On the conformal dimension of the Sierpiński carpet

Master's thesis
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Contents

1. Introduction and Notation

1.1. **Motivation.** One principal goal in mathematical research is to classify structures. In general, as one tries to understand a mathematical structure, often the first goal is to find a good stock of examples. If one is able to write a list that contains every instance of the structure, the understanding of the structure is in some sense complete. If this kind of characterization is obtained, the search for counterexamples, examination of the boundaries of one's intuition and testing hypotheses becomes almost effortless. Generally this is rather ambitious goal, but one way to obtain almost complete classifications is to consider equivalence classes. In this kind of research, the notion of invariants becomes indispensable.

For our purposes, invariant property is some property P that is unchanged if we transform the structure into something that is in some sense equivalent to the original structure. When one tries to find useful invariants, one seeks to find properties that more often than not differ for nonequivalent spaces, while being easy to measure. This is because invariant properties can be equal for nonequivalent spaces, and whenever the property is unequal, one should be able to measure it to conclude the inequality. However, the most useful invariants tend to be such properties that can be measured, but the calculation is not easy.

The main subject of this thesis, conformal dimension, is an invariant property under certain transformations called quasisymmetric mappings. Thus, if we know that a space X has some quasisymmetrically invariant property equal to P and for some other space X' the same property equals $P' \neq P$, we can be sure that there does not exist any quasisymmetry between the spaces. In this case we can say that X and X' are not equivalent. Hence, if we are able to give a detailed description of the set of all spaces that are quasisymmetrically equivalent to X, we obtain a certain classification for all the instances that are equivalent to X up to a quasisymmetry.

Historically, there have been many different notions of dimension, all natural in some sense and useful in different respects. One basic idea, for instance, has been to think about a dimension of a space or geometrical object being a representative of how many coordinates one needs to specify a point on the space. This is used, for example, in manifold theory. On the other hand, dimension can also be thought of as representing the size of a space. In this thesis, we will focus on this notion of dimension and we shall use a couple of slightly different, but in some sense related, definitions of dimension. Main motivation for using these particular definitions is the fact that we will be dealing with fractal spaces. Our main focus will be on one of the most natural and intuitive of these dimensions, namely, the Hausdorff dimension. In particular, we are interested how this dimension is changed when we transform the space with a quasisymmetry. In some sense, by studying this question, we can find what is the "optimal form" of the space under investigation.

Moreover, conformal dimension is related to a famous conjecture in the field of geometric group theory called Cannon's conjecture, which predicts that boundaries of hyperbolic groups with 2-dimensional spheres as boundaries are in fact quasisymmetric spheres, which are metric spaces that are quasisymmetrically equivalent to 2-dimensional spheres. In 2000, Kapovich and Kleiner made a related conjecture on hyperbolic groups but with boundary homeomorphic to Sierpiński carpet [17, Conjecture 6]. Interestingly, to prove both of these conjectures at one stroke it suffices to show that the conformal Hausdorff dimension in a certain canonical set of metric spaces is attained by some space in the set [25, 17].

Although the notion of conformal dimension is relatively new, the roots of the underlying concepts, questions and ideas date back more than 100 years to complex function theory. In 1928, the notion of quasiconformal mappings was found by trying to find suitable generalizations to fundamental mappings in complex analysis, namely, the conformal mappings. It turned out be a fruitful generalization of conformal mappings, which led to powerful results. Later, in 1956, Ahlfors and Beurling introduced the notion of quasisymmetric mappings [2], but it wasn't until 1966 when Kelingos was the first to use the term "quasisymmetric" mapping [19]. After these initial investigations, in 1980, Tukia and Väisälä took this generalization a step further and provided a detailed study of the properties of quasisymmetric mappings and showed that the right generalization of quasiconformal mappings from Euclidean spaces to arbitrary metric spaces were in fact the quasisymmetric mappings [39]. These mappings have since been under extensive research amongst the collection of metric analysts. Hence, studying properties that are invariant under quasisymmetric mappings can eventually help us complete our understanding of the complex function theory by filling in the intricate details and finishing the big picture of the puzzle.

Now that we have motivated our subject of interest, we shall introduce the notation used in this thesis as well as the prerequisites for understanding the theory of conformal dimension.

1.2. **Notation and Prerequisites.** We have given a descriptive name and an obvious enumeration for every result, definition and example. When we cite results or definitions given in this thesis, we write them in italics. For example, "see Theorem 1.24 *Quasisymmetry Equivalence*" refers to the 24th result/definition/example given in the Chapter 1. Most of the following notation and results can be found in [35]. Hence, in the rest of this chapter, if we do not explicitly mention any reference, the book by Rudin should be the first place to consult for more information. Occasionally, we will not give a reference for a simple argument, which indicates that the argument is so elementary, that it is common knowledge.

We let $\mathbb{N} = \{1, 2, 3, ...\}$ to denote the *natural numbers* and $\mathbb{R}$ the *real numbers.*

Constants are usually denoted by C, C', c, K, but in this work we are never interested in the specific value of a constant. For this reason, even if a constant changes during a chain of inequalities, we may still denote it by the same letter. We also write $C(\mu)$ if it is important to note that the constant depends only on the data μ.

We denote the *set complement* for $A \subset X$ by $A^c := \{x \in X : x \notin A\}$ and the *set difference* as $A \backslash B = \{x \in A : x \notin B\}$.

If A is a set, then $|A|$ denotes its *cardinality*, or in other words, the number of elements A contains. Note that $|A| \in \mathbb{N} \cup \{\infty\}$.

We denote the *closure* of a set A by $\overline{A}$.

If X is a set, then $\mathcal{P}(X)$ denotes the set of all subsets of X, that is, the full *power set* of X.

Support of a function $f : X \to Y$ is the closure of the set

$$\{x \in X : f(x) \neq 0\}.$$

If the support is a compact set, we say that the function is *compactly supported.*

The *space of all continuous functions* whose domain is a set X is denoted by $C(X)$. If in addition, the functions are compactly supported, we write $C_c(X)$.

We call a function $\chi_A : X \to Y$, $A \subset X$, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

the *characteristic function* of the set A.

Definition 1.1. (Metric Space). We say that a set X equipped with a function $d : X \times X \to \mathbb{R}^+$, called a *metric*, is a *metric space* if the function d satisfies the following axioms

(i) $d(x, y) = 0$ if and only if $x = y$ (Definiteness)
(ii) $0 \leq d(x, y) < \infty$ for every $x, y \in X$ (Positivity and Finiteness)
(iii) $d(x, y) = d(y, x)$ for every $x, y \in X$ (Symmetry)
(iiv) $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \in X$ (Triangle Inequality)

Note that the axiom of positivity can be omitted, since it can be deduced from the other three axioms.

Distance between two sets A, B is denoted by $\operatorname{dist}(A, B)$ and defined as $\operatorname{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. If $A = \emptyset$ or $B = \emptyset$, then we set $d(A, B) = 0$. If $A = \{a\}$ a one-element set, then $\operatorname{dist}(A, B) := \operatorname{dist}(a, B) := \inf\{d(a, x) : x \in B\}$.

Metric space is said to be *locally compact* if for every point $x \in X$ there exists an open set $U \subset X$ and compact set $K \subset X$ such that $x \in U \subset K$. This is the same as saying every point in X has a locally compact neighborhood.

An open (resp. closed) *ball* centered at x and having a radius $r > 0$ in a metric space X is denoted by $B(x, r)$ (resp. $\overline{B}(x, r)$) and defined as $B(x, r) := \{z \in X : d(z, x) < r\}$ (resp. $\overline{B}(x, r) := \{z \in X : d(z, x) \leq r\}$). Note that in an arbitrary metric space it may be the case that neither the center nor radius of a ball is unique. To see this, consider the discrete metric space, where the distance between distinct points is equal to 1. To make notation less cumbersome, we sometimes write $B_i = B(x_i, r_i)$ and $CB_i = B(x_i, Cr_i)$, if C is a constant.

A *neighborhood* in a metric space for a point x (or a subset) is defined to be any set U satisfying $x \in B(x, r) \subset U$.

Definition 1.2. (Topological Hausdorff Space). We recall that a *topology* τ on a set X is a set of subsets satisfying

(1) $\emptyset \in \tau$ and $X \in \tau$,
(2) $\bigcap_{i=1}^{n} V_i \in \tau$ whenever $V_i \in \tau$ for any finite n,
(3) $\bigcup_{\alpha \in I} V_\alpha \in \tau$ whenever $V_\alpha \in \tau$ for every $\alpha \in I$ where I is an arbitrary index set.

A pair (X, τ) is called a *topological space.* If there exists an extra structure on (X, τ), namely, the separation structure: for any $u, v \in X$, there exists some subsets $N_u, N_v \in \tau$ containing u and v, respectively, and satisfying

$$N_u \cap N_v = \emptyset,$$

then we call the space *topological Hausdorff space.* A topological space is called *separable* if it contains a countable dense subset. *Dense* subset $A \subset X$ means that for any $x \in X$ there exists a sequence $\{a_j\} \subset A$ such that $a_j \to x$. A neighborhood of a point p or a set B in a topological space is just any open set containing p or B, respectively.

Definition 1.3. (Sigma-algebra and Measure). A *σ-algebra* $\mathfrak{M}$ on a set X is a set of subsets of X satisfying

(1) $X \in \mathfrak{M}$,
(2) $A \in \mathfrak{M}$ implies $A^c \in \mathcal{M}$,
(3) $\bigcup_{i\in I}^{\infty} A_i \in \mathfrak{M}$ whenever $A_i \in \mathfrak{M}$ for every $i \in I$, where I is at most a countable index set.

Measure in this thesis is a countably subadditive function $\mu : \mathcal{P}(X) \to [0, \infty]$ that vanishes on the empty set, that is,

$$\mu\left(\bigcup_{i}^{\infty} A_i\right) \leq \sum_{i}^{\infty} \mu(A_i)$$

and

$$\mu(\emptyset) = 0.$$

Also, we assume that $\mu(A) < \infty$ for at least one $A \in \mathcal{P}(X)$. A *measure space* is a space X equipped with a measure μ and denoted by (X, μ).

We call a set $A \in \mathcal{P}(X)$ a *μ-measurable set*, if for every $F \in \mathcal{P}(X)$ we have

$$\mu(F) = \mu(F \cap A) + \mu(F \cap A^c).$$

Restriction of a measure to the set of μ-measurable sets, gives a *countably additive measure* defined on the σ-algebra of the μ-measurable sets, that is, in addition to the above properties, we have

$$\mu\left(\bigcup_{i}^{\infty} A_i\right) = \sum_{i}^{\infty} \mu(A_i)$$

whenever A_i is μ-measurable and $A_i \cap A_j = \emptyset$ for every $i \neq j$.

Note that every measure (respectively countably additive measure) is monotonic, that is, if $A, B \in \mathcal{P}(X)$ (respectively in $\mathfrak{M}$) and $A \subset B$, then $\mu(A) \leq \mu(B)$. Moreover, the continuity properties of measures are well-known and by these we mean the following facts,

(1) If $A = \bigcup_{n=1}^{\infty} A_n$, A_n is μ-measurable for every n, and $A_i \subset A_{i+1}$, then $\lim_n \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$.
(2) If $A = \bigcap_{n=1}^{\infty} A_n$, $\mu(A_1) < \infty$, A_n is μ-measurable for every n, and $A_i \supset A_{i+1}$, then $\lim_n \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$.

We say that a property P holds μ almost everywhere (μ-a.e.) if the set where P does not hold has μ-measure zero.

A measure space is called *σ-finite*, if it can be written as a countable union of measurable sets each having finite measure. All measure spaces in this thesis are assumed to be σ-finite.

We call a set B *σ-compact*, if B is a countable union of compact sets.

If we have a product space $X \times Y$ of two measure spaces $(X, \mu, \mathfrak{M})$ and $(Y, \lambda, \mathfrak{B})$, then we define a *product measure* on $X \times Y$ as $(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B)$ whenever $A \in \mathfrak{M}$ and $B \in \mathfrak{B}$. See [35, Chapter 8] for the definition of the product σ-algbera and the related technical details.

Next we give the definition of the most used notion of measure and dimension in this thesis. Given a set $B \subset X$, a *cover* for B is a set of subsets $\{A_\alpha\}$ of X satisfying $B \subset \bigcup_\alpha A_\alpha$.

Let (Z,d) be a metric space and $\mathcal{F} \subset \mathcal{P}(Z)$. Then for each $0 < \delta \leq \infty$ and $s \geq 0$ we define a set function

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} A_i)^s : A \subset \bigcup_{i=1}^{\infty} A_i,\ \operatorname{diam} A_i < \delta,\ A_i \in \mathcal{F}\right\}$$

whenever $A \subset Z$. Infimum is taken over all countable covers of A. These are called *s-dimensional Hausdorff pre-measures*. The *s-dimensional Hausdorff measure* is denoted by $\mathcal{H}^s$ and defined as

$$\mathcal{H}^s(A) := \lim_{\delta \to 0+} \mathcal{H}^\alpha_\delta(A) = \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(A).$$

Note that the limit exists since we are in $[0,\infty]$ and $\delta \mapsto \mathcal{H}^\alpha_\delta(A)$ is monotonically nonincreasing: if δ increases, the set of covers increases and hence the infimum can only decrease or stay the same. By monotone convergence theorem the limit tends to its supremum.

The set function $\mathcal{H}^s_\infty$ is called the *s-dimensional Hausdorff content*.

We will need some basic properties of the measure and content, which can be proven directly from the definitions. For example, we have $\mathcal{H}^1(I) = \mathcal{H}^1_\infty(I) = \operatorname{diam} I$ for intervals in $I \subset \mathbb{R}$. Next we define a couple of two useful versions of Hausdorff measures and briefly explore their relation to the original Hausdorff measure.

If we restrict the set of possible covers $\mathcal{F}$ to be the set of all metric balls or all dyadic cubes, we obtain the so-called *spherical* and *dyadic* Hausdorff measures, respectively.

Theorem 1.4. (Spherical Hausdorff Measure). *In the definition 1.3 of Hausdorff pre-measures, let $\mathcal{F}$ be the family of all metric balls in (Z,d). Define this version of Hausdorff pre-measure to be spherical Hausdorff pre-measure, and denote it by Φ^s_δ. Then*

$$\mathcal{H}^s_\delta(A) \leq \Phi^s_\delta(A) \leq 2^s \mathcal{H}^s_{2\delta}(A)$$

for all $A \subset Z$ and all $\delta \geq 0$.

Proof. Let $A \subset Z$ and $\delta > 0$ (the case $\delta = 0$ follows by sending $\delta \to 0$). Now for $\mathcal{H}^s_\delta$ we consider all possible covers consisting of subsets of Z, but for Φ^s_δ consider only covers consisting of metric balls. If we let $\{B_i\}$ be a countable set of balls such that $A \subset \bigcup_{i=1}^\infty B_i$, and then $\{A_i\}$ be an arbitrary countable cover of A (in particular, these $\{A_i\}$-covers contain every metric ball cover), then it is obvious that

$$\begin{aligned}\mathcal{H}^s_\delta(A) &= \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} A_i)^s : A \subset \bigcup_{i=1}^{\infty} A_i, \operatorname{diam} A_i < \delta, A_i \subset Z\right\}\\ &\leq \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} B_i)^s : A \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam} B_i < \delta, B_i \in \mathcal{F}\right\}\\ &= \Phi^s_\delta(A).\end{aligned}$$

Let $\epsilon > 0$. Since A_i is an arbitrary countable cover for A, we have that $A_i \subset B_i = B(x_i, (\operatorname{diam} A_i)(1+\epsilon))$ if $x \in A_i$. Thus, $\operatorname{diam} B_i \leq 2 \operatorname{diam} A_i < 2\delta$. Raising these to the power of s and summing over all indexes we find $\sum_i (\operatorname{diam} B_i)^s \leq 2^s \sum_i (\operatorname{diam} A_i)^s$. Then letting A_i run through all possible covers and taking the infimum of these sums we find,

$$\begin{aligned}
\Phi^s_\delta(A) &= \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} B_i)^s : A \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam} B_i < 2\delta, B_i \in \mathcal{F}\right\} \\
&\le \inf\left\{2^s\sum_{i=1}^{\infty}(\operatorname{diam} A_i)^s : A \subset \bigcup_{i=1}^{\infty} A_i, \operatorname{diam} A_i < \delta, A_i \subset Z\right\} \\
&= 2^s \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} A_i)^s : A \subset \bigcup_{i=1}^{\infty} A_i, \operatorname{diam} A_i < \delta, A_i \subset Z\right\} \\
&= 2^s \mathcal{H}^s_{2\delta}(A).
\end{aligned}$$

□

Remarks. These are called *spherical Hausdorff measures* for obvious reasons. Below we prove a similar result for *dyadic Hausdorff measures* on $\mathbb{R}^n$.

Definition 1.5. (Dyadic Cubes in $\mathbb{R}^n$). Let $a = (\alpha_1, .., \alpha_n) \in \mathbb{R}^n$. Define $\mathcal{D}_k$ to be the set of half-open cubes $\{(x_1, ..., x_n) \in \mathbb{R}^n : \alpha_i \le x_i < \alpha_i + \frac{1}{2^k}, 1 \le i \le n\}$ each having a side length of $\frac{1}{2^k}$ and the point a as one of the corners of the cube. These corners belong to the set

$$\{2^{-k}(x_1, ..., x_n) : x_i \in \mathbb{Z}\},$$

Then the union $\bigcup_{k=1}^{\infty} \mathcal{D}_k = \mathcal{D}$ is called the set of all *half-open dyadic cubes.*

Remarks. The most important property, at least for our purposes, is that the set $\mathcal{D}_k$ partitions $\mathbb{R}^n$. That is, we can write $\mathbb{R}^n$ as a pairwise disjoint union $\bigcup_{Q \in \mathcal{D}_k} Q$.

Theorem 1.6. (Dyadic Hausdorff Measure). *In the definition 1.3 of Hausdorff pre-measures, let $\mathcal{F}$ be the set of all dyadic cubes in $\mathbb{R}^n$. Define this version of Hausdorff pre-measure to be dyadic Hausdorff pre-measure, and denote it by Ψ^s_δ. Then*

$$\mathcal{H}^s_\delta(A) \le \Psi^s_\delta(A) \le (4\sqrt{n})^s \mathcal{H}^s_{4\sqrt{n}\delta}(A)$$

for all $A \subset \mathbb{R}^n$ and all $\delta \ge 0$.

Proof. Let $A \subset \mathbb{R}^n$ and $\delta > 0$. Suppose $\{C_i\}$ is a countable set of dyadic cubes such that $A \subset \bigcup_{i=1}^{\infty} C_i$. Let $\{A_i\}$ be an arbitrary countable cover of A. As for the spherical measure, we note that these covers $\{A_i\}$ contain every dyadic cover, and hence,

$$\begin{aligned}
\mathcal{H}^s_\delta(A) &= \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} A_i)^s : A \subset \bigcup_{i=1}^{\infty} A_i, \operatorname{diam} A_i < \delta, A_i \subset Z\right\} \\
&\le \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} C_i)^s : A \subset \bigcup_{i=1}^{\infty} C_i, \operatorname{diam} C_i < \delta, C_i \in \mathcal{F}\right\} \\
&= \Psi^s_\delta(A).
\end{aligned}$$

Then we note that $A_i \subset C_i$ where $\operatorname{diam} C_i = 2\sqrt{n} \operatorname{diam} A_i$ for some dyadic point. Thus $\operatorname{diam} C_i \le 4\sqrt{n} \operatorname{diam} A_i$. Raising these to the power of s and summing over all indexes we find $\sum_i (\operatorname{diam} C_i)^s \le (4\sqrt{n})^s \sum_i (\operatorname{diam} A_i)^s$. Then letting A_i run through all possible covers and taking the infimum of these sums we find,

$$\begin{aligned}\Psi^s_\delta(A) &= \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} C_i)^s : A \subset \bigcup_{i=1}^\infty C_i, \operatorname{diam} C_i < \delta, C_i \in \mathcal{F}\right\}\\ &\le \inf\left\{2^s\sum_{i=1}^\infty (\operatorname{diam} A_i)^s : A \subset \bigcup_{i=1}^\infty A_i, \operatorname{diam} A_i < 4\sqrt{n}\delta, A_i \subset Z\right\}\\ &= (4\sqrt{n})^s \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} A_i)^s : A \subset \bigcup_{i=1}^\infty A_i, \operatorname{diam} A_i < 4\sqrt{n}\delta, A_i \subset Z\right\}\\ &= (4\sqrt{n})^s \mathcal{H}^s_{4\sqrt{n}\delta}(A).\end{aligned}$$

□

Definition 1.7. (Hausdorff Dimension). *Hausdorff dimension* of a set $A \subset (Z,d)$, denoted by $\dim A$, is the unique value satisfying

$$\dim A := \inf\{s \ge 0 : \mathcal{H}^s(A) = 0\} = \sup\{s \ge 0 : \mathcal{H}^s(A) = \infty\}$$

Next we will prove a couple of basic properties of the Hausdorff dimension to show how the dimension behaves. They are the monotonicity and countable stability.

Theorem 1.8. (Properties of Hausdorff Dimension). *Let Z be a metric space. Then we have the following.*

(1) *If $A \subset B$, then* $\dim A \le \dim B$. (Monotonicity)
(2) *If $A_i \subset Z$ for every $i \in I$, where I is at most countable index set, then* $\dim\left(\bigcup_{i\in I}^\infty A_i\right) = \sup_{i\in I}\dim A_i$. (Countable Stability)

Proof.

(1) If there exists s such that $\mathcal{H}^s(B) = 0$, then surely $\mathcal{H}^s(A) = 0$ since every cover of B also covers A. Hence, if $\mathcal{H}^s(B) = 0$, then $\mathcal{H}^s(A) = 0$, which implies $\{s \ge 0 : \mathcal{H}^s(B) = 0\} \subset \{s \ge 0 : \mathcal{H}^s(A) = 0\}$. Now take infimums to conclude $\dim A = \inf\{s \ge 0 : \mathcal{H}^s(A) = 0\} \le \inf\{s \ge 0 : \mathcal{H}^s(B) = 0\} = \dim B$.
(2) Suppose $\sup_i \operatorname{diam} A_i < \infty$ and write $d = \dim\left(\bigcup_{i\in I}^\infty A_i\right)$. By the monotonicity of the dimension, we have $\dim A_k \le d$ for every k since $A_k \subset \bigcup_{i\in I}^\infty A_i$.

    Next let $\epsilon > 0$ and write $t = d - \epsilon$. For the sake of contradiction suppose there exists s satisfying $t < s < d$ and having the property that $\mathcal{H}^s(A_i) = 0$ for every i. This means that for every i there exists a cover $\{A_{i_j}\}$ of A_i such that when taking the infimum of the sum $\sum_j (\operatorname{diam} A_{i_j})^s$ for covers with smaller and smaller diameters, the sum tends to zero.

    To be more precise, let $\delta > 0$ and then choose $0 < \eta < \delta 2^{-i}$. Next choose a cover $\{A_{i_j}\}$ of A_i such that $\operatorname{diam} A_{i_j} < \eta 2^{-j}$, that is, $A_i \subset \bigcup_j A_{i_j}$. If we do this for every i we find that the union $\bigcup_j A_{i_j}$ also covers $\bigcup_i A_i$, that is, $\bigcup_i A_i \subset \bigcup_i \bigcup_j A_{i_j}$. Hence, $\sum_i (\operatorname{diam} A_i)^s \le \sum_i \sum_j (\operatorname{diam} A_{i_j})^s \to 0$ as $\delta \to 0$. Therefore, we find $\mathcal{H}^s\left(\bigcup_i A_i\right) = 0$. But this is a contradiction since $s < d$. Thus, we conclude that there must exist at least one A_i with the property that $\mathcal{H}^s(A_i) \ne 0$. This in turn implies $s \le \dim A_i$ which in turn shows that there exists at least one set A_i with the property $d - \epsilon < \dim A_i \le d$ for every $\epsilon > 0$. This proves that $d = \sup_i \dim A_i$. □

Since the Hausdorff measure of a set can be bounded by the spherical and dyadic Hausdorff measures, as we proved above in the Theorem 1.4 *Spherical Hausdorff Measure* and the Theorem

1.6 *Dyadic Hausdorff Measure*, we can replace the Hausdorff measure by the spherical or dyadic version in the definition of the dimension. One useful basic result relating a curve's 1-dimensional Haudorff content and its diameter is used in the proof of our first main result for finding lower bounds for conformal dimension. Namely,

Theorem 1.9. (Bound for Connected Space). *If (Z,d) is a connected metric space, then* $\operatorname{diam} Z \leq \mathcal{H}^1_\infty(Z)$.

To prove this we use the following interesting result about Hausdorff dimension under Lipschitz maps. Recall that a *homeomorphism* is any continuous bijection whose inverse is also continuous. If $f:(Z,d)\to(Z',d')$ is a map satisfying

$$d'(f(x),f(y)) \leq Ld(x,y)$$

for every $x,y\in Z$ and for some fixed $0<L<\infty$, then f is *L-Lipschitz*. If, in addition, f is a homeomorphism, and f^{-1} is also L-Lipschitz, we say that f is L-bi-Lipschitz.

Theorem 1.10. (Dimensions under Mappings). *If $f:(Z,d)\to(Z',d')$ is $L-$Lipschitz, then $\mathcal{H}^s_{L\delta}(f(A)) \leq L^s\mathcal{H}^s_\delta(A)$ for all $A\subset Z$ and $0\leq\delta\leq\infty$.*

Proof. Fix $0\leq\delta\leq\infty$, $s\geq 0$ and $A\subset Z$. Let $\{A_i\}$ be a countable cover for A with $\operatorname{diam} A_i<\delta$. Then $\{f(A_i)\}$ is a cover for $f(A)$ such that $\operatorname{diam} f(A_i)\leq L\operatorname{diam} A_i < L\delta$. From this inequality we obtain $\sum(\operatorname{diam} f(A_i))^s \leq \sum(L\operatorname{diam} A_i)^s$. It is clear that $\inf\sum(\operatorname{diam} C_i)^s \leq \sum(\operatorname{diam} f(A_i))^s \leq L^s\inf\sum(\operatorname{diam} A_i)^s$ where each C_i satisfies $\operatorname{diam} C_i < L\delta$ and $f(A)\subset\bigcup_i C_i$, and the infimums are taken over all countable covers. This proves the result. □

Remarks. In particular, if f is L-bi-Lipschitz, then $\dim f(A)=\dim A$.

Now we can give a proof of the Theorem 1.9 *Bound for Connected Space*, adapted from [25, Chapter 1].

Proof of Bound for Connected Space. First suppose that Z is unbounded. It is elementary to verify that $f:Z\to\mathbb{R}, f(z)=d(z,y)$, $y\in Z$ fixed, is 1-Lipschitz. Then $f(Z)=[0,\infty]$, since the domain contains z_0 and is unbounded. By the Theorem 1.10 *Dimensions under Mappings*,

$$\infty = \mathcal{H}^1_\infty([0,\infty]) = \mathcal{H}^1_\infty(f(Z)) \leq \mathcal{H}^1_\infty(Z).$$

Next assume Z is bounded and pick $\epsilon>0$. Then there exists $y_0\in Z$ such that

$$\operatorname{diam} Z - \epsilon \leq d(z,y_0).$$

Define $f(z):=d(z,y_0)$. Since Z is connected, $f(Z)$ is an interval containing 0. Since $f(y_0)=0$,

$$\operatorname{diam} Z - \epsilon \leq d(z,y_0) = |f(z)-f(y_0)| \leq \mathcal{H}^1_\infty(f(Z)) \leq \mathcal{H}^1_\infty(Z)$$

by application of the Theorem 1.8. *Dimension under Mappings.* Arbitrariness of ϵ establishes the assertion. □

Borel sets are sets belonging to the *Borel σ-algebra*, which is the smallest σ-algebra that contains all the open sets of the space. When we say that μ is a Borel measure we mean that every Borel set is μ-measurable. Metric space in which we have a positive Borel measure μ with $\mu(B(x,r))<\infty$ for every $x\in X$ and $r>0$ is called a *metric measure space* and denoted by (X,d,μ).

Definition 1.11. (Doubling Spaces). Let (Z, d) be a metric space. Then Z is called N-*doubling* if for every ball $B(z, r) \subset Z$ there exists some constant $N < \infty$ and some finite set of points $\{x_i\}_{i=1}^N$ such that $\overline{B(z,r)} \subset \bigcup_{i=1}^N B(x_i, \frac{r}{2})$. If Z is N-doubling for some N, we say Z is *metric doubling.*

Measured doubling space is a metric measure space (X, d, μ) such that there exists a constant $C(\mu)$ with the property $\mu(B(x, 2r)) \leq C(\mu)\mu(B(x, r))$ for every ball $B(x, r)$ in X.

In other words, if we can cover any ball of radius r with at most N balls having radii $\frac{r}{2}$, then the space satisfies this metric doubling property. For example, $\mathbb{R}$ is a doubling metric measure space endowed with the usual Euclidean metric and with 1-dimensional Hausdorff measure. It satisfies the metric as well as the measured doubling condition with constants $N = C(\mu) = 2$. For a simple example of a non-doubling space, one can look at the aforementioned discrete metric space.

If ambiguity between these two notions is possible, we differentiate between them by pointing out explicitly that we are referring to the *metric* or *measured* doubling condition. However, if our space is complete, then this distinction is redundant: complete and metric doubling space (X, d) admits a doubling measure, and if we have a metric measure space (X, d, μ) equipped with a doubling measure μ, then it is metric doubling [23].

In this thesis, we will use the Lebesgue theory of integration with arbitrary measures. This theory is presented in the abstract form in [35]. Thus, we only state the results that we use in this thesis without further discussion. It should be noted, that we do not always explicitly state the domain of integration, and hence we sometimes write $\int f$ instead of $\int_X f$, whenever the domain of integration is clear from the context.

Given a measure space $(X, \mathfrak{M}, \mu)$ and a topological space (Y, τ_Y), we say that a function $f : X \to Y$ is called *measurable* if $f^{-1}(U) \in \mathfrak{M}$ whenever $U \in \tau_Y$.

Definition 1.12. (L^p-space). Let (X, μ) be any measure space and $1 \leq p < \infty$. Then the vector space of all functions $f : X \to \mathbb{C}$ that satisfy

$$\left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} < \infty,$$

is called the L^p-space and denoted by $L^p(X, \mu)$, $L^p(\mu)$ or just L^p.

If we identify functions that are equal only μ almost everywhere, then this space becomes a metric space, and in fact, a Banach space [35, Chapter 3]. The elements are not precisely functions, but equivalence classes. However, for simplicity purposes we can think of them as functions and speak of a "function space".

