-the-money</td>
<td>3 976</td>
<td>28 791</td>
<td>4 878</td>
<td>37 645</td>
</tr>
<tr>
<td>Total</td>
<td>65 132</td>
<td>154 668</td>
<td>15 240</td>
<td>235 040</td>
</tr>
</tbody>
</table>

Moneyness of the option is defined as presented earlier. Maturities in three sub-samples are 0-30 days (short), 31-120 days (medium), 121-180 days (long).

From the Table 2 it can be seen that most of the options (~56 %) included in the analysis are in-the-money options and approximately 16 % exhibit out-of-the-money options. In addition, options with TTM as medium represent majority while options with TTM as long are minority (~6.5 %).

An illustrative figure about how call options amount in the sample depending on their moneyness and maturity is presented next. The Figure 3 is a density graph which was derived during a regression where implied volatility surface is defined as dependent variable, while maturity and moneyness are explanatory variables (more detailed description of method follows in chapter 6.5).
Figure 3. Density of the S&P 500 call options.

From the sample density figure above we can see that most of the options are clustering around moneyness of unity and with time-to-maturity shorter than 0.17 years (or 2 months), which represent the most actively traded and with higher open interest options. In addition, this kind of distribution was expected since options under investigation are calls and, in this particular case, written on an index that is expected to have upward trending development. Excluding call options with value less than $0.5 can be seen as one explanation for the shortage of short maturity OTM options.
6 EMPIRICAL ANALYSIS AND RESULTS

In this chapter empirical results are presented. Analysis follows procedure presented in Hurn et al. (forthcoming) which starts with presenting results for volatility calculated from historical returns, continues with statistical distribution of an error term and then possible biasedness property of the BS model is examined. After that result for heteroscedasticity test is presented and finally result for volatility smile is showed. Differing from Hurn et al. analysis of volatility surface is complementary.

6.1 Volatility of historical returns

Volatility can be estimated from historical returns of an underlying asset. In this case we have plotted in the following figure evolution of the underlying index and logarithmic returns to emphasize how S&P 500 index has emerged in 2014 and how much variation can be seen in daily returns. From the data we first calculate variance of daily returns and finally estimate annualized volatility based on 252 daily observations, which will be a reference point for further evaluation of characteristics affecting volatility estimate when using the Black-Scholes model.

Figure 4. S&P 500 index and its daily logarithmic returns in 2014.
From the Figure 4 it is noticeable that although the price of the index is not stationary but upward trending, the logarithmic return series exhibits stationary property and is hovering at a constant level. There exist few large drops in the price index that cause negative returns in short time periods, which in turn can be seen as stimulants for increased volatility.

Skewness of the return sample is -0.4368 and together with negative sign of the statistic it emphasizes not normally distributed returns. The fourth moment regarding abnormal returns is illustrated with sample kurtosis, which derived from S&P 500 returns in 2014 exhibits kurtosis of 4.3425. Since this is different and greater compared to normal distribution’s kurtosis (=3), returns exhibit excess kurtosis and that there are more extreme returns in the data which cannot be predicted by the normal distribution.

Assumption about independent returns and that no correlation between samples exist is evaluated next. Figure 5 below represents the partial autocorrelation between a return at time $t$ and a lagged return. There seems not to be any relationship between lagged returns as the correlation appears randomly, although some of the lags get relatively high negative values. The autocorrelation for lag 0 is omitted (would have been fixed at 1 by convention).
Figure 5. Correlation between returns at time $t$ and lagged returns.

In addition, further analyzing of squared returns could reveal dependency between lags and show sign of autocorrelation. That could be interpreted as a fact that the volatility might be clustering, indicating that absolute returns today are actually correlating with the past absolute returns as mentioned in chapter 2.3 Time-series characteristics of returns.

