Analysis of Nonlinear Dynamics in a Classical Transmon Circuit

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Abstract

The focus of this thesis is on classical dynamics of a transmon qubit. First we introduce the basic concepts of the classical circuit analysis and use this knowledge to derive the Lagrangians and Hamiltonians of an LC circuit, a Cooper-pair box, and ultimately we derive Hamiltonian for a transmon qubit. The transmon Hamiltonian is used to derive the equations of motion and also the meaning of these equations is discussed. Finally, the thesis is ended with some numerical results for the transmon equations of motion with a brief interpretation included.

1 Introduction

Quantum computing is an active field in physics and computer science that rests on totally new way to approach computational problems. Classical computer has its foundation in binary logic which manifests itself as bits (smallest information unit that is usually represented as "0" and "1"). On the other hand, quantum computer is based on "quantum" logic, which benefits from exploiting the quantum mechanical nature of reality. The information is stored as the quantum superpositions of the classical states, "0" and "1". This information unit is known as a quantum bit, also often called as a qubit.

The idea of quantum computing has existed for some time. It was introduced in the 1980s, famously by Richard Feynman. He argued that computational simulations based on nature became much more efficient and accurate if executed by using quantum algorithms, since nature itself behaves, fundamentally, in a quantum mechanical way [4]. Peter Shor, in 1994, developed the quantum logic algorithm for integer factorization [5], which performs much more efficiently than any other classical counterpart. This was the first major application designed for quantum computers. Nevertheless, the realization of practical quantum computer was thought to be impossible for a long time, but from beginning of 2000s till recent moment, the development of physical implementations for qubits has been immense. This has lifted quantum computing to be one of the most influential research fields today.

To this day, there have been lots of proposals for building a qubit. Many of these designs are founded on some obvious two-state quantum systems like spin of the electron or nucleus, for example. This thesis concentrates on less predictable design of superconducting qubits [6], and especially on transmon qubits [7]. Superconducting qubits rely on macroscopic quantum phenomena, superconductivity. This design has some crucial benefits like adjustable parameters which are defined by the components used in the circuit, and the already existing semiconductor industry which can be utilized to build large scale quantum computers [6,8]. Because of these characteristics the superconducting qubit is often called as artificial atom. The adjustability of a parameter can be problematic as well since the manufacturing process will inevitably produce some statistical errors into the parameters of a qubit, but fortunately these
errors can be measured and consequently eliminated [6]. Due to their large size, compared to other qubit candidates, the coupling with the environment is much stronger [9]. This can be seen as an advantage, since strong coupling is needed for efficient control and readout, but on the other hand, it does have a drawback since it can also couple strongly with harmful environmental sources. These notorious noise sources are leading to short coherence time, which has been the Achilles’ heel for many superconducting qubit designs [7]. Coherence time [10] refers to the duration of the coherent time period while qubit preserves the information of its state before leaking it into the environment through coupling. Fortunately, the clever design of a transmon qubit avoids this coupling to the main sources of environmental noise but still maintains a strong coupling to control signals [7].

Visualization of the simple transmon design is shown in Fig. 1. A resonator cavity is represented by a long partly sinuating transmission line. There are two transmon qubits in this design, one on the right and other on the left, which are both coupled to the resonator. At the bottom, there is an enlargement of the actual qubit where the middle part of the qubit is the Josephson junction and the saw-toothed metal plates represent the shunt capacitance. Explanations for each of these parts is given in later parts of the thesis.

The main objective of this thesis is to describe the behavior of the transmon qubit with classical equations of motion. This view on the qubit does not take quantum mechanics into account, except the nonlinearity which is a result of Josephson effect and therefore it is a result of quantum mechanics. The nonlinear sinusoidal potential term of the Josephson junction can be used in the equations of motion of transmon, but the behavior these equations are inducing is still classical. This reference frame will allow us to forget quantum effects like quantization and entanglement, and this way we can purely focus on investigating the nonlinear dynamics of the system. Effects of the nonlinearity
will be explicitly demonstrated with some numerical solutions of the equations of motion in the end part of the thesis. However, the thesis will begin by introducing basic concepts of classical circuit analysis, and after that we will concentrate on Lagrangians and Hamiltonians of electric circuits.

2 Classical network theory

In classical mechanics, there are three different ways to describe an evolving system. These approaches are called Newtonian, Lagrangian and Hamiltonian formalisms. In principle these are equivalent theories, but the Hamiltonian approach provides the smoothest way of describing transmon qubits and quantum systems in general. This is because of following reasons:

1. Hamiltonian $H$ describes the mechanical energy of the system and therefore it does have straightforward physical meaning
2. Transition to quantum mechanics happens naturally in Hamiltonian description.

Because of the reasons stated above, I will be using Hamiltonian description, although the easiest way to get there is to develop Lagrangian of the system first and then proceed to Hamiltonian description.

In derivation of the Hamiltonian description, I will be using the work of Vool and Devoret [12] as a guideline. This thesis will include only parts that are relevant for the classical analysis of the transmon qubit, but more thoroughly introduction can be found from Refs. [12,13], for example.

2.1 From electromagnetic fields to circuit elements

Transmon qubit consists of electromagnetic components, which are cooled down into the superconductive temperatures. These components are separable enough that system can be represented by using lumped element approximation, which lumps the attributes (capacitances, inductances, ...) of the components into a one discrete point in space. In general, this kind of approximation is valid only at the wavelengths that are considerably larger than the length separating elements from each other ($\lambda \gg a$). In the case of transmon this constraint isn’t a problem, because it is operated only at the two lowest energy levels, which naturally corresponds to two longest wavelengths that the system is able to support, and also a transmon qubit in itself is small compared to the wavelengths.

At this point it is useful to define couple concepts concerning on circuit topology. The most essential concepts are circuit branches and nodes. Node is defined as a point in a circuit, that connects two or more elements to each other. On the other hand, circuit branches are defined as the connecting paths between two nodes. An example circuit diagram is shown in the figure 2 from which one can see visually the meaning of circuit nodes and branches.
Figure 2: RLC circuit diagram. Dots $a$, $c$, $d$ labels the nodes of the circuit, where the node $d$ has been chosen as the ground node. Arrow $b$ represents the branch between nodes $a$ and $c$.

The general and accurate way to calculate the dynamics of the electromagnetic system is to use Maxwell’s equations, which uses electric fields $E(r, t)$ and magnetic fields $B(r, t)$ as variables. Now the lumped element approximation is very useful because it allows the usage of much simpler branch variables instead. Branch variables can be defined, for example, as branch voltage $v_b$ and branch current $i_b$. Transition from electromagnetic fields to these new variables is justified with equations

$$v(t)_b = \int_a^c E(r, t)dx$$

where the integration path $x$ goes along the branch $b$ (see figure 2), and

$$i(t)_b = \frac{1}{\mu_0} \oint_s B(r, t)ds$$

where the closed integration path $s$ goes around the branch (conductor).

