ORBIT-AVERAGED PERTURBATION EQUATIONS OF CELESTIAL MECHANICS WITH APPLICATIONS TO SATURN’S E RING PARTICLES

A Bachelor’s Thesis for the Physics Degree Program

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Contents

1 Introduction 4
  1.1 Conic Sections and Unperturbed Keplerian Orbits ............................. 4
  1.2 Orbital Elements .............................................................................. 6
  1.3 Orbital Elements Expressed in Terms of the Orbital Energy $E$ and Angular
      Momentum $H$ .................................................................................. 7

2 Derivation of the Perturbation Equations 9
  2.1 Introduction ....................................................................................... 9
  2.2 Perturbed Orbits ............................................................................... 9
      2.2.1 Perturbation Equation for the Semi-major Axis, $a$ ..................... 10
      2.2.2 Perturbation Equation for the Eccentricity, $e$ ............................ 11
      2.2.3 Perturbation Equation for the Inclination, $i$ ............................... 11
      2.2.4 Perturbation Equation for the Longitude of Ascending Node, $\Omega$ .... 12
      2.2.5 Perturbation Equation for the Argument of Pericenter, $\omega$ .......... 13
      2.2.6 Perturbation Equation for the Time of the Pericenter Passage, $\tau$ ... 13
  2.3 Orbit-averaged Perturbation Equations ............................................. 15
      2.3.1 Time-average of a Perturbation Equation Over One Orbit .......... 15
      2.3.2 Benefits of Orbit-averaging ....................................................... 16

3 Aspects of the Dynamics of Particles in Saturn’s E Ring 17
  3.1 Introduction to the E Ring Particles ................................................... 17
  3.2 Orbit-averaged Evolution Equations for $a$, $e$, and $\omega$ ................. 19
      3.2.1 Orbit-averaged Time-derivative of the Semi-major Axis, $a$ .......... 20
      3.2.2 Orbit-averaged Time-derivative of the Eccentricity, $e$ ............... 21
      3.2.3 Orbit-averaged Time-derivative of the Longitude of Pericenter, $\omega$ .. 21
  3.3 Numerical Modelling ........................................................................ 22
  3.4 Correction to Horanyi et al. (1992) .................................................. 23
  3.5 Comparing Theory to Observations of the E Ring ......................... 24
4 Conclusion

References

Appendix A: The Code
Abstract

Following Burns 1976[1], we study the effect of a variety of perturbing forces on a set of orbital elements—semi-major axis $a$, eccentricity $e$, inclination $i$, the longitude of pericenter $\omega$, the longitude of the ascending node $\Omega$, and the time of pericenter passage $\tau$. Using elementary dynamics, we can derive the time rates of change of these quantities to produce the perturbation equations of celestial mechanics, which are written in terms of the perturbing forces.

If the perturbing forces on a dust particle are small in comparison to a planet’s gravitational attraction, the change in (the first five) orbital elements is slow and on timescales much longer than the dust particle’s orbital period. We can average the effects of perturbations over a single Keplerian orbit (assumed constant). This “orbit-averaging” has both analytical and numerical advantages over non-averaged perturbation equations, which can be seen for example in processing times of computerised orbital models.

We can sum the individual perturbation equations of perturbing forces to account for the cumulative effect of all perturbations on an orbital element[2]:

$$\left\langle \frac{d\Psi}{dt} \right\rangle_{\text{total}} = \sum_j \left\langle \frac{d\Psi}{dt} \right\rangle_j$$

where $\Psi$ is any one of the six osculating orbital elements. These orbit-averaged equations are on the order of hundreds of times faster to numerically integrate than the Newtonian equations.

To demonstrate the orbit-averaged equations, we can use the orbit-averaged perturbation equations to model paths of dust particles in Saturn’s E Ring. [3] Saturn’s moon Enceladus’ orbit is approximately at the same distance from Saturn as the E Ring, and it has been suggested[4] that the E Ring—made of icy dust—originates from cryovolcanic activity on Enceladus’ south pole[5].

Following Horanyi et al. 1992[3], we will explore the effects of higher order gravity, radiation pressure, and electromagnetic forces as perturbing forces in the Saturnian system to show the individual effects of perturbing forces on Enceladus-originated ice dust, as well as the cumulative effect of these perturbing forces on orbital equations of the dust.
1 Introduction

1.1 Conic Sections and Unperturbed Keplerian Orbits

In 1687, Isaac Newton defined the relative motion of two spherical bodies under mutual gravitational attraction as conic sections. For bound orbits the shape of the orbit is an ellipse, and for unbound trajectories the trajectories are described by parabolae and hyperbolae, as seen in figure 1.

To define an orbit, we can derive the equations of the two-body problem from Newton’s universal law of gravity:[6]

$$F_{Mm} = -\frac{GMm}{r^2} \hat{r}, \quad (1.1.1)$$

where $F_{Mm}$ is the gravitational force exerted on a particle $m$ by a mass $M$ in its gravitational field, $\mathbf{r}$ is the position vector from $M$ to $m$, $r$ is the magnitude of $\mathbf{r}$, $\hat{r} = \mathbf{r}/r$ is the unit vector, $G$ is the universal gravitational constant, and the dot marks differentiation with respect to time. An equivalent equation is written for the motion of $M$ in the field of $m$. The equation for their
relative motion reads:

\[ \mathbf{F} = \ddot{r} = \frac{\mu}{r^3}, \quad (1.1.2) \]

where \( \mu = G (M + m) \).

For the centre of mass of the system, we have:

\[ \mathbf{F}_\Sigma = \ddot{r}_\Sigma = \mathbf{F}_{Mm} + \mathbf{F}_{mM} = \mathbf{F}_{Mm} - \mathbf{F}_{Mm} = 0. \quad (1.1.3) \]

Thus the centre of mass of the two-body system is either at rest or it moves at constant velocity. Hence, the equation of motion (1.1.2) is not affected by the movement of the two-body system as a whole.

The radial nature of (1.1.2) leads to the cross product \( \mathbf{r} \times \dot{\mathbf{r}} \equiv 0 \), which in turn implies that the angular momentum \( \mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}} \) is a constant vector.\[1\] This implies that the orbital motion takes place in a plane and that the modulus of the angular momentum vector is also conserved:

\[ H = r^2 \dot{\theta}, \quad (1.1.4) \]

where \( \theta \) is a positional angle measured in the orbital plane from a fixed line in the plane.