As next Figure 6 illustrates the well documented phenomenon, realized returns are not normally distributed but exhibit negative skewness. That is, the distribution has thicker left tail meaning that large negative returns are more likely to occur. Even from this aspect it is obvious that returns are not normally distributed, which might suggest that assumption regarding normal distribution on the BS model is too strict.
Figure 6. Distribution of daily return on S&P 500 index during 2014.

Annualized volatility estimate from daily realized returns on S&P 500 index in year 2014 is 0.1132128, or 11.32%.

Volatility estimate based on historical returns is expected to be lower than an estimate which has memory effect of the previous change. When applying the constant mean model for returns with a GARCH(1,1) conditional variance described in equations (10) and (11), the annualized volatility estimate based on the long-run GARCH estimate now becomes as 0.1173758, or 11.74%, which is slightly higher but very close to previous constant mean model estimate.

6.2 Volatility based on option data and normality of residual

Furthermore, when applying maximum likelihood method with log-likelihood function presented in equation (14) and using an iterative gradient algorithm, the estimated implied volatility becomes:

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{0.01822} = 0.13498, \text{ or } 13.5\%.$$
This implied volatility derived from observed option prices is approximately 17% higher than previous ones that were estimated from returns of the underlying. The value of the log-likelihood function is -3.328, which is referred as $\log(\hat{\theta}_0)$ when calculating significance of the bias in chapter 6.3.

Next normality of pricing error, or residual, is evaluated. Hypothesis is that, under the Black-Scholes residual $\epsilon_i$ should have and exhibit normal distribution. Statistics show that while pricing error vary between -16.4 and 40.1, it has mean of 1.51 and median of 1.55. This suggest that at least for vast amount of options the BS model is insufficient. Figure 7 shows distribution of the error term derived from the call options (range of x-axis is limited to 25 due to illustrative purposes although maximum error was 40.1).

![Distribution of the error term covering options of all time-to-maturities.](image)

It can be seen that the empirical distribution exhibit really heavy left tail, as the estimate for error term is widely negative:

$$\hat{\epsilon}_i = c_i - c_i^{BS}(\hat{\sigma}) < 0$$
This in turn indicate that the BS model overprices a vast amount of options compared to the observed market prices than would be predicted by the normal distribution.

Conclusion about abnormal distribution is supported also by the Jarque-Bera normality test yielding in value of 22 859 and with corresponding $p$-value of 0.000. This provides empirical evidence for a clear rejection of the null hypothesis about normality, which was one of the core assumptions of the Black-Scholes model.

### 6.3 Unbiasedness

If the base BS model described in equation (13) is relaxed such that it allows for the possibility of the observed option price being biased, the empirical model is extended according to equation (15) and an additional intercept $\beta_0$ is added to the model.

Maximization with three unknown parameters $\hat{\theta} = \{\hat{\sigma}^2, \hat{\omega}^2, \hat{\beta}_0\}$ with initial values 0.01, 10 and 1 respectively, converges after 78 iterations. Estimate for $\hat{\beta}_0$ is 3.399 with standard error of 0.01881. Significance of the intercept can be tested by $t$-statistic resulting in value of 180.7, which in turn establishes significant bias and rejection of the BS model. Positive bias estimate suggest that the original BS model underestimates option prices on average by 3.399.

This result suggests that when investors approximate option prices with the Black-Scholes formula, a more illustrative price on average is achieved by computing the BS price first and then augment it by an additional 3.399 dollars.

### Table 3. ML estimates for unbiasedness test.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard error</th>
<th>$t$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\sigma}^2$</td>
<td>0.01458***</td>
<td>0.00002609</td>
<td>559.0</td>
</tr>
<tr>
<td>$\hat{\omega}^2$</td>
<td>40.13***</td>
<td>0.09268</td>
<td>433.0</td>
</tr>
<tr>
<td>$\hat{\beta}_0$</td>
<td>3.399***</td>
<td>0.01881</td>
<td>180.7</td>
</tr>
</tbody>
</table>

Estimation is based on BFGS maximization method with three variables resulting in log-likelihood value of -767 804.3. Results are achieved after 78 iterations with successful convergence, initial values being 0.01, 10 and 1 respectively. Significance codes are: *** $p<0.001$, ** $p<0.01$, * $p<0.05$. 
The same maximization procedure returns also estimate for the volatility. Now volatility estimate $\hat{\sigma}$ seems to be $\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{0.01457} = 0.1207$ or approximately 12.1%, which is lower than 13.5% when incorrectly assuming unbiasedness.