The concept of energy plays a crucial role in the Hamiltonian and Lagrangian description. Thus it is important to note, that the energy loss in the component (resistor) or energy stored in the component (capacitor, inductor) by using a branch voltage and current is defined as

$$E_b(t) = \int_{-\infty}^{t} v_b(t')i_b(t')dt'$$

where $-\infty$ must be interpreted as some time in the past when circuit was at rest. The general sign convention is that branch voltages and currents are chosen to be opposite signs.
2.2 Generalized flux and charge

In Hamiltonian mechanics one should introduce some set of canonical coordinate pairs, which represent the position and the momentum of the system and are also independent from each other [12]. By taking this into account, voltage (1) and current (2) are not suitable variables so one needs to substitute these variables into

$$\Phi_b(t) = \int_{-\infty}^{t} v_b(t') dt'$$  (4)

and

$$Q_b(t) = \int_{-\infty}^{t} i_b(t') dt',$$  (5)

where $\Phi_b(t)$ is called generalized branch flux and $Q_b(t)$ is called generalized branch charge.

It will be assumed, for the moment, that all capacitive and inductive elements in the circuit are linear, which means that $C = \frac{dQ_b}{dv_b}$ and $L = \frac{d\Phi_b}{di_b}$ are constants. This is, for the most part, a reasonable assumption for the circuits associated with transmon (leaving out the nonlinear inductance in Josephson junction, see section 4). Under this assumption, variables in equation (3) can be easily replaced by branch flux and branch charges. For a capacitive element the energy is

$$E_b(t) = \int_{-\infty}^{t} v_b(t')i_b(t') dt'$$

$$= \int_{-\infty}^{t} v_b(t') \frac{dQ_b(t')}{dt'} dt'$$

$$= v_b(t)(Q_b(t) - Q_0) - \int_{-\infty}^{t} \frac{dv_b(t')}{dt'}Q_b(t') dt'$$

$$= \frac{(Q_b(t) - Q_0)^2}{C} - \frac{(Q_b(t) - Q_0)^2}{2C}$$

$$= \frac{(Q_b(t) - Q_0)^2}{2C},$$  (6)

where second row uses equation (5), third row uses integration by parts and the fourth row uses the relation for linear capacitance. $Q_0$ refers to static charge in the branch, which is caused by external source. In a similar way one can derive energy relation for a linear inductor

$$E_b(t) = \frac{(\Phi_b(t) - \Phi_0)^2}{2L},$$  (7)

where $\Phi_0$ refers to a constant offset flux created by some external source.

For the energy of a capacitive element (6) one can derive an alternative form

$$E_b(t) = \frac{C}{2}(\Phi_b(t) - V_0)^2,$$  (8)
which is acquired by using relation for a linear capacitance, and constant $V_0$ refers to external voltage source ($V = \dot{\Phi}$). This form is used to construct the Lagrangian of a system.

Although, we are mainly interested in the Hamiltonian, the Lagrangian acts as a handy stepping stone in the process of constructing the Hamiltonian. This method is called Legendre transform and it is widely used way to make transition form Lagrangian description to Hamiltonian description (see section 3).

2.3 Node variables as degrees of freedom

The energy of the system can be presented by using branch variables $\Phi_b$ and $Q_b$, but these are not suitable variables for the Hamiltonian. Hamiltonian and Lagrangian needs to be formed by using variables that represent the degrees of freedom of the system. This means that there are no surplus variables and these variables are not dependent on each other.

There are two general ways to find the degrees of freedom of the electrical circuit. The derivation of these variables happens by using Kirchhoff’s laws: Method of nodes and method of loops. This thesis will be using only the method of nodes

$$\sum_b Q_b = \tilde{Q}_n,$$

where summation is over every branch $b$ connected to node $n$ and $\tilde{Q}_n$ is a constant.

In the Method of nodes, the branch fluxes are expressed by using node fluxes

$$\Phi_b = \phi_n - \phi_{n'},$$

where $\phi_n$ and $\phi_{n'}$ are the node fluxes that are connected to the branch $b$. The number of node variables can be cut down by choosing a reference node, also called as a ground node, where flux remains constant.

These node variables are a degrees of freedom of the system and therefore they can be used to construct the Lagrangian for the system. The energy equations (7) and (8) obtain forms

$$E_b(t) = \frac{(\phi_n - \phi_{n'} - \Phi_0)^2}{2L}$$

and

$$E_b(t) = \frac{C}{2} (\dot{\phi}_n - \dot{\phi}_{n'} - V_0)^2.$$

The choice of using flux variables over charge variables means, that a flux must be interpreted as position-like variable and therefore charge is interpreted as momentum-like variable. This implies that inductive energy (11) acts as potential energy and capacitive energy (12) is the kinetic energy of the system. Respectively, I could have done this by using charge as position-like variable (see reference [14]).
3 Hamiltonians for electric circuits

The purpose of this chapter is to show how the Hamiltonians are derived in circuits relevant for transmon qubit and also to introduce some key components in these circuits.

3.1 LC Circuit and DC voltage source

First step in deriving the equations of motion for LC circuit is to form the corresponding Lagrangian. For this we need to find expressions for the kinetic and potential energy. By using branch variables, one gets

\[ T = \frac{1}{2} C \dot{\Phi}^2 \]  
(13)

for kinetic energy and

\[ V = \Phi^2 / 2L \]  
(14)

for potential energy.

Equations (13) and (14) hold in the case of conservative LC circuits, which means that there are no external sources or dissipation affecting the LC circuit. This implies, that the LC circuit will oscillate without any energy losses endlessly. Obviously, any realistic system does not behave in this way, but in the case of superconducting circuits, conservative circuit can give very good approximations due to very low dissipation. However, in many cases it is useful to have some way to affect the behavior of the system in a controlled way. This can be achieved, for example, by using a voltage source (see figure 3).

There is specific way to implement a DC voltage source in the Lagrangian and Hamiltonian description. This is done by modeling the source with capacitance \( C \to \infty \), which has been charged to \( Q \to \infty \). The voltage in a DC source is constant, therefore we assume that \( V = \frac{Q}{C} \). At first, this may look like a strange
definition because it means that this element has an infinite energy. Fortunately, the energy of the voltage source is not a relevant factor, since only the coupling term between the source and rest of the circuit is relevant.