As the rotation of \( \mathbf{F} \) is \( \nabla \times \mathbf{F} = 0 \), the total energy per unit mass \( E \) can be derived from (1.1.2):

\[ E = \frac{\dot{r} \cdot \ddot{r}}{2} - \frac{\mu}{r}. \quad (1.1.5) \]

We can write the dependence \( r = r(\theta) \) as a solution of a conic section as:

\[ r = \frac{p}{1 + e \cos (\theta - \omega)} = \frac{a (1 - e^2)}{1 + e \cos f}, \quad (1.1.6) \]

where the eccentricity \( e \) and the argument of pericenter \( \omega \) are constants defined by initial conditions, the argument of the cosine is the true anomaly \( f \equiv \theta - \omega \), and the parameter \( p \) is:

\[ p = \frac{H^2}{\mu} \equiv a (1 - e^2), \quad (1.1.7) \]
the right-hand side of which defines the semi-major axis $a$.

1.2 Orbital Elements

We can now describe elliptic, parabolic, and hyperbolic conic sections in the orbital plane. If we further define the longitude of ascending node $\Omega$ as an angle from the point at which the orbit passes an arbitrary reference plane on its upward path to the line of nodes (see figure 2), we define the inclination $i$ as the angle between this reference plane and the orbital plane. Choosing a reference time — the time of pericenter passage $\tau$ — we now have a set of six orbital elements:

1. the semi-major axis $a$,
2. the eccentricity $e$,
3. the inclination $i$,
4. the argument of pericenter $\omega$,
5. the longitude of ascending node $\Omega$, and
6. the time of pericenter passage $\tau$.

Figure 2: A diagram of the orbital elements, with the line of nodes denoting the intersection between the orbital and reference planes.[7]
Using these six orbital elements, we can define an elliptic orbit with respect to our reference plane. We also define the longitude of pericenter, \( \varpi = \omega + \Omega \), for later use. The instantaneous position of the mass \( m \) is specified by the angle \( \theta \), or equivalently, by the true anomaly \( f \).

### 1.3 Orbital Elements Expressed in Terms of the Orbital Energy \( E \) and Angular Momentum \( H \)

Taking the time-derivative of (1.1.6) and replacing the time-derivative of the true anomaly \( f \) with the help of (1.1.4) gives us the radial velocity of particle \( m \):

\[
\dot{r} = \frac{H}{p} \sin f = \frac{H}{a (1 - e^2)} e \sin f.
\]

The transverse velocity is:

\[
r \dot{\theta} = \frac{H}{p} \left(1 + e \cos f\right) = \frac{H}{a (1 - e^2)} \left(1 + e \cos f\right).
\]

Another helpful redefined equation is the equation of the orbital energy \( E \) as a function of \( a \), which we get by substituting (1.1.6), (1.1.7), (1.3.1), and (1.3.2) into (1.1.5):

\[
E = -\frac{\mu}{2a}.
\]

The above redefined equations for \( H \) and \( E \) let us define the orbital eccentricity \( e \) in terms of energy per unit mass and orbital angular momentum per unit mass; from (1.1.7) and (1.3.3):

\[
e = \sqrt{1 + \frac{2H^2E}{\mu^2}}.
\]

We can define the inclination \( i \) and the longitude of the ascending node \( \Omega \) in terms of the
components and magnitude of $H$:

\[
\cos i = \frac{H_z}{H}, \quad \text{and} \quad (1.3.5)
\]

\[
\tan \Omega = -\frac{H_x}{H_y}, \quad (1.3.6)
\]
2 Derivation of the Perturbation Equations

2.1 Introduction

A fundamental problem of celestial mechanics is to find out how unperturbed orbits are reshaped under perturbing forces, such as higher order central gravity, radiation force (including solar “light pressure”), electromagnetic forces, as well as drag forces exerted by gas and plasma. Perturbing forces are also exerted by other bodies orbiting the primary body.

The ultimate purpose of defining the perturbing forces mathematically is to find out how the orbit of the particle $m$ evolves over time under perturbations. This is an important difference to the unperturbed case discussed in chapter [1] because unperturbed Keplerian orbits do not intrinsically hold true for real two-body systems.

In this chapter, we first introduce a small perturbing force $dF$, after which we derive perturbation equations for the six-element set of orbital elements $(a, e, i, \omega, \Omega, \tau)$. Finally, we discuss time-averaging perturbation equations over one orbit, so called “orbit-averaged perturbation equations”.

2.2 Perturbed Orbits

Following Burns (1976)[1], we define an orbit, that experiences a small perturbing force $dF$, in addition to the dominant gravitational force of the spherical primary mass $M$:

$$F \gg dF = R + T + N = R\dot{e}_R + T\dot{e}_T + N\dot{e}_N,$$  \hspace{1cm} (2.2.1)

where $R$ is the component of the perturbing force radially outwards along $r$, $T$ is the component perpendicular to $r$ and lying in the orbital plane, and $N$ is perpendicular to both $R$ and $T$, normal to the orbital plane, as seen in figure 3 on page 10.

As mentioned before, an unperturbed Keplerian orbit would take the shape of a conic section, whereas an orbit perturbed by $dF$ can best be described with osculating orbits. An osculating orbit is the elliptical orbit at time $t$ which the particle $m$ would take if $dF$ would vanish instantaneously and $F$ would remain the only acting force on the particle.

The osculating orbit touches the real orbit at time $t$, so by specifying osculating elements
(the orbital elements of the osculating orbit), we can define the position and velocity of the particle at any given time. Our goal then is to find the time rates of change for the orbital element set \((a, e, i, \omega, \Omega)\) caused by \(dF\).

### 2.2.1 Perturbation Equation for the Semi-major Axis, \(a\)

Differentiating \((1.3.3)\), we get our first perturbation equation:

\[
\dot{a} = \frac{2a^2}{\mu} \dot{E}, \tag{2.2.2}
\]

We note that perturbing forces that cause energy dissipation, \(\dot{E} < 0\), cause shrinking in orbits, \(\dot{a} < 0\). To convert \((2.2.2)\) to a form where the perturbing elements of \((2.2.1)\) appear, we note that \(\dot{E}\) is the work done by the perturbing forces per unit mass on the body per unit time, \(\dot{E} \equiv \dot{r} \cdot dF\), and write down an equation for \(\dot{E}\) using \((2.2.1)\):

\[
\dot{E} \equiv \dot{r} \cdot dF = \dot{r} R + r \dot{\theta} T. \tag{2.2.3}
\]

where radial velocity \(\dot{r} \parallel R\) and transverse velocity \(r \dot{\theta} \parallel T\). The component \(N\) does not appear
because the change in $a$ takes place in the two-dimensional orbit plane and only forces in the orbit plane can change the orbit size.