Theorem 1.13. (Basic Properties of the Integral). *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $f, g : X \to [-\infty, \infty]$ are in $L^p(X, \mathfrak{M}, \mu)$ and $A, B \subset X$ are measurable sets. Then we have:*

1. *If $f \leq g$, then $\int_A f \, d\mu \leq \int_A g \, d\mu$.*
2. $\left| \int_A f \, d\mu \right| \leq \int_A |f| \, d\mu$
3. *If $A \subset B$, then $\int_A f \, d\mu \leq \int_B f \, d\mu$*
4. *If $A \cap B = \emptyset$, $f \geq 0$, then $\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$*
5. *If $C \in \mathbb{R}$, then $\int C(f + g) \, d\mu = C \int f \, d\mu + C \int g \, d\mu$.*
6. *If $f = g$ μ-almost everywhere, then $\int_A f \, d\mu = \int_A g \, d\mu$.*
7. *If $\mu(A) = 0$, then $\int_A f \, d\mu = 0$.*
8. $\int_A f \, d\mu = \int_X \chi_A f \, d\mu$.

Theorem 1.14. (Lebesgue's Monotone Convergence Theorem). *Let $\{f_n\}$ be a sequence of measurable functions in a measure space (X, μ). If the sequence is non-negative, monotonic and convergent pointwise, that is,*

(1) $0 \leq f_1(x) \leq f_2(x) \leq \cdots$ *for every* $x \in X$ *and*
(2) $f_n(x) \to f(x)$ *for every* $x \in X$,

then the limit function is measurable, and we have

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

We will sometimes use a corollary to this result, which is stated as follows. *If $\{f_n\}$ is a sequence of non-negative measurable functions in a measure space (X, μ) and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for every $x \in X$, then*

$$\sum_{n=1}^{\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Closely related to L^p-space are the fundamental result called Hölder's inequality, which emerges in the daily work of an analyst as well as in many proofs in this thesis, and the notion of duality. The *dual space* of $L^p(X, \mu)$ is $L^{\frac{p}{p-1}}(X, \mu)$. We do not need the specific notion of duality in this thesis and therefore we refer the interested reader to the Theorem 6.16 in [35] for more information on this relation between $L^p(X, \mu)$ and $L^{\frac{p}{p-1}}(X, \mu)$.

Theorem 1.15. (Hölder Inequality). *Let f, g be measurable functions in (X, μ), and $1 < p < \infty$. Then*

$$\int fg \, d\mu \leq \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}} \left(\int |g|^{\frac{p}{p-1}} \, d\mu \right)^{\frac{p-1}{p}}.$$

Theorem 1.16. (L^p Duality). *If $1 < p < \infty$, then for any $u \in L^p(X, \mu)$ we have*

$$||u||_p = \sup_{||g||_{\frac{p}{p-1}}=1} \left| \int ug \, d\mu \right|,$$

where the supremum is taken over all functions $g \in L^{\frac{p}{p-1}}$ with norm 1.

Remark. The proof of the $L^p(X, \mu)$-duality is sketched in the Appendix 11.3 of [20] and can be found in fully rigorous form in [37, Chapter 1].

Definition 1.17. (Average of a Function). We define the average of a function $f \in L^p(A, \mu)$ over a set A, with $0 < \mu(A) < \infty$, as

$$⨍_A f \, d\mu := \frac{1}{\mu(A)} \int_A f \, d\mu.$$

Definition 1.18. (Maximal Function). If $f : X \to Y$ has for every $p \in X$ a neighborhood N_p for which $f \in L^p(N)$, we say that f is locally p-integrable. Then we define the centered *maximal function*, Mf, as

$$M(f)(x) := \sup_{r>0} ⨍_{B(x,r)} |f| \, \mu.$$

The following result and its proof is the content of [16, Chapter 2].

Theorem 1.19. (Maximal Function Theorem). *Let (X, d, μ) be a doubling metric measure space and $p > 1$. Then the maximal operator sends L^p to L^p, and we have,*

$$\int |M(f)|^p \, d\mu \leq C(p, C(\mu)) \int |f|^p \, d\mu.$$

Theorem 1.20. (Fubini's Theorem). *Suppose we have a measure space $(X \times Y, \mu \times \lambda)$ and $\mu \times \lambda$-measurable f. If*

$$\int_{X \times Y} |f| \, d(\mu \times \lambda) < \infty,$$

then

$$\int_X \left(\int_Y f(x, y) \, d\lambda(y) \right) d\mu(x) = \int_{X \times Y} f \, d(\mu \times \lambda) = \int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\lambda(y).$$

Definition 1.21. (Ahflors Regular Space). If a metric measure space (Z, d, μ) equipped with a Borel measure μ satisfies

$$\frac{1}{C} r^s \leq \mu(\overline{B}(z, r)) \leq C r^s$$

for all balls $B(z, r)$ in Z with $r < \operatorname{diam} Z$ and for some $s > 0$, where $C < \infty$, we say that (Z, d) is *Ahlfors s-regular* and μ is an *Ahlfors regular measure* on (Z, d). In addition, (Z, d) is *Ahlfors regular* if it is Ahlfors s-regular for some $s > 0$.

Intuitively, this definition gives a rough measure of the self-similarity of the space, in the sense that every ball with fixed radius must be comparable in size, and scale like the fixed power of the fixed radius. *Self-similarity* means, informally speaking, that no matter how close into the space or how far away from the space one zooms, or which location one decides to look at, the space always looks the same.

Theorem 1.22. (Ahlfors s-Regular $\mathcal{H}^s$). *If a metric space is Ahlfors s-regular equipped a Borel measure μ, then it is Ahlfors s-regular for the s-dimensional Hausdorff measure.*

Proof. Suppose (Z, d) is Ahlfors s-regular for a Borel measure μ. Let $\{A_i\}$ be any countable cover for an arbitrary ball $B(x, r) \subset Z$. Then by the Ahlfors-regularity of μ,

$$\frac{1}{C} r^s \leq \mu(B(x, r)) \leq \sum_i \mu(A_i) \leq \sum_i \mu(B(x_i, \operatorname{diam} A_i)) \leq C \sum_i (\operatorname{diam} A_i)^s,$$

where each $x_i \in A_i$. Hence, for some constant $K := C^2$, $\frac{1}{K} r^s$ is a lower bound for $\sum_i (\operatorname{diam} A_i)^s$. But $\mathcal{H}^s_\infty(B(x, r))$ is the greatest lower bound for arbitrarily sized covers. Thus, $\frac{1}{K} r^s \leq \mathcal{H}^s_\infty(B(x, r)) \leq \mathcal{H}^s(B(x, r))$.

For the other direction, note that we have $\frac{1}{C} r^s \leq \mu(B(z, r))$. We shall apply the Theorem 2.10 *5r Covering Lemma* by extracting from arbitrary ball cover of $B(x, r)$ a countable ball cover $\{B_i := B(x_i, 5r_i)\}$ of $B(x, r)$ whose radii satisfy $r_i < \delta$ for every i and $\frac{1}{5} B_i \cap \frac{1}{5} B_j = \emptyset$ whenever $i \neq j$. Then by the definition of the Hausdorff pre-measure,

$$\begin{aligned}
\mathcal{H}^s_\delta(B(x, r)) &\leq \sum_i (\operatorname{diam} B_i)^s \leq \sum_i (2r_i)^s = 10^s \sum_i \left(\frac{1}{5} r_i \right)^s \leq 10^s C \sum_i \mu \left(\frac{1}{5} B_i \right) \\
&\leq C' \mu(B(x, 2r)) \leq C'' r^s.
\end{aligned}$$

Sending $\delta \to 0$, we obtain $\mathcal{H}^s(B(x,r)) \le C'\mu(B(x,r))$. In conclusion, we have proved

$$\frac{1}{C}\mathcal{H}^s(B(x,r)) \le \mu(B(x,r)) \le C\mathcal{H}^s(B(x,r))$$

and

$$\frac{1}{C}r^s \le \mathcal{H}^s(\overline{B}(z,r)) \le Cr^s.$$

□

Remarks. This shows that we do not lose too much information if we use the s-dimensional Hausdorff measure instead of μ. Thus, in an Ahlfors s-regular space we can justifiably call the s-dimensional Hausdorff measure a *s-Ahlfors regular Hausdorff measure.*

We finish this introduction with a definition and some fundamental properties of the most important mappings used in this thesis.

Definition 1.23. (Quasisymmetry). We call a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ a *control function.* Let $f : (Z,d) \to (Z',d')$ be a homeomorphsim and η a control function. The map f is said to be *η-quasisymmetric* if

$$\frac{d'(f(x),f(y))}{d'(f(x),f(z))} \le \eta\left(\frac{d(x,y)}{d(x,z)}\right) \tag{1}$$

whenever $x,y,z \in Z$ and $x \ne z$. We say f is *quasisymmetric* if it is η-quasisymmetric for some control function η.

Note that if $f : Z \to Z'$ is an η-quasisymmetry, and we have $x,y,z \in Z$ and $t \ge 0$ satisfying $\frac{d(x,y)}{d(x,z)} \le t$, then $\frac{d'(f(x),f(y))}{d'(f(x),f(z))} \le \eta\left(\frac{d(x,y)}{d(x,z)}\right) \le \eta(t)$. Hence, $d'(f(x),f(y)) \le \eta(t)d'(f(x),f(z))$. Conversely, suppose $f : Z \to f(Z) \subset Z'$ is a homeomorphism and $\frac{d(x,y)}{d(x,z)} \le t$ implies $\frac{d'(f(x),f(y))}{d'(f(x),f(z))} \le \eta(t)$. Then let $x,y,z \in Z$ be arbitrary and write $0 \le \frac{d(x,y)}{d(x,z)} = t$. Thus, we obtain $\frac{d'(f(x),f(y))}{d'(f(x),f(y))} \le \eta(t) = \eta\left(\frac{d(x,y)}{d(x,z)}\right)$. Therefore we have shown that an embedding f is an η-quasisymmetry if and only if $d(x,y) \le td(x,z)$ implies $d'(f(x),f(y)) \le \eta(t)d'(f(x),f(z))$ for every $t \ge 0$. This characterization of quasisymmetries will be implicitly applied in many instances. For example, in the Theorem 1.25 *Basic Properties of Quasisymmetries.*

We say that Z and Z' are *quasisymmetrically equivalent* if there exists a quasisymmetry $f : Z \to Z'$. This relation is shown to be an equivalence relation below; thus justifying the terminology.

Sometimes it will be useful to have a weaker notion of quasisymmetry, namely, we say that $f(Z,d) \to (Z',d')$ is K-quasisymmetric if the condition (1) in the Definition 1.23 *Quasisymmetry* holds for some control function η satisfying $\eta(1)\eta(1/K) \le 1$. Note that the use of the word "weak" is justified in the sense that if f is a quasisymmetry, then f is always K-quasisymmetric for some $K < \infty$.

Every result about the properties of quasisymmetries discussed in this thesis is based on [39].

Theorem 1.24. (Quasisymmetry Equivalence). *The existence of quasisymmetry between metric spaces is an equivalence relation.*

Proof. Write $Z \cong Z'$ if there exists a quasisymmetry from (Z,d) and (Z',d'). The identity map $f:(Z,d)\to(Z,d), f(z)=z$ of Z to itself is a homeomorphism. Let x,y,z with $x\neq z$. Then

$$\frac{d(f(x),f(y))}{d(f(x),f(z))}=\frac{d(x,y)}{d(x,z)}=\eta\left(\frac{d(x,y)}{d(x,z)}\right)$$

With control function $\eta(t)=t$. Thus, $Z\cong Z$ and we have shown reflexivity.

Next suppose $Z\cong Z'$. Then let f be an η-quasisymmetry map between the spaces. We claim that the inverse of an η-quasisymmetric homeomorphism is $\tilde{\eta}$-quasisymmetric with control function $\tilde{\eta}(t)=\frac{1}{\eta^{-1}(\frac{1}{t})}$. To see this let $f:(Z,d)\to(Z',d')$. Then $f^{-1}:(Z',d')\to(Z,d)$. Let $x,y,z\in(Z',d'), x\neq z$. Write $f(x')=x, f(y')=y, f(z')=z$. Now we have

$$\frac{d'(x,z)}{d'(x,y)}=\frac{d'(f(x'),f(z'))}{d'(f(x'),f(y'))}\leq\eta\left[\frac{d(x',z')}{d(x',y')}\right],$$

which implies, since η^{-1} is (strictly) increasing being the inverse of a homeomorphism from positive reals to itself, that

$$\eta^{-1}\left[\frac{d'(x,z)}{d'(x,y)}\right]\leq\frac{d(x',z')}{d(x',y')}.$$

Taking reciprocals we find

$$\frac{d(x',y')}{d(x',z')}=\frac{d(f^{-1}(x),f^{-1}(y))}{d(f^{-1}(x),f^{-1}(y))}\leq\frac{1}{\eta^{-1}\left[\dfrac{d'(x,z)}{d'(x,y)}\right]}=\tilde{\eta}\left[\frac{d'(x,y)}{d'(x,z)}\right]$$

where $\tilde{\eta}(t)=\frac{1}{\eta^{-1}(\frac{1}{t})}$. Thus $Z'\cong Z$ and we have shown symmetry.

Lastly suppose $Z\cong Z'$ and $Z'\cong Z''$. Let $f_1:Z\to Z'$ and $f_2:Z'\to Z''$ be η_1- and η_2-quasisymmetries, respectively. Suppose $x,y,z\in(Z,d)$ with $z\neq x$. Then

$$\frac{d''(f_2(f_1(x)),f_2(f_1(y)))}{d''(f_2(f_1(x)),f_2(f_1(z)))}\leq\eta_2\left[\frac{d'(f_1(x),f_1(y))}{d'(f_1(x),f_1(z))}\right]\leq\eta_2\left[\eta_1\left[\frac{d(x,y}{d(x,z}\right]\right].$$

Hence, $f_2\circ f_1:Z\to Z''$ is an $\eta_2\circ\eta_1$-quasisymmetry. Since compositions of homeomorphisms are always homeomorphic, we have $Z\cong Z''$ and thus we find that transitity holds. Further this shows that quasisymmetry is in fact an equivalence relation between metric spaces. Thus, we may say that metric spaces are equivalent up to a quasisymmetry. ☐

Note that the above proof also shows that the set of quasisymmetries is a group of transformations with identity function playing the role of the identity in the group structure.

Theorem 1.25. (Basic Properties of Quasisymmetries).

(1) *If $f:(Z,d)\to(Z',d')$ is a quasisymmetry and $A\subset Z$, then $f\big|_A:A\to f(A)$ is a quasisymmetry.*

(2) *Let (Z,d) and (Z',d') be metric spaces and $A\subset B\subset Z$ with $\operatorname{diam}A>0$ and $\operatorname{diam}B<\infty$. If $f:Z\to Z'$ is an η-quasisymmetric map, then $\operatorname{diam}f(B)<\infty$, and we have the following inequality*

$$\frac{1}{2\eta(\frac{\operatorname{diam}B}{\operatorname{diam}A})}\leq\frac{\operatorname{diam}f(A)}{\operatorname{diam}f(B)}\leq\eta\left(\frac{2\operatorname{diam}A}{\operatorname{diam}B}\right). \tag{2.1}$$

Proof. (1) Restriction of a homeomorphism to a subset of $A \subset Z$ and to its image $f(A)$ is clearly again a homeomorphism. Let the metric on A and $f(A)$ be the inherited metric from Z and Z', respectively. Next suppose $x, y, z \in A \subset Z$, $x \neq z$. Then $f(x), f(y), f(z) \in f(A) \subset Z'$. Thus,

$$\frac{d'(f(x), f(y))}{d'(f(x), f(z))} \leq \eta\left(\frac{d(x,y)}{d(x,z)}\right).$$

(2) Choose two sequences $b_i, b_i' \subset B$ satisfying

$$\frac{1}{2}\operatorname{diam} B \leq d(b_i, b_i') \to \operatorname{diam} B \qquad \text{as } i \to \infty.$$

First we show that $f(B)$ is bounded. To do this pick any $b \in B$. We shall show that for a fixed point $f(b_1) \in f(B)$, the distance from $f(b)$ is always bounded above. Now, $d(b, b_1) \leq \operatorname{diam} B \leq 2d(b_1, b_1')$. Since f is an η-quasisymmetry, this implies $d'(f(b), f(b_1)) \leq \eta(2)d'(f(b_1), f(b_1')) < \infty$. Note that $d'(f(b_1), f(b_1')) < \infty$, since f is continuous and b_1, b_1' are fixed points in a bounded set.

Next we prove the inequality (2.1). We begin by showing that the rightmost inequality holds. Choose any $x, y \in A$. Then by the triangle inequality

$$d(b_i, b_i') \leq d(b_i, y) + d(y, b_i').$$

Hence, $d(b_i, b_i') \leq 2\max\{d(y, b_i'), d(y, b_i)\}$. Suppose $d(b_i, b_i') \leq 2d(y, b_i)$. This implies

$$\begin{aligned} d'(f(x), f(y)) \leq \eta\left(\frac{d(x,y)}{d(b_i,y)}\right) d'(f(b_i), f(y)) &\leq \eta\left(\frac{2\operatorname{diam} A}{d(b_i, b_i')}\right)\operatorname{diam} f(B) \\ &\to \eta\left(\frac{2\operatorname{diam} A}{\operatorname{diam} B}\right)\operatorname{diam} f(B), \end{aligned}$$

as $i \to \infty$. Recall that since η is a homemomorphism from positive reals to positive reals, it is strictly increasing. If $d(b_i, b_i') \leq 2d(y, b_i')$, the same reasoning works. Just replace b_i by b_i' in the above sequence of inequalities. This proves the right-side inequality in (2.1).

The left-side inequality in (2.1) follows from the fact that $f^{-1} : Z' \to Z$ is a $\hat{\eta}(t)-$quasisymmetry, where $\hat{\eta}(t) = \frac{1}{\eta^{-1}(\frac{1}{t})}$ (see the Theorem 1.24 *Quasisymmetry Equivalence* for details). We will apply the right-side inequality to f^{-1}. Hence,

$$\frac{\operatorname{diam} f^{-1}(f(A))}{\operatorname{diam} f^{-1}(f(B))} \leq \hat{\eta}\left(\frac{2\operatorname{diam} f(A)}{\operatorname{diam} f(B)}\right) = \frac{1}{\eta^{-1}\left(\frac{\operatorname{diam} f(B)}{2\operatorname{diam} f(A)}\right)}$$

which implies after taking reciprocals and operating with η on both sides,

$$\begin{aligned} & \frac{\operatorname{diam} A}{\operatorname{diam} B} \leq \frac{1}{\eta^{-1}\left(\frac{\operatorname{diam} f(B)}{2\operatorname{diam} f(A)}\right)} \\ \iff \quad & \frac{\operatorname{diam} f(B)}{2\operatorname{diam} f(A)} \leq \eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right) \\ \iff \quad & \frac{1}{2\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right)} \leq \frac{\operatorname{diam} f(A)}{\operatorname{diam} f(B)}. \end{aligned}$$

□

Definition 1.26. (Quasiconformal Mappings). Suppose $F : \Omega \to \Omega'$, where $\Omega, \Omega' \subset \mathbb{R}^n$, is a homeomorphism. Then define the *linear dilatation* of F to be a function $L_F : \Omega \to \mathbb{R}$,

$$L_F := \limsup_{r \to 0} \frac{\sup\{|F(x) - F(y)| : |x - y| = r\}}{\inf\{|F(x) - F(z)| : |x - z| = r\}}.$$

If there exists $1 \leq H < \infty$ such that $L_F(x) \leq H$ for every $x \in \Omega$, then we say that F is H-quasiconformal.

As was mentioned before, quasisymmetries are generalizations of quasiconformal mappings.

Theorem 1.27. (Quasiconformal-Quasisymmetrical Relation). *If F is η-quasisymmetric, then it is H-quasiconformal with $H = \eta(1)$.*

Proof. Suppose $F : \mathbb{R}^n \supset \Omega \to \Omega' \subset \mathbb{R}^n$ is η-quasisymmetry. Then we let $x \in \Omega$ be arbitrary and choose arbitrary $y, z \in \Omega$ such that $|x - y| = |x - z| = r$. Then we find

$$\frac{|F(x) - F(y)|}{|F(x) - F(z)|} \leq \eta\left(\frac{|x - y|}{|x - z|}\right) = \eta(1)$$

$$\iff \quad \frac{\sup\{|F(x) - F(y)| : |x - y| = r\}}{\inf\{|F(x) - F(z)| : |x - z| = r\}} \leq \eta(1).$$

Hence, the dilatation function L_F is uniformly bounded for every $x \in \Omega$ by the constant $\eta(1) = H$. Letting $r \to 0$, we find that the limsup is less than $\eta(1)$, and therefore we conclude F is H-quasiconformal. □

Remarks. Note that this does not prove that the notions are equivalent. The other direction in the equivalence relation depends on the geometry of Ω and Ω', and, in the general case, is a deep result in function theory. In fact, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal, $n \geq 2$, then F is η-quasisymmetric. The restriction of the dimension to $n \geq 2$ is necessary. For a counterexample one can consider $f : \mathbb{R} \to \mathbb{R}, f(x) = x + e^x$. A thorough discussion of this relation can be found in [40, Section 34].

The outline for this thesis is the following. First, we will survey the basic theory of conformal dimension beginning with definitions and some fundamental properties. Then we will give an instructive example by calculating the conformal dimension of the middle thirds Cantor set.

After this, we will delve into the theory in more detail and examine three results for obtaining lower bounds for the conformal dimension. One of these results will be used in the study of the Sierpiński carpet. Applications for the rest of these lower bound results can be found, for instance, in [25].

In the third chapter, we will shortly focus on concepts from metric geometry in order to discuss the relationship between conformal dimension and tangent spaces. This chapter will culminate in a powerful result, namely, the Theorem of Keith and Laakso, which is then used to study our main example, the Sierpiński carpet.

Finally, we will briefly mention one recent approach to the conformal dimension of the Sierpiński carpet as well as some related investigations.

2. Theory of Conformal Dimension

In this chapter we will define the notion of conformal Hausdorff dimension and discuss the main results in the theory that are currently known. This concept of "main results" is of course subjective and we have not aimed for exhaustive presentation, but merely have selected interesting results whose proofs are of special interest as well. As was mentioned in the previous chapter, this notion of dimension is relatively new. In fact, conformal dimension was first introduced by Pansu in 1989 [30].

2.1. Fundamentals of Conformal Dimension.

This section will focus on introducing and discussing the basic properties of the conformal dimension. These properties include monotonicity, finite stability and range of the dimension. We begin with the definition.

The language for discussing conformal dimension of a metric space is based on the notion of *conformal gauge*. This language was introduced by Heinonen [16], and it is particularly useful since it forces the conformal dimension to be a quasisymmetrical invariant by definition.

Definition 2.1. (Conformal Gauge). Given a metric space (Z, d), we say that the set $\mathcal{G}(Z)$ of all metric spaces that are quasisymmetrically equivalent to Z is the *conformal gauge* of Z.

Remark. Recall that the relation that there exists a quasisymmetric map between two metric spaces is indeed an equivalence relation. See the Theorem 1.24 *Quasisymmetry Equivalence*.

Definition 2.2. (Conformal Dimension). Let $\mathcal{G}$ be a conformal gauge. The infimal Hausdorff dimension of all the spaces in the gauge is called the *conformal dimension*. We write $\operatorname{Cdim} \mathcal{G}$ for the conformal dimension of the gauge. Symbolically this can be written as

$$\operatorname{Cdim} \mathcal{G} := \inf\{\dim Z : Z \in \mathcal{G}\}.$$

Remark. If $Z \in \mathcal{G}$, then we abuse notation and write $\operatorname{Cdim} Z$ instead of $\operatorname{Cdim} \mathcal{G}(Z)$, when we are discussing the conformal dimension of the metric space Z.

Definition 2.3. (Minimality of Conformal Dimension). If (Z, d) is a metric space, we say that Z is *minimal for conformal dimension* if

$$\operatorname{Cdim} Z = \dim Z.$$

In other words, a metric space Z is minimal for Cdim if the conformal dimension is attained by the space Z. Later, we will show that there are conformal gauges for which this minimality condition cannot be satisfied, that is, there is no space in the gauge that attains the infimal Hausdorff dimension. Note the connection to the Cannon's conjecture mentioned in the introduction.

There are at least three points why this notion of dimension is important for the study of metric spaces. First of all, one can ask why we are taking the infimum and not the supremum? This can be easily settled by investigating the behavior of Hausdorff dimension under *snowflaking* the space. This means that we transform a metric space (Z, d) to another metric space called snowflaked metric space, (Z, d^ϵ) where $d^\epsilon(x, y) := d(x, y)^\epsilon$ and $0 < \epsilon < 1$.

Now we claim that d^ϵ is indeed a metric. Positive definiteness is clear when we look at the definition of the map, $x : [0, \infty) \to [0, \infty), x \mapsto x^\epsilon$. Symmetry is also clear, since $d^\epsilon(x, y) = [d(x, y)]^\epsilon = [d(y, x)]^\epsilon = d^\epsilon(y, x)$. To see why the triangle inequality holds, let $x, y, z \in (Z, d^\epsilon)$. Then

$$d^\epsilon(x, y) = [d(x, y)]^\epsilon \leq [d(x, z) + d(z, y)]^\epsilon \leq [d(x, z)]^\epsilon + [d(z, y)]^\epsilon.$$

To see why this inequality holds, we consider the mapping $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as,

$$f(x) = x^{\varepsilon} + (d(z,y))^{\varepsilon} - [x + d(z,y)]^{\epsilon}.$$

Clearly, $f(0) = 0$ and

$$f'(x) = \epsilon[x^{\epsilon-1} - [x + d(z,y)]^{\epsilon-1}] > 0,$$

for every $x \in \mathbb{R}_{\geq 0}$. Thus $f \geq 0$, and we obtain

$$0 \leq [d(x,z)]^{\epsilon} + [d(z,y)]^{\epsilon} - [d(x,z) + d(z,y)]^{\epsilon},$$

which gives us the desired inequality.

Observe that snowflaking increases the Hausdorff dimension of a space by a multiplicative factor of $\frac{1}{\epsilon}$. In fact, suppose we have a metric space (X,d) with $\dim X = s$. We know that s is the unique value that satisfies

$$\inf\{s \geq 0 : \mathcal{H}^s(X) = 0\}.$$

Hence, when we calculate the dimension of (X, d^{ε}), the infimum of all $\alpha \geq 0$ such that

$$\liminf_{\delta \to 0} \sum_i ((\operatorname{diam} X_i)^{\epsilon})^{\alpha} = 0,$$

with $X \subset \bigcup_i X_i, \operatorname{diam} X_i < \delta$, is obtained when $\varepsilon\alpha = s$. Thus, the Hausdorff dimension of the snowflaked space is $\alpha = \frac{s}{\epsilon}$.

Hence, in general, there is no restriction of increasing Hausdorff dimension by a quasisymmetrical map. Therefore, there is no point asking what is the supremal Hausdorff dimension of the gauge, since it is always ∞ or 0. But this only justifies the use of the infimum in the definition. We still need to ask why quasisymmetrical maps are the correct tool for studying the distortion of dimension under mappings?

Traditionally we are interested in how the shape, or form, of the space changes when we transform the space into some image space with some transformation. One may ask why this is an interesting question? The simplest answer is that studying this question enriches our understanding of the space and its properties, and in some sense shows what is the "best shape" of the space. At this point, one must ask what are the most suitable or natural transformations of metric spaces that should be used to study these kind of phenomena? Isometric mappings clearly preserve all the metric properties, but they are so restrictive that the set of isometrically equivalent spaces becomes dull. Hence, we need something less restrictive. On the other side of the spectrum, homeomorphisms preserve topological properties and thus also topological dimension, denoted by $\dim_T$. We know that

$$\dim_T Z = \inf \dim Z',$$

where the infimum is taken over all homeomorphic spaces Z' to the given space Z. For a proof and for more information on $\dim_T$, see, for instance, [16, Chapter 8]. Hence, there is never any restriction for lowering the dimension by a homeomorphism. On the other hand, we have already seen that bi-Lipschitz mappings preserve Hausdorff dimension. Interestingly, it turns out that quasisymmetric maps strike a good balance between these mappings. One reason for this is that quasisymmetries are a generalization of bi-Lipschitz mappings and quasiconformal mappings.

Theorem 2.4. (Quasisymmetry Generalizes Bi-Lipschitz). *Every L-bi-Lipschitz map $f : (Z,d) \to (Z',d')$ is η-quasisymmetric with control function $\eta(t) = L^2 t$.*

Proof. Suppose f is L-bi-Lipschitz. First, we note that since $0 < L < \infty$, the function $\eta(t) = L^2 t$ satisfies the requirements of being a control function, since its co-domain is the nonnegative real axis and it is a homeomorphism. Second, we note that f is a homeomorphism between Z and Z' since it is bi-Lipschitz.

Let x, y, z with $x \neq z$. Since $f^{-1} : (Z', d') \to (Z, d)$ is L-Lipschitz, we have $f^{-1}(u) = x, f^{-1}(v) = z \iff f(x) = u, f(z) = v$, and hence, we find

$$\begin{aligned} & d(f^{-1}(u), f^{-1}(v)) \leq Ld'(u, v) \\ \iff & d(x, z) \leq Ld'(f(x), f(z)) \\ \iff & \frac{1}{Ld'(f(x), f(z))} \leq \frac{1}{d(x, z)} \\ \iff & \frac{1}{d'(f(x), f(z))} \leq \frac{L}{d(x, z)} \end{aligned}$$

and thus

$$\frac{d'(f(x), f(y))}{d'(f(x), f(z))} \leq L^2 \frac{d(x, y)}{d(x, z)} = \eta\left(\frac{d(x, y)}{d(x, z)}\right).$$

This proves f is an η-quasisymmetry with $\eta(t) = L^2 t$. □

In some sense, the use of quasisymmetric images in the comparison of Hausdorff dimensions answers the question what is the "best shape" of the metric space under investigation. For example, every metric space that is quasisymmetrically equivalent to $\mathbb{S}^1$ is in some respect similar to the unit circle. In fact, they share the invariant properties, and hence, every metric space in the conformal gauge $\mathcal{G}$ for which $\mathbb{S}^1 \in \mathcal{G}$, is called a *quasicircle*. We have a profound characterization of this gauge: *Any metric space is a quasicircle precisely when it is doubling and has bounded turning.* A metric space (Z, d) is said to be of *bounded turning*, if there exists $C < \infty$ such that for every two points $x, y \in Z$ we can find a connected set $x, y \in A \subset Z$ satisfying $\operatorname{diam} A \leq Cd(x, y)$. For more details see the Theorem 4.9 in [39]. Note that there may still exist some properties that distinguish $\mathbb{S}^1$ and a quasisymmetrically equivalent metric space. However, this characterization gives us a rough idea that the space under investigation must be as simple as the unit circle.

The third point motivating the use of quasisymmetric maps follows from the Theorem 1.24 *Quasisymmetry Equivalence* proven in the introduction. This provides a way to classify spaces and hence, in particular, to distinguish spaces up to a quasisymmetry. As mentioned in the Definition 2.2 *Conformal Dimension*, we know that Cdim is invariant under quasisymmetric maps. Therefore, if we have two spaces whose conformal dimensions do not agree, then there cannot exist a quasisymmetry between them, and we can justifiably call them non-equivalent spaces.