Now it is possible to construct the Lagrangian, corresponding the LC circuit in figure 3, by using the definition $L = T - V$.

$$L = \frac{1}{2} C(\dot{\phi}_C - V_g)^2 + \frac{1}{2} C_g V_g^2 - \frac{\Phi_L^2}{2L},$$

(15)

where $C_g$ is an infinite capacitance modeling the voltage source.

As discussed before in section 3, the branch variables $\phi_C$ and $\phi_L$ are not the degrees of freedom of the circuit, and therefore they need to be replaced by node variables

$$L = \frac{1}{2} C(\dot{\phi}_a - V_g)^2 - \frac{\phi_a^2}{2L},$$

(16)

where $V_g = \dot{\phi}_b$. The second term in equation (15) describes the energy of the voltage source so it can be ignored.

First step in converting Lagrangian to Hamiltonian is to solve expression for a canonical conjugate $q_n$ (node charge). Canonical conjugate follows from the expression [1]

$$q_n = \frac{\partial L}{\partial \dot{\phi}_n},$$

(17)

As mentioned before, the node $\phi_b$ voltage is equal to $V_g$. The DC voltage $V_g$ is constant, which means that the only variable is $\phi_a$. Therefore, equation (17) gives a conjugate momentum

$$q_a = \frac{\partial L}{\partial \dot{\phi}_a} = C(\dot{\phi}_a - V_g)$$

(18)

$$\Leftrightarrow \dot{\phi}_a = \frac{q_a + CV_g}{C}.$$  

(19)

The goal is to express the Hamiltonian by using canonical variables $q_n$ (node charge) and $\phi_n$ (node flux).

We can transform the Lagrangian to Hamiltonian by using Legendre transformation [1]

$$H = \sum_n q_n \dot{\phi}_n - L.$$  

(20)

By placing equations (16) and (18) into the Legendre transformation (20) we get

$$H = C(\dot{\phi}_a - V_g)\dot{\phi}_a - \frac{1}{2} C(\dot{\phi}_a - V_g)^2 + \frac{\phi_a^2}{2L}$$

$$= C(\dot{\phi}_a - V_g)\dot{\phi}_a - \frac{1}{2} C \dot{\phi}_a^2 + CV_g \dot{\phi}_a - \frac{1}{2} CV_g^2 + \frac{\phi_a^2}{2L}$$

$$= \frac{1}{2} C \dot{\phi}_a^2 - \frac{1}{2} CV_g^2 + \frac{\phi_a^2}{2L}.$$  

(21)
Now equation (19) can be easily inserted in the Hamiltonian, which takes a form

\[
\mathcal{H} = \frac{(q_a + CV_g)^2}{2C} + \frac{\phi_a^2}{2L} - \frac{1}{2} CV_g^2
\]

\[
= \frac{q_a^2}{2C} + \frac{\phi_a^2}{2L} + V_g q_a
\]

The first two terms, in this result, forms the Hamiltonian of the harmonic oscillator. The third term describes the contribution that voltage source $V_g$ makes into rest of the circuit.

### 3.2 Cooper-Pair Box

Figure 4: Lumped element model of a Cooper-pair box. The symbol X inside the square represents a Josephson junction in parallel with a capacitance $C_J$.

Next step towards transmon qubit, is to introduce a Cooper-pair box, also known as a superconducting island. The simple version of the Cooper-pair box circuit is presented in figure 4. Only upgrade compared to LC circuit is Josephson junction, which replaces the conventional inductor.

#### 3.2.1 Josephson junction

Josephson junction is a fundamental element in all superconducting qubit designs. It introduces the much needed nonlinearity element into qubit energy levels, making it to an effective two-state quantum system.

Basically, Josephson junction is just two superconductors separated from each other by a tiny insulating barrier. In a sense, Josephson junction acts like a nonlinear inductor, which can be approximated as linear inductor if the flux in the junction is small. Josephson junction stores energy as the supercurrent is tunneled through the barrier, but it doesn’t create magnetic field like inductors do.
The nonlinearity of the junction stems from the coherent wave function describing all the Cooper-pairs in the superconductor. As Cooper-pairs tunnel through the junction, the phase of the wave function changes and this phase difference over the junction can be interpreted as the flux in the junction. These qualities cause some peculiar relations between supercurrent and flux

\[ I_S = I_c \sin \delta, \]  

where \( I_c \) refers to critical current, and relation between voltage and flux

\[ \dot{\delta} = \frac{2e}{\hbar} V. \]

The potential energy stored in the junction can be acquired by inserting relations (23) and (24) into energy equation (3). Hence, we get equation

\[ U = -E_J \cos \left( \frac{2\pi \phi}{\Phi_0} \right) = -E_J \cos(\delta), \]

where \( E_J = \hbar I_c/2e \) is the Josephson energy of the junction, \( \Phi_0 = \hbar/2e \) is the magnetic flux quantum and \( \delta = \frac{2\pi}{\Phi_0} \phi \) is the phase difference between the supercurrents across the junction. [2]

3.2.2 Dynamics of the Cooper-pair box

Dynamics of the Cooper-pair box differ somewhat from the case of LC circuit. Now we have two linear capacitances connected to each other, and also linear inductance of the LC circuit have been replaced by non-linear inductance of Josephson junction.

In same way as derived for the LC circuit in the earlier sections, we can construct the Lagrangian of the Cooper-pair box (see figure 4) by using node variables

\[ \mathcal{L} = \frac{1}{2} C_g (\dot{\phi}_J - V_g)^2 + \frac{1}{2} C_J \dot{\phi}_J^2 + E_J \cos \left( \frac{2\pi \phi}{\Phi_0} \right), \]

where the conjugate momentum is obtained form equation (17).

\[ q_J = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_J} = C_g (\dot{\phi}_J - V_g) + C_J \dot{\phi}_J = C_\Sigma \dot{\phi}_J - C_g V_g \]

\[ \Leftrightarrow \dot{\phi}_J = \frac{q_J + C_g V_g}{C_J + C_g} = \frac{q_J + C_g V_g}{C_\Sigma}, \]

where \( C_\Sigma = C_J + C_g \). Hamiltonian is obtained by using the Legendre transfor-
\( \mathcal{H} = (C_\Sigma \dot{\phi}_J - C_g V_g) \dot{\phi}_J - \frac{1}{2} C_g (\dot{\phi}_J - V_g)^2 - \frac{1}{2} C_J \dot{\phi}_J^2 - E_J \cos \left( \frac{2\pi \Phi}{\Phi_0} \phi_J \right) \)