Substituting (1.3.1) and (1.3.2) into (2.2.3), and further substituting the result into (2.2.2), we are left with an equation for $\dot{a}$ with perturbing forces lying only in the orbit plane\(^1\):

$$\dot{a} = \sqrt{\frac{2a^3}{\mu(1-e^2)}} \left[ R(e \sin f) + T(1 + e \cos f) \right]. \quad (2.2.4)$$

2.2.2 Perturbation Equation for the Eccentricity, $e$

Taking the derivative of (1.3.4), we get:

$$\dot{\varepsilon} = \frac{1}{2e} (e^2 - 1) \left[ 2 \frac{\dot{H}}{H} + \frac{\dot{E}}{E} \right]. \quad (2.2.5)$$

$\dot{H}$ and $\dot{E}$ can be either positive or negative, so the terms possibly rival each other in defining the shape of the orbit. The change in angular momentum equals the applied torque, $\dot{H} \equiv r \times dF$, which requires that the magnitude of $H$ changes according to\(^1\):

$$\dot{H} \equiv rT. \quad (2.2.6)$$

As with the semi-major axis $a$, we can rewrite (2.2.5) by substituting (1.1.7), (1.3.3), (2.2.6), and (2.2.3):

$$\dot{\varepsilon} = \sqrt{\frac{a(1-e^2)}{\mu}} \left[ T \left( \frac{e + \cos f}{1 + e \cos f} + \cos f \right) + R \sin f \right]. \quad (2.2.7)$$

2.2.3 Perturbation Equation for the Inclination, $i$

Taking the derivative of (1.3.5), we get\(^2\)

$$\frac{d\dot{i}}{dt} = \frac{H \dot{H} - H \dot{R}}{\sqrt{\left( \frac{H}{R} \right)^2 - 1}} \quad (2.2.8)$$

\(^1\)In Burns (1976), there is a typo in (2.2.4); the prefactor of $T$ is written down as $(1 + e \sin f)$, whereas it should be $(1 + e \cos f)$.\(^8\)\(^9\)

\(^2\)For clarity, we use Leibniz’s notation for the time-derivative of inclination $i$ instead of Newton’s notation.
For the components of the angular momentum $\mathbf{H}$, we find from elementary geometry (Figure 3) that:

\begin{align}
\dot{H}_x &= r \left( T \sin i \sin \Omega + N \left[ \sin \theta \cos \Omega + \cos \theta \cos i \sin \Omega \right] \right) \\
\dot{H}_y &= r \left( -T \sin i \cos \Omega + N \left[ \sin \theta \sin \Omega - \cos \theta \cos i \cos \Omega \right] \right) \\
\dot{H}_z &= r T \cos i - r N \cos \theta \sin i
\end{align}

Using (1.1.4), (1.3.5), (2.2.6), and (2.2.11), we can further rewrite (2.2.8) as:

\[
\frac{di}{dt} = r N \cos \theta \frac{H_z}{H} = \sqrt{\frac{a (1 - e^2)}{\mu} \frac{N \cos \theta}{1 + e \cos f}},
\]

(2.2.12)

We can deduce from the absence of $T$ and $R$, that perturbing forces in the orbital plane do not change the angle between the orbital plane and the reference plane.

### 2.2.4 Perturbation Equation for the Longitude of Ascending Node, $\Omega$

Taking the derivative of (1.3.6), we get:

\[
\dot{\Omega} = \frac{H_x \dot{H}_y - \dot{H}_x H_y}{H^2 - H_z^2}.
\]

(2.2.13)

Substituting (1.3.5) and (1.3.6) into the derivative, we get:

\[
\dot{\Omega} = \frac{\sin \Omega \dot{H}_y + \cos \Omega \dot{H}_x}{H \sin i}.
\]

(2.2.14)

Using (2.2.9) and (2.2.10), which we derived from Figure 3 and (1.1.7), we can rewrite (2.2.14) as:

\[
\dot{\Omega} = \sqrt{\frac{a (1 - e^2)}{\mu} \frac{N \sin \theta}{\sin i (1 + e \cos f)}} \equiv \frac{r N \sin \theta}{H \sin i},
\]

(2.2.15)

where in the rightmost form of $\dot{\Omega}$ we see in the denominator the angular momentum component in the $(x, y)$-plane, and the numerator representing the torque on the orbit.
2.2.5 Perturbation Equation for the Argument of Pericenter, $\omega$

Because the argument of pericenter $\omega$ and the time of the pericenter passage $\tau$ are not explicit functions of $E$ and $H$, to get the time-derivatives of $\omega$ and $\tau$ we must reconsider (1.1.6).

In a situation where $d\mathbf{F}$ is only applied for an instant, the angular momentum $H$, the orbital energy $E$, and the argument of pericenter $\omega$ change, but $r$ does not change as the particle $m$ instantaneously stays in the same position. Treating $r$ as a constant, rewriting (1.1.6) by substituting (1.1.7) and (1.3.4) yields:

$$H^2 = \mu r \left( 1 + \sqrt{1 + 2H^2E\mu^2 \cos f} \right), \quad (2.2.16)$$

where we recall that $f = \theta - \omega$, which we can differentiate to get $\dot{\omega}$:

$$\dot{\omega} = \dot{\theta} + \frac{2HH'}{e\mu \sin (\theta - \omega)} \left( \frac{1}{r} - \frac{E}{e\mu} \cos (\theta - \omega) \right) - \frac{H^2E}{e^2\mu^2} \cot (\theta - \omega). \quad (2.2.17)$$

Substituting (2.2.3) and (2.2.6) in (2.2.17), we get the final form of $\dot{\omega}$:

$$\dot{\omega} = \frac{\sqrt{a(1-e^2)}}{e\sqrt{\mu}} \left[ -R \cos f + T \sin f \left( \frac{2 + e \cos f}{1 + e \cos f} \right) \right] - \Omega \cos i \quad (2.2.18)$$

2.2.6 Perturbation Equation for the Time of the Pericenter Passage, $\tau$

To define the time-derivative of the time pericenter passage, $\tau$, we need to compare it to an orbital equation that explicitly contains time $t$. One equation that fills this criterion is Kepler’s third law:

$$\frac{P}{2\pi} = \frac{1}{n} = \sqrt{\frac{a^3}{\mu}}, \quad (2.2.19)$$

where $P$ is one orbital period and $n$ is the mean motion.

\[^3\text{There is a typo in Burns (1976): the prefactor of } T \sin f \text{ is written as } (2 + e \cos f), \text{ whereas it should be } \left( \frac{2 + e \cos f}{1 + e \cos f} \right).\]
Integrating (1.1.4) from the time of the pericenter passage $\tau$ to a general time $t$, we get:

$$H \int_{\tau}^{t} \, dt = \int_{0}^{f} r^2 \, df.$$  \hspace{1cm} (2.2.20)

An equivalent solution of a conic section to (1.1.6) is [8]:

$$r = a \left(1 - e \cos \epsilon\right), \hspace{1cm} (2.2.21)$$

where $\epsilon$ is the eccentric anomaly.