Now we have given some motivation for the definition of conformal dimension and are ready to delve into our first result which states some basic properties of this dimension. We give proofs for parts 1 and 2, but the proof of 3 is beyond the scope of this thesis. It is due to Leonid Kovalev [21]. Proof of the property (2) is adapted from [25, Chapter 5].

Theorem 2.5. (Properties of Conformal Dimension). *Let (Z, d) be a metric space.*

(1) *If $A, B \subset Z$ and $A \subset B$, then $\operatorname{Cdim} A \leq \operatorname{Cdim} B$. (Monotonicity)*
(2) *If $A, B \subset Z$ are disjoint and compact, then $\operatorname{Cdim}(A \cup B) = \max\{\operatorname{Cdim} A, \operatorname{Cdim} B\}$. In particular, if $A_1, ..., A_n$ are pairwise disjoint and compact subsets of Z, then $\operatorname{Cdim}\left(\bigcup_i^n A_i\right) = \max_{1 \leq i \leq n}\{\operatorname{Cdim} A_i\}$. (Finite Stability)*

(3) *For any metric space Z, we have* $\operatorname{Cdim} Z \in \{0\} \cup [1,\infty)$. *In other words, the range of conformal dimension is the union of 0 and the infinite interval* $[1,\infty)$. *Thus, if* $\dim Z < 1$, *then* $\operatorname{Cdim} Z = 0$. *(Range)*

Proof.

(1) Assume $A, B \subset Z$ and $A \subset B$. Let $\mathcal{G}_B$ be the conformal gauge of B. Pick one $B' \in \mathcal{G}_B$. Then for some quasisymmetry f, we have $f(B) = B'$. Since $A \subset B$, we have $f(A) \subset B'$, and by the Theorem 1.25 *Basic Properties of Quasisymmetries* $f\big|_A : A \to f(A)$ is a quasisymmetry. By monotonicity of Hausdorff dimension (see the Theorem 1.8 *Properties of Hausdorff Dimension*), we have $\dim f(A) \le \dim B'$. Since $B' \in \mathcal{G}_B$ was arbitrary, this holds for every $B' \in \mathcal{G}_B$. Precisely, for every $B' \in \mathcal{G}_B$, there exists $A' \in \mathcal{G}_A$ so that $\dim A' \le \dim B'$. This shows that the Hausdorff dimension of quasisymmetric images of B is bounded below by quasisymmetric images of A, hence, the claim.

(2) Suppose $A, B \subset Z$ are disjoint and compact. Since we proved above that Cdim is monotone, we have $\max\{\operatorname{Cdim} A, \operatorname{Cdim} B\} \le \operatorname{Cdim}(A \cup B)$. Hence, to prove the claim, it suffices to show $\max\{\operatorname{Cdim} A, \operatorname{Cdim} B\} \ge \operatorname{Cdim}(A \cup B)$.

Let $\epsilon > 0$ and $f_1 : A \to (A', d_1)$, $f_2 : A \to (B', d_2)$ be quasisymmetries such that $\dim A' < \operatorname{Cdim} A + \epsilon$ and $\dim B' < \operatorname{Cdim} B + \epsilon$. Compactness of A, B and continuity of f_1, f_2 implies that A', B' are compact.

Next we define a metric δ to the disjoint union space $A' \cup B'$. Let

(a) $\delta\big|_{A'} = d_1$,
(b) $\delta\big|_{B'} = d_2$
(c) $\delta(y_1, y_2) = \operatorname{diam} A' + \operatorname{diam} B'$ whenever $y_1 \in A'$ and $y_2 \in B'$

Elementary verification of three cases shows that the axioms for metric hold. Also, by the Theorem 1.8 (2) *Properties of Hausdorff Dimension*,

$$\dim(A' \cup B', \delta) = \max\{\dim A', \dim B'\} < \max\{\operatorname{Cdim} A, \operatorname{Cdim} B\} + \epsilon.$$

Now we will prove the claim by showing that $f : A \cup B \to A' \cup B'$ defined as the restriction $f\big|_A = f_1$ and $f\big|_B = f_2$ is a quasisymmetry. We will do this by showing that if $b = \operatorname{dist}(A, B) > 0$ and η is a common control function for f_1, f_2, then f is $\tilde{\eta}$-quasisymmetric for some $\tilde{\eta} = \tilde{\eta}(\eta, b, \operatorname{diam} A, \operatorname{diam} B, \operatorname{diam} A', \operatorname{diam} B')$. Note that this combined with the arbitrariness of ϵ implies that $\operatorname{Cdim}(A' \cup B') \le \max_i\{\operatorname{Cdim} A, \operatorname{Cdim} B\}$ and thus invariance of Cdim under quasisymmetries gives the result.

Let $x, y, z \in (A \cup B)$, $x \ne z$. To show that $\frac{\delta(f(x), f(y))}{\delta(f(x), f(z))} \le \tilde{\eta}\left(\frac{d(x,y)}{d(x,z)}\right)$ holds, it suffices to study the three cases:

(i) $x, y \in A$, $z \in B$
(ii) $x, z \in A$, $y \in B$
(iii) $y, z \in A$, $x \in B$

Note that without loss of generality we can study only these, since changing the roles of A and B does not require any changes in the arguments. Let $L = \operatorname{diam} A + \operatorname{diam} B + b$.

Case (i): Suppose $x, y \in A$, $z \in B$. Then we have

$$\frac{\delta(f(x), f(y))}{\delta(f(x), f(z))} = \frac{d_1(f_1(x), f_1(y))}{\operatorname{diam} A' + \operatorname{diam} B'} \qquad \text{and} \qquad \frac{d(x,y)}{d(x,z)} \ge \frac{d(x,y)}{L}.$$

Next we note that $X = \{x, y\} \subset A$ and $\operatorname{diam} X = d(x, y)$. Thus, we may apply the right-hand side inequality in the Theorem 1.25 *Basic Properties of Quasisymmetries* (2) to find,

$$\frac{d_1(f_1(x),f_1(y))}{\operatorname{diam} A' + \operatorname{diam} B'} = \frac{d_1(f_1(x),f_1(y)) \operatorname{diam} A'}{\operatorname{diam} A'(\operatorname{diam} A' + \operatorname{diam} B')} \le \eta\left(\frac{2d(x,y)}{\operatorname{diam} A}\right)\frac{\operatorname{diam} A'}{\operatorname{diam} A' + \operatorname{diam} B'}$$
$$= \eta\left(\frac{L2d(x,y)}{L \operatorname{diam} A}\right)\frac{\operatorname{diam} A'}{\operatorname{diam} A' + \operatorname{diam} B'}$$
$$\le \eta\left(\frac{d(x,y)}{d(x,z)}\frac{2L}{\operatorname{diam} A}\right)\frac{\operatorname{diam} A'}{\operatorname{diam} A' + \operatorname{diam} B'}.$$

Case (ii): Assume $x, z \in A$, $y \in B$. Then

$$\frac{\delta(f(x),f(y))}{\delta(f(x),f(z))} = \frac{\operatorname{diam} A' + \operatorname{diam} B'}{d_1(f_1(x),f_1(z))} \qquad \text{and} \qquad \frac{d(x,y)}{d(x,z)} \ge \frac{b}{d(x,z)}.$$

Next we proceed similarly as in the case (i). This time we need to apply the left-hand side inequality in the Theorem 1.25 *Basic Properties of Quasisymmetries* (2), since the distances corresponding to the subset $\{x,z\} \subset A$ are in the denominator. Thus,

$$\frac{\operatorname{diam} A' + \operatorname{diam} B'}{d_1(f_1(x),f_1(z))} = \frac{\operatorname{diam} A'(\operatorname{diam} A' + \operatorname{diam} B')}{d_1(f_1(x),f_1(z)) \operatorname{diam} A'}$$
$$\le 2\eta\left(\frac{\operatorname{diam} A}{d(x,z)}\right)\frac{\operatorname{diam} A' + \operatorname{diam} B'}{\operatorname{diam} A'}$$
$$= 2\eta\left(\frac{b}{b}\frac{\operatorname{diam} A}{d(x,z)}\right)\frac{\operatorname{diam} A' + \operatorname{diam} B'}{\operatorname{diam} A'}$$
$$\le \eta\left(\frac{d(x,y)}{d(x,z)}\frac{\operatorname{diam} A}{b}\right)\frac{2(\operatorname{diam} A' + \operatorname{diam})}{\operatorname{diam} A'}.$$

Case (iii): Suppose $y, z \in A$, $x \in B$. Then

$$\frac{\delta(f(x),f(y))}{\delta(f(x),f(z))} = 1 \qquad \text{and} \qquad \frac{d(x,y)}{d(x,z)} \ge \frac{b}{L}.$$

In this case, we need only to choose sufficiently large constants,

$$C_i = C_i(b, \operatorname{diam} A, \operatorname{diam} B, \operatorname{diam} A', \operatorname{diam} B') \qquad i = 1, 2$$

such that $\tilde{\eta}(t) = C_1\eta(C_2 t)$ and $\tilde{\eta}$ satisfies

$$\tilde{\eta}(t) = \max\left\{\frac{\operatorname{diam} A'}{\operatorname{diam} A' + \operatorname{diam} B'}\eta\left(\frac{2L}{\operatorname{diam} A}t\right), \frac{2(\operatorname{diam} A' + \operatorname{diam} B')}{\operatorname{diam} A'}\eta\left(\frac{\operatorname{diam} A}{b}t\right)\right\}$$

for all $t > 0$, and also

$$\tilde{\eta}\left(\frac{b}{L}\right) \ge 1.$$

Since we are free to choose C_1 and C_2, this is possible. This finishes the proof. □

Remark I. Without the disjointness and compactness assumptions, conformal dimension is not finitely stabile. See, for instance, the application of Tukia's example in [25, Chapter 5].

Remark II. The case for conformal dimension of product spaces is unclear to some extent. The difficulty lurks in the definition of conformal gauges for product spaces. That is, the concept of conformal dimension is not well defined in terms of the conformal gauges of the original gauges on the factor spaces. See [25, Pages 73-76] for more information on the open questions.

Now that we have introduced the conformal dimension, have given hopefully sufficient motivation for the definition and discussed three fundamental properties, we will shortly focus on one interesting case study of the conformal dimension, namely, the conformal dimension of the middle-thirds Cantor set.

Example 2.6. (Conformal Dimension of the Cantor Set). The middle-thirds Cantor set can be constructed from the unit interval $[0,1] \subset \mathbb{R}$, which we denote by E_0. On the first step, remove the middle third. Then we are left with,

$$\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Write $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next we proceed by removing the middle thirds from these two intervals. Thus, $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continuing this procedure iteratively indefinitely, we obtain the set $C = \bigcap_{i=1}^{\infty} E_i$ the *Cantor set*. For more details of this construction and the basic properties of Cantor set we refer the reader to [34, Pages 41-42, 309].

For our case study, we need a slightly different representation of the Cantor set. Let $a = \{x_n\}_{n=1}^{\infty}$ be a binary sequence where $x_n \in \{0,2\}$ and denote the set of all such sequences by A. For example, a could be the binary sequence $a = (0,2,2,2,0,0,0,2,0,2,...)$. Then to each a we associate the real number $x(a)$ as

$$x(a) = \sum_{n=1}^{\infty} \frac{x_n}{3^n}.$$

Define $T := \{x(a) : a \in A\}$. We claim that $T = C$.

Observe that the representation for a real number x, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, is a ternary number representation. We recall that every number in $[0,1]$ has a representation in the usual decimal form $x = 0.y_1y_2y_3... = \sum \frac{y_n}{10^n}$, but we can as easily represent this number in a ternary form where the base ten is replaced by three, $x = 0.x_1x_2x_3... = \sum \frac{x_n}{3^n}$, where $x_n \in \{0,1,2\}$. But now we throw away the number 1 and choose the coefficients only from $\{0,2\}$. See Figure 1. We call the 0's, 1's and 2's by the common term *digit*.

For any $x \in T$, the first digit must be 0 or 2, and hence we have the representation $x = 0.0a_1a_2a_3...$ or $x = 0.2a_1a_2a_3...$. Similarly, for the second digit we have $x = 0.00a_2a_3..., x = 0.02a_2a_3..., x = 0.22a_2a_3...$, or $x = 0.20a_2a_3...$. In general, in the nth step of the construction of T, the nth digit of x cannot be equal to 1. Excluding the digit 1 corresponds to excluding the middle third segment as is illustrated in Figure 1. Therefore, $T \subset C$.

Conversely, let $x \in C$. If x is one of the endpoints of some interval in the construction step E_j, for some j, then x is a rational number and we can represent x with an infinite string of the form $0.a_1a_2...a_j\overline{yy}...$, where $a_i \in \{0,2\}$ and $y = 0$ or $y = 2$. Thus, it terminates in finitely many digits and ends with an infinite string of 0's or 2's. Hence, $x \in T$.

Therefore, suppose x is not an endpoint of for any of construction intervals E_j. Now every $x \in [0,1]$ can be expressed in the ternary representation, $x = \frac{a_0}{3} + \frac{a_1}{3^2} + \cdots + \frac{a_{i-1}}{3^i} + \cdots$, as was mentioned above. If we multiply x by 3^i for some $1 \le i \le k$, then we obtain $3^i x = n + \frac{a_i}{3} + \frac{a_{i+1}}{3^2} + \cdots = n + y$, where $n \in \mathbb{N}$ and $y \in E_{k-i}$.

Next, for a moment, suppose that for some i we have $a_i = 1$. Choose k such that $1 \le i \le k$. Thus, $3^i x = n + \frac{1}{3} + \frac{a_{i+1}}{3^2} + \cdots = n + y$. From this we find $\frac{1}{3} \le y = 3^i x - n \le \frac{1}{3} + \sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{2}{3}$. But $3^i x - n \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, since $y \in E_{k-i}$ and hence, $y = \frac{1}{3}$ or $y = \frac{2}{3}$. These imply that

$x = \frac{n}{3^m} + \frac{1}{3^{m+1}} = \frac{3n+1}{3^{m+1}}$ or $x = \frac{3n+2}{3^{m+1}}$, which further implies x is an endpoint for some construction-interval. This is a contradiction. Thus, $a_i \neq 1$ for all i. This line of reasoning combined with the previous paragraphs we conclude $C \subset T$ and further $T = C$.

If we identify the binary sequences for rational points, the representation for any $x \in C$ is unique. Let ∂C be the set of all end points of the construction intervals. Assume there exists two different binary sequences for a point $x \in C \backslash \partial C$. But now, as is illustrated in Figure 1, these sequences necessarily coincide. In fact, if they differed at some digit, then they would necessarily lie in different parts of the Cantor set. Thus, we shall call the binary sequence consisting of 0's and 2's corresponding to every $x \in C$ the *address* of x.

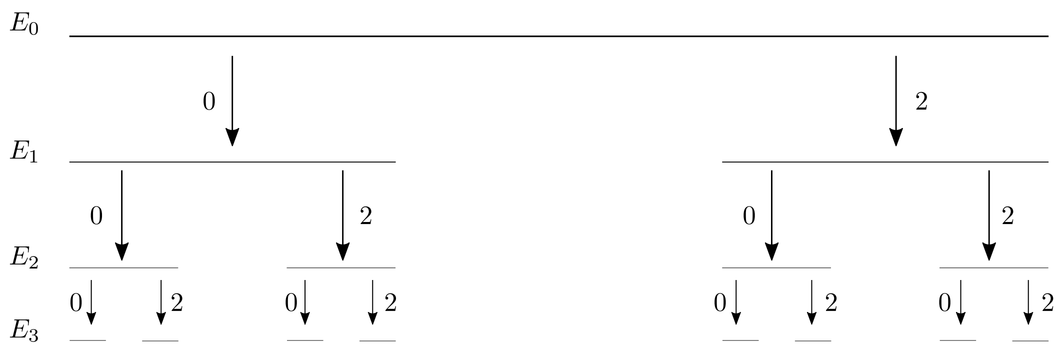

FIGURE 1. At each step of the construction choose either 0 or 2 depending on whether the point $x \in C$ is in the left or right part of the construction.

Finally we can begin this small detour in earnest. We shall calculate the conformal Hausdorff dimension of the ternary Cantor set C. Our approach will be to define such a metric on the Cantor set that transforming quasisymmetrically the original Cantor metric space back into itself, but with a different metric, it will be easy to compute the Hausdorff dimension, and hence the conformal dimension. We begin with simple results and definition.

Theorem 2.7. (QUASISYMMETRIC METRIC). *If $f : (X, d) \to (X', d')$ is a quasisymmetry, then $d_f(x, y) := d'(f(x), f(y))$ is a metric in the source space X. Moreover, $g : (X, d) \to (X, d_f), g(x) = x$ is a quasisymmetry.*

Proof. Suppose $f : X \to X'$ is a quasisymmetry. Now we verify the axioms of a metric. Suppose $d_f(x, y) = 0$. By definition, $d'(f(x), f(y)) = 0$ which implies $f(x) = f(y)$. Injectivity of f implies $x = y$. Hence, d_f is definite. Next assume $x, y \in X$. Then $d_f(x, y) = d'(f(x), f(y)) = d'(f(y), f(x)) = d_f(y, x) \geq 0$ which verifies positivity and symmetry at once. Triangle inequality is verified as easily; $x, y, z \in X$ implies $d_f(x, y) = d'(f(x), f(y)) \leq d'(f(x), f(z)) + d'(f(z), f(y)) = d_f(x, z) + d_f(z, y)$.

Quasisymmetricity of the identity function g is easily seen by letting $x, y, z \in X$, $x \neq z$. Then

$$\frac{d_f(g(x), g(y))}{d_f(g(x), g(z))} = \frac{d_f(x,y)}{d_f(x,z)} = \frac{d'(f(x), f(y))}{d'(f(x), f(z))} \leq \frac{d(x,y)}{d(x,z)},$$

where we applied the assumption that $f : X \to X'$ is a quasisymmetry. □

Next we make an important observation. If we define (X, d) where $X = C$ and d is the usual Euclidean metric on the real line, then the Hausdorff dimension cannot be decreased by any quasisymmetry $f : (X, d) \to (X, d^\alpha)$. The metric d^α is the snowflake metric when $\alpha \leq 1$, and for $\alpha > 1$ it is not a metric since the triangle inequality fails to hold: $|1 - 0|^\alpha = 1 > |1 - \frac{1}{3}|^\alpha + |\frac{1}{3} - 0|^\alpha$ for $\alpha > 1$. During the motivation for the definition of conformal dimension we observed that snowflake increases Hausdorff dimension. To circumvent this problem, we define another useful notion of a metric.

Definition 2.8. (Ultrametric). If (X, d) is a metric space where the metric satisfies the stronger version of the triangle inequality,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for every $x, y, z \in X$, then we say that (X, d) is *ultrametric space*. In this case, we denote the metric by d_∞. Note that every ultrametric is also a metric, hence the word stronger.

Suppose that (X, d_∞) is an ultrametric space. Then we claim that d_∞^α is a metric for every $\alpha > 0$. Positive definiteness and symmetry are clear as is the ultrametric version of the triangle inequality: $[d_\infty(x, y)]^\alpha \leq [\max\{d_\infty(x, z), d_\infty(z, y)\}]^\alpha$.

We need to define an ultrametric to C, by letting

$$d_\infty(x, y) := \begin{cases} \max\{3^{-n} : x_n \neq y_n\} \\ 0 \quad \text{if } x = y \end{cases}$$

where x_n and y_n are the unique binary sequences corresponding to x and y, respectively. To verify that this is indeed an ultrametric, we note that positive definiteness and symmetry are again clear. The ultra triangle inequality $d_\infty(x, y) \leq \max\{d_\infty(x, z), d_\infty(z, y)\}$ follows from considering three cases. Fix $x, y \in C$. Denote by $k_{x,y}$ the position of the digit that is different for x and y in their ternary representations. That is, $k_{x,y}$ denotes the first digit that is different in the addresses of x and y. For example, if $x = 0.00202\overline{00}$ and $y = 0.00202\overline{200}$, then $k_{x,y} = 5$. First, suppose that $k_{x,z} \leq k_{x,y}$. Then $d_\infty(x, y) = 3^{-k_{x,y}} < 3^{-k_{x,z}} = d_\infty(x, z)$. Note that this case covers also the second case $k_{y,z} \leq k_{x,y}$. Third, assume that $k_{x,z} = k_{x,y} \leq k_{z,y}$. Then, $d_\infty(x, z) = d_\infty(x, y)$. This proves that we have ultrametric.

Hence, by the observation in the previous paragraph and Theorem 2.7 *Quasisymmetric Metric*, we find that $f : (C, d_\infty) \to (C, d_\infty^\alpha), f(x) = x$ is a quasisymmetry. Thus, to verify that d and d_∞^α are quasisymmetric, it suffices to show that d and d_∞ are bi-Lipschitz equivalent, since this automatically makes them also quasisymmetrically equivalent (recall that every bi-Lipschitz function is quasisymmetric). Hence, if d is quasisymmetrically equivalent to d_∞ and d_∞ is quasisymmetrically equivalent to d_∞^α, then by the Theorem 1.24 *Quasisymmetry Equivalence* d is quasisymmetrically equivalent to d_∞^α.

To do this, we let $x, y \in C$ and n to represent the step of the construction in which the ternary representations for x and y have a different digit for the first time. Clearly $\frac{1}{3^n} \leq d(x, y) \leq \frac{1}{3^{n-1}}$, since x and y are in different components of E_n. Therefore,

$$\frac{1}{3}d_\infty(x,y) \leq \max\{3^{-n} : x_n \neq y_n\} \leq d(x,y) \leq 3\max\{3^{-n} : x_n \neq y_n\} = 3d_\infty(x,y),$$

which shows d_∞ is bi-Lipschitz equivalent to d.

To finish our calculation of $\operatorname{Cdim} C$, we observe that at the nth step of the construction we can cover (C, d_∞^α) with 2^n sets each having a diameter of $\frac{1}{3^{n\alpha}}$, since we are now in the space (C, d_∞^α), $\alpha > 0$. Write $X := (C, d_\infty^\alpha)$.

Let $s > 0$ be arbitrary and choose $\alpha \in \mathbb{R}_{\geq 0}$ such that $\alpha > \frac{\log_3 2}{s}$. Recall that the s-dimensional Hausdorff content of a metric space vanishes simultaneously with the s-dimensional Hausdorff measure [16, Chapter 8].

An upper bound for the s-dimensional Hausdorff content of X is given by,

$$\sum_{i=1}^{2^n} (3^{-n})^{s\alpha} \geq \mathcal{H}^s_\infty(X).$$

But we have $\frac{2^n}{3^{ns\alpha}} \to 0$ as $n \to \infty$, since $\alpha s > \log_3 2 \iff \frac{2}{3^{\alpha s}} < 1$. This implies $\mathcal{H}^s(X) = 0$. Recalling the definition of Hausdorff dimension, this means $\dim X \leq s$. Arbitrariness of s forces $\dim X = 0$, which yields $\operatorname{Cdim} X = 0$ by the definition of Cdim. ■

Remark I. We refer the reader to [24, Chapters 1, 15] for more details on Hausdorff dimension of trees embedded in Euclidean spaces.

This example shows that the computation of conformal dimension can be a bit involved. Moreover, it illustrates the remarkability of the Kovalev's result the Theorem 2.5 *Properties of Conformal Dimension* (3), since applying Kovalev would have saved us a lot of labor. We note that this holds since $\dim C = \frac{\log 2}{\log 3} < 1$ (see [11] for details).

Remark II. It should be noted that there exists a slightly simpler version of the direct argument showing that $\operatorname{Cdim} C = 0$. See Example 2.2.3(2) in [25].

Remark III. Also interesting is to note that C is not minimal for Cdim. In fact, every quasisymmetric image of C is uniformly perfect and hence, has a nonzero Hausdorff dimension. Thus, the conformal dimension is not attained by any image space in the conformal gauge.

Remark IV. The idea of the proof in this example was suggested to us by Sylvester Eriksson-Bique.

2.2. **Lower bounds for Conformal Dimension.** It is easy to see that upper bounds for the Hausdorff dimension of a space are more effortless to obtain than the lower bounds. For example, one merely needs to find a suitable cover for the space to obtain an upper bound for its dimension. On the other hand, the lower bound question is more intricate, since one needs to study arbitrary covers. But luckily, there have been plenty of research in this area, and we now have a collection of techniques, methods, ideas and powerful results for approximating the Hausdorff dimension from below.

Looking at the definition of conformal dimension, it should not come as a surprise that those same difficulties that exists for Hausdorff dimension also exist for conformal dimension. However, the search for lower bounds for Cdim is even more difficult question than lower bounds for its counterpart dim, at least in the interesting cases. Not to make things easier, the state-of-the-art research is still in its infancy due to the young age of the concept. In this section we discuss some of the main tools for obtaining lower bounds.

The common heuristic for obtaining lower bounds for the conformal dimension is to show the existence of abundance of curves, which is then used to deduce lower bounds. We will discuss three main tools where one can apply this heuristic quantitatively to obtain a lower bound for Cdim. Their proofs are of independent interest since they illustrate how the basic tools and techniques are used in these kind of arguments [25]. However, it should be noted, that for many interesting metric spaces, such as most fractals, the existence of abundant curve family fails [16]. This turns out to be the case for our main example, the Sierpiński carpet. Fortunately we have other tools to study the lower bound for its dimension. In fact, as we will briefly note in the Chapter 4, this problem can be circumvented in some cases. Since our main interest is in the theory of conformal dimension, we turn now to discussing the fundamental tools that can be used for studying the lower bounds.

One can make the notion of "abundant curve family" precise in several ways, for example, using Poincaré inequalities. In this thesis, instead of Poincaré inequalities, we make this concept "lot of curves" formal by assuming the existence of a diffuse measure on the curve family, and after that, by applying the definition of the p-modulus of a curve family. First we begin by defining the concept of curves.

2.2.1. *Curves.* Let (X, d, μ) be a doubling metric space. We say that $\gamma : \mathbb{R} \supset I \to X$ is a *curve* in X if γ is a continuous mapping of an interval $I \subset \mathbb{R}$ into X. In some cases it is useful to distinguish between the image of γ and the mapping itself, but for our purposes we do not need this specification and thus we call both the mapping and its image a curve. A curve γ is called *rectifiable* if it has finite length. Formally this means that

$$\ell(\gamma) := \sup \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) < \infty,$$

where the supremum is taken over all finite partitions of I and $\ell(\gamma)$ denotes the length of the curve γ. Otherwise the curve is called *non-rectifiable.*

The key idea behind our first lower bound result is that we need to have a set of rectifiable curves that is uniformly bounded below, and a sufficiently diffuse measure defined on the set of curves. In general, this set of sets need not be curves but any family of connected subsets of X suffices (see the statement of the Theorem 2.12 *Lower Bound Result I*).

2.2.2. *Lower Bound Result I.* We begin by discussing some preliminary results which lead to our first main result. First, we prove an important covering result. To obtain this result in a somewhat general form, our proof will be nonconstructive and depends on the Zorn's lemma. For the sake of completeness we state and prove the Zorn's lemma. For this purpose we need to define a choice function and assume the axiom of choice.

If $\mathcal{P}$ is any set of sets, we define a *choice function* to be any function that satisfies $f(A) = a \in A \in \mathcal{P}$. Simply put, this function selects one element from a set, that is, it maps the set A to one of the elements of A.

The axiom of choice states that,

For any set $\mathcal{P}$ of nonempty sets, there exists a choice function f defined on $\mathcal{P}$.

Next we shortly discuss some required notions. *Partially ordered set* $\mathcal{P}$ is a set with a binary relation $\leq$ that is reflexive, antisymmetric and transitive. That is if $a, b, c \in \mathcal{P}$, then

$$\begin{aligned} &a \leq a && \text{(reflexivity)} \\ &a \leq b \text{ and } b \leq a \text{ implies } b = a && \text{(antisymmetry)} \\ &a \leq b \text{ and } b \leq c \text{ implies } a \leq c && \text{(transitivity).} \end{aligned}$$

This is intended to capture the notion that not every pair of elements in $\mathcal{P}$ is comparable. Stronger notion related to this is the *totally ordered set* $\mathcal{T}$ where every two elements are comparable. In this case we add to the above list the so-called property of *totality.* That is, for every $a, b \in \mathcal{T}$ we have $a \leq b$ or $b \leq a$. For any totally ordered set $\mathcal{T}$, we can further define the notion of *well order*, which means that any nonempty $A \subset \mathcal{T}$ has a least element with respect to the total order relation. Moreover, a *maximal element* $m \in \mathcal{P}$ is an element such that there are no $x \in \mathcal{P}$ satisfying $x > m$. This means that if there exists $p \in \mathcal{P}$ such that $m \leq p$, then we must have $m = p$. For instance, if the binary relation is the set inclusion, and A is a maximal element of some partially ordered set $\mathcal{P}$, then there are no sets in $\mathcal{P}$ that strictly contains A.

To prove the Zorn's Lemma we will use the notion of *ordinal numbers.* We define a set S to be an ordinal if and only if S is strictly well-ordered with respect to set membership and every element of S is also a subset of S. For instance, the first ordinals are $1, 2, 3, ..$ if considered as sets: $1 := \{\emptyset\}, 2 := 1 \cup \{1\}, 3 := 2 \cup \{2\}, ...$ Heuristically, ordinals can be thought of as a generalization and an extension of natural numbers in the following sense

$$1, 2, 3, ..., \omega, \omega + 1, \omega + 2, ..., \omega + \omega =: 2\omega, 2\omega + 1, ...$$

where $\omega := \mathbb{N}$ is the first countable ordinal. Given any ordinal α, there always exists a successor ordinal, namely, $\alpha + 1 := \alpha \cup \{\alpha\}$. We obtain a well-ordering of all ordinals if we consider the relation

$$\alpha \leq \beta \iff \alpha \in \beta \iff \alpha \subset \beta$$

Observe that the set of all ordinals is not a set. In fact, any *set* $\mathcal{P}$ of ordinals has a least upper bound: just take the union of all $\alpha \in \mathcal{P}$. If the set of all ordinals was a set, then it would have a least upper bound. This least upper bound would be the greatest ordinal, which contradicts the existence of the successor for every given ordinal.

We will apply the principle of *transfinite induction* and *recursion*,

Transfinite induction means that any property P that holds for ordinal α whenever P holds for all ordinals $\beta < \alpha$, is true for all ordinals. The principle of transfinite recursion states that we can construct sequences indexed by ordinals.

For more details on ordinals we refer the reader to the classic [15, Section 19].

With these notions in mind, we are ready to give a proof of Zorn's Lemma. This simple and intuitive proof is adapted from [6].

Theorem 2.9. (Zorn's Lemma). *Let $\mathcal{P}$ be a nonempty partially ordered set. Suppose that every totally ordered $\mathcal{T}$, $\mathcal{T} \subset \mathcal{P}$, has an upper bound. Then $\mathcal{P}$ has a maximal element.*

Proof. Suppose a nonempty partially ordered set $\mathcal{P}$ has the property that every totally ordered $\mathcal{T} \subset \mathcal{P}$ is bounded above, but $\mathcal{P}$ has no maximal element.