As $t$-statistic resulted in rejection of null hypothesis about unbiasedness, the likelihood ratio (LR) test gives the same conclusion:

$$LR = -2N \left( \log L(\hat{\theta}_0) - \log L(\hat{\theta}_1) \right) = -2 \times 235040 \left( -\frac{782218.9}{235040} + \frac{767804.3}{235040} \right)$$

results in the likelihood ratio of 28 829.2 with $p$-value of 0.000.

Further analysis regarding biasedness property is based on option’s moneyness-category with five unknown parameters $\hat{\theta} = \{\hat{\sigma}^2, \hat{\omega}^2, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2\}$ and it follows the equation (16). Initial values for iteration were 0.01, 10, -1, 1 and 5, respectively. Successful convergence is achieved after 105 iterations. Results are presented in the following Table 4.

### Table 4. ML estimates for biasedness property based on moneyness.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard error</th>
<th>t-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\sigma}^2$</td>
<td>0.02086***</td>
<td>0.00003161</td>
<td>659.77</td>
</tr>
<tr>
<td>$\hat{\omega}^2$</td>
<td>24.82***</td>
<td>0.06554</td>
<td>378.69</td>
</tr>
<tr>
<td>$\hat{\beta}_0$</td>
<td>-6.504***</td>
<td>0.03720</td>
<td>-174.83</td>
</tr>
<tr>
<td>$\hat{\beta}_1$</td>
<td>1.916***</td>
<td>0.03855</td>
<td>49.69</td>
</tr>
<tr>
<td>$\hat{\beta}_2$</td>
<td>10.95***</td>
<td>0.03601</td>
<td>303.99</td>
</tr>
</tbody>
</table>

Estimation is based on BFGS maximization method with five variables resulting in log-likelihood value of -710 930.6. Results are achieved after 105 iterations with successful convergence, initial values being 0.01, 10, -1, 1 and 5, respectively. Significance codes are: *** $p<0.001$, ** $p<0.01$, * $p<0.05$.

Now the estimated parameters regarding biasedness property for the three moneyness-based groups are all showing their statistical significance. Negative sign of $\hat{\beta}_0 = -6.504$ indicates that the BS model overprices out-of-the-money options and the same applies for at-the-money ($\hat{\beta}_0 + \hat{\beta}_1$) options as the coefficient gets value of -4.588. But for in-the-money ($\hat{\beta}_0 + \hat{\beta}_2$) options the BS model seems to underestimate prices as sum for the coefficient results in 4.446.
Wald test statistic for the restriction $\beta_1 = \beta_2 = 0$ results in value of 6 335 514 053 which is distributed asymptotically as $\chi^2$ under the null hypothesis. $p$-value of 0.000 shows that assumption about unbiasedness can be easily rejected.

### 6.4 Heteroscedasticity

Heteroscedasticity testing is divided into two categories depending on the model. Evaluated characteristics are option’s maturity and moneyness. Empirical models and evaluation follow equations (17) and (18).