\( = (C_J + C_g) \dot{\phi}_J^2 - C_g V_g \dot{\phi}_J - \frac{1}{2} C_g \dot{\phi}_J^2 + C_g V_g \dot{\phi}_J - \frac{1}{2} C_g V_g^2 - \frac{1}{2} C_J \dot{\phi}_J^2 - E_J \cos \left( \frac{2\pi \Phi}{\Phi_0} \phi_J \right) \)

\( = \frac{1}{2} C_J \dot{\phi}_J^2 + \frac{1}{2} C_g \dot{\phi}_J^2 - \frac{1}{2} C_g V_g^2 - E_J \cos \left( \frac{2\pi \Phi}{\Phi_0} \phi_J \right) \)

\( = \frac{1}{2} C_\Sigma \dot{\phi}_J^2 - \frac{1}{2} C_g V_g^2 - E_J \cos \left( \frac{2\pi \Phi}{\Phi_0} \phi_J \right). \)

By using conjugate momentum \( q_J \), Hamiltonian takes the form

\[ \mathcal{H} = \frac{(q_J + C_g V_g)^2}{2C_\Sigma} - \frac{1}{2} C_g V_g^2 - E_J \cos \left( \frac{2\pi \Phi}{\Phi_0} \phi_J \right) \]

\[ = \frac{q_J^2}{2C_\Sigma} + \beta V_g q_J + \frac{1}{2} \beta C_g V_g^2 - \frac{1}{2} C_g V_g^2 - E_J \cos \left( \frac{2\pi \Phi}{\Phi_0} \phi_J \right), \]

where \( \beta = C_g/C_\Sigma. \)

### 3.3 Transmon qubit

Now we are ready to start examining the main issue of this thesis, the transmon qubit. The circuit network of transmon (figure 5) has many similarities with the Cooper-pair box circuit (see Fig.4). Only two big differences between these two are the cavity resonator and shunt capacitance \( C_B \).

#### 3.3.1 Cavity resonator

Cavity resonator is represented by LC circuit, which is derived from a continuous, one-dimensional, transmission line by using a lumped element approximation. The resonator is placed in between the gate voltage source \( V_g \) and the qubit part (Cooper-pair box). It provides a coherent way to control qubit states without wave function collapse and also enables some methods for the measurement of the qubit state. [7] Similar cavity resonators are also used in other superconducting qubit designs. [6]

#### 3.3.2 Shunt capacitance \( C_B \)

The shunt capacitance \( C_B \), on the other hand, is used specifically in transmon qubits. It is an element that separates transmon from a similar qubit design,
charge qubit (also known as Cooper-pair box qubit). In transmon design, the charging energy $E_C$ of the superconducting island, between Josephson junction and $C_g$, is

$$E_C = \frac{e^2}{2(C_J + C_B + C_g)}, \quad (31)$$

which clearly states that larger $C_B$ makes $E_C$ smaller. Because $C_B$ does not affect Josephson energy $E_J$ at all, this implies that in transmon qubit, $E_J$ has a more crucial role in determining the energy levels of the transmon. Decreased charging energy means that the qubit is not that sensitive to charge dispersion anymore, which has been the major source of decoherence in charge qubit designs. In fact, transmon qubit can be made practically immune to charge dispersion. [7]

Although, the increase of ratio $E_J/E_C$ makes transmon coherence times much longer, it does have some negative effects. The anharmonicity of the qubit is decreased because the oscillations get smaller with respect to the size of the sinusoidal potential well. Fortunately, sufficient amount of anharmonicity can be found around at ratio $20 < E_J/E_C < 100$, often called as transmon regime. Besides from these restrictions for the $E_J/E_C$, charge dispersion can be decreased low enough to make it insignificant. [7]

![Figure 5: Transmon qubit with a coupled resonator and external voltage source. This circuit diagram corresponds to standard design used for transmon qubits.](image)

### 3.3.3 Transmon Lagrangian

Derivation of the Hamiltonian of the configuration consisting of a transmon and a resonator (see figure 5) will turn out to be more complicated than the examples leading to this transmon design. This design will have two relevant degrees of freedom, $\phi_J$ and $\phi_r$, instead of one.

Like we have done previous examples, it is pretty straightforward to construct equation for the Lagrangian. First we find relations for the kinetic energy and
the potential energy,

\[ T = \frac{1}{2} C_B \dot{\phi}_J^2 + \frac{1}{2} C_r \dot{\phi}_r^2 + \frac{1}{2} C_g (\dot{\phi}_J - \dot{\phi}_r)^2 + \frac{1}{2} C_{in} (\dot{\phi}_r - V_g)^2 \]  

(32)

and

\[ V = \frac{\phi_r^2}{2L_r} - E_J \cos \left( \frac{2\pi}{\Phi_0} \phi_J \right). \]  

(33)

Here has been used the common practice, where the capacitance \( C_J \) is lumped into the much bigger shunt capacitance \( C_B \). The Lagrangian of the system is

\[
\mathcal{L} = \frac{1}{2} C_B \dot{\phi}_J^2 + \frac{1}{2} C_r \dot{\phi}_r^2 + \frac{1}{2} C_g (\dot{\phi}_J - \dot{\phi}_r)^2 \\
+ \frac{1}{2} C_{in} (\dot{\phi}_r - V_g)^2 - \frac{\phi_r^2}{2L_r} + E_J \cos \left( \frac{2\pi}{\Phi_0} \phi_J \right). 
\]  

(34)

This Lagrangian is more complicated compared to previous examples, and similar way of solving Hamiltonian will turn out to be more tedious. For this system we will use a matrix method [12,13] that simplifies the problem and it can be used as a general method to convert complicated Lagrangians into corresponding Hamiltonians.

3.3.4 Matrix notation in the Legendre transformation

The first step is to present the Lagrangian in a matrix form. There is a general way to construct matrices for capacitance and inverted inductance. Diagonal elements for a capacitance matrix \( C_{ii} \), where \( i = 1, 2, \ldots, n \) and \( n \) is the number of nodes, are obtained by taking a sum over every capacitance straightly connected to a corresponding node. Capacitances between the nodes are used to fulfill elements \( C_{i,i+1} \) and \( C_{i+1,i} \), and values of these elements are \( C_{i,i+1} = C_{i+1,i} = -C_{i,i+1} \) where \( C_{i,i+1} \) is the capacitance between nodes \( i \) and \( i+1 \). The rules for obtaining inverted inductance matrix \( L^{-1} \) are similar, but it will not be needed in this particular case.