Axiom of choice provides a successor operator on $\mathcal{P}$, namely, $x \mapsto x^+$ such that $x^+ > x$, since $\{y : y > x\} \neq \emptyset$. Moreover, we may assume, again by invoking the axiom of choice, that we have a function $h : B \to b$ that maps any set B to one of its upper bounds b for every totally ordered set B having an upper bound.

We shall define recursively an increasing sequence indexed by the ordinals. Suppose α is an ordinal and x_δ has been chosen for all δ for which $\delta < \alpha$. Then $\alpha = \delta+1 = \delta \cup \{\delta\}$ or $\alpha = \{\delta : \delta < \alpha\}$. If $\alpha = \delta + 1$, we define $x_\alpha = x_\delta^+$. If $\alpha > \delta + 1$ for all δ, then we set $x_\alpha = h(\{x_\beta\}_{\beta<\alpha})$, that is, α is set to be equal to the upper bound provided by the map h of the totally ordered bounded set $\{x_\beta\}_{\beta<\alpha}$.

Observe that the sequence $\{x_\beta\}$ is strictly increasing if and only if $\mathcal{P}$ has no maximal element. If $\{x_\beta\}$ is strictly increasing, then this sequence would be in one-to-one correspondence with the set of all ordinals. Clearly this contradicts the fundamental property of ordinals that one cannot form the set of all ordinals. This proves the lemma. □

Now we can prove the 5r covering lemma. This proof can be found, for example, in [16, Chapter 1].

Theorem 2.10. (5r Covering Lemma). *Let $\mathcal{F}$ be a nonempty set of balls in a metric space (X, d) so that $X = \bigcup_{B(x,r)\in\mathcal{F}} B(x, r)$. Assume that $C := \sup\{r : B(x, r) \in \mathcal{F}\} < \infty$ for every $B(x, r) \in \mathcal{F}$. Then there exists a subset $\mathcal{G} \subset \mathcal{F}$ so that the elements of $\mathcal{G}$ are pairwise disjoint and $X = \bigcup_{B(x,r)\in\mathcal{G}} B(x, 5r)$.*

Proof. Let $\mathcal{P}$ be the partially ordered set (with the partial order relation $\subset$) that contains all subsets $\mathcal{S} \subset \mathcal{F}$ that are pairwise disjoint and satisfy the property P: if $B(x, r) \in \mathcal{F}$ and $B(x, r) \cap B(x', r') \neq \emptyset$ for some $B(x', r') \in \mathcal{S}$, then there exists a ball $B(x'', r'') \in \mathcal{S}$ such that $B(x, r) \cap B(x'', r'') \neq \emptyset$ with $2r'' \geq r$.

To apply Zorn's lemma we need to show $\mathcal{P}$ is nonempty: just choose any ball $B(x, r) \in \mathcal{F}$ with $r \geq \frac{C}{2}$. Then the set $\{B(x, r)\} \in \mathcal{P}$ shows $\mathcal{P}$ is not empty.

Write $\mathcal{T}$ for an arbitrary totally ordered set in $\mathcal{P}$. Consider $\bigcup_{A\in\mathcal{T}} A$, which is an upper bound for $\mathcal{T}$. To invoke Zorn's lemma, we need only to show that $\bigcup_{A\in\mathcal{T}} A \in \mathcal{P}$. Suppose there exists $B(x, r) \in \mathcal{F}$ such that $B(x, r) \cap B(x', r') \neq \emptyset$ for some $B(x', r') \in \bigcup_{A\in\mathcal{T}} A$. Since $B(x', r') \in A$ for some $A \in \mathcal{T}$, and A satisfies the property P, there exists a ball $B(x'', r'') \in A \subset \bigcup_{A\in\mathcal{T}} A$ such that $B(x, r) \cap B(x'', r'') \neq \emptyset$ with $2r'' \geq r$.

Thus, $\bigcup_{A\in\mathcal{T}} A \in \mathcal{P}$ and therefore Zorn's lemma implies there exists a maximal element of $\mathcal{P}$. We call this maximal element $\mathcal{G}$.

Our next goal is to show that every $B(x, r) \in \mathcal{F}$ meets at least one $B(x', r') \in \mathcal{G}$. Thus, for the sake of contradiction suppose there exists a ball $B(x, r) \in \mathcal{F}$ such that $B(x, r) \cap \bigcup_{B(x',r')\in\mathcal{G}} B(x', r') = \emptyset$. Denote by $\mathcal{B} := \{B(x, r) : B(x, r) \cap B(x', r') = \emptyset, \forall\, B(x', r') \in \mathcal{G}\}$ the set of all such balls and choose $B(x, r) \in \mathcal{B}$ such that $2r \geq r'$ for every $B(x', r') \in \mathcal{B}$. Such a ball can be chosen, since the radii of the balls in $\mathcal{F}$ are uniformly bounded and we are choosing a ball whose radius is at least half of the supremum: $\sup\{r : B(x, r) \in \mathcal{B}\}$. Write $\mathcal{K} = \mathcal{G} \cup \{B(x, r)\}$.

We will verify that $\mathcal{K}$ satisfies the property P and hence $\mathcal{K} \in \mathcal{P}$. Suppose there exists $B(x'', r'') \in \mathcal{F}$ such that $B(x'', r'') \cap B(x^*, r^*) \neq \emptyset$ for some $B(x^*, r^*) \in \mathcal{K}$. Now it cannot be that $B(x^*, r^*) = B(x, r)$ since $B(x, r)$ does not intersect any of the balls in $\mathcal{F}$. Hence, $B(x^*, r^*) \in \mathcal{G}$ and since $\mathcal{G} \in \mathcal{P}$, there exists some ball $B(x^{**}, r^{**}) \in \mathcal{G} \subset \mathcal{K}$ such that $B(x^*, r^*) \cap B(x^{**}, r^{**}) \neq \emptyset$ with $2r^{**} \geq r^*$. Thus, $\mathcal{K}$ has the property P. But this means $\mathcal{G}$ is not maximal. This is a contradiction. Hence, every ball $B(y, r_0) \in \mathcal{F}$ meets at least one ball $B(y', r_0') \in \mathcal{G}$ such that $r_0 \leq 2r_0'$.

Finally, choose arbitrary $B(x, r) \in \mathcal{F}$. Since $\mathcal{G}$ satisfies P and every ball in $\mathcal{F}$ meets at least one ball in $\mathcal{G}$, there exists $B(x', r') \in \mathcal{G}$ such that $B(x, r) \cap B(x', r') \neq \emptyset$ and $r \leq 2r'$. Since the balls overlap, the distance from the center of $B(x', r')$ that covers $B(x, r)$ is at most $r' + 2r$. In other words, for $y \in B(x,r)$ we have $d(x',y) \le r' + 2r \le 5r'$. Therefore, $X = \bigcup_{B(x,r)\in\mathcal{F}} B(x,r) \subset \bigcup_{B(x,r)\in\mathcal{G}} B(x,5r)$ implies $X = \bigcup_{B(x,r)\in\mathcal{G}} B(x,5r)$. □

Next, we obtain a useful characterization of quasisymmetric mappings in terms of distortion of metric annulus. By a *metric annulus* we mean a region $B(x,R)\backslash B(x,r)$ where $0 < r < R$.

Theorem 2.11. (Characterization of Quasisymmetry). *Let (Z,d) and (Z',d') be doubling spaces. Homeomorphism $f : (Z,d) \to (Z',d')$ is η-quasisymmetric if and only if whenever $B_1 = B(z,r_1) \subset B_2 = B(z,r_2)$ are concentric balls in Z (that is balls with common center), then there exists concentric balls $B_1' = B'(f(z),r_1') \subset B_2' = B'(f(z),r_2')$ so that $B_1' \subset f(B_1) \subset f(B_2) \subset B_2'$ and $\frac{r_2'}{r_1'} \le \eta\left(\frac{r_2}{r_1}\right)$.*

Proof. Suppose $f : Z \to Z'$ is η-quasisymmetric and $B_1 = B(z,r_1) \subset B_2 = B(z,r_2)$ are concentric balls in Z. Let $r_1' = \inf_y\{d'(f(x),y) : y \in f(B_1)^c\}$ and $r_2' = \sup_y\{d'(f(x),y) : y \in f(B_2)\}$. Clearly $B'(f(x),r_1') \subset f(B_1) \subset f(B_2) \subset B'(f(x),r_2')$. Since we are in a doubling space, there exists $f(z) \in Z'$ and $f(y) \in Z'$ such that $d'(f(x),f(z)) = r_1'$ and $d'(f(x),f(y)) = r_2'$. Then we find

$$\frac{r_2'}{r_1'} = \frac{d'(f(z),y)}{d'(f(z),x)} \le \eta\left(\frac{d(z,f^{-1}(y))}{d(z,f^{-1}(x))}\right) = \eta\left(\frac{r_2}{r_1}\right).$$

Conversely, suppose that whenever $B_1 = B(x,r_1) \subset B_2 = B(x,r_2)$ are concentric balls in Z, then there exists concentric balls $B_1' = B'(f(x),r_1') \subset B_2' = B'(f(x),r_2')$ so that $B_1' \subset f(B_1) \subset f(B_2) \subset B_2'$ and $\frac{r_2'}{r_1'} \le \eta\left(\frac{r_2}{r_1}\right)$.

Pick any three points $x,y,z \in Z$ and without loss of generality assume $x \ne z$ and $d(x,z) < d(x,y)$. Then let $B_1 = B(x,r_1), B_2 = B(x,r_2)$ where $r_1 = d(x,z), r_2 = d(x,y)$. Applying our hypothesis we have $d'(f(x),f(z)) \ge r_1'$ and $d'(f(x),f(y)) \le r_2'$. Thus

$$\frac{d'(f(x),f(y))}{d'(f(x),f(z))} \le \frac{r_2'}{r_1'} \le \eta\left(\frac{r_2}{r_1}\right) = \eta\left(\frac{d(x,y)}{d(x,z)}\right).$$

This shows f is η-quasisymmetric. □

Figure 2 illustrates this characterization.

Before we delve into our main result, we briefly mention some preliminary definitions. Let (Z,d,μ) be a compact and doubling metric measure space and assume there exists a set $\Gamma \subset \{E \subset Z : E \text{ is connected }\}$ of connected subsets of Z. We need to define a way to measure the distance between these sets.

The *Hausdorff distance*, $d_H(A,B)$, between compact subsets $A, B \subset Z$ is defined as

$$d_H(A,B) := \inf\{r > 0 : A \subset N_r(B) \text{ and } B \subset N_r(A)\}$$

where $N_r(A)$ is the *r-neighborhood* of A, $N_r(A) := \bigcup_{x\in A} B(x,r)$. Here $B(x,r)$ are open balls with respect to the ambient metric d in Z. Note that $N_r(A) \subset Z$ is open.

Let $K(Z)$ be the set of all compact sets in Z. Then d_H is a metric on $K(Z)$, and hence $(K(Z),d_H)$ becomes a metric space [7]. Given $A \in K(X)$, and defining $\mathcal{B}_r(A) := \{D \in K(Z) : d_H(A,D) < r\}$, we have a definition of an open ball $\mathcal{B}_r(A)$ in $(K(X),d_H)$ centered at A with radius $r > 0$. The set of all such open balls generates a topology on $K(X)$. Note that in the metric space $(K(Z),d_H)$, elements are compact subsets of Z.

Finally, we are ready to state and prove our first main result for obtaining lower bounds for conformal dimension. This argument is adapted from [25], but the techniques and ideas originate from the work of Pansu [30, 31] and Bourdon [5].

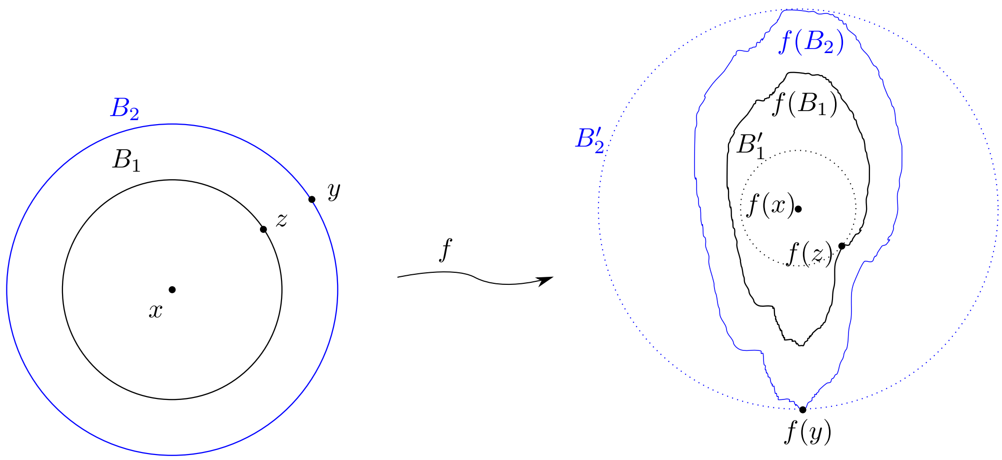

FIGURE 2. Idea of the quasisymmetry characterization with metric balls.

Theorem 2.12. (LOWER BOUND RESULT I). *Let (Z, d, μ) be a compact and doubling metric measure space and $p \in (1, \infty)$ and q its exponent conjugate (that is $pq = p + q$). Suppose Γ is a set of connected sets in Z. Suppose there exists a Borel probability measure $\mathbb{P}$, with respect to the Hausdorff distance, on Γ such that*

(1) *there exists $c > 0$ such that $\operatorname{diam} E \geq c$ whenever $E \in \Gamma$ and*
(2) *there exists $C < \infty$ and $r_0 > 0$ such that*

$$\mathbb{P}\{E \in \Gamma : E \cap B \neq \emptyset\} \leq C\mu(B)^{\frac{1}{q}}$$

for every $B(x, r) \subset Z$ with $r \leq r_0$. Then $p \leq \operatorname{Cdim} Z$.

Proof. For the sake of contradiction suppose there exists a quasisymmetry $f : (Z, d) \to (Z', d')$ where Z' satisfies $\dim Z' < p$. Note that compactness of Z shows Z' is compact, because f is continuous. By assumption (1) and the Theorem 1.25 *Basic Properties of Quasisymmetries*, there exists $c' > 0$ such that $\operatorname{diam} f(E) \geq c' > 0$ holds uniformly for every E.

Next let $\epsilon > 0$ and $\{B(x'_\alpha, r'_\alpha)\}$ be an arbitrary set of open balls that satisfies $\bigcup B_\alpha = Z'$ and $\sum (r'_\alpha)^p < \epsilon$. Applying the Theorem 2.10 *5r Covering Lemma* we can choose a subfamily of pairwise disjoint balls with $\bigcup B(x'_i, 5r'_i) = Z'$ and $\sum (r'_i)^p < \epsilon$.

Apply the Theorem 2.11 *Characterization of Quasisymmetry* to f^{-1} and to the concentric balls $B(x'_i, r'_i) \subset B(x'_i, 5r'_i)$. Doing so, we obtain a set of balls $\{B(x_i, r_i)\} \subset Z$ with $B(x_i, r_i) \subset f^{-1}(B(x'_i, r'_i)) \subset f^{-1}(B(x'_i, 5r'_i)) \subset B(x_i, Kr_i)$ for some constant K. Since the balls $B(x'_i, r'_i)$ are disjoint, the bijective images $f^{-1}(B(x'_i, r'_i))$ are disjoint, which in turn makes the set $\{B(x_i, r_i)\}$ pairwise disjoint.

Next define a set function $g_i : \Gamma \to \{0, 1\}$ corresponding to each ball $B(x_i, r_i)$ by

$$g_i(E) = \begin{cases} 1 & \text{if } f(E) \cap B(x'_i, 5r'_i) \neq \emptyset \\ 0 & \text{elsewhere.} \end{cases}$$

First, we shall verify that each g_i is Borel-measurable. To accomplish this we consider the topology induced by the Hausdorff distance on Z.

To show Borel measurability of g, it suffices to consider the openness of $g_i^{-1}\{1\}$ in subspace topology on Γ, since if this preimage is a Borel set, then the complement $g_i^{-1}\{0\}$ is also a Borel set.

Write $g_i^{-1}\{1\} = \{E \in \Gamma : E \cap f^{-1}(5B_i') \neq \emptyset\} =: U$. Fix $E \in U$. Since $E \cap f^{-1}(5B_i') \neq \emptyset$ and $f^{-1}(5B_i')$ is open, for every $x \in E \cap f^{-1}(5B_i')$, there exists $\epsilon = \epsilon(x) > 0$ such that $B(x,\epsilon) \subset f^{-1}(5B_i')$. Then every $K \in \mathcal{B}_\epsilon(E)$ meets $f^{-1}(5B_i')$. In fact, $d_H(K,E) < \epsilon$ implies by definition that $x \in E \subset \bigcup_{y\in K} B(y,\epsilon)$. Hence, $d(x,K) < \epsilon$. This in turn means that there exists some $y \in K$ such that $y \in B(x,\epsilon) \subset f^{-1}(5B_i')$.

Hence, $\mathcal{B}_\epsilon(E) \cap \Gamma$ is open in Γ (in the relative topology) and $\mathcal{B}_\epsilon(E) \cap \Gamma \subset U$. Thus, U is relatively open.

Next, we will use the Theorem 1.9 *Bound for Connected Space* and the definition of $\mathcal{H}^1_\infty(f(E))$ which states that it is the infimum of the sums $\sum \operatorname{diam} B_i$ where B_i cover $f(E)$. Hence,

$$
\begin{aligned}
c' &\leq \operatorname{diam} f(E) \leq \mathcal{H}^1_\infty(f(E)) \leq \sum_{f(E)\cap B(x_i',5r_i')\neq\emptyset} \operatorname{diam}(5B_i') \leq \sum_{f(E)\cap B(x_i',5r_i')\neq\emptyset} 10r_i' \\
&= \sum_i 10r_i' g_i(E).
\end{aligned}
$$

Note that since the sequence $\{|r_i' g_i(E)|\}$ satisfies the assumptions of the corollary to the Theorem 1.14 *Lebesgue's Monotone Convergence Theorem*, we can interchange the sum and integral symbols. Also note that because f is a homeomorphism and $f^{-1}(B(x_i',5r_i')) \subset B(x_i,Kr_i)$, we find $E \in \{F \in \Gamma : f(F) \cap B(x_i',5r_i') \neq \emptyset\}$ implies $E \in \{F \in \Gamma : F \cap B(x_i',Kr_i') \neq \emptyset\}$. With these in mind, we obtain the following inequality

$$
\begin{aligned}
\frac{c'}{10} &= \frac{c'}{10}\mathbb{P}(\Gamma) = \int_\Gamma \frac{c'}{10}\, d\mathbb{P}(E) \leq \int_\Gamma \sum_i r_i' g_i(E)\, d\mathbb{P}(E) = \sum_i \int_\Gamma r_i' g_i(E)\, d\mathbb{P}(E) \\
&= \sum_i r_i' \mathbb{P}(\{E \in \Gamma : f(E) \cap B(x_i',5r_i') \neq \emptyset\}) \leq \sum_i r_i' \mathbb{P}(\{E \in \Gamma : E \cap B(x_i',Kr_i') \neq \emptyset\}) \\
&\leq C \sum_i r_i' \mu(B(x_i,Kr_i))^{\frac{1}{q}} \leq C \left(\sum_i (r_i')^p\right)^{\frac{1}{p}} \left(\sum_i \mu(B(x_i,Kr_i))\right)^{\frac{1}{q}} \\
&\leq C' \left(\sum_i (r_i')^p\right)^{\frac{1}{p}} \left(\sum_i \mu(B(x_i,r_i))\right)^{\frac{1}{q}} \leq C' \epsilon^{\frac{1}{p}} \mu(Z)^{\frac{1}{q}}
\end{aligned}
$$

where we applied the assumption (2), and basic facts such as $\int_\Gamma \chi_A(E)\, d\mathbb{P}(E) = \mathbb{P}(A)$, monotonicity of measure, and the Theorem 1.15 *Hölder Inequality* (on the third last step) as well as the fact that μ is doubling (on the second last step).

Arbitrariness of ϵ forces $c' = 0$, which is a contradiction. Thus, $\dim Z' \geq p$ for every metric space Z', which is quasisymmetrically equivalent to Z. Hence, we obtain $\operatorname{Cdim} Z \geq p$ as asserted. □

Remarks. The assumption (2) is our definition for the *diffuse measure*. Intuition for this nomenclature originates from chemistry. The condition (2) essentially means that the measure is not *concentrated* on a set with small μ-measure. That is, most of the sets in Γ must have been spread out (or diffused) to a "large" set. Formally, we say that $\mathbb{P}$ is concentrated on a measurable set A if

$\mathbb{P}(A \cap E) = \mathbb{P}(E)$ for every $E \in \Gamma$. There can be multiple concentration-sets, and, in particular, $\mathbb{P}$ is always concentrated on Γ.

2.2.3. *Lower Bound Result II.* Now we approach the notion of abundance of curves from the perspective of geometric function theory. For this purpose we introduce the definition of p-modulus of a curve family. It is a measure on the set of all curve families in a given space, and therefore it can be thought of as an approximate measure of the "size" of a curve family. However, in practice, one finds counterintuitive examples, if one thinks of this as an "absolute measure" for the size of the curve family. For example, if we take take any family of curves Γ_1, and then construct another family Γ_2 by taking a subcurve from each curve in Γ_1, we will find the p-modulus of Γ_1 is at most the p-modulus of Γ_2, that is, $\mathrm{Mod_p}\,\Gamma_1 \leq \mathrm{Mod_p}\,\Gamma_2$. Hence, it is more intuitive to think of p-modulus as a tool that gives a rough estimate of the "richness" of the family.

It may be helpful to think that p-modulus gives a qualitative estimate for the "abundance" of the family in the sense that nontrivial p-modulus shows there exists "plenty" of curves (at least one rectifiable curve). Moreover, good intuition for p-modulus of a curve family connecting two sets is to remember that p-modulus is small when there are only few long curves and large when there are many short curves.

We are going to give only the minimal definitions required for this thesis, and for a thorough treatment of line integrals and modulus, we refer the reader to [16, Chapter 7] and [40, Chapter 1].

Let (X, d, μ) be a metric measure space. We call a Borel measurable function $\rho : X \to [0, \infty]$ *admissible* for the curve family Γ if for all rectifiable curves $\gamma \in \Gamma$ we have

$$\int_\gamma \rho\, ds := \int_0^{\ell(\gamma)} \rho \circ \gamma_s(t)\, dt \geq 1$$

where $\gamma_s : [0, \ell(\gamma)] \to X$ is the arc-length parametrization. Recall that $\ell(\gamma)$ denotes the length of the curve. We denote by $\mathrm{Adm}\Gamma$ the *set of all admissible functions* for the family Γ.

For any $1 \leq p < \infty$ the *p-modulus*, $\mathrm{Mod_p}$, is defined as

$$\mathrm{Mod_p}\,\Gamma := \inf \int_X \rho^p\, d\mu$$

where the infimum is taken over all admissible functions ρ for the family Γ. If there are no rectifiable curves in Γ, we define $\mathrm{Mod_p}\,\Gamma = 0$.

Next we will prove some fundamental properties of the p-modulus. The parts (I)-(III) show that $\mathrm{Mod_p}$ is indeed a measure. Proofs of (I)-(III) and (V) are adapted from [40] and (IV) was originally proved in [42].

Theorem 2.13. (Properties of $\mathrm{Mod_p}$). *Let (Z, d, μ) be a metric measure space and let $p \geq 1$. Then the following hold;*

- I $\mathrm{Mod_p}\,\emptyset = 0$,
- II *if* $\Gamma_1 \subset \Gamma_2$, *then* $\mathrm{Mod_p}\,\Gamma_1 \leq \mathrm{Mod_p}\,\Gamma_2$,
- III $\mathrm{Mod_p} \bigcup_i \Gamma_i \leq \sum_i \mathrm{Mod_p}\,\Gamma_i$ *for any countable set of sets* Γ_i,
- IV *if* $p > 1$, *and* $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \cdots$, *then* $\mathrm{Mod_p}\left(\bigcup_i \Gamma_i\right) = \lim_{i\to\infty} \mathrm{Mod_p}\,\Gamma_i$*; and*
- V *if* Γ_1, Γ_2 *are curve families such that to each curve* $\gamma_1 \in \Gamma_1$ *corresponds a subcurve* γ_2 *which belongs to* Γ_2, *then* $\mathrm{Mod_p}\,\Gamma_1 \leq \mathrm{Mod_p}\,\Gamma_2$,

Proof.

I Since there are no rectifiable curves in $\emptyset$, by definition we have $\mathrm{Mod_p}\,\emptyset = 0$.

II Suppose $\Gamma_1 \subset \Gamma_2$. Thus, $\mathrm{Adm}\Gamma_2 \subset \mathrm{Adm}\Gamma_1$. In fact, $\rho \in \mathrm{Adm}\Gamma_2$ means that the path integral of ρ over every $\gamma \in \Gamma_2$ is at least 1. Choose any $\gamma_1 \in \Gamma_1$. Then the assumption $\Gamma_1 \subset \Gamma_2$ gives $\gamma_1 \in \Gamma_2$ and hence,

$$\int_{\gamma_1} \rho \, ds \geq 1.$$

Thus, $\rho \in \mathrm{Adm}\Gamma_1$.

Therefore, $\{\int_Z \rho^p \, d\mu : \rho \in \mathrm{Adm}\Gamma_2\} \subset \{\int_Z \rho^p \, d\mu : \rho \in \mathrm{Adm}\Gamma_1\}$, which implies that $\inf\{\int_Z \rho^p \, d\mu : \rho \in \mathrm{Adm}\Gamma_1\} \leq \inf\{\int_Z \rho^p \, d\mu : \rho \in \mathrm{Adm}\Gamma_2\}$.

III Let $\{\Gamma_i\}$ be a countable set of curve families. Considering the ℓ^p-sum, $(\sum_i \rho_i^p)^{\frac{1}{p}}$, of an arbitrary sequence of Γ_i-admissible functions, where we pick one admissible ρ_i from each $\mathrm{Adm}\Gamma_i$, we find $(\sum_i \rho_i^p)^{\frac{1}{p}}$ is in $\mathrm{Adm}\left(\bigcup_i \Gamma_i\right)$. In fact, if $\gamma \in \bigcup_i \Gamma_i$, then surely $\gamma \in \Gamma_i$ for some i. Hence,

$$\int_\gamma \left(\sum_j \rho_j^p\right)^{\frac{1}{p}} ds \geq \int_\gamma \rho_i \, ds \geq 1.$$

Since the ℓ^p-sum is admissible for $\bigcup_i \Gamma_i$, we find

$$\mathrm{Mod_p}\left(\bigcup_i \Gamma_i\right) \leq \int_Z \left(\left(\sum_i \rho_i^p\right)^{\frac{1}{p}}\right)^p d\mu = \sum_i \int_Z \rho_i^p \, d\mu$$

which implies countable subadditivity after taking infimums of the righ-hand-side.

IV We shall execute this proof by constructing a Cauchy sequence of the admissible functions for $\{\Gamma_i\}$, and then use a result in Fuglede's paper [14, Theorem 3].

First, observe that the assumption $\Gamma_k \subset \bigcup_i \Gamma_i$, for every k, combined with the monotonicity of p-modulus imply $\lim_{i\to\infty} \mathrm{Mod_p}\,\Gamma_i \leq \mathrm{Mod_p}\left(\bigcup_i \Gamma_i\right)$. If $\lim_{i\to\infty} \mathrm{Mod_p}\,\Gamma_i = \infty$, we are done. Hence, assume $\lim_{i\to\infty} \mathrm{Mod_p}\,\Gamma_i < \infty$.

Uniform convexity of L^p, $1 < p < \infty$, means that for every $0 < \epsilon \leq 2$ there exists some $\delta(\epsilon) > 0$ such that

$$\frac{1}{2}\,||\rho_1 + \rho_2|| \leq 1 - \delta \quad \text{ whenever } ||\rho_1|| \leq 1,\ ||\rho_2|| \leq 1 \text{ and } ||\rho_1 - \rho_2|| \geq \epsilon.$$

Geometrically uniform convexity of a normed space means that the unit ball of the space is convex in the usual sense, but, for example, squares are not considered as uniformly convex sets. For more information on uniform convexity of Banach spaces see, for instance, [10].

Uniform convexity of L^p implies that any convex, closed and nonempty set contains an element of minimal norm. For every i, there exists a closed, convex and nonempty set of Borel functions satisfying $\int_\gamma \rho \, ds \geq 1$ for almost every $\gamma \in \Gamma$. Hence, if we omit a subset having null $\mathrm{Mod_p}$-measure, we obtain a subset $\Gamma_i' \subset \Gamma_i$ such that there exists a Borel function $\rho_i \in \mathrm{Adm}\Gamma_i'$ that minimizes the p-modulus: $\mathrm{Mod_p}\,\Gamma_i' = \mathrm{Mod_p}\,\Gamma_i$ and

$$\int_Z \rho_i^p \, d\mu = \mathrm{Mod_p}\,\Gamma_i. \tag{1}$$

For details see [14, Pages 181-182].

Let $i, j \in \mathbb{N}$ and without loss of generality assume $i < j$. Since $\Gamma_i \subset \Gamma_j$, and the only requirement for the corresponding Γ_i' and Γ_j' is that they are families that contain almost every curve in Γ_i and Γ_j, we can arrange them so that $\Gamma_i' \subset \Gamma_j'$.

Let $\rho_i \in \mathrm{Adm}\Gamma_i'$ and $\rho_j \in \mathrm{Adm}\Gamma_j'$ be the corresponding minimizing functions. Thus,

$$\int_{\gamma\in\Gamma_i'} \frac{1}{2}(\rho_i + \rho_j)\, ds \geq 1.$$

Thus, $2^{-1}(\rho_i + \rho_j) \in \mathrm{Adm}\Gamma_i'$. Consequently,

$$\begin{aligned} &\mathrm{Mod_p}\, \Gamma_i \leq 2^{-p} \int_Z (\rho_i + \rho_j)^p \, d\mu \\ \iff \qquad & 2^{\frac{p}{p-1}} (\mathrm{Mod_p}\, \Gamma_i)^{\frac{1}{p-1}} \leq \left(\int_Z (\rho_i + \rho_j)^p \, d\mu \right)^{\frac{1}{p-1}}. \end{aligned}$$

First, suppose $p \geq 2$. Using the above inequality, monotonicity of p-modulus and the following inequality (3) in [10, Theorem 3];

$$||x + y||^p + ||x - y||^p \leq 2^{p-1}(||x||^p + ||y||^p) \quad (\text{for all } x, y \in L^p),$$

we obtain the following estimate,

$$\begin{aligned} 2^p \, \mathrm{Mod_p}\, \Gamma_i + ||\rho_i - \rho_j||_p^p &\leq ||\rho_i + \rho_j||_p^p + ||\rho_i - \rho_j||_p^p \\ &\leq 2^{p-1}(||\rho_i||_p^p + ||\rho_j||_p^p) \\ &\leq 2^p \lim_{i\to\infty} \mathrm{Mod_p}\, \Gamma_i \end{aligned}$$

Hence, $||\rho_i - \rho_j|| \leq 2^p(\lim_{i\to\infty} \mathrm{Mod_p}\, \Gamma_i - \mathrm{Mod_p}\, \Gamma_i) \to 0$ as $i \to \infty$.