First maturity based dependency is evaluated. Based on three maturity (short, medium, long) categories, estimates for the unknown parameters using maximum likelihood method are as follow:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard error</th>
<th>t-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\sigma}^2$</td>
<td>0.01394***</td>
<td>0.00002566</td>
<td>543.3</td>
</tr>
<tr>
<td>$\hat{\beta}_0$</td>
<td>2.755***</td>
<td>0.01587</td>
<td>173.6</td>
</tr>
<tr>
<td>$\hat{\alpha}_0$</td>
<td>4.994***</td>
<td>0.01142</td>
<td>437.4</td>
</tr>
<tr>
<td>$\hat{\alpha}_1$</td>
<td>-2.321***</td>
<td>0.01310</td>
<td>-177.2</td>
</tr>
<tr>
<td>$\hat{\alpha}_2$</td>
<td>-1.260***</td>
<td>0.01191</td>
<td>-105.8</td>
</tr>
</tbody>
</table>

Estimation is based on BFGS maximization method with five variables resulting in log-likelihood value of -747 704.2. Results are achieved after 190 iterations with successful convergence, initial values being 0.01, 2, 2, -2 and -1, respectively. Significance codes are: *** p<0.001, ** p<0.01, * p<0.05.

Wald test statistic for the restriction $\alpha_1 = \alpha_2 = 0$ results in value of 37 461.51 which is distributed asymptotically as $\chi^2$ under the null hypothesis. $p$-value of 0.000 shows that assumption about homoscedasticity when categorized according to option’s maturity is easily rejected at the 5% significance level.

As the Wald test resulted in rejection of null hypothesis about homoscedasticity, the likelihood ratio (LR) test gives the same conclusion:

$$LR = -2N \left( \log L(\hat{\theta}_0) - \log L(\hat{\theta}_1) \right) = -2 * 235040 \left( - \frac{782218.9}{235040} + \frac{747704.2}{235040} \right)$$
results in value of 69 029.4 for the likelihood ratio statistic. Once again this is distributed asymptotically as $\chi^2$ under the null hypothesis. The $p$-value of 0.000 shows that assumption about homoscedasticity is easily rejected at the 5 % level using likelihood ratio test.

Result indicates that there is strong dependency in option price regarding if options are categorized according to maturity. Further analysis about other characteristics affecting distribution of an error term could be based on option’s moneyness. When estimation is based on option’s moneyness, analysis follow equation (18) and the result are presented in Table 6 below.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard error</th>
<th>t-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^2$</td>
<td>0.01498***</td>
<td>0.00003238</td>
<td>462.72</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>3.666***</td>
<td>0.01931</td>
<td>189.87</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>3.828***</td>
<td>0.009824</td>
<td>389.71</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.06122***</td>
<td>0.009833</td>
<td>6.23</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>-0.2781***</td>
<td>0.01294</td>
<td>-21.49</td>
</tr>
</tbody>
</table>

Estimation is based on BFGS maximization method with five variables resulting in log-likelihood value of $-767078$. Results are achieved after 120 iterations with successful convergence, initial values being 0.01, 1, 1, 0.1 and 0.1, respectively. Significance codes are: *** $p<0.001$, ** $p<0.01$, * $p<0.05$.

Wald test statistic for the restriction $\alpha_1 = \alpha_2 = 0$ results in value of 1 354.01 which is distributed asymptotically as $\chi^2$ under the null hypothesis. $p$-value of 0.000 shows that assumption about homoscedasticity based on option’s moneyness can be similarly rejected as based on maturity.

6.5 Varying volatility and a volatility smile

Previous chapters already evaluated empirically the Black-Scholes model for certain inconsistent restrictions that are related to asset price process. Obviously the last assumption regarding constant volatility across moneyness is about to fail as well, and for that we define unknown parameters according to chapter 4.5 and equation (19).