Comparing the two solutions of a conic section (1.1.6) and (2.2.21), we can see that the relationship between the true anomaly $f$ and the eccentric anomaly $\epsilon$ is:

$$\cos \epsilon = \frac{e + \cos f}{1 + e \cos f}, \hspace{1cm} (2.2.22)$$

which we can rewrite into:

$$df = \frac{\sqrt{1 - e^2}}{1 - e \cos \epsilon} \, d\epsilon. \hspace{1cm} (2.2.23)$$

Substituting (1.1.7), (2.2.21), and (2.2.22) into (2.2.20), we find Kepler’s equation:

$$n(t - \tau) = \epsilon - e \sin \epsilon. \hspace{1cm} (2.2.24)$$

If we define $\chi \equiv n\tau$, we can derive $\dot{\chi}$ from (2.2.24) by substituting $\dot{\epsilon}$ differentiated from (2.2.21):

$$\dot{\chi} = \left[ -\frac{3}{2} nt + \frac{(1 - e^2)^{\frac{3}{2}} (2e - \cos f - e \cos^2 f)}{2e^2 \sin f (1 + e \cos f)} \right] \frac{\dot{E}}{E} - \frac{(1 - e^2)^{\frac{3}{2}}}{e^2} \cot f \frac{\dot{H}}{H}. \hspace{1cm} (2.2.25)$$
Further solving for $\dot{r}$ from (2.2.25), we find that:

$$\dot{r} = \left[ 3 (\tau - t) \frac{\sqrt{\mu}}{\sqrt{1 - e^2}} e \sin f + \frac{a^2}{\mu} (1 - e^2) \left( \frac{2}{1 + e \cos f} - \frac{\cos f}{e} \right) \right] R$$

$$+ \left[ 3 (\tau - t) \frac{\sqrt{\mu}}{\sqrt{1 - e^2}} (1 + e \cos f) + \frac{a^2}{\mu} (1 - e^2) \frac{\sin f (2 + e \cos f)}{e (1 + e \cos f)} \right] T.$$  

(2.2.26)

### 2.3 Orbit-averaged Perturbation Equations

#### 2.3.1 Time-average of a Perturbation Equation Over One Orbit

The time-average of a function can be found by evaluating the integral:

$$\langle f (t) \rangle = \frac{1}{\Delta T} \int_{t}^{t+\Delta T} f (t') dt'.$$  

(2.3.1)

To orbit-average our perturbation equations, we solve for the time-average of a perturbation equation over one orbit[2]:

$$\langle \Psi (t) \rangle = \frac{1}{P} \int_{0}^{P} \Psi (t) dt,$$  

(2.3.2)

where $\Psi$ is any one perturbation equation, and the period $P$ and the mean motion $n$ are:

$$P = \frac{2\pi}{n},$$

$$n = \sqrt{\frac{\mu}{a^3}}.$$

We can also express (2.3.2) as an integral expressed in terms of a positional angle. Using (1.1.4), (1.1.6), (1.3.2), and (2.3.2), we get:

$$\langle \Psi (t) \rangle = \frac{(1 - e^2)^2}{2\pi} \int_{0}^{2\pi} \frac{\Psi (t)}{(1 + e \cos f)^2} df,$$  

(2.3.3)

where we integrate $f$ over $[0, 2\pi]$, a single full orbit in terms of the true anomaly $f$.

We apply this method of orbit-averaging and discuss an example case in chapter 3.
2.3.2 Benefits of Orbit-averaging

![Oscillating semi-major axis $a$ (expressed in central planet radii, $R_P$) plotted against time for (a) the full Newtonian and (b) orbit-averaged equations of motion. Source: Hamilton (1993)[2]

Orbit-averaging perturbation equations yields benefits in numerical integration of osculating orbital elements, typically resulting in several hundred times faster computations when compared to the full Newtonian equations.[2]

The downside of orbit-averaging is, to an extent, loss of accuracy, shown in figure 4 where the orbital evolution of a circumplanetary dust grain under perturbing forces, by the oblateness of the primary mass and radiation pressure, is plotted. As seen in the figure, short-term variations in the elements get lost in the averaging.

On long-term, the evolution of the orbital elements ideally results in the same evolution of the elements with reduced numerical load, but there also exist cases where the orbit-averaging method does not work. On page 9 we discussed $dF$ being a comparatively small perturbing force when compared to the higher-order gravity of the primary mass. In case of strong perturbations, these may accumulate and render the orbit-averaging method invalid.

Additionally, there are interactions that cannot be expressed as Gaussian perturbation equations, such as close gravitational encounter three-body interactions, for which the orbit-averaging method does not work, and perturbations complex enough, that they cannot be orbit-averaged at all. But when the perturbations can be orbit-averaged, the overall benefit in numerical modelling is notable.
3 \textbf{Aspects of the Dynamics of Particles in Saturn’s E Ring}

Next we demonstrate the methods of orbit-averaging shown in section 2.3 along with the perturbation equations derived in chapter 2. We then take the orbit-averaged time-evolutions of the perturbation equations and solve them numerically.

We study as an example the trajectories of dust particles under perturbing forces in Saturn’s dusty E Ring. Micron-sized dust particles forming the diffuse E Ring originate from cryovolcanic activity on the south pole of Saturn’s icy moon Enceladus (figure 5).

3.1 \textbf{Introduction to the E Ring Particles}

Observations of Enceladus made by the Cassini spacecraft imply that a global subsurface ocean exists underneath Enceladus’ ice crust. It has been shown by Choblet, et al. that more than 10 GW can be generated by tidal friction inside the rocky core of Enceladus, which would supply the high heat power required to maintain the subsurface ocean in liquid state.

The rocky core is claimed to be permeable, causing hot ($> 363$ K) upwellings from the seafloor towards the ice crust. Simulated water circulation suggests that the ice crust is on average from 20 km to 25 km thick, but due to the geometry of the rocky core, the ice crust
is less than $5 \text{ km}$ thick at the south pole of Enceladus.[11] The water penetrates the thinner ice crust at the south pole through cracks to create ice geysers or jets at the south pole. These geysers are the sources of plumes of vapour and micrometer-sized icy dust[5], from which a fraction is ejected to orbit Enceladus (figure 6).