Next, assume $1 < p < 2$, which means $\frac{p}{p-1} > 2$. Therefore, similar argumentation as above, but with the following inequality (5) in [10, Theorem 3];

$$||x + y||^p + ||x - y||^p \leq 2(||x||^{\frac{p}{p-1}} + ||y||^{\frac{p}{p-1}})^{p-1} \quad (\text{for all } x, y \in L^p)$$

shows,

$$\begin{aligned} 2^{\frac{p}{p-1}} (\mathrm{Mod_p}\, \Gamma_i)^{\frac{1}{p-1}} + ||\rho_i - \rho_j||_p^{\frac{p}{p-1}} &\leq ||\rho_i + \rho_j||_p^{\frac{p}{p-1}} + ||\rho_i - \rho_j||_p^{\frac{p}{p-1}} \\ &\leq 2(||\rho_i||_p^p + ||\rho_j||_p^p)^{\frac{p}{p-1}-1} \\ &= 2 \cdot 2^{\frac{1}{p-1}} (2^{-1}(||\rho_i||_p^p + ||\rho_j||_p^p))^{\frac{1}{p-1}} \\ &\leq 2^{\frac{p}{p-1}} (\lim_{i\to\infty} \mathrm{Mod_p}\, \Gamma_i)^{\frac{1}{p-1}} \end{aligned}$$

Thus, the sequence $\{\rho_i\}$ is Cauchy for $1 < p < \infty$. Completeness of L^p shows that there exists $\rho \in L^p$ such that $\rho_i \to \rho$.

To finish this proof, we apply part (f) in [14, Theorem 3], which gives us a L^1 convergent subsequence $\{\rho_{i,k}\}$ of our L^p convergent sequence $\{\rho_i\}$. Moreover, we can choose this subsequence so that there exists a subset $\Gamma^* \subset \Gamma' = \bigcup_i \Gamma_i'$ for which

$$\mathrm{Mod_p}\, \Gamma^* = \mathrm{Mod_p}\, \Gamma' = \mathrm{Mod_p} \left(\bigcup_i \Gamma_i \right)$$

and the subsequence converges in the following sense,

$$\lim_{k\to\infty}\int_{\gamma}|\rho_{i,k}-\rho|\,ds=0 \tag{2}$$

for every $\gamma\in\Gamma^*$. Note that $\Gamma^*\subset\Gamma'$ means that every $\gamma\in\Gamma^*$ is in some Γ'_m. If $k\geq m$, then $\rho_{i,k}\in\mathrm{Adm}\Gamma'_m$. Since this holds for every k, by (2) we have

$$\int_{\gamma\in\Gamma^*}\rho\,ds=\int_{\gamma\in\Gamma^*}\lim_{k}\rho_{i,k}\,ds=\int_{\gamma\in\Gamma'_m}\lim_{k}\rho_{i,k}\,ds\geq 1.$$

Hence, $\rho\in\mathrm{Adm}\Gamma^*$. Therefore,

$$\begin{aligned}\mathrm{Mod_p}\left(\bigcup_i\Gamma_i\right)&=\mathrm{Mod_p}\,\Gamma^*\leq\int_Z\rho^p\,d\mu=\lim_{k\to\infty}\int_Z\rho_{i,k}^p\,d\mu=\lim_{i\to\infty}\int_Z\rho_i^p\,d\mu\\&=\lim_{i\to\infty}\mathrm{Mod_p}\,\Gamma_i,\end{aligned}$$

since (1) holds for every i. This proves part (IV).

V Let $\rho\in\mathrm{Adm}\Gamma_2$ and pick any $\gamma_1\in\Gamma_1$. Denote by γ_2 the subcurve of γ_1 in Γ_2. Then

$$\int_{\gamma_1}\rho\,ds=\int_{\gamma_2}\rho\,ds+\int_{\gamma_1\setminus\gamma_2}\rho\,ds\geq\int_{\gamma_2}\rho\,ds\geq 1.$$

Thus, $\rho\in\mathrm{Adm}\Gamma_1$, which implies $\mathrm{Adm}\Gamma_2\subset\mathrm{Adm}\Gamma_1$. Hence,

$$\inf_{\rho\in\mathrm{Adm}\Gamma_1}\int_Z\rho\leq\inf_{\rho\in\mathrm{Adm}\Gamma_2}\int_Z\rho.$$

□

Remark. Clearly, (II) in the above theorem is just a special case of (V). Also note that property (IV) is stronger than what we usually have for measures, since we do not require any measurability of the curve families nor any regularity properties of $\mathrm{Mod_p}$.

Final tool that we will need for our next main result is the following lemma, sometimes called Bojarski's lemma. The original source of this result seems to be [38]. However, our argument is adapted from [20, Lemma 2.8], but there seems to be many ways to prove this fundamental inequality in harmonic analysis. We will state the lemma in our setting for convenience of the reader, but clearly this proof generalizes to more general settings.

Theorem 2.14. (Bojarski's lemma). *If $1\leq p<\infty$, $\{B(x_i,r_i)\}$ is a countable collection of balls in a doubling measure space (Z,μ) and $\{r'_i\geq 0\}$ is a countable collection of non-negative real numbers, then*

$$\int_Z\left(\sum_i\frac{r'_i}{r_i}\chi_{B(x_i,2Kr_i)}\right)^p d\mu\leq C(2K,p,\mu)\int_Z\left(\sum_i\frac{r'_i}{r_i}\chi_{B(x_i,r_i)}\right)^p d\mu. \tag{1}$$

for $K>1$.

Proof. The case $p=1$ follows directly from the Theorem 1.20 *Fubini's Theorem* and the doubling property of the measure. Thus, assume $1<p<\infty$.

To prove (3) we recall the notion of duality between L^p and $L^{p'}$ given in the introduction (Theorem 1.16 L^p *Duality*),

$$\left|\left|\sum_i \frac{r_i'}{r_i}\chi_{B(x_i,2Kr_i)}\right|\right|_p = \sup_{||g||_{p'}=1}\left|\int \sum_i \frac{r_i'}{r_i}\chi_{B(x_i,2Kr_i)}g\right|.$$

In the proof of the inequality (1), we will replace the average with maximal function, recall the Definition 1.18 *Maximal Function*, over a smaller ball with a nice trick. Note that $y \in B(x, r_i)$ implies $B(y, r_i) \subset B(x_i, 2Kr_i) \subset B(y, 4Kr_i)$ and thus

$$\begin{aligned}
⨍_{B(x_i,2Kr_i)} |g|\, d\mu &= \frac{1}{\mu(B(x_i,2Kr_i))}\int_{B(x_i,2Kr_i)} |g|\, d\mu \\
&\le \frac{\mu(B(x_i,4Kr_i))}{\mu(B(x_i,4Kr_i))}\frac{1}{\mu(B(x_i,2Kr_i))}\int_{B(y,4Kr_i)} |g|\, d\mu \\
&\le C(\mu,K)Mg(y)
\end{aligned}$$

Then integrating over every $y \in B(x_i, r_i)$ we obtain

$$\mu(B(x_i,r_i))⨍_{B(x_i,2Kr_i)} |g|\, d\mu \le C(\mu,K)\int_{B(x_i,r_i)} Mg(y)\, d\mu(y).$$

Using this fact, the Theorem 1.14 *Lebesgue's Monotone Convergence Theorem* to interchange the sum and integral, and the estimate for maximal function given in the Theorem 1.19 *Maximal Function Theorem*, we find

$$\begin{aligned}
\left|\int \sum_i \frac{r_i'}{r_i}\chi_{B(x_i,2Kr_i)}g\, d\mu\right| &\le \sum_i \frac{r_i'}{r_i}\int_{B(x_i,2Kr_i)} |g|\ d\mu \\
&= \sum_i \frac{r_i'}{r_i}\frac{\mu(B(x_i,2Kr_i))}{\mu(B(x_i,2Kr_i))}\int_{B(x_i,2Kr_i)} |g|\ d\mu \\
&\le \sum_i \frac{r_i'}{r_i}C(2K,\mu)\mu(B(x_i,r_i))⨍_{B(x_i,2Kr_i)} |g|\, d\mu \\
&\le C(2K,\mu)\sum_i \frac{r_i'}{r_i}\int_{B(x_i,r_i)} Mg(y)\, d\mu(y) \\
&= C(2K,\mu)\int \sum_i \frac{r_i'}{r_i}\chi_{B(x_i,r_i)}(y)Mg(y)\, d\mu(y) \\
&\le C(2K,\mu)\left|\left|\sum_i \frac{r_i'}{r_i}\chi_{B(x_i,r_i)}\right|\right|_p ||Mg||_{p'} \\
&\le C(2K,p,\mu)\left|\left|\sum_i \frac{r_i'}{r_i}\chi_{B(x_i,r_i)}\right|\right|_p ||g||_{p'}
\end{aligned}$$

After taking supremums of both sides over all $g \in L^{p'}$ with $||g||_{p'} = 1$ we obtain (1). Note that we applied basic properties of the Lebsgue integral such as $\int_X \chi_A g = \int_A g$ and as always, the doubling property of μ as well as Hölder's inequality on the second to last line.

□

With these tools under our belt we can proceed to our second main result. This proof is adapted from [25, Chapter 4]

Theorem 2.15. (LOWER BOUND RESULT II). *Let (Z, d, μ) be a compact, doubling metric measure space satisfying the upper mass bound*

$$\mu(B(z, r)) \leq Cr^Q \tag{1}$$

for all balls $B(z, r)$ in Z, for some fixed constant C. Suppose that Z contains a curve family Γ with $\mathrm{Mod_p}\,\Gamma > 0$ for some $1 < p \leq Q$. Then $\mathrm{Cdim}\, Z \geq Q$.

Proof. Let $f : (Z, d) \to (Z', d')$ be a quasisymmetry. For the sake of contradiction suppose $\dim Z' < Q$.

Define $\Gamma_i := \{\gamma \in \Gamma : \mathrm{diam}\,\gamma > \frac{1}{i}\}$. Then $\mathrm{Mod_p}\,\Gamma = \mathrm{Mod_p} \bigcup_i \Gamma_i = \lim_i \mathrm{Mod_p}\,\Gamma_i > 0$ by the Theorem 2.13 *Properties of* $\mathrm{Mod_p}$ (IV). Thus, there exists $m \in \mathbb{N}$ such that $\mathrm{Mod_p}\,\Gamma_m > 0$. To see this, note that if there wasn't such an m, then it would mean $\lim_i \mathrm{Mod_p}\,\Gamma_i = 0$. It is noteworthy to mention that this curve family having a uniform lower bound on the diameter of the curves can also be extracted via the subadditivity of the $\mathrm{Mod_p}$.

Now $\mathrm{diam}\,\gamma \geq \frac{1}{m} > 0$ for every $\gamma \in \Gamma_m$. Write $c = \frac{1}{m}$.

We can argue exactly as in the proof of the Theorem 2.12 *Lower Bound Result I* to obtain $c' > 0$ such that $\mathrm{diam}\, f(\gamma) \geq c'$ for every $\gamma \in \Gamma_m$.

Let $\epsilon > 0$. Again we refer the reader to the proof of the previous lower bound result for details of the following lines. Suppose we have chosen $\{B(x_i, r_i)\}$ and $\{B(x_i', r_i')\}$ so that they are pairwise disjoint sets of balls in Z and Z', respectively, and the radii r_i' satisfy $\sum_i (r_i')^Q < \epsilon$. Since we applied the same reasoning as in the proof of Theorem 2.12, we have

$$B_i \subset f^{-1}(B_i') \subset f^{-1}(5B_i') \subset KB_i, \tag{2}$$

for some $K \geq 5$.

Define a Borel function $\rho : Z \to [0, \infty]$,

$$\rho(x) = \sum_i \frac{r_i'}{r_i} \chi_{B(x_i, 2Kr_i)}(x).$$

Now for any $\gamma \in \Gamma_m$ we obtain,

$$\int_\gamma \rho\, ds = \sum_i \frac{r_i'}{r_i} \ell(\gamma \cap 2KB_i) \geq \frac{1}{10} \sum_{\substack{i \\ f(\gamma) \cap 5B_i' \neq \emptyset}} 10r_i' \frac{\ell(\gamma \cap 2KB_i)}{r_i}.$$

The inequality holds, since we did nothing but added restrictions to the summing indexes. Next, suppose we have chosen ϵ small enough so that $8Kr_i < c$ for every i. Thus, $\mathrm{diam}(2KB_i) \leq 4Kr_i < \frac{c}{2} < \mathrm{diam}\,\gamma$. This implies $\gamma \not\subset 2KB_i$. Note that (2) in turn implies that if for some i we have $f(\gamma) \cap 5B_i' \neq \emptyset$, then $\gamma \cap KB_i \neq \emptyset$. Putting these together, we find that when summing over i, we have $\ell(\gamma \cap 2KB_i) \geq Kr_i$. Now, by the Theorem 1.9 *Bound for Connected Space*,

$$\int_\gamma \rho\, ds \geq \frac{K}{10} \sum_{\substack{i \\ f(\gamma) \cap 5B_i' \neq \emptyset}} 10r_i' \geq \mathcal{H}^1_\infty(f(\gamma)) \geq \mathrm{diam}\, f(\gamma) \geq \frac{Kc'}{10}$$

This shows that $K_0\rho$ is Γ_m-admissible, where $K_0 = \frac{10}{Kc'}$.

Finally, with these derived facts and the Theorem 2.14 *Bojarski's lemma* we can show our assumption $\dim Z' < Q$ leads to a contradiction. Lets estimate,

$$\begin{aligned}
\operatorname{Mod_p}\Gamma &\le \int_Z K_0^p \rho^p \, d\mu = K_0^p \int_Z \left(\sum_i \frac{r_i'}{r_i} \chi_{B(x_i, 2Kr_i)} \right)^p \, d\mu \\
&\le K_0^p C(2K, p, \mu) \int_Z \left(\sum_i \frac{r_i'}{r_i} \chi_{B(x_i, r_i)} \right)^p \, d\mu = C' \sum_i \left(\frac{r_i'}{r_i} \right)^p \mu(B(x_i, r_i)) \\
&\le C' \sum_i (r_i')^p \, \mu(B(x_i, r_i))^{1-\frac{p}{Q}} \le C' \left(\sum_i (r_i)^Q \right)^{p/Q} \left(\sum_i \mu(B_i) \right)^{1-\frac{p}{Q}} \\
&\le C' \epsilon^{p/Q} \mu(Z)^{1-\frac{p}{Q}},
\end{aligned}$$

where we used the assumption (1) and other basic facts used, for example, in the proof of the Theorem 2.14 *Bojarski's lemma*. Note that the constant changes during the chain of inequalities, but it is always independent of ϵ. Thus, we have $\operatorname{Mod_p}\Gamma \le \epsilon$ for every $\epsilon > 0$. This is a contradiction, since we assumed $\operatorname{Mod_p}\Gamma > 0$. □

Remarks. The key assumption is that the modulus is nonzero. This qualitative statement for the curve family is relatively easy to obtain and thus makes the result powerful. On the other hand, we also need the upper mass bound for the volume measure μ to be comparable to the arc length measure, which is the usual one-dimensional Hausdorff measure. The following result illustrates nicely the application of this result.

This result gives us an interesting corollary which we shall call Tyson's Theorem. It states that for every real number there always exists a metric space that is minimal for the conformal dimension (with the obvious exclusion of $(0, 1)$). Namely

Theorem 2.16. (Tyson's Theorem). *There exist metric spaces Z with* $\operatorname{Cdim} Z = \dim Z = Q$ *for any* $Q \ge 1$.

Proof. Let $(X, d, \mathcal{H}^{Q-1})$ be a compact metric measure space where $\mathcal{H}^{Q-1}$ is the Ahlfors $(Q-1)$-regular measure.

Note that there exists such a space for any $Q - 1 \ge 0$. To see this, consider the central Cantor set C_a for $a \in (0, \frac{1}{2})$. This set can be constructed similarly as the middle-thirds Cantor set, but now instead of removing $\frac{1}{3}$ from the center of the unit interval, we remove a segment having length of $1 - 2a$. Then we continue this process iteratively as in the construction of the middle-thirds Cantor set. Hausdorff dimension of C_a is $\log(2)/\log(1/a)$, which can be proven similarly as for the middle thirds Cantor set. If we take the Cartesian product of C_a with itself n times, this product space is compact and Ahlfors $n\log(2)/\log(1/a)$-regular. Choosing a and n appropriately, we find $Q - 1 = n\log(2)/\log(1/a)$.

Next define $Z = X \times [0, 1]$ where we consider the product measure $\mu = \mathcal{H}^{Q-1} \times \mathcal{H}^1$ and the product metric. Then for any $B(x, r) \subset Z$ we have

$$\mu(B(x, r)) = \mathcal{H}^{Q-1}(B^{Q-1}(x, r))\mathcal{H}^1(B^1(x, r)) \le C\mathcal{H}^Q(B(x, r)) = C' r^Q.$$

Hence, μ satisfies the mass bound condition of the Theorem 2.15 *Lower Bound Result II* (1). Since we are taking the Cartesian product of X with the unit interval, we always have curve families with positive p-modulus in Z.

To see this, consider a set $A \subset X$ such that $\mathcal{H}^{Q-1}(A) > 0$ and define a curve family $\Gamma = \{\gamma_y(t) : y \in A, 0 \le t \le 1\}$. Then pick any $\rho \in \mathrm{Adm}\Gamma$. Now applying the fact that $\int_{\gamma_y} \rho\, ds \ge 1$, and then using Fubini and Hölder, in this order, we find

$$\begin{aligned}\mathcal{H}^{Q-1}(A) &\le \int_A \int_{\gamma_y} \rho\, ds\, d\mathcal{H}^{Q-1} = \int_{A\times[0,1]} \rho\, d\mu \le \left(\int_Z \rho^p\, d\mu\right)^{\frac{1}{p}} \left(\int_{A\times[0,1]} d\mu\right)^{\frac{1}{p'}} \\ &= \left(\int_Z \rho^p\, d\mu\right)^{\frac{1}{p}} \left(\mathcal{H}^{Q-1}(A)\right)^{\frac{1}{p'}},\end{aligned}$$

which gives us $\mathcal{H}^{Q-1}(A)^{\frac{1}{p}} = \mathcal{H}^{Q-1}(A)^{1-\frac{1}{p'}} \le \left(\int_Z \rho^p\, d\mu\right)^{\frac{1}{p}}$. Thus, $\mathcal{H}^{Q-1}(A)$ is a lower bound for all $\int_Z \rho^p\, d\mu$.

Although not necessary for this result, we will show $\mathcal{H}^{Q-1}(A)$ is the greatest lower bound. Consider the function $\chi_{A\times[0,1]}$, which is Γ-admissible because,

$$\int_{\gamma_y} \chi_{A\times[0,1]}\, ds = \int_0^{\ell(\gamma_y)} \chi_{A\times[0,1]}(\gamma_{y_s}(t))\, dt = 1$$

for every $\gamma_y \in \Gamma$. This gives us

$$\int_Z \chi^p_{A\times[0,1]}\, d\mu = \int_{A\times[0,1]} d\mu = \mathcal{H}^{Q-1}(A).$$

Hence, $\mathrm{Mod_p}\,\Gamma = \mathcal{H}^{Q-1}(A) > 0$.

Now the Theorem 2.15 *Lower Bound Result II* implies that $\mathrm{Cdim}\, Z \ge Q$. Since μ is Q-regular, we have $\dim Z = Q$. Hence, Z attains the infimal Hausdorff dimension, that is, Z is minimal for Cdim. □

Remark. If X is the Cantor set, then the set Z in the above theorem is called the *Cantor comb*. We will obtain a generalization of this result later.

2.2.4. *Lower Bound Result III.* Now we have arrived at the beginning of our third goal for this chapter. To get there, we need to construct some machinery.

The key idea in the proof of the Theorem 2.23 *Lower Bound Result III* will be to take advantage of the geometry of Euclidean spaces. By this we mean that instead of arbitrary covers or spherical covers we use dyadic cube covers. Recall that these are just half-open cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes and vertices at rational coordinates belonging to the set $\{2^{-k}(x_1, ..., x_n) : x_i \in \mathbb{Z}\}$.

The real usefulness of this becomes evident when we want to obtain a disjoint set of covering sets. This is in general done with covering theorems, but now we do not need to use them, because dyadic cubes are disjoint by construction. See the definition 1.5 *Dyadic Cubes in* $\mathbb{R}^n$. Before giving the proof we need a few auxiliary results from geometric measure theory. First of these is a well-known result in the field, namely, *Frostman's lemma* and the two other results are fundamental properties of Hausdorff dimension.

Theorem 2.17. (Frostman's Lemma and Mass Distribution Principle). *Let* $0 \le s \le n$. *For any compact set* $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) > 0$ *if and only if there exists a positive Borel measure* μ *supported on* A *satisfying* $\mu(B(x,r)) \le r^s$ *for all balls* $B(x,r)$ *in* $\mathbb{R}^n$.

Remarks. By the *support of a measure* we mean the set where the measure "lives". In formal language the support of μ is defined to be the set $\operatorname{supp}\mu := \{x \in X : \mu(B(x,r)) > 0 \text{ for every } r > 0\}$.

The forward implication gets to be called Frostman's lemma, while the converse is sometimes called *Mass Distribution Principle*. It should be noted that there exists a more general version of the Frostman's lemma for arbitrary Borel sets. The proof is more involved, and since we only need this less general version, we refer the interested reader to [8]. Note that since every metric space is a Hausdorff topological space, and in every Hausdorff space compact sets are Borel sets, this is indeed a special case of the more general result.

Both of these results are principal tools for obtaining bounds for Hausdorff dimension. These results have been shown to be of crucial importance in bounding Hausdorff dimension from below, since it only suffices to show the existence of a Borel measure which is supported on the Borel set A and has the upper mass bound [26]. First we prove the mass distribution principle.

Proof of Mass Distribution Principle. Suppose A is compact, μ is supported on A and for any $B(x,r) \subset Z$ we have $\mu((Bx,r)) \leq r^s$. Let $B(x_i,r_i)$ be any spherical cover for A. Then

$$0 < \mu(A) \leq \mu\left(\bigcup_i B_i(z,r)\right) \leq \sum_i \mu(B_i(z,r)) \leq \sum_i r_i^s \leq \sum_i (\operatorname{diam} B_i)^s.$$

Since this holds for every cover B_i, we have

$$\begin{aligned} 0 &< \inf\{\sum_i (\operatorname{diam} B_i)^s : A \subset \bigcup_i B_i, \operatorname{diam} B_i < \infty\} = \mathcal{H}^s_\infty(A) \\ &\leq \lim_{\delta\to 0} \inf\{\sum_i (\operatorname{diam} B_i)^s : A \subset \bigcup_i B_i, \operatorname{diam} B_i < \delta\} = \mathcal{H}^s(A) \end{aligned}$$

since $\delta \mapsto \inf\{\sum_i (\operatorname{diam} B_i)^s : A \subset \bigcup_i B_i, \operatorname{diam} B_i < \delta\}$ is nonincreasing. Hence, as δ decreases, the value of the output cannot decrease. □

Remarks. Note that this result can be strengthened at least by a small amount since we also gain that $\mathcal{H}^s(A) \geq \mu(A)$. To see this note that

$$\sum_i (\operatorname{diam} U_i)^s \geq \sum \mu(B(x_i, \operatorname{diam} U_i)) \geq \mu(A).$$

where $\{U_i\}$ is any countable cover of A. Infimizing this over all covers $\{U_i\}$ shows $\mathcal{H}^s(A) \geq \mu(A)$.

To prove the Frostman's lemma, we need some tools. We begin by discussing some preliminary definitions and results. The following discussion is based on [26] and [29].

Definition 2.18. (Weak Convergence of Measures). Let (X,d) be a metric space equipped with a sequence of Borel measures $\{\mu_j\}$ satisfying $\mu_j(K) < \infty$ for every compact set $K \subset X$. If for $f \in C_c(X)$, $f : X \to \mathbb{C}$, we have

$$\lim_i \int_X f\, d\mu_i = \int_X f\, d\mu$$

we say that μ_i *converges weakly* to μ and we write $\mu_i \rightharpoonup \mu$.

The next inequalities are required in the proof we give for Frostman's lemma.

Theorem 2.19. (Inequalities for Weak Convergence). *Suppose we have a weakly convergent sequence of Borel measures μ_i which converges to μ in a locally compact metric space X such that the limit also satisfies $\mu(K) < \infty$ for every compact set $K \subset X$. Then for any compact $K \subset X$ and σ-compact but open $U \subset X$, we have the following*

$$\limsup_j \mu_j(K) \le \mu(K) \qquad \textit{and} \qquad \mu(U) \le \liminf_j \mu_j(U).$$

Proof. Let $\epsilon > 0$ and $K \subset X$ be a compact subset. Since X is locally compact, we can choose a subset $\{x_\alpha \in K\}$ of points of K such that each point has an open neighborhood U_α included in some compact set C_α, and $\bigcup U_\alpha$ covers K. Since K is compact, there exists a finite sub-cover $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ of K.

Taking the union of these open sets $\bigcup_{i=1}^n U_{\alpha_i}$ and the corresponding union of the compact sets $\bigcup_{i=1}^n C_{\alpha_i}$ we find that $K \subset \bigcup_{i=1}^n U_{\alpha_i} \subset \bigcup_{i=1}^n C_{\alpha_i}$. Since $\bigcup_{i=1}^n C_{\alpha_i}$ is compact and $\mu(C) < \infty$ for all compact sets C by our hypothesis, we find that $\mu(U') < \infty$, where $U' := \bigcup_{i=1}^n U_{\alpha_i}$. Since we are dealing with Borel measures, the finiteness of $\mu(U')$ implies that there exists an open set U with μ-measure arbitrarily close to $\mu(K)$. In particular, there exists open $U \supset K$ with $\mu(K) \ge \mu(U) - \epsilon$. See [27, Theorem 1.10].

Next, let $f \in C_c(X)$, $f : X \to \mathbb{C}$ satisfying

$$\chi_K \le f \le \chi_U. \tag{1}$$

To show there exists at least one such function, consider for instance

$$f(x) = \max\left\{0, 1 - \frac{\text{dist}(x, K)}{\text{dist}(K, U^c)}\right\}.$$

Applying our assumption and the definition of weak convergence, we find

$$\begin{aligned} \mu(K) &\ge \mu(U) - \epsilon \ge \int_X f - \epsilon = \lim_j \int_X f \, \mu_j - \epsilon \ge \limsup_j \int_X \chi_K \, \mu_j - \epsilon \\ &= \limsup_j \mu_j(K) - \epsilon. \end{aligned}$$

Arbitrariness of ϵ proves the first inequality.

To prove the second inequality, note that U is σ-compact. Suppose $U = \bigcup_{i=1}^\infty K_i$, where each K_i is compact. Write $C_1 := K_1$ and in general $C_m := \bigcup_{i=1}^m K_i$. Then each C_m is compact, $C_i \subset C_{i+1}$ and $U = \bigcup_{i=1}^\infty K_i = \bigcup_{i=1}^\infty C_i$.

For every C_i choose some $f \in C_c(X)$ that satisfies $\chi_{C_i} \le f \le \chi_U$. Then we have

$$\mu(C_i) \le \int f \, d\mu = \lim_n \int f \, d\mu_n \le \lim_n \int \chi_U \, d\mu_n = \liminf_n \mu_n(U).$$

When we let $i \to \infty$, we find by continuity of measures (see the notes below Definition 1.2) that the left-hand-side tends to $\mu\left(\bigcup_i C_i\right) = \mu(U)$. Thus, $\mu(U) \le \liminf_n \mu_n(U)$ as asserted. □

Next we need one basic result concerning the weak convergence of measures. Namely that every uniformly bounded sequence has a convergent subsequence. Recall that the space $C_c(X)$ is separable whenever X is locally compact.

Theorem 2.20. (Convergent Subsequence of Measures). *If $\{\mu_i\}_{i\in I}$, $I \subset \mathbb{N}$, is a sequence of Borel measures on $\mathbb{R}^d$ which is uniformly bounded, that is, $\sup_i \mu_i(\mathbb{R}^d) < \infty$, then there exists a subsequence $\{\mu_{i_j}\}_{j\in J}$, $J \subset \mathbb{N}$, that converges weakly.*

Proof. Let $(C_c(\mathbb{R}^d), ||\cdot||_\infty)$ be a normed space where $C_c(\mathbb{R}^d)$ is equipped with the supremum norm $||f||_\infty := \sup\{|f(x)| : x \in \mathbb{R}^d\}$. Local compactness gives a dense sequence of functions $\{f_k\} \subset C_c(\mathbb{R}^d)$.

For every k, there exists a subsequence $\{i_j^k\}_j$ of the index set $i \in I$ such that

$$c_k := \lim_j \int f_k \, d\mu_{i_j^k}$$

is well defined, since each f_k has a compact support and our sequence of measures is uniformly bounded.

Clearly we can choose these index subsequences $\{i_j^k\}_j$ such that $\{i_j^{k+1}\}_j$ is a subsequence of $\{i_j^k\}_j$. Just pick the sequence $\{i_j^k\}_j$ first and then pick the sequence $\{i_j^{k+1}\}_j$. Because of this, for $m \geq k$ we have $i_m^m \in \{i_k^j\}_j$. In other words, for every k we have $\{i_m^m\}_{m\geq k} \subset \{i_j^k\}_j$.

Hence, in fact, we have the following

$$c_k = \lim_j \int f_k \, d\mu_{i_j^j}.$$

Next choose any $\varphi \in C_c(\mathbb{R}^d)$. Since $\{f_k\}$ is dense in $C_c(\mathbb{R}^d)$, there exists a subsequence $\{f_{k_n}\}_n$ such that $\lim_n ||\varphi - f_{k_n}||_\infty = 0$.

Next, we will show that c_{k_n} converges by appealing to the completeness of $\mathbb{R}^d$. Let $n, m \in \mathbb{N}$ and then we find

$$\begin{aligned}
|c_{k_n} - c_{k_m}| &\leq \limsup_m \left| c_{k_n} - c_{k_m} + \int \varphi \, d\mu_{i_j^j} - \int \varphi \, d\mu_{i_j^j} \right| \\
&= \limsup_m \left| \int f_{k_n} \, d\mu_{i_j^j} - \int f_{k_m} \, d\mu_{i_j^j} + \int \varphi \, d\mu_{i_j^j} - \int \varphi \, d\mu_{i_j^j} \right| \\
&\leq \limsup_m \left(\left| \int f_{k_n} \, d\mu_{i_j^j} - \int \varphi \, d\mu_{i_j^j} \right| + \left| \int \varphi \, d\mu_{i_j^j} - \int f_{k_m} \, d\mu_{i_j^j} \right| \right) \\
&\leq \sup_{m\geq 0} \mu(\mathbb{R}^d) \left(||f_{k_n} - \varphi|| + ||\varphi - f_{k_m}|| \right) \to 0
\end{aligned}$$

as $n, m \to \infty$. Hence, c_{k_i} is Cauchy. By completeness of $\mathbb{R}^d$, there exists a limit $c = \lim_i c_{k_i}$.