Thus, unknown parameters are now $\theta = \{\sigma^2, \beta_0, \gamma_0, \gamma_1, \gamma_2\}$ and dummy variables are respectively depended on the option’s moneyness category.
Based on option’s three different categories of moneyness, estimation based on Newton-Raphson method results in the following estimated parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard error</th>
<th>t-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\omega}^2$</td>
<td>24.924***</td>
<td>0.065536</td>
<td>380.3</td>
</tr>
<tr>
<td>$\hat{\beta}_0$</td>
<td>2.236***</td>
<td>0.015707</td>
<td>142.3</td>
</tr>
<tr>
<td>$\hat{\gamma}_0$</td>
<td>-4.655***</td>
<td>0.003764</td>
<td>-1236.9</td>
</tr>
<tr>
<td>$\hat{\gamma}_1$</td>
<td>0.408***</td>
<td>0.003390</td>
<td>110.6</td>
</tr>
<tr>
<td>$\hat{\gamma}_2$</td>
<td>1.015***</td>
<td>0.003683</td>
<td>275.5</td>
</tr>
</tbody>
</table>

Table 7. ML estimates for volatility smile based on option’s moneyness.

Estimation is based on Newton-Raphson maximization method with five variables resulting in log-likelihood value of -711 425.4. Results are achieved after 25 iterations as function values successively are within tolerance limits and initial values being 10, 1, -1, 0.1 and 0.1, respectively. Significance codes are: *** p<0.001, ** p<0.01, * p<0.05.

To recall from the empirical model that the variance was defined as

$$\sigma_i^2 = \exp(\gamma_0 + \gamma_1 DUM_i^{ATM} + \gamma_2 DUM_i^{ITM})$$

we get volatility estimates as follow:

$$\hat{\sigma}_i(OTM) = \sqrt{\exp(-4.4655101)} = 0.0975344 \approx 9.8 \%$$
$$\hat{\sigma}_i(ATM) = \sqrt{\exp(-4.4655101 + 0.408209)} = 0.1196187 \approx 12.0 \%$$
$$\hat{\sigma}_i(ITM) = \sqrt{\exp(-4.4655101 + 1.014726)} = 0.1619954 \approx 16.2 \%$$

Given that volatility monotonically decreases as options move from in-the-money to out-of-the-money, the volatility smile is actually more like a volatility smirk. This is in line with earlier empirical studies that in case of index options the volatility tend to exhibit a smirk.

Wald test statistic for the restriction $\gamma_1 = \gamma_2 = 0$ results in value of 157 433.1 which is distributed asymptotically as $\chi^2$ under the null hypothesis. p-value of 0.000 shows that assumption about constant volatility based on option’s moneyness category can be easily rejected.
Since results show dependency on moneyness and earlier empirical evidence adds that another significant factor is maturity, we next draw an illustrative graphs of volatility surface. Regression is done with \textit{loess}-function\textsuperscript{10} in R that fits a polynomial surface determined by one or more numerical predictors. Now implied volatility is dependent variable, while maturity and moneyness are explanatory variables. Assumption according previously presented results is that the volatility should achieve its highest value for the options that are at the same time both very deep in-the-money and exhibit short maturity.

For every option with necessary price related information available in the data, an implied volatility (IV) estimate is calculated with respect to the Black-Scholes formula. Note that in most commercial data sources the implied volatility information might be directly available for trading purpose, but it is still good to verify that when applying a price calculated as an average of best-bid and best-offer, the resulted IV equals to the provided data. This was also confirmed with the analyzed data by calculating IVs for several randomly chosen mid-prices with different strike price and maturity.

As an outcome, the figures on the following page are representing how the estimated implied volatility vary depending on option’s characteristics. Moneyness for call option is defined as price of the underlying relative to option’s strike price ($S_0/K$), and maturity is expressed in years.

The Figure 8 expresses IV in two-dimensional format including contours that exhibit volatilities with their respective values. In addition, color range covers the same information as darker blue describes lower implied volatility while dark red high IV.

\textsuperscript{10} More information about Local Polynomial Regression Fitting with R can be found from several sources or simply by applying “\texttt{?loess}” in R console.
Figure 8. Implied volatility with respective contours as a function of moneyness and maturity.

Figure 9. Regressed implied volatility surface in 3D format.