When the icy dust particles are expelled from the satellite, a part of them ends on orbits with a larger semi-major axis, with a slower orbital motion, trailing Enceladus. Another fraction of particles populates orbits with a smaller semi-major axis, with faster orbital motion, leading Enceladus. This positioning of particles in orbit around Enceladus can be seen as a spectacle known as “Enceladus Fingers”, as seen in figure 7.

Together with Enceladus, the particles then orbit Saturn under the perturbations of higher order central gravity, solar radiation forces, and electromagnetic forces, with the particles eventually diffusing under these perturbations around the entire orbit, creating Saturn’s E Ring (figure 8).
3.2 Orbit-averaged Evolution Equations for $a$, $e$, and $\varpi$

Following Horanyi et al. (1992)\cite{Horanyi1992}, we solve the evolution equations for the semi-major axis $a$ (2.2.4), the eccentricity $e$ (2.2.7), and the longitude of pericenter $\varpi$ (2.2.15) and (2.2.18) (recalling that: $\varpi = \omega + \Omega$). In this case, we follow the orbital evolution of a Saturnian dust grain.
with a constant charge, which is perturbed by planetary oblateness, solar radiation pressure, and the Lorentz force.

For simplicity, we ignore the planet’s motion around the Sun (the orbital period of Saturn around the Sun is \( \sim 10^4 \times \) the orbital period of the grain around Saturn) and the planetary shadow. We describe the orbital path in a planar case, i.e. for inclination \( i = 0 \), so we are only interested in the size, shape, and orientation of the orbit in the plane, which we can define with three osculating orbital elements: \( \dot{a}, \dot{e} \) and \( \dot{\omega} \).

### 3.2.1 Orbit-averaged Time-derivative of the Semi-major Axis, \( \dot{a} \)

The major perturbation contributing to the perturbing force component \( T \) in (2.2.1) is solar light pressure in the dust particle’s orbit around Saturn. Measuring the azimuthal angle from the anti-sun direction we have for the components \( R \) and \( T \) of the perturbation force:

\[
R_{lp} = f_{sol} \cos \theta \\
T_{lp} = f_{sol} \sin \theta.
\]

where the strength \( f_{sol} \) of the radiation force is given by:[3]

\[
f_{sol} \equiv \frac{3J_0Q_{pr}}{4\rho cd_S^2 r_g}.
\]

Here, \( J_0 \) is the solar radiation energy flux at 1 AU (\( J_0 = 1.36 \times 10^6 \) ergs cm\(^{-1}\)s\(^{-1}\)[3]), \( Q_{pr} \) is the radiation pressure coefficient (\( Q_{pr} \approx 1 \) for 1\( \mu \)m dust grains[3]), \( \rho \) is the dust grain’s density (\( \rho = 1 \) g cm\(^{-3}\)[3]), \( c \) is the speed of light (\( c = 299 792 458 \) ms\(^{-1}\)), \( d_S \) is the distance of Saturn from the Sun in Astronomical Units (\( d_S = 9.58 \) AU[3]), and \( r_g \) is the radius of the circumplanetary dust grain.

In our example, where we do not consider the planetary shadow, the Sun accelerates and decelerates the dust particle equally over one orbit, so we can deduce that: \( \langle T_{lp} \rangle \equiv 0 \). In the case of small eccentricity (\( e \ll 1 \)), retaining only the lowest order terms in \( e \), approximating an instantaneous near-circular orbit, and with the above consideration of the effect of \( T \), the
right-hand side of (2.2.4) reduces to zero, so using (2.3.3) we get:

\[ \langle \ddot{a} \rangle = 0. \tag{3.2.4} \]

### 3.2.2 Orbit-averaged Time-derivative of the Eccentricity, \( e \)

Using the same initial assumptions as we had for \( \langle \dot{a} \rangle \), using (2.2.7) and (2.3.3) we get:

\[
\langle \dot{e} \rangle = \sqrt{\frac{a (1 - e^2)}{\mu}} \left[ T \left( \frac{e + \cos f}{1 + e \cos f} + \cos f \right) + R \sin \beta \right] \\
= \sqrt{\frac{a (1 - e^2)}{\mu}} f_{\text{sol}} \frac{\sqrt{1 - e^2}}{2\pi} \int_{0}^{2\pi} (3 \sin f \cos f \cos \varpi + 3 \cos^2 f \sin \varpi - \sin \varpi) \, df \\
= \frac{3}{2} \sqrt{\frac{\alpha}{\mu}} f_{\text{sol}} \sin \varpi, 
\]

\[ \tag{3.2.5} \]

where we have neglected \( O(e^2) \), \( \varpi \equiv \omega \) when \( i = 0 \), and \( f_{\text{sol}} \) is taken as a constant factor of both \( R \) and \( T \) from (3.2.1) and (3.2.2). For clarity, we rewrite (3.2.5) as:

\[ \langle \dot{e} \rangle = \beta \sin \varpi, \tag{3.2.6} \]

where:

\[ \beta = \frac{3 H f_{\text{sol}}}{2} \equiv \frac{3}{2} f_{\text{sol}} \sqrt{\frac{\alpha}{\mu}} \tag{3.2.7} \]

### 3.2.3 Orbit-averaged Time-derivative of the Longitude of Pericenter, \( \varpi \)

Continuing with the same initial assumptions, from (2.2.18) and (2.3.3) we get:

\[ \langle \dot{\varpi} \rangle = \frac{3 f_{\text{sol}}}{2} e \sqrt{\frac{\alpha}{\mu}} \cos(\varpi) + \gamma, \tag{3.2.8} \]

where \( \gamma \) describes the precession rate of the longitude of pericenter, \( \varpi \), due to Saturn’s oblate-
ness and the Lorentz force:

\[
\gamma = \frac{3}{2} \omega_k J_2 \left( \frac{R_{\text{Saturn}}}{a} \right)^2 - 2 \frac{Q B_0}{m c} \left( \frac{R_{\text{Saturn}}}{a} \right)^3 \approx 51.4 \left( \frac{R_{\text{Saturn}}}{a} \right)^{3.5} \frac{\circ}{\text{d}} - 25.5 \left( \frac{R_{\text{Saturn}}}{a} \right)^3 \frac{\circ}{\text{d}},
\]  

(3.2.9)

where \( \omega_k \) is the Keplerian angular velocity \( (\omega_k^2 \equiv \mu/a^3) \), \( J_2 \) describes the departure of Saturn’s gravitational field from spherical symmetry (for Saturn, \( J_2 = 16290.71 \pm 0.27 \) [15]), \( Q \) is the dust grain’s charge calculated from the equilibrium surface potential, \( B_0 \) is the magnetic field strength on Saturn’s surface \( (B_0 = -0.2 \text{G} \text{ for the magnetic field strength at Saturn’s surface}) \), \( m \) is the mass of the dust grain, \( c \) is the speed of light, \( R_{\text{Saturn}} \) is the radius of Saturn \( (R_{\text{Saturn}} = 60300 \text{km}) \), and \( \circ/\text{d} \) is degrees per day.