Let $\varepsilon > 0$ be arbitrary. Then there exists N such that for all $k_i \geq N$ we have $|c - c_{k_i}| < \epsilon$. Therefore,

$$\begin{aligned}
0 &\leq \limsup_j \left| c - \int \varphi \, d\mu_{i_j^j} \right| = \limsup_j \left| c - \int \varphi \, d\mu_{i_j^j} + c_{k_i} - c_{k_i} \right| \\
&\leq \limsup_j \left| \int f_{k_i} \, d\mu_{i_j^j} - \int \varphi \, d\mu_{i_j^j} \right| + \epsilon \leq \limsup_j \mu_{i_j^j}(\mathbb{R}^d) ||f_{k_j} - \varphi|| + \epsilon = \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we find the $\Lambda(f) := \lim_j \int f \, d\mu_{i_j^j}$ is well defined.

Our final move will be to invoke the powerful Riesz Representation Theorem which is used to represent measures uniquely in terms of linear functionals. See, for instance, the Theorem 2.14 in [35] for more details.

Since the functional $\Lambda(f) = \lim_j \int f \, d\mu_{i_j^j}$ is clearly linear and positive, Riesz tells us that there exists a unique positive Borel measure that represents the functional Λ in the following sense,

$$\Lambda(f) = \int f \, \mu = \lim_i \int f \, \mu_{i_j^j},$$

for every $f \in C_c(X)$. Hence, the subsequence $\mu_{i_j^j}$ is the asserted weakly convergent subsequence. $\square$

Now we have the required machinery for proving the Frostman's lemma. Recall that the statement is: *For any compact set $E \subset \mathbb{R}^n$, with $\mathcal{H}^s(E) > 0$, there exists a positive Borel measure μ supported on E satisfying*

$$\mu(B(x,r)) \leq r^s \tag{1}$$

for all balls $B(x,r)$ in $\mathbb{R}^n$.

Proof of Frostman's Lemma. We will construct inductively a sequence of measures $\{\mu_k\}$ whose weak limit is the desired measure μ. Translating and scaling E, we may assume that E is contained in the unit cube $[0,1]^n$. Recall the Definition 1.5 *Dyadic Cubes in* $\mathbb{R}^n$. As in the definition, let $\mathcal{D}_k$ denote the dyadic cubes of side length 2^{-k} for $k = 1, 2, 3, ...$

Since $\mathcal{H}^s(E) > 0$, there exists $c > 0$ such that

$$\sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s \geq c \tag{2}$$

for every covering $\{A_j\}$ of E.

Let $k \in \mathbb{N}$ and define a Borel measure $\mu_{k,1}$ on $\mathbb{R}^n$, such that on the dyadic cubes $Q \in \mathcal{D}_k$ it satisfies

$$\mu_{k,1}\big|_Q = \begin{cases} \frac{(\operatorname{diam} Q)^s}{\mathcal{H}^n(Q)} \mathcal{H}^n\big|_Q & \text{if } Q \cap E \neq \emptyset \\ 0 & \text{if } Q \cap E = \emptyset \end{cases}.$$

Now this measure $\mu_{k,1}$ controls any $B(x,r)$ with $\operatorname{diam} B(x,r) < 2^{-k}$, but for balls with greater radii we have no control over. Thus, we will modify $\mu_{k,1}$ to construct $\mu_{k,2}$ so that we can measure cubes with greater size, that is, cubes $Q \in \mathcal{D}_{k-1}$.

Let $Q \in \mathcal{D}_{k-1}$. Define $\mu_{k,2}$ as

$$\mu_{k,2}\big|_Q = \begin{cases} \mu_{k,1}\big|_Q & \text{if } \mu_{k,1}(Q) \leq (\operatorname{diam} Q)^s \\ (\operatorname{diam} Q)^s [\mu_{k,1}(Q)]^{-1} \mu_{k,1}^n\big|_Q & \text{else} \end{cases}.$$

Hence, now $\mu_{k,2}$ controls balls with diameter less than 2^{1-k}. Since E is contained in the unit cube, we continue this process until we reach the measure $\mu_{k,k}$, which controls the cube $[0,1]^n$. Set $\mu_k := \mu_{k,k}$.

Note that we have

$$\mu_{k,i}(Q) \leq \mu_{k,j}(Q)$$

for all $j < i \leq k$. In other words, the measure of a dyadic cube does not increase during the construction. Hence, for any $Q \in \bigcup_{i=0}^k \mathcal{D}_i$, we have the bound

$$\mu_k(Q) \leq \mu_{k,m}(Q) \leq (\operatorname{diam} Q)^s, \tag{3}$$

for all $1 \leq m \leq k$. Thus, we have a sequence of measures $\{\mu_k\}$ which control the dyadic cubes of all sizes less than or equal to the size of $[0,1]^n$.

Choose a point $x \in E$ and let Q^i to denote that $Q^i \in \mathcal{D}_i$. Then for all $1 \leq m \leq k$ there exists some cube $Q^m \in \mathcal{D}_m$ such that $x \in Q^m \subset [0,1]^n$. Thus, suppose $x \in Q^k$. Then

$$\mu_{k,1}(Q^k) = (\operatorname{diam} Q^k)^s$$

implies that there exists greatest m such that

$$\mu_{k,m}(Q^{k-m}) = \operatorname{diam}(Q^{k-m})^s.$$

Write $Q = Q^{k-m}$. Then by the definition of $\mu_{k,i}$ we have

$$\mu_k(Q) = \mu_{k,k}(Q) = \mu_{k,k-1}\big|_Q(Q) = \mu_{k,k-1}(Q) = \cdots = \mu_{k,m}(Q) = (\operatorname{diam} Q)^s.$$

Thus, any $x \in E$ is in some subcube Q of $[0,1]^n$ satisfying

$$\mu_k(Q) = (\operatorname{diam} Q)^s.$$

Picking for each $x \in E$ the largest such cube, we obtain a pairwise disjoint and finite collection of cubes $\{Q\} := \mathcal{M}$ that covers E.

The definition of μ_k, pairwise disjointness of $\mathcal{M}$ and (2) yields

$$\mu_k(\mathbb{R}^n) = \sum_{Q\in\mathcal{M}} \mu_k(Q) = \sum_{Q\in\mathcal{M}} (\operatorname{diam} Q)^s \geq c,$$

which shows $\mu_k \not\equiv 0$.

Now we are three steps away from the finish line. The sequence $\{\mu_k\}$ satisfies the uniform bound in the assumption of the Theorem 2.20 *Convergent Subsequence of Measures*. Hence, passing to a subsequence, if required, the sequence $\{\mu_k\}$ converges weakly to some positive Borel measure μ.

Since any ball in a Euclidean space is contained in some union of finitely many cubes $\bigcup_{i=1}^{k(n)} Q_i$, where $k(n)$ depends on the dimension of $\mathbb{R}^n$, we have by the Theorem 2.19 *Inequalities for Weak Convergence*,

$$\mu(B(x,r)) \leq \mu\left(\bigcup_{i=1}^{k(n)} Q_i\right) \leq \liminf_k \mu_k\left(\bigcup_{i=1}^{k(n)} Q_i\right) \leq C(n)(\operatorname{diam} B(x,r))^s$$

where $C(n)$ depends on the dimension of $\mathbb{R}^n$

Finally, recall that E is compact and each μ_k is supported on the closure of the union $\bigcup_k\{Q : Q \cap E \neq \emptyset, Q \in \mathcal{D}_k\} := A_k$. Hence, $E \subset A_k$. Choose $x \in E^c$. Since E^c is open, there exists some open ball B centered at x and entirely contained in E^c. Because there exists some positive distance between B and E, we can choose k large enough so that $A_k \cap \bigcup_k\{Q : Q \cap B \neq \emptyset, Q \in \mathcal{D}_k\} = \emptyset$. Therefore,

$$\mu(\mathbb{R}^n \backslash E) \leq \liminf_k \mu_k(\mathbb{R}^n \backslash E) = 0.$$

Thus, $\operatorname{supp} \mu \subset E$. This finishes the proof. □

As was mentioned above, we will need Frostman's lemma in the proof of our third main result. Additionally, we will later require one fundamental result concerning the Hausdorff dimension of product sets. This can be proven easily with the aid of Frostman's lemma. But first we need another notion of dimension defined in the next paragraph. This dimension is not under principal investigation in this thesis, but instead it is used a couple of times to find a convenient argument in a proof. Therefore, we refer the interested reader to [11, Chapter 3] for more information. We simply state its definition and use it whenever needed.

Definition 2.21. (Upper Box-counting Dimension). Let A be any set. We define $N_\delta(A)$ to be the minimum number of sets that have diameter δ and that cover a set A. Thus, it is just the smallest cardinality taken over all covers with a fixed size. Next we write $\overline{\dim}$ for the *upper box-counting dimension* and define it as,

$$\overline{\dim}F := \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$$

whenever $F \subset \mathbb{R}^n$ is bounded. Note that $N_\delta(F)$ we have $N_\delta(F) \leq C\delta^{-\overline{\dim}F}$ for some positive C.

The next proof is from [11, Chapter 7]. We do not require the most general form of the following result and hence we define *nice set* to mean a set whose upper box-counting dimension agrees with its Hausdorff dimension. For clarity, this means that for a nice set B we have $\dim B = \overline{\dim}B$. Instances of nice sets include bounded intervals and the ternary Cantor set.

Theorem 2.22. (Dimension of Product Sets). *If $A, B \subset \mathbb{R}^n$ are compact, B is a nice set, and $s, t \geq 0$, then*

$$\mathcal{H}^{s+t}(A \times B) \geq \mathcal{H}(A)^s\mathcal{H}(B)^t. \tag{1}$$

In particular,

$$\dim(A \times B) = \dim A + \dim B. \tag{2}$$

Proof. If $\mathcal{H}^s(A) = 0$ or $\mathcal{H}^t(B) = 0$, then the proof is trivial. Hence, suppose $\mathcal{H}^s(A) > 0$ and $\mathcal{H}^t(B) > 0$. By Frostman's lemma, there exists Borel measures μ_A and μ_B supported on subsets of A and B, respectively, satisfying the bounds $\mu_A(B(x,r)) \leq r^s$ and $\mu_B(B(x,r)) \leq r^t$ for every ball.

Next write $\lambda = \mu_A \times \mu_B$ for the product measure. This vanishes outside of $\operatorname{supp}\mu_A \times \operatorname{supp}\mu_B$. To see this, consider $\lambda(\mathbb{R}^n \backslash (\operatorname{supp}\mu_A \times \operatorname{supp}\mu_B))$. For any ball $B((x,y),r) \subset \mathbb{R}^{2n}$, we have $B((x,y),r) \subset B(x,r) \times B(y,r)$. Hence, $\lambda(B((x,y),r)) \leq \mu_A(B(x,r))\mu_B(B(y,r)) \leq r^{s+t}$. Then the Theorem 2.17 *Frostman's Lemma and Mass Distribution Principle* implies (see the remarks after the *Proof of Mass Distribution Principle*)

$$\mathcal{H}^s(A)\mathcal{H}^t(B) \leq \mu_A(A)\mu(B) = \lambda(A \times B) \leq \mathcal{H}^{s+t}(A \times B).$$

This proves (1).

To prove (2), we choose s and t such that $s < \dim A$ and $t < \dim B$. By the definition of Hausdorff dimension, this implies $\mathcal{H}^s(A) > 0$ and $\mathcal{H}^t(B) > 0$. By (1) we have $0 < \mathcal{H}^t(B)\mathcal{H}^s(A) \leq \mathcal{H}^{s+t}(A\times B)$. Again using the definition of Hausdorff dimension this implies $\dim(A \times B) \geq s + t$. Since this holds for every s, t with $s < \dim A$ and $t < \dim B$, we conclude $\dim(A \times B) \geq \dim A + \dim B$.

Next choose s and t such that $s > \dim A$ and $t > \overline{\dim}B$. By the definition of $\overline{\dim}$, B can be covered with $N_\delta(B) \leq \delta^{-t}$ sets with diameter $\delta \leq \delta_0$ for some δ_0. Since $\mathcal{H}^s(A) = 0$, we can choose a cover $\{U_i\}$ such that $\operatorname{diam} U_i < \delta$ for every i and $\sum_i(\operatorname{diam} U_i)^s < 1$. Pick i and define $\{U_{i,j}\}_{j\in J_i}$ to be a cover of B by $N_{\operatorname{diam} U_i}(B)$ sets such that $\operatorname{diam} U_{i,j} = \operatorname{diam} U_{i,j}$ for every j.

Then $U_i \times B$ can be covered by $N_{\operatorname{diam} U_i}(B)$ product sets of the form $U_i \times U_{i,j}$. Since $\{U_i\}$ covers A, we have $A \times B \subset \bigcup_i \bigcup_j (U_i \times U_{i,j})$. By the definition of Hausdorff pre-measure we find

$$\begin{aligned}\mathcal{H}^{s+t}_{\delta\sqrt{2}}(A \times B) &\leq \sum_i \sum_j (\operatorname{diam}(U_i \times U_{i,j}))^{s+t} \leq \sum_i N_{\operatorname{diam} U_i}(B) 2^{s+t}(\operatorname{diam} U_i)^{s+t} \\ &\leq 2^{s+t}\sum_i(\operatorname{diam} U_i)^{-t}(\operatorname{diam} U_i)^{s+t} = 2^{s+t}\sum_i(\operatorname{diam} U_i)^s < 2^{s+t},\end{aligned}$$

where we used the derived fact that B can be covered with $\{U_i\}$ such that each set in the cover satisfies $\operatorname{diam} U_i < \delta_0$ and the cover has $N_{\operatorname{diam} U_i}(B) \leq (\operatorname{diam} U_i)^{-t}$ elements, as well as the fact that $\sum_i (\operatorname{diam} U_i)^s < 1$. Since B is nice, we have shown $\dim A + \dim B = \dim A + \overline{\dim} B \geq \dim(A \times B) \geq \dim A + \dim B$ which proves the result. □

Thus, if $B = [0,1]$ and A is any compact set in $\mathbb{R}^n$, then $\dim(A \times [0,1]) = \dim A + 1$.

Finally we can prove our third main result, which closely resembles the result above. It is fundamental and well-known result in the theory of conformal dimension. It was first proven by Bishop and Tyson in 2001 [3]. In some sense this generalizes the Theorem 2.16 *Tyson's Theorem*, because this result shows that we can omit the assumption of Ahlfors-regularity. Moreover, it turns out to be a corollary to the Theorem 2.15 *Lower Bound Result II.*

Theorem 2.23. (Lower Bound Result III). *If $Y \subset \mathbb{R}^{n-1}$ is compact, then the product space $Z = Y \times [0,1]$ is minimal for conformal dimension.*

Proof. Let $\mathcal{D}(n)$ be the family of all dyadic cubes in $\mathbb{R}^n$. Recall the Definition 1.5 *Dyadic Cubes in* $\mathbb{R}^n$. Then pick any arbitrary $d < \dim Y$. This implies by the definition of Hausdorff dimension that $\mathcal{H}^d(Y) > 0$, which in turn implies by the Theorem 2.17 *Frostman's Lemma and Mass Distribution Principle* that there exists a positive Borel measure supported on Y and satisfying the upper mass bound

$$\mu(Q) \leq \operatorname{diam}(Q)^d,$$

for every $Q \in \mathcal{Q}(n-1)$.

Then we define a product measure ν on $Z = Y \times [0,1]$ by $\nu = \mu \times \mathcal{H}^1$. Note that $\mu(Y) > 0$, since μ is supported on Y.

Let Q_n denote that $Q_n \in \mathcal{Q}(n)$. Then by the definition of product measure we have

$$\nu(Q_n) = \mu(Q_{n-1})\mathcal{H}^1(Q_1) \leq \operatorname{diam}(Q_n)^{d+1}$$

for every $Q_n \in \mathcal{Q}(n)$.

Define a curve family $\Gamma = \{\gamma_y : y \in Y\}$ with $\gamma_y(t) = (y,t), 0 \leq t \leq 1$. This family consists of the "straight line segment" curves, if we visualize the product space as a rectangle, where we have exactly one curve corresponding to every element of Y.

Then arguing line by line as we did in the Theorem 2.16 *Tyson's Theorem* for the curve family Γ, we find

$$\operatorname{Mod_p} \Gamma = \mu(Y) > 0, \tag{1}$$

for every $p \geq 1$, in particular, for $p = d+1$. Thus, the Theorem 2.15 *Lower Bound Result II* gives the asserted result.

□

Example 2.24. (Conformal Dimension of Square). What is the conformal dimension of a unit square Q in $\mathbb{R}^2$? The Theorem 2.23 *Lower Bound Result III* answers this, since we can write $Q = [0,1] \times [0,1]$. Now, $\dim([0,1] \times [0,1]) = 1 + 1$ and since this set is minimal for conformal dimension, we find $\operatorname{Cdim} Q = 2$. ■

Remark. Note that the same holds for any bounded square in the plane.

Next we will go off on a short tangent, and discuss a couple of interesting facts of conformal dimension. The following short discussion is based on [3]. We mention some intriguing results, which are corollaries to a theorem they proved, namely,

Theorem 2.25. (Theorem of Bishop and Tyson). *If* $1 \leq \alpha < d$ *and* $K < \infty$*, then there exists a compact and totally disconnected set* $X \subset \mathbb{R}^d$ *with* $\dim X = \alpha$*. Additionally, we have*

(1) $\dim f(X) \geq \alpha$ *for every* K*-quasisymmetric map* $f : X \to f(X)$*,*
(2) *for every* $\epsilon > 0$ *there exists a quasiconformal map* $g : \mathbb{R}^d \to \mathbb{R}^d$ *with* $\dim g(X) < \epsilon$*. Thus,* $\operatorname{Cdim} X = 0$*.*

We do not discuss the proof, but the underlying ideas are similar to the ideas we used in the proof of the Theorem 2.23 *Lower Bound Result III.* Of course their proof differs in that they do a careful construction of such a space. Note that in (2) we can think of the mapping to be a quasisymmetry (see the Theorem 1.27 *Quasiconformal-Quasisymmetrical Relation*). This result is evidently local. But we can make it global by the following corollary.

Theorem 2.26. (First Corollary). *Given* $\alpha \in [1, n]$*, there exists a compact and totally disconnected set* $X \subset \mathbb{R}^n$ *with* $\operatorname{Cdim} X = \dim X = \alpha$*.*

Proof. Pick X_1 as in the Theorem 2.25 *Theorem of Bishop and Tyson* with $\alpha_1 := \alpha$ and $K_1 < \infty$. Next choose X_2, with $\alpha_1 = \alpha_2$, where $K_1 < K_2$. Then continue choosing X_n, with $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \cdots = \alpha$, where $K_1 < K_2 < \cdots < K_n < \cdots \to \infty$. We can choose the sets X_n such that $\operatorname{diam} X_n \to 0$ as $n \to \infty$ and with the property that one of the sets, say, X_1, contains an accumulation point of the sequence $\{X_n\}$. This makes the union $X = \bigcup_n X_n$ compact and totally disconnected, by the accumulation point characterization of compactness.

Since Hausdorff dimension is countably stable by Theorem 1.8 *Properties of Hausdorff Dimension*, we have $\dim X = \dim \bigcup_n X_n = \sup_n \dim X_n = \alpha$. Thus, $\operatorname{Cdim} X \leq \alpha$.

Each α_n is a lower bound for $\dim f(X_n)$, over all K_n-quasisymmetric maps. Thus,

$$\alpha = \alpha_n \leq \dim f(X_n).$$

Let $f : X \to Y$ be an arbitrary quasisymmetric map. Then there exists some $K_n < \infty$ such that f is K_n-quasisymmetric. Therefore,

$$\dim f(X) = \dim f\left(\bigcup_i X_i\right) = \dim \bigcup_i f(X_i) = \sup_i \{\dim f(X_i)\} \geq \alpha_n = \alpha,$$

Thus, we have $\alpha \leq \operatorname{Cdim} X$, and hence the claim. □

In other words, for any given real number, we now know that there always exists some totally disconnected space which is minimal for conformal dimension. This is interesting in its own right, since it gives us a myriad of examples of spaces that are minimal for conformal dimension. Before this result, there was only a rather small collection of such examples. But this result becomes even more interesting if we consider totally disconnected and compact spaces (such as Cantor sets constructed on the unit interval), whose dimension lies in the range $(0, 1)$. We know that they are not minimal for conformal dimension, since the range of Cdim is $\{0\} \cup (1, \infty)$. However, this result shows that if we move to other spaces with dimension greater than 1, there will always exist spaces falling to the category of being totally disconnected, compact and minimal for Cdim.

Juha Heinonen posed an interesting question in his monograph [16, Chapter 15], that whether the infimal Hausdorff dimension is always attained by some metric space in the conformal gauge. Perhaps surprisingly, the answer is *negative*, which is shown by the following corollary.

Theorem 2.27. (Second Corollary). *For any $\alpha \in [1, n)$, there exists a subset $X \subset \mathbb{R}^n$ with the properties that* $\operatorname{Cdim} X = \alpha$, *but* $\dim f(X) > \alpha$ *for every* $f(X)$ *in the conformal gauge of* X.

Proof. Construct a disjoint union of sets $\bigsqcup_n X_n$ by applying the Theorem 2.25 *Theorem of Bishop and Tyson* as in the Theorem 2.26 *First Corollary*, but this time choose α_n such that $\alpha_1 > \alpha_2 > \cdots$ and $\lim_n \alpha_n = \alpha$. We will show that $\bigsqcup_{i=1}^{\infty} X_i = X$ satisfies the asserted properties.

Exactly as in the proof of the Theorem 2.26 *First Corollary*, let $f : X \to Y$ be an arbitrary quasisymmetric map. Again, f is K_n-quasisymmetric for some K_n. Thus, we find

$$\dim f(X) = \sup_i \{\dim f(X_i)\} \geq \alpha_n > \alpha,$$

which yields $\dim f(X) > \alpha$ for every quasisymmetry f from X.

For a moment, suppose there exists some $\beta > \alpha$ satisfying $\beta \leq \dim f(X)$ for all quasisymmetric images of X. Choose m large enough so that $\alpha_m < \beta$. Then by property (2) of the Theorem 2.25 *Theorem of Bishop and Tyson*, for every $1 \leq i \leq m$ there exists a quasiconformal map $g_i :\to \mathbb{R}^d$ satisfying $\dim g_i(X_i) < \alpha_m$. For $i > m$, set $g_i : X_i \to g(X_i) = X_i$, that is, g_i is the identity map for all $i > m$.

Next construct a metric on the space $\bigsqcup_i g(X_i)$ as in the proof of the Theorem 2.5 *Properties of Conformal Dimension* (2). Arguing as in that proof, we find that $G : \bigsqcup_i X_i \to \bigsqcup_i g(X_i)$ is a quasisymmetry such that $G\big|_{X_i} = g_i : X_i \to g(X_i)$. Therefore,

$$\dim G(X) = \dim G\left(\bigsqcup_i X_i\right) = \dim \bigsqcup_i G(X_i) = \dim \bigsqcup_i g(X_i) = \sup_i \dim g(X_i) \leq \alpha_m.$$

which shows that β is not a lower bound for $\dim G(X)$. Hence, $\alpha = \operatorname{Cdim} X$. This proves the claim. □

Now we have introduced three principal tools for the study of lower bounds for conformal dimension of various spaces. It should be noted that the compactness of the spaces is not essential. However, it is necessary that the curve family is in some compact subset of the space. It is easy to obtain counterexamples, if the compactness is omitted. Consider, for example, a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, mapping the set of open discs with diameter ≥ 1 to open discs with diameter reciprocal of the source disc.

Our next goal is to discuss the important result of Keith and Laakso [18]. This will be crucial for our study of the Sierpiński carpet, since we will apply it to show that the carpet is *not* minimal for conformal dimension.

2.3. **Metric Geometry and Conformal Dimension.** In this section we will discuss convergence of metric spaces and tangent spaces and their relation to conformal dimension. This discussion will culminate in the interesting Theorem 2.39 *Keith-Laakso*

We shall begin with some intuitive notions from metric geometry which hopefully make the following discussion more digestible. For example, in order to discuss this theorem by Keith and Laakso, we need to understand what are tangent spaces and weak tangent spaces of a metric space and what is meant by convergence of subsets and metric spaces. The discussion in the next subsection is based on the book [7].

2.3.1. *Metric Geometry.* The intuition behind the need for metric geometry concepts can be most easily conceived by thinking of the distance between two mountain tops. The straight line distance, or "as the crow flies" distance, is the usual three-dimensional Euclidean distance between these two points. However, if we think of a flightless mammal moving from one mountain top to the other, the distance the animal moves is obviously greater than the distance the crow flies from one top to the other. In other words, the smallest Euclidean distance between these two points is not realized by any curve on the surface of the earth. Recall that in the definition of a curve $\gamma : \mathbb{R} \supset I \to X$, the interval I can be any connected subset of $\mathbb{R}$.

Definition 2.28. (Length Structures and Length Spaces). Suppose we have a set of curves $\Gamma = \{\gamma : \gamma : [a,b] \to X\}$, where the images of the curves live in a topological Hausdorff space X, satisfying

(i) $\Gamma \ni \gamma : [a,d] \to X$ implies $\Gamma \ni \gamma\big|_{[b,c]} : [b,c] \to X$, where $a \leq b \leq c \leq d$ (Closure Under Restrictions)
(ii) $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 : [a,b] \to X$ and $\gamma_2 : [c,d] \to X$, where $a \leq b \leq c \leq d$ implies $\gamma \in \Gamma$, where $\gamma : [a,d] \to X$ and γ_1 and γ_2 are restrictions of γ (Closure Under Concatenations)
(iii) If $\varphi : [a,b] \to [c,d]$ is a homeomorphism and $\Gamma \ni \gamma : [c,d] \to X$, then $\gamma \circ \varphi : [a,b] \to X$ is again in Γ (Closure Under Reparametrization)

and a function $L : \Gamma \to \mathbb{R}^+ \cup \{\infty\}$, where we call the output value of $L(\gamma)$ the length of the curve γ. We also require that L is additive, invariant under reparameterizations, the length of a subcurve continuously depends on the subcurve and the length structure is compatible with the topology. For details, see [7, Chapter 2].

Then we say that Γ is a set of *admissible curves* for L. Here L is called the *length function*. This pair (Γ, L) is called a *length structure* on a topological Hausdorff space X. Using this we can easily define a metric on any Hausdorff topological space that admits a length structure (Γ, L);

$$d_L(x,y) = \inf\{L(\gamma) : \gamma : [a,b] \to X, \gamma \in \Gamma, \gamma(a) = x, \gamma(b) = y\}$$

for every $x, y \in X$. It can be shown that L is indeed a metric by using the properties of the length structure. Then we call such a metric space (X, d_L) a *length space*.

As instructive examples we may consider a space consisting of two disconnected components and a planar space with one segment removed. The first example shows that between two points there may not be any paths. Hence, the infimum over all paths between these two points is infinite, that is, the metric on length spaces is not necessarily finite. The other example shows that the inifmum may not be attained by any path.

Definition 2.29. (Gromov-Hausdorff Convergence of Length Spaces). Suppose $A, B \subset (X, d_L)$ of a length space. The Hausdorff distance, $d_H(A,B)$, which was already defined right before 2.12 *Lower Bound Result I*, can equivalently be stated as,

$$d_H(A,B) := \max\{\sup_{a \in A0} \inf\{d_L(a,x) : x \in B\}, \sup_{b \in B} \inf\{d_L(b,y) : y \in A\}\}.$$

Looking at the definition, we see that the Hausdorff distance is the maximum of the distances one needs to travel in order to get from A to the farthest point in B and from B to the farthest point in A.

Next suppose (X, d_L) and (X', d'_L) are length spaces, and $\varphi : X \to M$, $\psi : X' \to M$ are some isometric mappings into a metric spaces M. We define the *Gromov-Hausdorff distance*, d_{GH}, of length spaces as

$$d_{GH}(X, X') := \inf\{r > 0 : d_H(\varphi(X), \psi(X')) < r\},$$

where the infimum is taken over all possible choices of M, φ and ψ satisfying the embedding relations. Thus, we embed the given length spaces isometrically into some ambient space, and then consider their distance as subsets in the Hausdorff sense. If we are given a sequence X_n of length spaces, we say that the sequence *converges* to X in the Gromov-Hausdorff sense if $\lim_n d_{GH}(X_n, X) = 0$. Moreover, if X is complete, then X is a length space. For details, examples and more interesting results, we refer the reader to [7].

This short note on some of the fundamental concepts of metric geometry is intended to give the reader some intuition on the underlying ideas. These notions will be discussed later in more detail, when we give alternative, but somewhat more abstract definitions for the convergence of metric spaces. Note that we will not use length spaces any more in this thesis, but they were introduced here since it is hoped that these preliminary definitions, although brief, will clarify the forthcoming discussion by putting the abstract ideas into a better perspective. Moreover, in the next subsection we give a detailed example, which aids in understanding these notions (see Example 2.35 *Tangent of Lattice*).

2.3.2. *Tangents and Conformal Dimension.* The material in this section is based on [25, Chapter 6] and [18]. We begin by introducing some new definitions which are essential for this section.

We say that a subset $A \subset (Z, d)$ is *ϵ-separated*, if $d(x, y) \geq \epsilon$ whenever $x, y \in A$ are distinct. Let $N(B, \epsilon)$ denote the supremal cardinality of ϵ-separated subsets $A \subset B$. For example, if $B = [0, 1] \subset \mathbb{R}$ and $\epsilon = \frac{1}{2}$, then $N(B, \epsilon) = 3$.

Suppose now that (Z, d) is a metric space. Then Z is metrically doubling if and only if there exists $C < \infty$ such that

$$N\left(B(x, r), \frac{r}{2}\right) \leq C, \tag{1}$$

for all balls in Z whose radius r satisfies $r < \operatorname{diam} Z$. Note that this is just a restatement of the Definition 1.11 *Doubling Spaces*. If, however, one proceeds inductively from (1), then we will find that there exists $C' = C'(C)$ and $s = s(C)$ such that

$$N\left(B(x, r), \epsilon r\right) \leq C' \epsilon^{-s} \tag{2}$$

for all $0 < \epsilon < 1$ and for all balls in Z whose radius r satisfies $r < \operatorname{diam} Z$. To see the existence, we will show explicitly that $s = \log_2 C$ works.

Suppose that Z is doubling. Let $0 < \varepsilon < 1$ and choose n large enough and a constant $K \geq 1$ so that $K \cdot 2^{-n} = \varepsilon$. Doubling property gives us a constant $C < \infty$ such that (1) holds. Moreover, since doubling property means that the maximal cardinality of $\frac{r}{2}$-separated set of $B(x, r)$ is bounded by C, for all $r < \operatorname{diam} Z$, clearly the maximal cardinality of $\frac{r}{4}$-separated set of $B\left(x, \frac{r}{2}\right)$ is bounded by C. Hence, repeating this line of reasoning n times, we obtain

$$N\left(B(x, r), \frac{r}{2^n}\right) \leq C^n = 2^{n \log_2 C} = \left(\frac{1}{2^n}\right)^{-\log_2 C}$$

Since $2^{-n} = \varepsilon \cdot K^{-1}$ we find,

$$N\left(B(x, r), \varepsilon r\right) \leq N\left(B(x, r), \frac{r}{2^n}\right) \leq \left(\frac{\varepsilon}{K}\right)^{-\log_2 C} = C' \varepsilon^{-s}$$

where $C' = K^s$ and $s = \log_2 C$. Thus, (2) holds.