Now we can write Lagrangian of the transmon (34) to the form of

\[
\mathcal{L} = \frac{1}{2} \Phi^T C \Phi - a^T \Phi + \frac{1}{2} C_{in} V_g^2 - \frac{\phi_r^2}{2L_r} + E_J \cos \left( \frac{2\pi}{\Phi_0} \phi_J \right), 
\]  

(35)

where

\[
C = \begin{bmatrix} C_B + C_g & -C_g \\ -C_g & C_r + C_g + C_{in} \end{bmatrix}, \quad \Phi = \begin{bmatrix} \dot{\phi}_J \\ \dot{\phi}_r \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ C_{in} V_g \end{bmatrix}.
\]

For the upcoming steps only the first two terms in (35) are relevant, so we drop the other terms for the moment.
Conjugate momenta can be obtained from the equation (17), which gives equations

\[ q = C\dot{\Phi} - a \]  
\[ \Leftrightarrow \Phi = C^{-1}(q - a). \]

where \( q = [q_J q_r]^T \). Hamiltonian is given by the Legendre transformation (20), which works little bit differently when using matrix notation. A convenient way to do it is with index notation

\[ H = \sum_l q_l \phi_l - \frac{1}{2} \sum_{l,k} \phi_l C_{lk} \phi_k + \sum_l a_l \phi_l \]
\[ = \sum_{l,k} \frac{1}{2} (q_l + a_l)(C^{-1})_{lk}(q_k + a_k) \]

Steps for this result are explicitly shown in the paper [13]. The result for the inverse of the capacitance matrix is

\[ C^{-1} = \frac{1}{C_s^2} \begin{bmatrix} C_r + C_g + C_{in} & C_g \\ C_g & C_B + C_g \end{bmatrix}, \]

where

\[ C_s^2 = C_B C_g + C_B C_{in} + C_B C_r + C_g C_{in} + C_g C_r. \]

3.3.5 Hamiltonian of transmon

Complete equation for the transmon Hamiltonian is obtained from equation (38) after the potential terms, neglected earlier from Lagrangian (35), have been subtracted from it (we also add the \( V_g \) term). Therefore the Hamiltonian reaches the form

\[ H = \frac{C_r + C_g + C_{in}}{2C_s^2} q_J^2 - E_J \cos \left( \frac{2\pi}{\Phi_0} \phi_J \right) \]
\[ + \frac{C_B + C_g}{2C_s^2} q_r^2 + \frac{\phi_r^2}{2L_r} \]
\[ + \frac{C_g q_r q_J}{C_s^2} + \frac{(C_B C_{in} + C_g C_{in})q_r V_g + C_g C_{in} q_J V_g}{C_s^2} \]
\[ + \frac{(C_B + C_g) C_{in} V_g^2}{2C_s^2} + \frac{1}{2} C_{in} V_g^2. \]

There is a meaning in the way how terms have been placed into different rows. The first row describes the energy of the qubit part, the second row describes the energy of the resonator and the third row describes the couplings between different parts of the system. The first term is the coupling between resonator
and qubit, and the second term tells how gate voltage couples to resonator and qubit. Terms in the fourth row are constants, so keeping them does not have effect into behavior of the system, and therefore these terms will be dropped out from analysis, for the moment. [7]

At this point Hamiltonian (39) seems cumbersome, but luckily we are able to do some simplifications. We can assume that resonator capacitance $C_r$ is much larger than the other capacitances in the circuit. This will let us approximate Hamiltonian to the form

$$\mathcal{H} \approx q_r^2 \frac{2}{2C_\Sigma} - E_J \cos \left( \frac{2\pi}{\Phi_0} \phi_J \right) + q_r^2 \frac{2}{2C_r} + \phi_r^2 \frac{2}{2L_r} + \frac{C_g q_r q_J}{C_r} + \frac{C_{in} q_r V_g}{C_r},$$

where $C_\Sigma = C_B + C_g$. The term describing the coupling between qubit and DC voltage $V_g$ disappears. This term is supposed to allow tuning of the offset charge on the gate capacitor, which sets the Cooper-pair box on the "sweet spot". [8] However, in the transmon qubit the dependence from charge noise has reduced to irrelevant levels. Therefore, there is no need for any fixing of the Hamiltonian (40). This is the form that will be used to derive the equations of motion for the transmon system.

4 Classical dynamics of transmon qubit

4.1 Equations of motion for transmon

Hamiltonian formalism tells us how to get the equations of motion, if the Hamiltonian is known. This is done by using Hamilton’s equations [1], which can be written as

$$\dot{q}_i = -\frac{\partial \mathcal{H}}{\partial \phi_i},$$

and

$$\dot{\phi}_i = \frac{\partial \mathcal{H}}{\partial q_i}.$$  

Now it is straightforward to solve the equations of motion for our transmon Hamiltonian. There are two degrees of freedom, so the dynamics of the system will be described by four equations,

$$\dot{\phi}_r = \frac{q_r}{C_r} + \beta_r \frac{q_J}{C_\Sigma} + \alpha V_g,$$

$$\dot{q}_r = -\frac{\phi_r}{L_r}.$$
\[
\dot{\phi}_J = \frac{q_J}{C_\Sigma} + \beta_J \frac{q_r}{C_r}
\]  
(45)

and

\[
\dot{\phi}_r = -\frac{2\pi E_J}{\Phi_0} \sin \left(\frac{2\pi}{\Phi_0} \phi_J\right),
\]  
(46)

where \(\beta_r = C_g/C_r\), \(\beta_J = C_g/C_\Sigma\) and \(\alpha = C_{in}/C_r\).

### 4.1.1 Relations with voltages

Defined parameters \(\beta_r\), \(\beta_J\) and \(\alpha\) can be thought as an impedance divider ratios, which determine how much of the voltage in question affects some other part of the system [8]. More specifically, the term with \(\beta_r\) tells us how much of the voltage in qubit part can be seen in the oscillator part of the system. The terms containing parameters \(\beta_J\) and \(\alpha\) can be interpreted in similar way. The connection of the equations (43) and (45) with voltage can be made more apparent by using the relation for linear capacitance \(C = Q/V\), which enables the more intuitive representation of these equations:

\[
\dot{\phi}_r = V_r + \beta_r V_J + \alpha V_g,
\]  
(47)

and

\[
\dot{\phi}_J = V_J + \beta_J V_r.
\]  
(48)

Now we can clearly see that \(\dot{\phi}_r\) and \(\dot{\phi}_J\) are the overall voltages in the corresponding node, which is exactly what one should expect when considering equation (4).