Again, for clarity, we rewrite (3.2.8) as:

\[
\langle \hat{\varpi} \rangle = \frac{\beta}{\gamma} \cos(\varpi) + \gamma.
\]  

(3.2.10)

### 3.3 Numerical Modelling

We now have a set of three orbit-averaged time-derivatives ((3.2.4), (3.2.6), and (3.2.10)):

\[
\langle \dot{a} \rangle = 0
\]  

(3.3.1)

\[
\langle \dot{e} \rangle = \beta \sin(\varpi)
\]  

(3.3.2)

\[
\langle \dot{\varpi} \rangle = \frac{\beta}{\gamma} \cos(\varpi) + \gamma.
\]  

(3.3.3)

We can solve (3.2.6) and (3.2.10) analytically through variable transformations \( p = e \sin \varpi \) and \( q = e \cos \varpi \):

\[
e = \frac{2\beta}{\gamma} \left| \sin \left( \frac{\gamma t}{2} \right) \right|
\]  

(3.3.4)

\[
\varpi = \left( \frac{\gamma}{2} \right) \text{mod} \pi + \frac{\pi}{2},
\]  

(3.3.5)

when our initial conditions stay the same, and \( \gamma \neq 0 \).

In figure 9, we have solved the evolution of (3.3.2) and (3.3.3) numerically, and com-
Figure 9: The upper plot shows the evolution of the eccentricity according to (3.2.5) and (3.3.4) over a time period of 100 years for dust grain radius $r_g = 3 \mu m$. Similarly, the lower plot shows the evolution of the pericenter angle according to (3.2.8) and (3.3.5) over the same time period.

pared the numerical solution to the analytical solutions (3.3.4) and (3.3.5) as per Horanyi et al. (1992) [3]. The numerical integration of the orbit-averaged equations is carried out in the programming language IDL, and the code can be found in Appendix A. As seen in (3.2.7), $\beta$ is a constant independent of the evolution of either $e$ or $\varpi$. We find excellent agreement between the numerical and analytical solutions, and the curves fall on top of each other.

### 3.4 Correction to Horanyi et al. (1992)

During my work, I came across a minor inconsistency in Horanyi et al. (1992) [3]. In figure 10, curve a) represents a curve made from scanned data points of the original, which fits the curve drawn according to the article’s nominally chosen $\beta$ ($\beta = 0.2 \text{ yr}^{-1}$). However, the article also gives the necessary values for calculating $\beta$, which gives a value that should not be rounded.
up but down ($\beta = 0.139 \text{yr}^{-1}$). This causes the more accurate maximum eccentricity curve to have a narrower peak than the one suggested by the article, and hence suggests faulty result in figure 2 of Horanyi et al. (1992)\textsuperscript{[3]}. All the curves have been capped at $e_{\max} = 0.65$, as set by the article, because grains with that eccentricity and a semi-major axis that corresponds to an Enceladus origin would intersect Saturn’s dense A ring, and thus be absorbed by that ring. The narrower peak suggests, that the range of dust grain potentials that would cause this A ring intersection, due to a higher maximum eccentricity $e_{\max}$, would be approximately $0.25 \text{V}$ higher at the lower end and $0.25 \text{V}$ lower at the upper end: $-6 \text{V} \lesssim \phi \lesssim 4 \text{V}$.

![Horanyi Figure 2 Correction](image)

Figure 10: Mistake in Horanyi et al. (1992): the value for $\beta$ given by the article ($a, b$); $\beta = 0.2 \text{yr}^{-1}$) does not correspond to the calculated value of $\beta$ by using the values given in the article ($c$); $\beta = 0.139 \text{yr}^{-1}$). All the curves have been capped at $e_{\max} = 0.65$, as set by the article, because grains with that eccentricity and a semi-major axis that corresponds to an Enceladus origin would intersect Saturn’s dense A ring, and thus be absorbed by that ring.

3.5 Comparing Theory to Observations of the E Ring

Based on observations of the E Ring and Enceladus, both observationally – by William Herschel in 1789 and others after him – and by different spacecraft – such as Pioneer 11, Voyager 1 and 2, Cassini – during flybys, Enceladus orbits Saturn at a distance of $a_{\text{Enceladus}} =$
238 000 km $\simeq 3.95 R_{\text{Saturn}}$. The E Ring was observed in the optical to extend from $a_{\text{ERing}} \simeq 3 R_{\text{Saturn}}$ to $\simeq 8 R_{\text{Saturn}}$, with an optical peak near the orbit of Enceladus (see figure 8 on page 19). In situ measurements with the Cassini Cosmic Dust Analyzer (CDA) found E ring particles out to Titan’s orbit close to 20$R_{\text{Saturn}}$ (Srama et al. (2006)[16]).

For the range of eccentricities seen in figure 9 on page 23, $0 \leq e < 1$, if we take Enceladus as the origin of the particles and use $a_{\text{Enceladus}}$ for the semi-major axis of the dust particles, using $r = a (1 + e)$ we get a range of distances between:

$$r < 7.9 R_{\text{Saturn}},$$

which is in reasonable agreement with the observations. In the Saturn system, the E ring grains follow paths with moderate inclinations (which we have neglected) and their evolution is terminated when their orbital nodes lie in the range of the dense rings ($r < 137 000 \text{ km} \simeq 2.27 R_{\text{Saturn}}$), where the particles get absorbed. This limits the maximum eccentricity of E ring dust grains to roughly $e_{\text{max}} \approx 0.65[3]$, which sets the inner ring boundary apparent in optical detections close to 3$R_{\text{Saturn}}$.

Comparing the values we have calculated from the orbit-averaging perturbation equations to the observed values, we see that the evolutions of the orbital elements of the E Ring dust particles overlap almost entirely. This is also in agreement with Hamilton (1993)[2] (see figure 4 on page 16).

The dynamics of E ring dust grains are affected by several processes which we have not included in this simple model. These are the effects induced by an initial size-distribution of the dust grains, the drag force induced by corotational plasma in the system, collisions and gravitational encounters of the grains with Saturnian satellites in the E ring, and plasma-sputtering gradually reducing the grain size[17].
4 Conclusion

In conclusion, we begun with a short introduction to unperturbed Keplerian orbits and defined a set of six orbital elements, \((a, e, i, \Omega, \omega, \tau)\), to describe orbits of two-body systems. We then defined a small perturbing force, \(dF\), used it to derive perturbation equations for the six orbital elements, \((\dot{a}, \dot{e}, \frac{\dot{i}}{d}, \dot{\Omega}, \dot{\omega}, \dot{\tau})\), as per Burns (1976)[1], and took a look at time-averaging these perturbation equations over one orbit to get orbit-averaged perturbation equations, \(\langle \Psi \rangle\), as per Hamilton (1993)[2].