This leads us to the definition of the Assouad dimension.

Definition 2.30. (Assouad Dimension). The *Assouad dimension*, denoted by $\dim_A Z$, of a doubling metric space Z is the infimal $s > 0$ for which (2) holds for some $C' < \infty$, for all balls in Z and $0 < \epsilon \leq 2$.

Remarks. Intuitively one can think that the problem of finding the Assouad dimension is equivalent to finding the infimal number of balls with radii ϵr that is required to cover a ball with radius r. The definition and discussion preceding it shows that a metric space is doubling if and only if it has a finite Assouad dimension. Our intuition implies that $\dim Z \leq \dim_A Z$, since the set of arbitrary covers is larger than the set of ball cover with radii ϵr. To be more precise, we proceed as follows.

Recall the Definition 2.21 *Upper Box-counting Dimension*,

$$\overline{\dim} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Note that although we defined it for Euclidean sets, the same definition can be applied to arbitrary metric spaces. Now we shall establish an important relationship between Assouad dimension and Hausdorff dimension. Intuitively, when one recalls that Hausdorff dimension considers arbitrary covers, and box-counting and Assouad dimension use some fixed sized covers, it seems that Hausdorff dimension should be the smallest of these three, since we are taking infimums. This turns out to be true. We prove the relationship between Hausdorff and Assouad dimension by passing through the box-counting dimension, since this gives interesting relationship between these three dimensions.

We may suppose F is a bounded set with Hausdorff dimension s. Recall that $N_\delta(F)$ denotes the smallest number of sets with diameter δ that is required to cover F. Then by the definition of Hausdorff measure we have,

$$\mathcal{H}^s_\delta(F) \leq N_\delta(F)\delta^s.$$

Suppose first that $\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) > 1$. Then $1 < N_\delta(F)\delta^s$. Thus, choosing δ small enough (< 1) so that $0 < \log(N_\delta(F)\delta^s) = \log N_\delta(F) + s \log \delta$, we obtain

$$s \leq \frac{N_\delta(F)}{-\log \delta}. \tag{1}$$

Therefore, letting $\delta \to 0$ in (1), we find $\dim F \leq \overline{\dim} F$. If, however, $0 < \mathcal{H}^s(F) \leq 1$, then we can choose $R > 0$ large enough so that $R^s \mathcal{H}^s(F) = \mathcal{H}^s(RF) > 1$ and apply the same argument. The case $\mathcal{H}^s(F) = 0$ then follows trivially.

Next, we note that the upper box-counting dimension can be expressed equivalently as

$$\overline{\dim} F = \inf \left\{ \alpha : \text{ there is } c > 0 \text{ such that } N_r(B(x,1) \cap F) \leq c \left(\frac{1}{r}\right)^\alpha \text{ for every } 0 < r < 1, x \in F \right\}$$

Let $\epsilon > 0$ and choose $x \in F$. Then there exists $c \geq 1$ for every r, $0 < r < \operatorname{diam} F$, such that

$$N_r(F) = N_r(B(x, \operatorname{diam} F) \cap F) \leq c \left(\frac{\operatorname{diam} F}{r} \right)^{\dim_A F + \epsilon}.$$

Thus, by the equivalent definition for $\overline{\dim} F$ given above, we have $\overline{\dim} F \leq \dim_A F + \epsilon$. Arbitrariness of ϵ yields $\overline{\dim} F \leq \dim_A F$, and, in particular, $\dim F \leq \dim_A F$.

Now we can define the principal notion of dimension used in this chapter, which is similar to Cdim.

Definition 2.31. (Conformal Assouad Dimension). If $\mathcal{G}$ is a conformal gauge, then the *conformal Assouad dimension* of $\mathcal{G}$ is

$$\mathrm{Cdim}_A \mathcal{G} := \inf\{\dim_A Z : Z \in \mathcal{G}\}.$$

As in the case of conformal Hausdorff dimension, we abuse notation and write also,

$$\mathrm{Cdim}_A Z = \inf_f \dim_A f(Z),$$

where the infimum is taken over all quasisymmetries f from Z into some metric space.

Note that considering the relationship between Assouad and Hausdorff dimension discussed in the remarks after the Definition 2.30 *Assouad Dimension*, we have

$$\mathrm{Cdim}\, Z \leq \mathrm{Cdim}_A Z.$$

Now that we have introduced some new definitions, we give a short description of the Theorem 2.39 *Keith-Laakso*, and then continue with the relation of metric geometry and conformal dimension in more detail.

In essence, the Theorem of Keith and Laakso establishes "almost perfect" sufficient and necessary conditions for minimality of conformal Assouad dimension. For our interests, this result will be useful, since for any metric space the conformal Assouad dimension is an upper bound for conformal Hausdorff dimension. One reason for using Assouad dimension lies in the fact that we can get rid of certain pathological cases, which will be discussed briefly in this chapter. Another reason is that there exists a deep connection between the moduli of a curve family and the conformal Assouad dimension. The following results will shed light to this connection. Note the similarity with conformal Hausdorff dimension: the lower bounds for $\mathrm{Cdim}\, Z$ mirror the connectivity properties of Z.

The outline for the rest of this section is the following. First, we will obtain results which show that the conformal Assouad dimension of a metric space is bounded below by the conformal Assouad dimension of a tangent space. Then comes in the Theorem 2.39 *Keith-Laakso*, which states that if some weak tangent space has a sufficiently rich curve family, then the Assouad dimension of the space cannot be lowered. Conversely, if the space is minimal for conformal Assouad dimension, then the space admits a weak tangent space that has nontrivial p-modulus. In this paragraph we used the word "tangent space" even though we have not yet defined it. So, let us begin by fixing this ambiguity. For this we need to define what it means for a sequence of metric spaces to converge to a metric space. Recall the beginning of this section, where we briefly discussed some metric geometry concepts.

Definition 2.32. (Convergence of Subsets). Let $\{F_m\}$ be a sequence of nonempty closed subsets in a metric space (Z, d). We say that F_m converges to a nonempty closed subset $F \subset Z$, if for every $q \in Z$ and $R > 0$ the following two conditions hold,

$$\lim_{m\to\infty} \sup_{x\in F_m \cap B(q,R)} \mathrm{dist}(x, F) = 0 \tag{1}$$

and

$$\lim_{m\to\infty} \sup_{x\in F \cap B(q,R)} \mathrm{dist}(x, F_m) = 0. \tag{2}$$

For this convergence definition, we set $\sup \emptyset = 0$. Note that both conditions (1) and (2) are essential. Think of a sequence of monotonically decreasing sets, that is, $F_i \supset F_{i+1}$, $\mathrm{diam}\, F_1 < \infty$. Suppose there exists some nonempty $F \subset F_i$, for every i, and some $r > 0$ for which the r-fattening

$B_r(F) = \bigcup_{x\in F} B(x,r)$ of F satisfies $B_r(F) \subset F_i$, but $B_r(F) \not\supset F_i$, for every i. Thus, F is strictly inside every F_i and there is some "room" between F and every F_i. Intuitively, it should be clear that $\lim_i F_i \neq F$. But now the condition (2) holds always. Thus, if we were to use only the condition (2) for convergence, then we would have to conclude that $\lim_i F_i = F$. However, (1) fails whenever $B(q,R)\cap F \neq \emptyset$ and $R < \operatorname{dist}(q,F)$. Thus, the definition shows that F_i does not converge to F.

Before we give the definition of convergence of metric spaces, we need one important definition of certain maps between metric spaces. We associate to each metric space (X,d) a *base point* $b \in X$ and we call the triple (X,d,b) a *pointed* metric space. If there exists a map $f : (X,d,b) \to (X',d',b')$ with $f(b) = b'$, then we say f is a *pointed map.* Recall that isometric embedding is an injective mapping onto its image that preserves distances. With these in mind we can give the definition of convergent sequences of metric measure spaces.

Definition 2.33. (Convergence of Metric Spaces). Let $\{(Z_n, d_n, b_n)\}$ be a sequence of complete pointed metric spaces and let (Z, d_Z, b_Z) be a complete pointed metric space. We say that $\{Z_n\}$ *converges* to Z if there exists pointed isometric embeddings $\varphi : Z \to M$, $\varphi_n : Z_n \to M$, where M is a pointed metric space (M, d_M, b_M) such that the closed nonempty subsets $\varphi_n(Z_n) \subset M$ converge to the closed nonempty subset $\varphi(Z) \subset M$.

Intuitively, this definition says that in order to study the convergence of abstract metric spaces $Z_n \to Z$, we need to find a suitable metric space M such that each Z_n and Z are isometric to some subsets of M, and then consider the convergence of these isometric copies in M using the previous Definition 2.32 *Convergence of Subsets.* Note that this definition depends on the choice of φ and φ_n. From now on, whenever we talk about convergence of metric spaces, we will think of these embeddings as fixed.

Definition 2.34. (Tangent Spaces). We say that a complete pointed metric space (W, d_W, b_W) is a *weak tangent space* of a metric space (Z,d), if $(\hat{Z}, \frac{d}{r_n}, b_n)$ converges to (W, d_W, b_W). Here $\hat{Z}$ denotes the completion of Z and r_n is a convergent sequence of positive real numbers with $0 < r_n < \operatorname{diam} Z$ and b_n is a sequence of (base) points in Z.

Furthermore, if $r_n \to 0$ and $b_n = b$ for every n, then we say that W is a *tangent space* of Z, or *tangent* to Z at b.

Next, we will illustrate these definitions with an instructive and detailed example. Moreover, this example shows how Hausdorff dimension of a tangent may act pathologically compared to the Hausdorff dimension of the original space. Thus, we need to work with the Assouad dimension since it behaves much better as we will see shortly.

Example 2.35. (Tangent of Lattice). Let $\mathbb{Z}^2$ denote the set of all pairs of integers, and $2^{-2m}\mathbb{Z}^2$, $m \in \mathbb{N}$, the scaled version of this space. Let $X_m = 2^{-2m}\mathbb{Z}^2 \cap [0, 2^{-m}]^2$. Hence, we scale the integer lattice by a factor of 2^{-2m} and then consider only those lattice points that lie in the scaled version of the unit square in $\mathbb{R}^2$. This is illustrated in Figure 3 for the first four X_i. Let $X = \bigcup_m X_m$.

Now consider the sequence of pointed metric spaces $\{Z_n\} := \{(X, 2^n d, (0,0))\}$. Note that we have set $r_n = \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$ and $b_n = b = (0,0)$ for the base point, in the Definition 2.34 *Tangent Spaces.* Since we scale the metric up, that is, we multiply the distances between any two points by integral powers of 2, we can think of moving along this sequence $\{Z_n\}$ as equivalent to zooming in to the space X under a microscope keeping the origin as the focus point. This is the idea that tangent spaces are aimed to capture.

Next let $M := (\mathbb{R}^2, d, (0,0))$ be a pointed Euclidean plane. We shall use this space M as the ambient space into which we embed the sequence $\{Z_n\}$ and the tangent space W. We claim that $W = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$. Define $\varphi : W \to M$ and $\varphi_n : Z_n \to M$ such that $\varphi_n(x) = x$ and $\varphi(x) = x$, for every n and every x. Note that these are pointed isometric embeddings.

Then appealing to the Definition 2.32 *Convergence of Subsets* we have

$$\lim_{n\to\infty} \sup_{x\in\varphi_n(Z_n)\cap B(q,R)} \operatorname{dist}(x, \varphi(W)) = 0$$

since for every n we have $\varphi_n(Z_n) \subset \varphi(W)$.

To verify the other condition in the definition, pick any $q \in M$ and $R > 0$, and consider $x \in \varphi(W) \cap B(q,R)$. We may assume $\varphi(W) \cap B(q,R) \neq \emptyset$. Since $||x|| < \infty$, there exists $m \in \mathbb{N}$ such that $||x|| \leq 2^m$. Now zoom m times into the space X. That is, consider $\varphi_m(Z_m)$. Note that for each m, the distance $d(x,y)$ between points $x, y \in X$ is increased by 2^m when we look at these same points x and y in the mth blow-up $\varphi_m(Z_m)$. Hence, $\varphi_m(Z_m)$ contains so dense unit square lattice, that when we zoom another m times, that is, we are in $\varphi_{2m}(Z_{2m})$, the grid size of $\varphi_{2m}(Z_{2m}) \subset M$ will be at most 1 in the square $[0, 2^m]^2 \subset M$.

Thus, $\operatorname{dist}(x, \varphi_{2m}(Z_{2m})) \leq \sqrt{2}$ for m. Moreover, $\operatorname{dist}(x, \varphi_{2m+1}(Z_{2m+1})) \leq \frac{\sqrt{2}}{2}$. Continuing this way we find that at the $(m+k)$th blowup we have $\operatorname{dist}(x, \varphi_{2m+k}(Z_{2m+k})) \leq \frac{\sqrt{2}}{2^k}$. Letting $k \to \infty$, we obtain

$$\lim_{n\to\infty} \sup_{x\in\varphi(W)\cap B(q,R)} \operatorname{dist}(x, \varphi(Z_n)) = 0.$$

Hence, $Z_m \to W$.

Clearly, this tangent contains an isometric copy of the unit square $[0,1]^2$. We already computed the conformal dimension of the unit square (See the Example 2.24 *Conformal Dimension of Square*), and therefore applying the monotonicity of Cdim, we deduce $2 = \operatorname{Cdim}([0,1]^2) \leq \operatorname{Cdim} W$. On the other hand, since W is the first quadrant of $\mathbb{R}^2$, we know that $\dim W = 2$. Combining these we conclude $\operatorname{Cdim} W = 2$. However, Z is countable which implies $\dim Z = 0$ and further $\operatorname{Cdim} Z = 0$. Hence, $\operatorname{Cdim} Z = 0 < 2 = \operatorname{Cdim} W$. ■

Remarks. As was discussed before, we see that conformal Hausdorff dimension of tangents may act pathologically. This is no longer the case when we consider conformal Assouad dimension, as we will see now.

Briefly, the idea of the next argument is to assume the negation, zoom into the blow-up to obtain a set with a desired cardinality and then zoom back to the original space, which immediately indicates a contradiction. This all boils down to the scale invariance of the Assouad dimension.

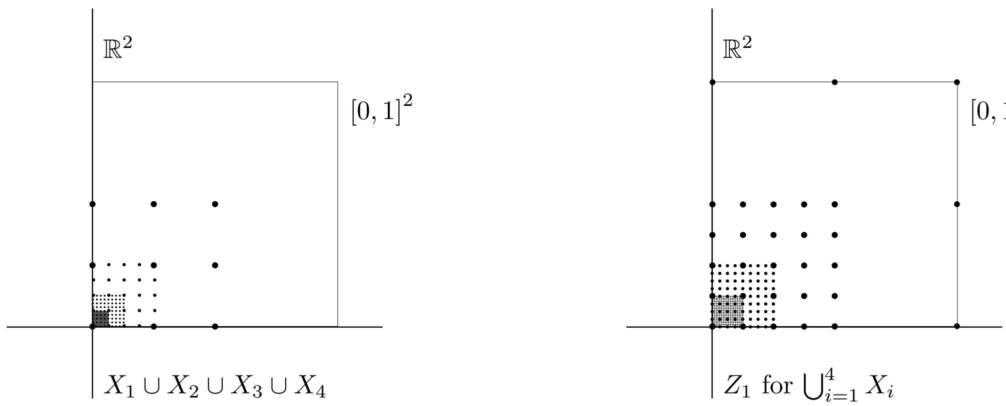

FIGURE 3. Illustration of the Example 2.35 *Tangent of Lattice*. Note that since in reality there exists infinitely many X_i and we zoom infinitely close to the origin, the whole first quadrant of the plane becomes filled with points.

Theorem 2.36. (BOUND FOR $\dim_A$ OF TANGENT). *Suppose W is a tangent to a complete metric space Z. Then $\dim_A W \leq \dim_A Z$.*

Proof. Suppose that W is a weak tangent to Z, but there exists α such that $\dim_A Z < \alpha < \dim_A W$. Then write δ_m corresponding to W for the sequence of real numbers scaling the metric (see the Definition 2.34 *Tangent Spaces*). Since $\alpha < \dim_A W$, by the Definition 2.30 *Assouad Dimension*, for all $c > 0$ there exists an $r > 0$ with $0 < r < R$ and an r-separated set $S \subset B(x, R) \subset W$, whose cardinality $|S|$ satisfies

$$|S| \geq c\left(\frac{R}{r}\right)^{\alpha}.$$

Since S is an r-separated subset of a tangent W, there exists some scaling number δ_m such that Z contains a set T with $|T| = |S|$ and such that T is at least $\frac{\delta_m r}{2}$-separated set and must be contained in a ball with larger radius than $\delta_m R$, say, in $B(x, 2\delta_m)$, for some x. Thus,

$$|T| = |S| \geq c\left(\frac{R}{r}\right)^{\alpha} = c\left(\frac{4\delta_m}{4\delta_m}\frac{R}{r}\right)^{\alpha} = 4^{-\alpha}c\left(\frac{2\delta_m R}{\frac{\delta_m r}{2}}\right)^{\alpha} \tag{1}$$

Recall that by the definition of $\dim_A Z$, we need to consider the infimum of all s in $N(B(x,R), r) \leq C\left(\frac{R}{r}\right)^s$. Now (1) implies that for sufficiently large c ($c = 4^{\alpha}C$), the number $4^{-\alpha}c\left(\frac{2\delta_m R}{\frac{\delta_m r}{2}}\right)^{\alpha}$ is not an upper bound for the cardinalities of $\frac{\delta_m r}{2}$-separated sets in $B(x, 2r\delta_m)$. Thus, $\dim_A Z \geq \alpha$. This is a contradiction to the assumption $\dim_A Z < \alpha$. That is, there cannot exist any α satisfying the inequality $\dim_A Z < \alpha < \dim_A W$ which implies we must have $\dim_A W \leq \dim_A Z$. □

Before jumping to the next result in the relationship between conformal dimension and tangents, we need one preliminary result, namely, Gromov's Compactness Theorem for pointed metric spaces.

Theorem 2.37. (Gromov's Compactness Theorem). *Let $\mathcal{X}$ be any set of complete pointed metric spaces. Assume that for every $r > 0$ and $\epsilon > 0$ there exists $C = C(r, \epsilon)$ such that for every $(X, p) \in \mathcal{X}$ the ball $B(p, r) \subset X$ does not admit an ϵ-net with more than C points. Then any sequence of spaces in $\mathcal{X}$ contains a converging subsequence.*

Remarks. Proof of this result can be found in [7, Chapter 8]. We say that a set S is an ϵ-net, $\epsilon > 0$, if for every $x \in X$, $\operatorname{dist}(x, S) \leq \epsilon$. Note the similarity to ϵ-separated set. Shortly, we will need the notion of *total boundedness*. Total boundedness merely means that there exists a finite ϵ-net for every $\epsilon > 0$.

Note that if we have a doubling space, and we re-scale the space, this re-scaled copy is again doubling with the same doubling constant N. Now every doubling space clearly satisfies the conditions of the Theorem 2.37 *Gromov's Compactness Theorem*, which then implies that every such sequence of pointed metric spaces contains a convergent subsequence. In other words, there exists at least one tangent space at every point of a doubling space.

Now we are going to discuss an important result, which shows that conformal Assouad dimension of a doubling metric measure space is bounded below by the conformal Hausdorff dimension of its tangent.

Theorem 2.38. (Tangent Bound for Cdim). *Suppose $(X_\infty, d_\infty, p_\infty)$ is a weak tangent of a complete doubling metric space (X, d). Then $\operatorname{Cdim}_A X_\infty \leq \operatorname{Cdim}_A X$. In particular, $\operatorname{Cdim} X_\infty \leq \operatorname{Cdim}_A X$.*

Proof. Let $f : (X, d) \to (X', d')$ be a quasisymmetry. Recall that the metric doubling property is quasisymmetrically invariant (see [25, Chapter 1]). Hence, X' is a doubling metric space.

As was discussed above, the doubling property of X implies that X_∞ is doubling. Moreover, every doubling space is separable: the Definition 1.11 *Doubling Spaces* states that any closed ball can be covered by finitely many balls. Thus, the metric doubling condition implies that any closed ball is totally bounded. Considering the union of all the finite $\frac{1}{n}$-nets, $n \in \mathbb{N}$, in any closed ball B, we see that B is separable. Since a doubling space is a countable union of such closed balls, the whole space X_∞ is separable.

Let $\{(X, r_m^{-1} d, p_m)\} =: \{(X_m, d_m, p_m)\}$ be the sequence of pointed metric spaces that converges to X_∞, and $\lim_m p_m = p_\infty$. Let (M, d_M, b_M) be the ambient space where we consider the convergence of X_m. If $X_\infty = \{p_\infty\}$, then the conclusion holds trivially. Hence, suppose there exists one more point $y \in X_\infty$. Let $R = d_\infty(y, p_\infty)$.

Define maps $f_m := f : X_m \to f(X_m) = (X', s_m^{-1} d', f(p_m)) =: (X'_m, d'_m, p'_m)$, where

$$s_m = \operatorname{diam} f(B(p_m, 2Rr_m)).$$

Let $y_m \in X_m$ such that $y_m \to y$. For large enough m, we have $y_m, p_m \in B(p_m, 2Rr_m)$, and thus $s_m \neq 0$. Since d_m and d'_m are just scaled versions of d and d', we have

$$\frac{d'_m(f_m(x), f_m(y))}{d'_m(f_m(x), f_m(z))} = \frac{d'(f(x), f(y))}{d'(f(x), f(z))} \leq \eta\left(\frac{d(x,y)}{d(x,z)}\right) = \eta\left(\frac{d_m(x,y)}{d_m(x,z)}\right),$$

for every $x, y, z \in X_m, x \neq z$. Hence, each f_m is a quasisymmetry.

As was mentioned above, every scaled version of a doubling space is doubling with the same doubling constant. Applying this we find that X'_m is doubling for every m. Hence, the Theorem 2.37 *Gromov's Compactness Theorem* implies that X'_m has a convergent subsequence. This means that there exists, by definition, an ambient space (M', d'_M, b'_M) where we consider the convergence of X'_m. See Figure 4.

Continue to denote the convergent subsequence by X'_m. Thus, $\lim_m X'_m =: (X'_\infty, d'_\infty, p'_\infty)$ is a weak tangent to X' at $\lim_m f_m(p_m)$. We will show the existence of a quasisymmetric mapping $G : X_\infty \to G(X_\infty) \subset X'_\infty$. This will give us the desired result: the Theorem 2.36 *Bound for* $\dim_A$ *of Tangent* then implies $\mathrm{Cdim}_A X_\infty = \mathrm{Cdim}_A G(X_\infty) \leq \mathrm{Cdim}_A X'_\infty \leq \dim_A X'_\infty \leq \dim_A X'$. Hence, $\mathrm{Cdim}_A X_\infty$ is a lower bound for Assouad dimension of every quasisymmetric image of X implying $\mathrm{Cdim}_A X_\infty \leq \mathrm{Cdim}_A X$. Moreover, $\mathrm{Cdim}\, X_\infty \leq \mathrm{Cdim}_A X_\infty \leq \mathrm{Cdim}_A X$, as asserted.

We begin by showing the existence of a quasisymmetric limit function on a dense subset of X_∞, and then extend it to the whole space X_∞.

We do this by applying the diagonal process used in the proof of the Arzela-Ascoli Theorem. Let E be a countable dense subset of X_∞ and $x_1, x_2, ...$ an enumeration of points of E.

Suppose we have chosen an infinite set $S_{k-1} \subset \mathbb{N}$. Since $X_m \to X_\infty$, also $\mathrm{dist}(X_m, X_\infty) \to 0$ as $m \to \infty$. Thus, we can choose $x_{k,m} \in X_m$ such that $d(x_k, x_{k,m}) \leq 2\,\mathrm{dist}(X_m, X_\infty) \to 0$ as $m \to \infty$. Let $A := \{f_m(x_{k,m}) : m \in S_{k-1}\}$. Recall that $f(y_m) \in f(B(p_m, 2Rr_m))$ for all m. Then for every m,

$$\begin{aligned} d'_m(f_m(x_{k,m}), f_m(p_m)) &= \frac{d'(f(x_{k,m}), f(p_m))}{s_m} = \frac{d'(f(x_{k,m}), f(p_m))}{\mathrm{diam}\, f(B(p_m, 2Rr_m))} \\ &\leq \frac{d'(f(p_m), f(x_{k,m}))}{d'(f(p_m), f(y_m))} \leq \eta\left(\frac{d(p_m, x_{k,m})}{d(p_m, y_m)}\right) \leq C', \end{aligned}$$

where we applied the fact that $\frac{d(p_m, x_{k,m})}{d(p_m, y_m)} \leq C$. This fact holds true, since $\frac{d(p_m, x_{k,m})}{d(p_m, y_m)} \to \frac{d(p_\infty, x_k)}{d(p_\infty, y)}$ as $m \to \infty$, and convergent sequences are bounded. Thus, $\{f_m(x_{k,m})\}$ is a family of pointwise bounded functions on E independent of m. This means A is a bounded set. This in turn implies A contains a convergent subsequence. Therefore, there exists an infinite index set $S_k \subset S_{k-1}$ such that $\lim_m f_m(x_{k,m}) =$ exists when $m \to \infty$ along S_k.

Continuing this construction indefinitely, we obtain a sequence of infinite index sets $\mathbb{N} \supset S_1 \supset S_2 \supset \cdots$ such that $\lim_m f_m(x_i)$ exists for every $1 \leq i \leq k$ if $m \to \infty$ along S_k.

Now let r_k denote the kth term of S_k, where we use the usual order relation of $\mathbb{N}$, and write $S = \{r_1, r_2, ...\}$. Clearly, for every k, S contains at most $k-1$ elements of S_k. Therefore, $\lim_m f_m(x)$ exists at every point of $x \in E$ when $m \to \infty$ along S. Define $g := \lim_m f_m$.

Next, we shall verify that the limit function is a quasisymmetry. Let $x, y, z \in E$, $x \neq z$. Then there exists sequences x_m, y_m, z_m converging to x, y, z such that for each m we have $\{x_m, y_m, z_m\} \subset X_m$ and . Thus,

$$\frac{d'_\infty(g(x), g(y))}{d'_\infty(g(x), g(z))} = \frac{\lim_m d'_m(f_m(x_m), f_m(y_m))}{\lim_m d'_m(f_m(x_m), f_m(z_m))} \leq \eta\left(\frac{\lim_m d(x_m, y_m)}{\lim_m d(x_m, z_m)}\right) = \eta\left(\frac{d(x,y)}{d(x,z)}\right)$$

by continuity of the distance function and η. Since g is a uniform limit by definition, it is a homeomorphism. Therefore, it is a quasisymmetry.

To finish our construction of the desired quasisymmetry, we need to show we can extend g from the dense set E to the whole space X_∞.

By the Definition 2.34 *Tangent spaces*, X'_∞ is complete. Recall that denseness is a topological property, which implies $g(E)$ is dense in X'_∞, and hence $\overline{g(E)} = X'_\infty$. Thus, $\overline{g(E)}$ is complete, which implies g can be extended into a quasisymmetric embedding $G : X_\infty = \overline{E} \to X'_\infty$ [39, Theorem 2.25]. Thus, we have $G(X_\infty) \subset X'_\infty$.

This finishes the proof. $\square$

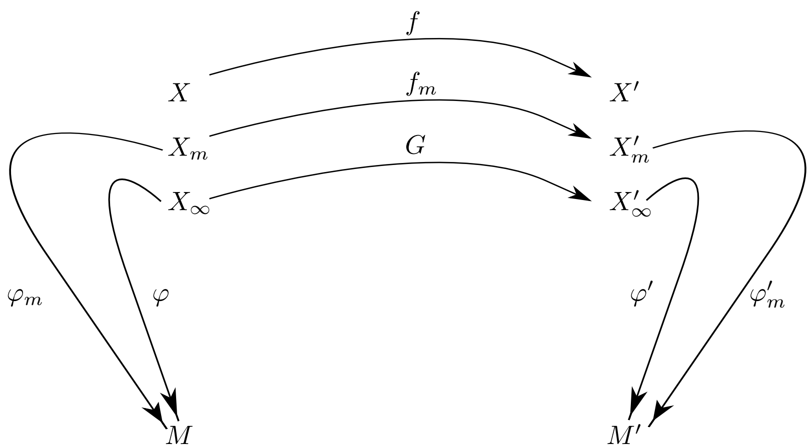

FIGURE 4. Diagram of the embedding and mapping relations. The curved arrows indicate which spaces are embedded in M and M' via the isometric embeddings $\varphi, \varphi', \varphi_m$ and φ'_m. Recall that these embeddings are provided by the Definition 2.34 *Tangent Spaces*.

Remark. Note that the embedding G is in fact a surjection, and hence $G(X_\infty) = X'_\infty$, which implies $\operatorname{Cdim}_A X_\infty = \operatorname{Cdim}_A X'_\infty$.

Finally, we have come to the main result of this section. Unfortunately, the proof is beyond the scope of this thesis, and therefore, we can do nothing but state the result and then apply it to the study of the Sierpiński carpet in the next chapter. A sketch of the proof can be found in [25, Chapter 6] and the complete proof in [18].

Theorem 2.39. (KEITH-LAAKSO). *Suppose (Z, d, μ) is a complete Ahlfors Q-regular metric measure space. Then Z is minimal for conformal Assouad dimension if and only if Z has a weak tangent space (W, d_w, μ_w) that admits a curve family Γ satisfying* $\operatorname{Mod_p} \Gamma > 0$*, for some $p \geq 1$.*

3. Sierpiński Carpet

The Sierpiński carpet was introduced in 1916 by Wacław Franciszek Sierpiński [36]. Its construction is similar to that of the Cantor set described previously in Chapter 2. Instead of $\mathbb{R}$ we are in $\mathbb{R}^2$ and we start with the unit square $[0,1]\times[0,1]$. At this point, it should be noted that there exists multiple generalizations and slight variations of these carpets, but this one is generally called *the* Sierpiński carpet. Divide the unit square into 9 squares of equal size, that is, each has side length $\frac{1}{3}$. Then remove the middle square. This is called the first iteration, E_1. Next, divide each of the 8 squares that are left into 9 equal squares each having side length $\frac{1}{3^2}$, and remove the middle one. This is the second iteration, E_2. These iterations as well as the four next ones are illustrated in Figure 5. The construction proceeds indefinitely and the Sierpiński carpet, S, is the set of points that are not removed, that is,

$$S = \bigcap_{j\in\mathbb{N}} E_j.$$

Clearly, S is bounded, and since it is an intersection of closed sets, it is closed. Thus, it is compact by Heine-Borel Theorem (see [34, Chapter 2]). This in turn implies that it is complete, when equipped with the Euclidean metric, and totally bounded. Moreover, basic properties of the Sierpiński carpet include, for instance, that its area (or its 2-dimensional Hausdorff measure) is 0, its interior is empty and its Hausdorff dimension is $\frac{\log 8}{\log 3}$. The first of these is easily seen by considering the area of the nth step in the construction and then sending n to infinity. Hence, it serves as an example of uncountable set with null measure. Likewise the emptyness of the interior is quickly concluded by considering the assumption that there exists a point in the interior. This implies the point must be inside a square, which is entirely in the carpet, but every such square contains holes. The Hausdorff dimension can be proven in many ways. For example, applying the Theorem 8.3.2 in [25, Chapter 8] we find that $\sum_i^8 \frac{1}{3^s} = 1$, where s is the Hausdorff dimension of the Sierpiński carpet. Solving for s, we find $s = \frac{\log 8}{\log 3}$. Alternatively, one can construct a measure on the carpet, or study appropriate covers using directly the definition, see [28] for details. Especially interesting to us are the local connectedness properties of S. Recall from Chapter 2 that the lower bounds for conformal dimension highly reflect the connectedness properties of the space.