Results are in line with assumption that the volatility in fact vary in respect to these two factors and that the restriction about constant volatility in the Black-Scholes model is misleading.
It was already Rubinstein (1994) who studied and reported typical implied volatility smile phenomenon before and after 1987 stock crash. Jackwerth and Rubinstein (1996) complemented the idea and concluded that those investors who are afraid of extreme market events should consider probability distributions with flexibility in the shape of the left-hand tail. Although they pointed out the absence of trading costs in their study, authors think expenses would exhibit less importance. Since investors can be even irrational and no assumptions regarding stochastic processes are required, it is the probability distribution that offers the solution for presence of volatility smile.

As Hull (2008: 397) points out, the volatility smile becomes less pronounced when the option maturity increases. Same phenomenon can be seen from the previous figures: volatility smirk is gently sloping for long maturity options while shorter maturities exhibit extreme steep slope.

6.6 Informativeness of cheap options

Beside the main analysis, a comparative analysis was executed where cheap and almost worthless options were included as well. In this case the 6th filter described in chapter 5.3 was removed and data now consisted of 254 587 call option prices of which 19 547 were having value of less than $0.5.

Since a cheap call option most likely is out-of-the-money and at the same time has short maturity, it is to exhibit risky asset with low probability eventually to be exercised in-the-money and thus should have high implied volatility. Some of the filters applied before the 6th one should also control for high implied volatility, like the second filter which removed options with IV higher than 100% and the 4th one which removes very deep out-of-the-money options.

As a summary it can be stated that cheap options did have a slight effect on biasedness test result and on heteroscedasticity based on moneyness. Bias $\beta_0$ presented in Table 3 changed from 3.399 to 2.904 (-14.6%) and in Table 4 $\beta_0$ increased from -6.504 to -4.185 while $\beta_1$ and $\beta_2$ decreased 62% and 18%, respectively. Biasedness based on
moneyness was expected to change since content of OTM option category changed when cheap options were included in the estimation.

Result of heteroscedasticity based on option’s maturity did not change much, so cheap options do not have significant effect on maturity-categorized analysis. The same phenomenon was seen in volatility smile analysis where cheap options surprisingly did not change outcome of volatilities significantly, although moneyness is one of the descriptive characteristics.

Common noticeable effects on each comparative analysis were that volatility estimate did not change much from the original one where cheap options were excluded, and that standard error of almost every estimate generally increased when cheap options were included.
7 CONCLUSIONS

This paper evaluates performance of the Black-Scholes option pricing model and empirically shows that too strict assumptions related to the model must be relaxed in accordance to incorporate with the observed market prices. The BS model is found to misprice US S&P 500 index options in 2014. The paper also shows that the error term cannot be assumed to exhibit normal distribution and that in case of volatility-affecting properties it is the assumption about homoscedasticity which in turn causes part of mispricing.

Findings regarding non-constant volatility are in line with prior literature, and assumption about constant volatility in the BS model is too strict and shown to fail also in this paper. Complementing the BS model with relaxations and estimating volatility with GARCH(1,1) model based on maximum likelihood method provided statistically significant results and despite its burdening computational need the model is found to be practical, although true accuracy of the method was left outside from the content. That could be one of the subsequent questions for further research to be analyzed, while another in parallel could be applying a cross-sectional analysis between stock exchanges with different option trading volume.

From investors’ perspective paper shows the expected result that the BS model exhibit biasedness property which in turn causes mispricing of the options. In addition, it is also evident from volatility smile analysis that implied volatility between options vary across their moneyness and maturity. Based on the sample data, the BS model is found to overprice OTM and ATM options while ITM options seem to be undervalued. Thus, it is worth to take into account lacks in the BS model and any acceptable alternative option pricing model must incorporate the volatility smile and maturity effects. Finally it is essential to emphasize that options analyzed in this paper were categorized according to author’s view, and that result might vary depending on how each individual investor eventually defines these moneyness and maturity categories. To conclude, the BS option pricing model is a simple framework to start with but defects involved in the model must be recognized by every market participant.
REFERENCES