We then followed Horanyi et al. (1992)[3] to study the aspects of the dynamics of particles in Saturn’s E Ring. We derived orbit-averaged equations for the semi-major axis \(\langle \dot{a} \rangle\), the eccentricity \(\langle \dot{e} \rangle\), and the longitude of pericenter \(\langle \dot{\omega} \rangle\) of an icy dust particle \(m\) in Saturn’s E Ring.

We then solved \(\langle \dot{e} \rangle\) and \(\langle \dot{\omega} \rangle\) analytically and numerically, and compared these results to the range of Saturn’s E ring, inferred from observations, which suggest:

\[
3R_{\text{Saturn}} \lesssim r \lesssim 8R_{\text{Saturn}}.
\]

This also agrees with the proposition by Showalter et al. (1991)[4] and Spahn et al. (2006)[5] that (the south pole of) Enceladus would be the source of E Ring particles.
References


Appendix A: The Code

; A Function for Solving Right-Hand Sides of Orbit-Averaged Horanyi (4b-4c)
;---------------------------------------------------------------------------
;---------------------------------------------------------------------------
FUNCTION RightHandSide, time, elements

; This function handles the right-hand sides of Horanyi’s orbit-averaged
; perturbation equations for the eccentricity ‘e’ and the longitude of
; pericenter ‘pomega’.
;
; While these few lines of code would seem to be better placed inside the
; ‘for’ loop in the main program, the IDL routine for the fourth order
; Runge-Kutta method *requires* to have a function for the derivatives,
; hence the function. Also due to the proprietary IDL routine, the current
; time has to be carried by the function.
;---------------------------------------------------------------------------

; The ‘common’ and ‘passvar’ keywords are used to distribute a shared
; structure ‘saved’:
;---------------------------------------------------------------------------
COMMON passvar, saved

; The current values of e and pomega are passed into the function inside a
; list ‘elements’, and saved constant values beta and gamma are read from a
; shared structure ‘saved’ (defined in the main program):
;---------------------------------------------------------------------------
e = elements(0)
pomega = elements(1)

; Horanyi defines e << 1, and furthermore sets beta to be constant. However,
; we are interested in eccentricity as a time-dependent variable, so we
; update beta at every step for one of the two sets of prints (saved.bet
; gives us the constant argument for calculating beta):
; ---------------------------------------------------------------------------------------------------------------------------------------------------------------------
IF saved.ematters EQ 1 THEN BEGIN
    bet = saved.bet * sqrt(1d0 - e*e)
ENDIF ELSE BEGIN
    bet = saved.bet
ENDELSE

; e and pomega are then used to calculate the orbit-averaged time-
; derivatives as per Horanyi (4b-c):
; ---------------------------------------------------------------------------------------------------------------------------------------------------------------------
dedt = bet * SIN(pomega) ; Horanyi (4b)
dpomegadt = (bet/e) * COS(pomega) + saved.gam ; Horanyi (4c)

RETURN, [dedt, dpomegadt]
END

;-------------------------------------------------------------------------------
;-------------------------------------------------------------------------------
;-------------------------------------------------------------------------------
; The Main Program
;-------------------------------------------------------------------------------
;-------------------------------------------------------------------------------
;-------------------------------------------------------------------------------
@psdirect
PRO OrbitAveraging, grainsize

;-- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
; This is the main program. In the following order, the main program:
;
; 1) Defines device parameters for outputting the plots and images,
; 2) Using initial conditions from Horanyi[1], it calculates needed
;    constant values for beta and gamma for Saturn’s E Ring particles,
3) Creates a common structure and saves beta and gamma into it,
4) Solves the analytical solution for the eccentricity and the
   longitude of pericenter [Horanyi (7a-b)],
5) Iterates the orbit-averaged time-derivatives for the eccentricity
   and the longitude of pericenter [Horanyi (4b-c)] using the fourth
   order Runge-Kutta method, and
6) Plots the numerical and analytical solution for comparison.

[1]: Mihaly Horanyi, Joseph A. Burns, Douglas P Hamilton: "The Dynamics of

; The 'common' and 'passvar' keywords are used to distribute a shared
; structure 'saved':
; -----------------------------------------------------------------------------
COMMON passvar, saved

; 1) The 'device' keyword and related parameters are used to define the
; graphic output of the main program (on screen and exporting):
; -----------------------------------------------------------------------------
DEVICE, decomposed=0, retain=2
TEK_COLOR
thick = 2
!x.thick = thick
!y.thick = thick
!p.thick = thick

; 2) Using initial conditions defined by Horanyi, we calculate beta and
; gamma:
; -----------------------------------------------------------------------------
J_0 = 1360d0 ; J m^-2 s^-1
Q_pr = 1d0 ; Radiation pressure coefficient, ~1
rho = 1000d0 ; Grain’s density, kg m^-3
v_c = 299792458d0 ; Speed of light, m s^-2
d_s = 9.58d0 ; Saturn distance from the Sun, AU
In case we want to vary the grain size, we define a keyword we can use in the call to the program using an IF-ELSE structure. If we do not specify the grain radius, then we use the value used by Horanyi (1 micron):

```plaintext
IF KEYWORD_SET(grainsize) THEN BEGIN
   gs = DOUBLE(grainsize); Keyword set -> defined grain size
ENDIF ELSE BEGIN
   gs = 1d0; Keyword not set -> 1um grain size
ENDELSE
r_g = 1d-6 * gs; Grain radius, microns
```

Sources for the following data:

[3]: Allen’s Astrophysical Quantities,
[4]: National Space Science Data Center
[5]: JPL Solar System Dynamics

```
R_Sat = 6.0268d7 ; Saturn’s radius, m
M_Sat = 5.68319d26 ; Saturn’s mass, kg
G = 6.674d-11 ; Gravitational constant, m^3 kg^-1 s^-2
a = 2.38040d8 ; Enceladus’s semi-major axis, m
psi = -5.6d0 ; Surface potential on micron-sized grain, V
```

We define constant ‘dpd2rps’ for translating angles and calculate the number of seconds in a year in ‘year’:

```
dpd2rps = ((2d0*!dpi) / 360d0) / (24d0 * 36d2); deg/day -> rad/sec
year = 365.25d0 * 24d0 * 36d2
```