### 4.1.2 Shunt resistances

So far, our analysis has been lossless. However, at this point I will add some dissipative elements to the circuit. These elements are resistances \(R_J\) and \(R_r\), which have the purpose of modeling the inevitable energy dissipation in the circuit. Resistances are connected in parallel with the corresponding capacitances. These leakage terms will be added to equations (44) and (46), which gives us

\[
\dot{q}_r = -\frac{\phi_r}{L_r} - \frac{\dot{\phi}_r}{R_r}
\]  
(49)

and

\[
\dot{q}_J = -\frac{2\pi E_J}{\Phi_0} \sin \left(\frac{2\pi}{\Phi_0} \phi_J\right) - \frac{\dot{\phi}_J}{R_J}
\]  
(50)

\[\approx -\frac{\phi_J}{L_J} - \frac{\dot{\phi}_J}{R_J},\]

where the approximation is reasonable when \(\phi_J\) is small with respect to \(\phi_0/2\pi\).
4.1.3 Linearized Josephson inductance

Approximation in (50) has been done by Taylor expanding the potential energy term of the Josephson junction (25) up to second order, and then we can solve the linear part of the Josephson inductance $L_J$ (see Ref. [2]), by comparing the second order term to expression $\phi^2/2L_J$. This gives us an approximation

$$L_J = \frac{\Phi_0}{2\pi I_c},$$

that does not depend on the flux $\phi_J$.

4.1.4 Relation with currents

Similarly we presented equations (47) and (48), we can use the relation for linear inductance $L = \Phi/I$ and Ohm's law, to express equations (49) and (50) by using currents:

$$\dot{q}_r = -I_r - I_{r\text{leak}},$$

and

$$\dot{q}_J = -I_c \sin\left(\frac{2\pi \Phi_0 \phi_J}{\Phi_0}\right) - I_{J\text{leak}}$$

$$\approx -I_J - I_{J\text{leak}}.$$

These equations tell us about the different currents located in the nodes.

4.2 Control and read-out signals

Until now, the only external source of control we have been discussing is the DC gate voltage $V_g$, which enables the tuning of the qubit, into the "sweet spot". This possibility being irrelevant in the transmon regime, we can use the same path of influence to couple AC signals with the system. These signals can be used to control and measure the state of the qubit.

4.2.1 Transmission line model

This section mainly follows the transmission line model introduced in reference [15].

External electromagnetic fields can be expressed as an infinite line of harmonic oscillators. In our case, we want use this information to develop a model that describes the AC signals, in way that fits into our Hamiltonian description of transmon qubit. This is achieved by coupling an infinite right-handed transmission line to rest of the circuit, through the capacitance $C_{in}$. Discrete right-handed transmission line is modeled by using capacitances and inductances, in a such
Figure 6: Infinite transmission line coupled to some system which is described by impedance $Z(\omega)$.

way (see figure 6), that it is described by a Lagrangian of a form

$$L_{tl} = \sum_{i=1}^{\infty} \Delta x \left[ c \phi_i^2 \left( \frac{\phi_{i+1} - \phi_i}{2} \right)^2 - \frac{1}{2l} \left( \frac{\partial \phi_i}{\partial x} \right)^2 \right], \quad (54)$$

where $c$ and $l$ are the capacitance and inductance per unit length, and $\Delta x$ is unit cell length. This transmission line can be made continuous by taking $\Delta x \to 0$, which gives us

$$L_{tl} = \int_{0}^{\infty} dx \left[ c \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2l} \left( \frac{\partial \phi}{\partial x} \right)^2 \right]. \quad (55)$$

Then we can use the Lagrangian density (square brackets in (55)) and the Lagrange’s equation [1], to get equation

$$\frac{\partial^2 \phi}{\partial t^2} - \nu^2 \frac{\partial^2 \phi}{\partial x^2} = 0. \quad (56)$$

This well known one dimensional wave equation, where $\nu = 1/\sqrt{cl}$ is the propagation velocity of the flux.

The transmission line has a uniform impedance $Z_0$, so the flux can propagate without any dispersion. Therefore the solution for the wave equation can be presented into a form

$$\phi(x,t) = \phi_{\text{in}} \left( \frac{x}{v} + t \right) + \phi_{\text{out}} \left( -\frac{x}{v} + t \right), \quad (57)$$

where $\phi_{\text{in}}$ is propagating towards the connecting impedance $Z$, and $\phi_{\text{out}}$ is propagating away from it [16].

Let’s assume that the AC signal, which is used to measure the qubit, has a wavelength that is much larger compared to the size of the circuit, neglecting the recently added transmission line. This allows us to approximate the system into a lumped element, that has an impedance $Z$ (see figure 6). In general, the
transmission line impedance $Z_0$ and the impedance $Z$ of the transmon and the resonator are not equal. This implicates, that when the propagating signal tries to cross the boundary between these impedances, part of the wave will bounce back and propagate towards the other end of the transmission line. Because the transmission line stretches infinitely to other direction, the wave that has bounced once will not reflect back. Now it is reasonable to define, that $\phi_{in}$ is the initial signal sent to the line, and $\phi_{out}$ is the reflected signal. We can solve voltage by taking the time derivative from equation (57), which gives us

$$\dot{\phi}(t) = V(t) = V_{in}(t) + V_{out}(t).$$  

(58)

The reflection coefficient is defined as the ratio of the reflected and the initial voltage

$$\Gamma(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)},$$  

(59)

but it can be also expressed by using impedances

$$\frac{V_{in} + V_{out}}{Z} = \frac{V_{in} - V_{out}}{Z_0} \Leftrightarrow \Gamma = \frac{Z - Z_0}{Z + Z_0},$$  

(60)

where the first equation comes from Kirchhoff’s current law. Latter relation for reflection coefficient assumes that the impedance $Z$ is linear, which does not hold for transmon because of the non-linear inductance of the Josephson junction. However, it is appropriate approximation when $V_{in}$ is small.

The reflection coefficient depends on the transmon and the resonator parameters which define impedance $Z$, along with the frequency of the input signal, and the transmission line parameters defining the impedance $Z_0$. This opens a possibilities to gain information about the transmon-resonator system under measurement by shifting the parameters while measuring the system. The information is usually gained from the reflected voltage $V_{out}$.