In 1957, Whyburn [41] gave a topological characterization of S. To state this characterization, we first define a *carpet* to be a metric space that is homeomorphic to S. Whyburn's characterization states that any metric space is a carpet if and only if it is a subset of the extended complex plane, nondegenerate, locally connected, compact, has topological dimension equal to one and doesn't contain local cut-points. A *local cut-point* p of a space X means that given any sufficiently small neighborhood U of p the set $U\backslash\{p\}$ is not connected. See [16, Chapter 8] for more information on topological dimension.

There has been a myriad of research focusing on various topological and analytical properties of this classical fractal, of which the properties mentioned above are just a glimpse. We cannot discuss all of them, but we shall mention a couple of instances of some recent and interesting findings.

In 2011, Mario Bonk gave a uniformization result for certain Sierpiński carpet like spaces [4], which states that these spaces can be mapped with a quasisymmetry to a round Sierpiński carpet. A *round* carpet is one whose peripheral circles are round, and *peripheral circle* T of a carpet S is a Jordan curve such that the set $S\backslash T$ is connected. For more details, see [4].

More recently, Pyörälä et al. showed that the carpet is in fact tube-null, which basically means that the Sierpiński carpet can be covered with countably many infinitely long tubes, but such that

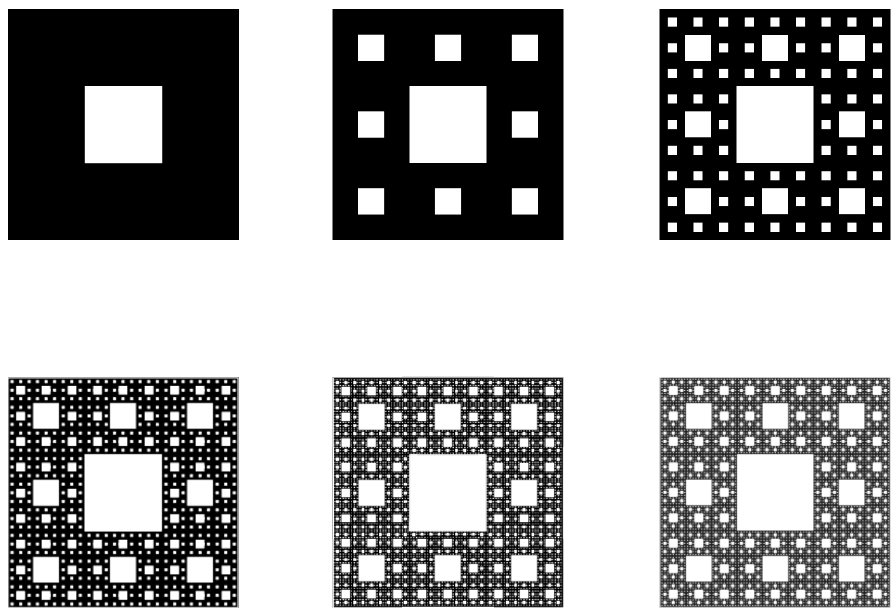

FIGURE 5. First six iterations of the construction of the Sierpiński carpet. As we continue to infinity, everything that stays black belongs to the carpet. It might seem there is nothing left as we let the construction proceed indefinitely. However, considering the Cantor comb, $[0,1] \times C \subset S$, where C is the Cantor middle-thirds set discussed in Chapter 2, we see that there exists uncountably many line segments in the carpet.

the sum of the widths of the tubes is less than any preassigned $\epsilon > 0$ [32]. This provides interesting further questions, for example, into the realm of harmonic analysis.

In addition, there has been interest on the analysis *on* the carpet. This analysis on the carpet seems to have been started from physical questions of diffusion and random walks on somewhat more complicated, but related, constructions called fractal percolations. After these questions turned out to be challenging, research has been focused on studying the Sierpiński carpet since it acts as a model fractal, in some sense (for more information on this see [1]).

Intriguingly, despite this intense research from many points of view, there is still much to accomplish until we can even wish to have a complete understanding. For example, the conformal Hausdorff dimension of the carpet is unknown. Next, we will apply some of the tools constructed in the previous chapter to obtain a lower bound for the conformal dimension of S.

First of all, as we showed above, the Hausdorff dimension of S is $\frac{\log 8}{\log 3}$. This is an upper bound for $\mathrm{Cdim}\, S$. For a lower bound, we can apply the Theorem 2.23 *Lower Bound Result III* or the Theorem 2.16 *Tyson's Theorem*. The lower bound will be immediate, as soon as we observe that S contains a Cantor's comb. To see this, note that the boundary of the unit square is never removed during the construction of the carpet. Hence, there exists a horizontal unit interval $[0,1] \subset S$. Next, if we look at the middle of S, we see that S contains a vertical middle-thirds Cantor set C. Their product, $[0,1] \times C$, is the aforementioned Cantor comb. Since the comb is minimal for Cdim and has dimension $1 + \frac{\log 2}{\log 3}$, by Theorem 2.22 *Dimension of Product Sets*, we obtain $\mathrm{Cdim}([0,1] \times C) = 1 + \frac{\log 2}{\log 3}$. By monotonicity of Cdim we further obtain $1 + \frac{\log 2}{\log 3} \leq \mathrm{Cdim}\, S$. Thus, in conclusion, we have

$$1.631... \approx 1 + \frac{\log 2}{\log 3} \leq \mathrm{Cdim}\, S \leq \frac{\log 8}{\log 3} \approx 1.892....$$

These bounds give a rough estimate of the dimension. The exact value is an open question. In Chapter 4, we sill shortly present some of the recent improvements of these bounds. What is interesting to us at this point, is the fact that $\mathrm{Cdim}\, S < \dim S$. That is, the Sierpiński carpet is not minimal for Cdim. This is obtained as a corollary to the Theorem 2.39 *Keith-Laakso*, which was discussed in the last section of Chapter 2.

First, we begin with some definitions and then proceed to prove some results, which are then combined with the Theorem 2.39 *Keith-Laakso* to show that S is not minimal for Cdim.

Definition 3.1. (λ-Similarity). Let (Z, d) be a metric space. We call a Borel mapping $f : Z \to Z$ a λ-similarity, if $d(f(x), f(y)) = \lambda d(x, y)$ for every $x, y \in Z$.

Definition 3.2. (Push-forward). If $f : (X, \mu) \to Y$, then we call the set function

$$f_{\#}\mu(A) := \mu(f^{-1}(A)),$$

for all $A \subset Y$, a *push-forward measure*. Clearly this makes the space Y into a measure space $(Y, f_{\#}\mu)$

Theorem 3.3. (Differentiation of Measures). *Let (Z, d, μ) be an Ahlfors Q-regular metric measure space. Then the Jacobian of any λ-similarity $f : Z \to Z$ is denoted by J_f and defined as*

$$J_{f(z)} = \frac{d(f_{\#}\mu)}{d\mu}(z) := \lim_{r \to 0} \frac{f_{\#}\mu(B(z,r))}{\mu(B(z,r))} := \lim_{r \to 0} \frac{\mu(f^{-1}(B(z,r))}{\mu(B(z,r))}$$
$$= \lim_{r \to 0} \frac{\mu(B(f^{-1}(z), \lambda^{-1}r))}{\mu(B(z,r))}.$$

This quantity exists and is finite μ-a.e. $z \in Z$.

Remarks. In other words, the Radon-Nikodym derivative of the push-forward measure $f_{\#}\mu(A)$ exists. This result can be proved as in [35, Chapter 7] or [27, Chapter 2], or see [12, Chapter 2] for a proof of a more general form. Note that since μ is a Borel measure and f a Borel function, $f_{\#}\mu$ is a Borel measure. If we switch the places of $df_{\#}\mu$ and $d\mu$ we obtain the definition of J_μ.

Note that the spaces Z and $f(Z)$ are essentially the same (modulo the scale), and the measures μ and $f_{\#}\mu$ are absolutely continuous with respect to each other, due to the Theorem 3.3 *Differentiation of Measures*. Thus, the Radon-Nikodym derivative of $f_{\#}\mu$ with respect to μ corresponds to a volume derivative, or the Jacobian J_f. Thus, we can apply the change of variables formula,

$$\int_Z \rho \circ f \, d\mu = \int_{f(Z)} \rho \, J_f \, d\mu = \int_{f(Z)} \rho \, df_{\#}\mu,$$

for any positive and measurable $\rho : f(Z) \to [0, \infty]$ [35, Theorem 7.26].

Theorem 3.4. (Moduli and λ-Similarities). *Suppose f is a λ-similarity defined on an Ahlfors Q-regular metric measure space (Z, d, μ), that is, $f : (Z, d, \mu) \to (Z, \lambda d, f_{\#}\mu)$. Then*

$$\frac{1}{C\lambda^Q} \leq J_f \leq \frac{C}{\lambda^Q} \tag{1}$$

for some $C > 0$. Moreover, if Γ is any curve family in Z and $p \geq 1$, then for some $C' > 0$ we have,

$$\frac{1}{C'} \lambda^{Q-p} \operatorname{Mod_p} \Gamma \leq \operatorname{Mod_p} f(\Gamma) \leq C' \lambda^{Q-p} \operatorname{Mod_p} \Gamma. \tag{2}$$

Proof. Note that Ahlfors Q-regularity of μ implies that

$$\frac{1}{Cr^Q} \leq \frac{1}{\mu(B(z,r))} \leq \frac{C}{r^Q}.$$

And further,

$$\frac{1}{Cr^Q\lambda^Q} r^Q \leq \frac{\mu(B(f^{-1}(z), \lambda^{-1}r))}{\mu(B(z,r))} \leq \frac{C}{r^Q\lambda^Q} r^Q.$$

Sending $r \to 0$, we obtain (1).

Observe that changing the roles of μ and $f_{\#}\mu$, we obtain

$$\frac{\lambda^Q}{C} \leq J_\mu \leq C\lambda^Q.$$

To prove (2), we first note that we have two spaces Z and $f(Z)$, which of the latter is just a re-scaled copy of the former. We also note that $f(\Gamma) = \{f \circ \gamma : \gamma \in \Gamma\}$.

Let $\rho \in \operatorname{Adm}(f(\Gamma))$. Since f is a λ-similarity, we have

$$\begin{aligned} 1 &\leq \int_{f\circ\gamma} \rho \, ds = \int_a^b (\rho \circ (f \circ \gamma(t))) \, d(f \circ \gamma)(t) \, dt \\ &= \int_a^b \lambda(\rho \circ (f \circ \gamma(t))) \, d\gamma(t) \, dt = \int_\gamma \lambda(\rho \circ f) \, ds \end{aligned}$$

for every $\gamma \in \Gamma$. Note that we used the simple calculation

$$d(f \circ \gamma)(t) = \lim_{h \to 0} \frac{d(f(\gamma(t+h)), f(\gamma(t)))}{h} = \lim_{h \to 0} \frac{\lambda d(\gamma(t+h), \gamma(t))}{h} = \lambda d\gamma(t),$$

which exists for a.e. point in $(a, b) \subset \mathbb{R}$. Thus, $\lambda(\rho \circ f) \in \operatorname{Adm}\Gamma$.

By applying the aforementioned change of variables formula,

$$\begin{aligned} \operatorname{Mod_p}(\Gamma) &\leq \int_Z (\lambda(\rho \circ f))^p \, d\mu = \lambda^p \int_Z (\rho \circ f)^p \, d\mu = \lambda^p \int_{f(Z)} \rho^p \, J_f \, d\mu \\ &\leq C' \lambda^{p-Q} \int_{f(Z)} \rho^p \, d\mu \end{aligned}$$

because $J_f \leq C\lambda^{-Q}$. Since this holds for every $\rho \in \operatorname{Adm}(f(\Gamma))$, we have

$$\operatorname{Mod_p}(\Gamma) \leq C' \lambda^{p-Q} \operatorname{Mod_p}(f(\Gamma)).$$

The second inequality in (2) follows similarly by observing the following facts: f^{-1} is λ^{-1}-similarity; if $\rho \in \mathrm{Adm}\Gamma$, then $\lambda^{-1}(\rho \circ f^{-1}) \in \mathrm{Adm}(f(\Gamma))$ over $f^{-1}(f(\gamma)) = \gamma \in \Gamma$; the change of variables formula works to the other direction if we use J_μ; and we have the bound $J_\mu \leq C\lambda^Q$. □

Theorem 3.5. (Triviality of $\mathrm{Mod_p}\,\Gamma_{lr}$ on S). *Suppose (S, μ) is the Sierpiński carpet equipped with the $\frac{\log 8}{\log 3}$-Ahlfors regular Hausdorff measure. Let Γ_{lr} be the curve family joining the left edge of S to the right edge of S. Assume $p \geq 1$. Then $\mathrm{Mod_p}\,\Gamma_{lr} = 0$.*

Proof. Consider the first step of the construction of S. It consists of 8 subsquares. Write $Q_{1,a}$ for the six subsquares that are adjacent to either the left or the right edge, and write $Q_{1,b}$ for the two subsquares that are adjacent to only either the top or the down edge. We normalize μ so that $\mu(S) = 1$. Thus, at the first iteration, for each of the eight subsquares $\{Q_{1,a}, Q_{1,b}\}$, we have $\mu(Q_{1,a}) = \mu(Q_{1,b}) = \frac{1}{8}$.

Next, define a Borel function $\rho_1 : S \to [0, \infty)$,

$$\rho_1(x) = \begin{cases} a & \text{if } x \in Q_{1,a} \\ b & \text{if } x \in Q_{1,b} \end{cases}.$$

This is illustrated in Figure 6.

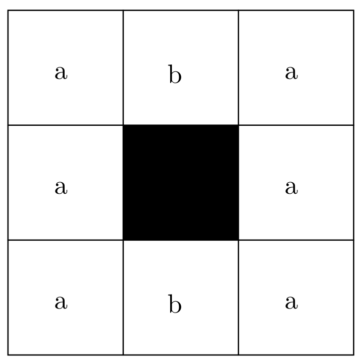

Figure 6. The Borel function ρ_1 maps every point on each of the subsquares adjacent to left or right edge to a and on the two subsquares in the middle to b.

Let $\gamma_1 \in \Gamma_{lr}$ be the straight line segment corresponding to the top edge joining the left edge to the right edge. Observe that if we choose weights a and b such that,

$$\frac{2a+b}{3} \geq 1, \tag{1}$$

then $\rho_1 \in \mathrm{Adm}\Gamma_{lr}$, since in this case for any $\gamma \in \Gamma_{lr}$

$$\int_\gamma \rho_1\, ds \geq \int_{\gamma_1} \rho_1\, ds = \frac{2a+b}{3} \geq 1,$$

Therefore, the p-modulus of Γ_{lr} is bounded above by

$$\int_S \rho_1^p\, d\mu = \frac{6}{8}a^p + \frac{2}{8}b^p = \frac{3a^p + b^p}{4}.$$

since μ is normalized as explained in the beginning of this proof.

Next, we define ρ_2 corresponding to the 2nd step of the construction of S. This time we take into account the values of ρ_1 on the squares $Q_{1,a}$ and $Q_{1,b}$ in the sense that if $\rho_1(x) = a$, then $\rho_2(x) = a^2$ or $\rho_2(x) = ab$ and if $\rho_1(x) = b$, then $\rho_2(x) = b^2$ or $\rho_2(x) = ab$ depending on the location of $x \in S$. More precisely,

$$\rho_2(x) = \begin{cases} a^2 & \text{if } x \in Q_{2,a} \cap Q_{1,a} \\ b^2 & \text{if } x \in Q_{2,b} \cap Q_{1,b} \\ ab & \text{if } x \in Q_{2,a} \cap Q_{1,b} \text{ or } x \in Q_{1,a} \cap Q_{2,b} \end{cases}.$$

For clarity, this is illustrated in the Figure 7.

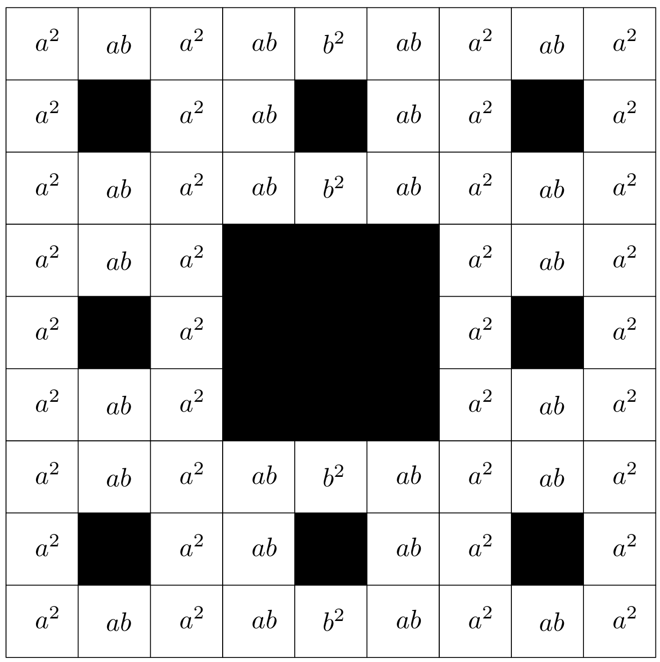

FIGURE 7. The Borel function ρ_2 is defined taking into account the values of ρ_1.

In general, we define ρ_n in a similar iterative manner. Thus, for each ρ_n, the admissibility holds, since (1) implies

$$\int_\gamma \rho_n \, ds \geq \int_{\gamma_1} \rho_n \, ds = \left(\frac{2a+b}{3}\right)^n \geq 1.$$

Similarly, we find that $\mathrm{Mod_p}\, \Gamma_{lr}$ is bounded above by

$$\int_S \rho_n^p \, d\mu = \left(\frac{3a^p + b^p}{4}\right)^n.$$

Hence, if we can show that there exists constants a, b satisfying

$$\frac{2a+b}{3} = 1 \tag{2}$$

and

$$\frac{3a^p + b^p}{4} < 1$$

for every $p \geq 1$, we will find that $\mathrm{Mod_p}\, \Gamma_{lr} = 0$ after we let $n \to \infty$.

If $p = 1$, then we can choose a and b such that $a < 1$, $b > 1$ and they satisfy (2). For example, choosing $a = \frac{1}{2}, b = 2$ yields

$$\frac{3a+b}{4} = \frac{7}{8} < 1.$$

If $p > 1$, then from [25, Page 61] we obtain solutions, by the method of Lagrange multipliers. Thus, we claim that $a = \left(\frac{8\lambda}{3p}\right)^{\frac{1}{p-1}}$ and $b = \left(\frac{4\lambda}{p}\right)^{\frac{1}{p-1}}$ where

$$\lambda = \frac{p3^p}{4(2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}})^{p-1}}.$$

work. We shall verify this, and then the proof is done.

First, we compute that

$$\frac{3a^p + b^p}{4} = \frac{3^{p+1}}{4(2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}})^{p-1}}.$$

Next we make some elementary observations familiar from calculus. Note that $2^{\frac{p}{p-1}}$ and $3^{\frac{1}{p-1}}$ are decreasing for $p > 1$ and $\lim_{p\to\infty} 2^{\frac{p}{p-1}} = 2$ and $\lim_{p\to\infty} 3^{\frac{1}{p-1}} = 1$. Thus, $2^{\frac{p}{p-1}} > 2$ and $3^{\frac{1}{p-1}} > 1$. Recall that $p < \infty$. Therefore,

$$\frac{3}{2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}}} < \frac{3}{3}.$$

Note that $\left(\frac{3}{2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}}}\right)^{p-1}$ is increasing in p, since $2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}}$ is decreasing, $p \geq 1$ and exponential of an increasing function is again increasing since $p \geq 1$. Then we compute,

$$\lim_{p\to\infty}\left(\frac{3}{2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}}}\right)^{p-1} = \frac{1}{2^{\frac{2}{3}}3^{\frac{1}{3}}}.$$

But now we have $243 < 256 \iff 3^5 < 2^8 \iff \frac{\sqrt[3]{3^5}}{\sqrt[3]{2^8}} < 1$, and hence, $\frac{\left(\frac{3}{2}\right)^2}{2^{\frac{2}{3}}3^{\frac{1}{3}}} < 1$. Therefore, we conclude,

$$\begin{aligned}\frac{3^{p+1}}{4(2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}})^{p-1}} &= \frac{3^{p+1-1+1}}{4(2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}})^{p-1}} \\ &= \left(\frac{3}{2}\right)^2 \left(\frac{3}{2^{\frac{p}{p-1}} + 3^{\frac{1}{p-1}}}\right)^{p-1} < 1.\end{aligned}$$

□

Remark. Note that the same result applies to the curve family $\Gamma_{tb} \subset S$, which consists of curves joining the top to bottom.

Next we apply these two lemmas and prove the following important result. This result in turn is used to show that the Sierpiński carpet is not minimal for conformal dimension.

Theorem 3.6. (Triviality of $\mathrm{Mod_p}\,\Gamma$ on S). *If μ is the $\frac{\log 8}{\log 3}$-Ahlfors regular Hausdorff measure on S and Γ is any curve family on S, then* $\mathrm{Mod_p}\,\Gamma = 0$.

Proof. Let $\epsilon > 0$. First we observe that the Theorem 2.13 *Properties of* $\mathrm{Mod_p}$ (III) implies that it suffices to prove this result for curve families Γ that satisfy $\inf_{\gamma\in\Gamma} \operatorname{diam}\gamma \geq \epsilon$. To see this more clearly, note that if we have proven the result for every $\epsilon > 0$, then, in particular, we have $\mathrm{Mod_p}\,\Gamma_j = 0$ for every j, where $\Gamma_j = \{\gamma \in \Gamma : \operatorname{diam}\gamma \geq \frac{1}{j}\}$. Clearly $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$, from which using the subadditivity of $\mathrm{Mod_p}$ we obtain $\mathrm{Mod_p}\,\Gamma = \mathrm{Mod_p}\left(\bigcup_j \Gamma_j\right) \leq \sum_j \mathrm{Mod_p}\,\Gamma_j = 0$.

Next, choose m such that $3^{1-m}\sqrt{2} < \epsilon$, and write $\Gamma = \bigcup_i \Gamma_i$, where $\Gamma_i := \{\gamma \in \Gamma : \gamma \cap Q_i \neq \emptyset\}$, where Q_i runs through all subsquares in the mth step of the construction of S. Again, the subadditivity of $\mathrm{Mod_p}$ implies that if $\mathrm{Mod_p}\,\Gamma_i = 0$ for every Q_i, then $\mathrm{Mod_p}\,\Gamma \leq \sum_i \mathrm{Mod_p}\,\Gamma_i = 0$. Thus, it suffices to prove the result for every Γ_i.

Since $\operatorname{diam}\gamma \geq \epsilon$, the choice of m implies that every curve in Γ_i is not entirely contained in $Q_i \cup A_{Q_i}$, where A_{Q_i} denotes all the subsquares that are adjacent to Q_i. Thus, every such curve contains a subcurve that joins the left (resp. bottom) edge of some subsquare $Q_{ij} \subset Q_i$ or the union $Q_{ij} \cup Q_{ik}$ of adjacent subsquares $Q_{ij}, Q_{ik} \subset Q_i$ to the right (resp. top) edge of Q_{ij} or $Q_{ij} \cup Q_{ik}$. But now we note that each Q_{ij} is just a scaled version of S due to the self-similarity, and the number of such subsquares Q_{ij} is countable. Hence, applying the Theorem 3.4 *Moduli and λ-Similarities* and choosing λ such that $f(Q_{ij}) = S$, we find that

$$\mathrm{Mod_p}\,\Gamma_{Q_{ij_{lr}}} \leq C'\lambda^{\frac{\log 8}{\log 3}-p}\,\mathrm{Mod_p}\, f(\Gamma_{Q_{ij_{lr}}}) = C'\lambda^{\frac{\log 8}{\log 3}-p}\,\mathrm{Mod_p}\,\Gamma_{lr} = 0$$

by the Theorem 3.5 *Triviality of* $\mathrm{Mod_p}\,\Gamma_{lr}$ *on S*, where $\Gamma_{Q_{ij_{lr}}}$ is the set of all subcurves that join the left side to the right side on Q_{ij} corresponding to curves in Γ_i, and Γ_{lr} is the set of curves that join the left side to the right side on S.

To finish our proof, we note that since every curve in Γ_i contains a subcurve in $\Gamma_{Q_{ij_{lr}}}$, the Theorem 2.13 *Properties of* $\mathrm{Mod_p}$ (V) implies $\mathrm{Mod_p}\,\Gamma_i \leq \mathrm{Mod_p}\,\Gamma_{Q_{ij_{lr}}} = 0$. ☐

Theorem 3.7. (Non-minimality of S for Cdim). *Let (S, d, μ) be a metric measure space, where S is the Sierpiński carpet, d is the two-dimensional Euclidean distance and μ is the Hausdorff Ahlfors $\frac{\log 8}{\log 3}$-regular measure. Then* $\operatorname{Cdim} S < \dim S$.

Proof. First we observe that at any given level m of the construction of S, the approximation S_m consists of 8 images of $\frac{1}{3}$-similarity mappings from S_{m-1}. Thus, if we zoom at some point on S, we will see a copy of S. Roughly speaking this is what self-similarity means.

If we re-scale the Euclidean metric on S, that is, only increase or decrease the distances between points, then the above self-similarity is still present. However, now zooming on the scaled S we obviously see scaled copies of S. Continuing the re-scaling process, the re-scaled S will always have this property.

Let $(S_\infty, d_\infty, b_\infty) =: S_\infty$ be an arbitrary weak tangent to S. By the Definition 2.34 *Tangent Spaces*, there exists a sequence $(S_n, r_n^{-1}d, b_n)$, an ambient space M in which S_n converges to S_∞ and isometric embeddings $\varphi_n : S_n \to M$ and $\varphi : S_\infty \to M$.

By the reasoning above, we conclude that given any location and scale on S_∞, it consists of scaled copies of S. Write $S_\infty = \bigcup_{\alpha\in I} S_\alpha$, where S_α are scaled copies of S. Note that the index set I is countable, since S is constructed in countably many steps.

Let Γ be any curve family on S_∞. Define Γ_α to be the *curve fragments* of curves in Γ that meet S_α, that is, $\Gamma_\alpha := \{\gamma \cap S_\alpha : \gamma \text{ goes inside } S_\alpha\}$. Now the Theorem 3.6 *Triviality of* $\operatorname{Mod_p}\Gamma$ *on* S implies that $\operatorname{Mod_p}\Gamma_\alpha = 0$ for every α. But now we have $\Gamma = \bigcup_\alpha \Gamma_\alpha$ and by subadditivity $\operatorname{Mod_p}\Gamma = 0$.

Note that S is Ahlfors $\frac{\log 8}{\log 3}$-regular and since it has the Euclidean metric it is complete. Therefore, the Theorem 2.39 *Keith-Laakso* implies that

$$\operatorname{Cdim} S \leq \operatorname{Cdim}_A S < \dim S.$$

□

4. Discussion

Recently, there has been noticeable improvements on the bounds of Cdim S. To discuss these, we need to define the notion of *Ahlfors regular conformal dimension*, ARCdim. For a given conformal gauge $\mathcal{G}$, we call ARCdim $\mathcal{G}$ the infimal Hausdorff dimension of all the Ahlfors regular spaces in $\mathcal{G}$. We have the following relationship,

$$\operatorname{Cdim} \mathcal{G} \leq \operatorname{Cdim}_A \mathcal{G} \leq \operatorname{ARCdim} \mathcal{G},$$

between these different conformal dimensions. See [25, Chapter 2] for details. Recall that the Sierpiński carpet is in fact Ahlfors $\frac{\log 8}{\log 3}$-regular.

In 2020, Kwapisz used a method called *p-resistance* to attack the problem of finding the conformal dimension of the Sierpiński carpet. This method is an alternative tool for the p-modulus in the theory of conformal dimension. It utilizes an approximation of the carpet with finite graphs. Without delving into details, we simply note that one defines the notion of *flow* from the bottom vertices to the top vertices of the finite graphs. Then, solving a certain minimization problem over all flows, one obtains the p-resistance. This is then used to define the so-called resistance dimension, which coincides with the Ahlfors regular conformal dimension. For detailed description, see the recent article by Kwapisz [22]. Using this method, he was able to compute that

$$1.765225 \leq \operatorname{ARCdim} S \leq 1.806703.$$

As we mentioned previously, there exists multiple generalizations and variations of the Sierpiński carpet. Just to mention a few of these investigations, for instance, Cheeger and Eriksson-Bique recently studied the so-called thin Loewner carpets, which are homoemorphic to the Sierpiński carpet, but with Hausdorff dimension in the range of $(1, 2)$. They were able to show that there exists carpets whose conformal dimension can be minimized. This is really interesting, since it provides concrete examples of carpets whose conformal dimension is possible to minimize [9].

Recently, Rossi and Suomala studied the aforementioned fractal percolation. Simply put, the construction resembles Sierpiński carpet, but during the construction, one flips a (weighted) coin to decide whether a square is removed or kept. Interestingly, they were able to show a similar result to the Theorem 3.7 *Non-minimality of S for* Cdim, since they proved that almost surely the fractal percolation is not minimal for conformal Hausdorff dimension [33]. On the other hand, fractal percolation is minimal for conformal Assouad dimension (see [13, Chapter 12] for details), which indicates that something interesting is happening with fractal percolations. In conclusion, almost surely, conditioned on non-extinction, we have,

$$\operatorname{Cdim} F < \dim F \leq \dim_A F = \operatorname{Cdim}_A F,$$

where F is the fractal percolation.

This thesis was only able to touch some aspects of the theory of conformal dimension, and hence our work was by no means exhaustive. For interesting aspects, such as, the classical investigations of conformal dimension of Gromov hyperbolic groups, the global quasiconformal dimension of a space or for more information on Ahlfors regular conformal dimension, we highly recommend the book by Mackay and Tyson [25] and the references therein.

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