Next we solve the last term of Horanyi (1), named here ‘f_sol’, and then...
; use this and the defined constants to calculate beta and gamma:
; -----------------------------------------------------------------------------------------------
f_sol = (3d0 * J_0 * Q_pr) / (4d0 * rho * v_c * d_s^2 * r_g)
mu  = G * M_Sat ; Saturn’s mass times gravitational constant

bet = 1.5d0 * f_sol * SQRT(a/mu)
gam = 51.4d0 * (R_Sat/a)^3.5 + 5.1d0 * psi * (R_Sat/a)^3

gam = gam * dpd2rps ; Angle translation, deg/day -> rad/sec

; 3) We define a common structure and save beta and gamma in it:
; -----------------------------------------------------------------------------------------------
saved = {bet:bet, gam:gam, ematters:0}

; Next we define our time interval. We choose a 100-year period, saved into 
; ‘timetab’, at intervals of ‘deltat’:
; -----------------------------------------------------------------------------------------------
timetab = 1d2 * DINDGEN(10000)/9999d0 * year
deltat = year * 1d2 / DOUBLE(N_ELEMENTS(timetab))

; 4) Using timetab, we solve the analytical solution Horanyi (7a-b):
; -----------------------------------------------------------------------------------------------
e_7a = ((2d0*bet)/ABS(gam)) * ABS( SIN((gam/2d0) * timetab) )
pomega_7b = ((gam/2d0) * timetab) MOD !dpi + !dpi/2d0

; 5) Next we iterate the orbit-averaged time-derivatives. First we define
; tables for e and pomega, as well as define the initial values (initial
; pomega is the same as it is for the analytical solution, but e has to be 
; fixed to a nonzero initial value):
; -----------------------------------------------------------------------------------------------
e_4b = timetab
pomega_4c = timetab

e_4b[0] = 1d-6
pomega_4c[0] = pomega_7b[0]
We want two sets of prints: one where \( e \) is set to constant as per Horanyi \((\text{ematters} = 0)\), and another where we let \( e \) be a variable \((\text{ematters} = 1)\). We accomplish this by running the iterations and printing into file within a two-step loop:

\[
\begin{align*}
\text{FOR ematters} &= 0, 1 \ \text{DO BEGIN} \\
\end{align*}
\]

\[
\begin{align*}
saved.\text{ematters} &= \text{ematters} \\
; \text{We define the initial values for variables } e \text{ and } pomega: \\
; \text{-----------------------------} \\
e &= e_{4b}[0] \\
pomega &= pomega_{4c}[0] \\
\text{FOR } i = 1d0, \text{N_ELEMENTS(timetab)}-1 \ \text{DO BEGIN} \\
; \text{To fulfil the requirements of the fourth order Runge-Kutta method} \\
; \text{routine 'RK4', we define a vector / list from } e \text{ and } pomega, \text{ as} \\
; \text{well as the current time 'time':} \\
; \text{-----------------------------} \\
elements &= [e, pomega] \\
time &= timetab[i-1] \\
; \text{Next we calculate time-derivatives of variables } e \text{ and } pomega: \\
; \text{-----------------------------} \\
dvecdt &= \text{RightHandSide}(time, elements) \\
; \text{We now process the data at this point in the loop using the fourth} \\
; \text{order Runge-Kutta method, outputting results into a new vector:} \\
; \text{-----------------------------} \\
elements\_new &= \text{RK4}(elements, dvecdt, time, deltat, 'RightHandSide') \\
; \text{Lastly, we read the processed } e \text{ and } pomega \text{ from elements\_new, fix} \\
; e > 0 \text{ and } |pomega| < \pi/2 \text{ (from definition), and save the values}
\end{align*}
\]
; into the lists e_4b and pomega_4c:
; ------------------------------------------------------------------
e   = elements_new(0)
pomega = elements_new(1)

IF e LE 1d-6 THEN e = ABS(e) ; fix e>0
IF pomega LE -0.5*!dpi THEN pomega = 0.5*!dpi ; fix pomega <= |pi/2|

   e_4b[i]   = e
   pomega_4c[i] = pomega

ENDFOR

; (6) Here we set keywords for plotting. The printing is handled by the
; ‘psdirect’ routine, written by Heikki Salo:
; ------------------------------------------------------------------
file_end = ‘’
IF ematters EQ 1 THEN file_end = ‘_e’

thisdir = getenv(‘PWD’) 
file = thisdir + ‘/’ + ‘OrbitAveraging’ + file_end
ps = 1
psdirect, file, ps, /vfont, /color

!p.charsize = 1.1
!p.charthick = 3
!p.thick = 3
!p.multi = [0,1,2]

titleend = ‘ (Beta is Constant)’
IF ematters EQ 1 THEN titleend = ‘ (Ecc. Changes Beta)’

m = MAX(e_7a) ; Set multiplier for eccentricity plot labels

; Plotting the analytical and numerical eccentricities:
; ------------------------------------------------------------------
nwin
PLOT, timetab/year, e_7a, title = 'Evolution of Eccentricity' + titleend, 
   xtype = 'Time, t (years)', ytitle = 'Eccentricity, e', /nodata
OPLOT, timetab/year, e_4b, col = 2
OPLOT, timetab/year, e_7a, col = 4, linestyle = 3
OPLOT, [2, 7], [0.17, 0.17]*m, col = 2
OPLOT, [2, 7], [0.07, 0.07]*m, col = 4, linestyle = 3
XYOUTS, 8, 0.15*m, "a) NUMERICAL ", col = 2, charsize = 0.5, charthick = 1
XYOUTS, 8, 0.05*m, "b) ANALYTICAL", col = 4, charsize = 0.5, charthick = 1

; Plotting the analytical and numerical longitudes of pericenter:
; ---------------------------------------------------------------

nwin
PLOT, timetab/year, pomega_7b, title = 'Evolution of Pericenter Angle' $ 
   + titleend, xtype = 'Time, t (years)', $
   ytitle = 'Pericenter angle, !7x!X', /nodata
OPLOT, timetab/year, pomega_4c, col = 2
OPLOT, timetab/year, pomega_7b, col = 4, linestyle = 3
OPLOT, [2, 7], [-1.32, -1.32], col = 2
OPLOT, [2, 7], [-1.72, -1.72], col = 4, linestyle = 3
XYOUTS, 8, -1.4, "c) NUMERICAL ", col = 2, charsize=0.5, charthick = 1
XYOUTS, 8, -1.8, "d) ANALYTICAL", col = 4, charsize=0.5, charthick = 1

; Closing the print routine:
; ---------------------------------------------------------------
!p.multi = 0
psdirect, file, ps, /vfont, /color, /stop

ENDFOR

END