4.2.2 Equations of motion for coupled transmission line

Transmission line introduction, presented above, did cover the coupling into the general impedance, instead of only accounting transmon. Achieving more precise model for transmon measurement and control, we need to implement the transmission line into the transmon Hamiltonian. The methods used to solve Hamiltonian of the LC oscillator (see section 3.1) can be easily used to transform transmission line Lagrangian (54) into a Hamiltonian

$$H_{tl} = \sum_{i=1}^{\infty} \frac{1}{2\Delta x} \left( \frac{q_{i+1}^2}{c} + \frac{(\phi_{i+1} - \phi_i)^2}{l} \right).$$  

(61)

Merging this part into full transmon Hamiltonian (39) does not cause much trouble, because we can replace the DC gate voltage $V_g$ with effective drive
voltage $V_d(t)$, which can be defined as $V_d(t) = q_1/C_{in}$. This definition is accurate if one assumes $C_r$ to be much larger compared to other capacitances and transmission line capacitance $c\Delta x$ to be much smaller than $C_{in}$. It is important to realize that the constant terms in Hamiltonian (39), are not constants after changing $V_g \rightarrow V_d(t)$, and should not be neglected. The new Hamiltonian takes a form

$$H \approx \frac{q_j^2}{2C_{\Sigma}} - E_J \cos \left( \frac{2\pi}{q_0} \phi_j \right) + \frac{q_r^2}{2C_r} + \phi_J^2 + C_g q_1 q_J \frac{q_1 + q_J = q_1}{C_{\Sigma} C_r} + q_r \frac{q_1 + q_J = q_1}{C_r} + \frac{q_j^2}{2C_{in}} \left( 1 + \frac{C_{in}}{C_r} \right) + \sum_{i=1}^{\infty} \frac{1}{2\Delta x} \left[ \frac{q_{i+1}^2}{c} + \frac{\left( \phi_{i+1} - \phi_i \right)^2}{l} \right].$$

(62)

We are particularly interested in the dynamics of the transmission line, therefore the only equations of motion we need to solve are for the node $\phi_1$. Hamilton’s equations (41) and (42) gives equations

$$\dot{\phi}_1 = \frac{q_1}{C_{in}} \left( 1 + \frac{C_{in}}{C_r} \right) + \frac{q_r}{C_r},$$

(63)

and

$$\dot{q}_1 = \frac{\phi_2 - \phi_1}{l\Delta x} \rightarrow \frac{1}{l} \frac{\partial \phi_1}{\partial x},$$

(64)

where we have taken limit $\Delta x \rightarrow 0$, at the boundary of transmission line.

These equations will prove to be useful in analyses of the voltages, at the boundary line. By using equation (57), we can write

$$v \frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_{in}}{\partial t} - \frac{\partial \phi_{out}}{\partial t}.$$  \[65\]

The facts that propagation velocity can be written as $v = \partial x/\partial t$ and $V_{in/out} = \partial \phi_{in/out}/\partial t$, combined with equations (63) and (65), gives us a relation that describes how the input voltage, the transmission line and the transmon interact between each other. This relation is

$$V_{out}(t) = -V_{in}(t) + \frac{q_1}{C_{in}} \left( 1 + \frac{C_{in}}{C_r} \right) + \frac{q_r}{C_r} + V_d(t) (1 + \alpha) + \frac{q_r}{C_r},$$

(66)

which gives the reflected voltage directly. This is an important equation when one is examining the transmon system by measuring reflected voltage.
The second equation of motion \((64)\), can be interpreted as a current at the boundary. Again, the equation \((57)\) is used, which gives us the relation

\[
\dot{q}_1 = \sqrt{\frac{l}{c}} (V_{in}(t) - V_{out}(t)) = \frac{V_{in}(t) - V_{out}(t)}{Z_0},
\]

(67)

where the transmission line impedance is \(Z_0 = \sqrt{l/c}\). This relation refers to the current at the boundary of the transmission line, and it is exactly what one would expect, just like we have figured out in equation \((60)\). This equation can be presented in a form

\[
\dot{V}_d(t) = \frac{V_{in}(t) - V_{out}(t)}{Z_0 C_{in}},
\]

(68)

which tells us the driving voltages rate of change with respect to time.

### 4.3 Quantum notation

We have already presented the equations of motion in section 4.1 but we can still develop our notation further. By defining resonance angular frequencies \(\omega_r = 1/\sqrt{L_rC_r}\) and \(\omega_J = 1/\sqrt{L_JC_Σ}\), and also relaxation frequencies \(\kappa_r = 1/(R_rC_r)\) and \(\kappa_J = 1/(R_JC_Σ)\), we can get much more intuition for the quantum case. The resonance angular frequencies \(\omega_J\) and \(\omega_r\) denote the energy level separation in the qubit and the resonator. On the other hand, relaxation frequencies \(\kappa_r\) and \(\kappa_J\) can be interpreted as inverse of the relaxation time of resonator and qubit (relaxation time of qubit \(T_J = \kappa_J^{-1}\)). Relaxation time tells us how long it takes for the exited system to return into equilibrium state. The transmission line equations \((66)\) and \((68)\), derived in previous section, are also required. This is because we need to solve the effective drive voltage \(V_d\) for realistic input voltage \(V_{in}\). For equation we define relaxation frequency \(\gamma = 1/(Z_0 C_{in})\). The equations of motion, in this "quantum" notation, takes the form

\[
\dot{\phi}_r = V_r + \beta_r V_J + \alpha V_d(t),
\]

(69)

\[
\dot{V}_r = -\omega_r^2 \phi_r - \kappa_r \dot{\phi}_r,
\]

(70)

\[
\dot{\phi}_J = V_J + \beta_J V_r,
\]

(71)

\[
\dot{V}_J = -\frac{\Phi_0}{2\pi} \omega_J^2 \sin \left(\frac{2\pi}{\Phi_0} \phi_J\right) - \kappa_J \dot{\phi}_J,
\]

(72)

\[
\dot{V}_d(t) = \gamma (2V_{in}(t) - V_d(t)(1 + \alpha) - V_r).
\]

(73)

Transition frequencies and relaxation times of the resonator and qubit are crucial parameters that define much of the physics concerning the quantum mechanics of the transmon. Now we have constructed classical analog for the quantum mechanical case, which can be used to separate purely classical effects from the quantum ones.
5 Numerical solutions for equations of motion

Table 1: Realistic parameters for the transmon.

<table>
<thead>
<tr>
<th>$\omega_J/2\pi$</th>
<th>$\omega_r/2\pi$</th>
<th>$\kappa_J$</th>
<th>$\kappa_r$</th>
<th>$\gamma$</th>
<th>$\beta_J$</th>
<th>$\beta_r$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 GHz</td>
<td>9 GHz</td>
<td>1/(60 µs)</td>
<td>1/(6 µs)</td>
<td>1/(12 µs)</td>
<td>1/15</td>
<td>1/10</td>
<td>1/50</td>
</tr>
</tbody>
</table>

Finally, we have reached the point where some concrete calculations can be made. The parameters required for the equations of motion are adopted from a typical realistic transmon design [7], and the values for parameters are given in the table 1.

5.1 Design parameters of the transmon

![Graph showing voltage ratio $V_J/V_{in}$ as a function of input signal frequency $f = \omega_{in}/2\pi$. Calculations are done by using input voltage amplitude $V_{in} = 10^{-3}$ V and the parameters shown in table 1. Peaks appear at the eigenfrequencies.](image)

The parameters in the table 1 are fully determined by the design of the circuit, and for that reason, in theory, their values should stay unchanged. These parameters can be divided into three categories: Natural frequencies, coupling and damping. Parameters $\omega_J$ and $\omega_r$ are the natural angular frequencies of
the qubit and the resonator, and parameters $\alpha$, $\beta_J$ and $\beta_r$ determine the coupling strength between individual systems. Together these values compose the eigenfrequencies and normal modes of the coupled resonator and qubit. In a linear case, eigenfrequencies can be solved analytically by using the method of small oscillations [1]. Transmon does not behave linearly, in general, so eigenfrequencies need to be determined numerically. In figure 7, the voltage ratio $V_J/V_{in}$ is shown as a function of input signal frequency $\omega_{in}$. If the input signal frequency is the same as some eigenfrequency of the system, then the amplitude will be at local maximum. This leads into the conclusion, that locations of the spikes in figure 7 correspond to the eigenfrequencies of the system. The parameters $\kappa_J$, $\kappa_r$ and $\gamma$ are the damping parameters, which describe energy dissipation in the system. Damping parameters are affecting the height and steepness of peaks in figure 7. For larger damping parameter values, the peaks will be lower and wider [17].

5.2 Resonance shift at nonlinear regime

![Graph](image)

Figure 8: Voltage ratio $V_J/V_{in}$ as a function of input signal frequency $\omega_{in}$. Calculations are done by using input voltage amplitude $V_{in} = 5$ V and the parameters shown in table 1. Peaks appear at the eigenfrequencies.

The parameters in table 1 are describing almost every meaningful aspect that
has an effect to the behavior of the transmon, and in linear circuits that would be all, but the sinusoidal potential of transmon makes system more complicated, because now eigenfrequencies depend on amplitude of the oscillations.

Figure 8 shows the same relation between $V_{J}/V_{in}$ and $\omega_{in}$ as presented in figure 7 where the input signal voltage has the same form $V_{in}(t) = V_{in} \cos(\omega_{in} t)$, but the difference is in the amplitude, which is $V_{in} = 5$ V. Compared to earlier voltage amplitude $10^{-3}$ V, this larger voltage takes the transmon easily into the nonlinear zone. The nonlinearity is manifesting itself by distorting the frequency peaks. The phenomena is sometimes called as foldover effect [3], which refers to hysteresis effect that appears at the frequencies where amplitude drops suddenly (hysteresis effect also implies sudden amplitude jump when approaching the discontinuity from another direction, although the exact frequency will be different. Amplitude jump is not observed in figure 8).

![Figure 9: Voltage ratio $V_{J}/V_{in}$ for different input voltage frequencies and amplitudes. Colorbar values are given in log10-scale.](image)

Same foldover effect is also observed in figure 9, which includes only the lower eigenfrequency. In this case, the calculations are done over 2-dimensional parameter space, where the added dimension is input voltage amplitude. Voltage ratio $V_{J}/V_{in}$ is represented as the color of the plot. The figure shows clearly that
Figure 10: Voltage $V_J$ as a function of time for different input voltage amplitudes. The frequency of the used input voltage is $\omega_{in}/2\pi = 8$ GHz.

Figure 11: Comparison between a sinusoidal and a quadratic potential
the strongest frequency response is received in the linear regime \( V_{in} \ll 10^{-1} \text{ V} \). Nonlinear effects are appearing at voltage \( V_{in} \gg 10^{-1} \text{ V} \). This affects the eigenfrequency so that it starts to shift towards lower frequencies and also maximum amplitude decreases. Also the discontinuities in the voltage ratio (consequent to hysteresis effect) start to appear along the other nonlinear effects. This effect is explicitly shown by the sharp contrast in color.

Yet another way to examine nonlinear effects in transmon qubit is provided in figure 10. It shows how voltage \( V_J \) in transmon qubit changes as a function of time for different input voltage amplitudes. Clearly the nonlinearity makes the oscillations much harder to predict and interpret.

One can rationalize the observed nonlinear effects by comparing the potential energies of linear inductor and Josephson junction. Figure 11 shows how the potentials differ from each other when oscillations grow larger. The steeper slope of the quadratic potential leads to larger oscillation frequency compared to sinusoidal potential. Actually, the sinusoidal potential of the Josephson junction is exactly analogous with mechanical rotor \([7]\) (or pendulum). Because of transmon having a sinusoidal potential, it is clearly possible for flux variable to escape from the initial potential well. This happens if the driving voltage is increased over certain threshold (it appears around \( V_{in} > 10^3 \text{ V} \)). When the threshold is surpassed the behavior of the transmon changes radically. The system starts to jump between potential wells in a chaotic way (see lower right corner in figure 10). This behavior is to be avoided, and in general it shouldn’t be a problem because only small nonlinearity \([7]\) is needed for operation of transmon qubit.

6 Conclusions

We have now discussed about classical circuit analysis, Lagrangian and Hamiltonian formalism in electric circuits, classical equations of motion of transmon and transmission line, and finally we calculated some concrete results by using the model for transmon qubit we derived. The central results we acquired are the transmon-resonator Hamiltonian \((40)\), the equations of motion for the coupled transmon qubit, resonator and transmission line (see eqs. \((69)\) to \((73)\)), and numerically calculated figures 7, 8, 9 and 10 demonstrating linear and nonlinear regimes of transmon.

We found out that for large driving forces the dynamical behavior of the system is significantly changed, but due to optimal balance between charge dispersion and anharmonicity, called as transmon regime, the overall nonlinearity will be small. Nevertheless, this small nonlinearity is crucial for transmon (and for superconducting qubits in general). When the Hamiltonian \((40)\) is quantized \([12]\) (or more general treatment in \([18]\)), it is precisely this nonlinearity that induces the uneven energy levels, which makes transmon an effective two-state system.

There is much more to be considered when studying transmons in quantum
description, which are beyond the scope of this thesis. In references, there are some good texts which cover this subject. Particularly, the work [7] is pioneering text about transmon qubit, how it differs from other superconducting qubits and how transmon is affected by known sources of decoherence. Also the Ref. [6,8] might be helpful sources.
References


